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Vacuum energy from an extra dimension with UV/IR connection

Florian Bauer [a], Gerhart Seidl [b]

[a] *Physik-Department, Technische Universität München, James-Franck-Straße, D-85748 Garching, Germany*
[b] *Department of Physics, Oklahoma State University, Stillwater, OK 74075, USA*

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Abstract

We propose a lower limit on the size of a single discrete gravitational extra dimension in the context of an effective field theory for massive gravitons. The limit arises in this setup from the requirement that the Casimir energy density of quantum fields is in agreement with the observed dark energy density of the universe $\rho_{\text{obs}} \simeq 10^{-47}$ GeV^4. The Casimir energy densities can be exponentially suppressed to an almost arbitrarily small value by the masses of heavy bulk fields, thereby allowing a tiny size of the extra dimension. This suppression is only restricted by the strong coupling scale of the theory, which is known to be related to the compactification scale via an UV/IR connection for local gravitational theory spaces. We thus obtain a lower limit on the size of the discrete gravitational extra dimension in the range $(10^{12}\ \text{GeV})^{-1} \cdots (10^{7}\ \text{GeV})^{-1}$, while the strong coupling scale is by a factor $\sim 10^2$ larger than the compactification scale. We also comment on a possible cancelation of the gravitational contribution to the quantum effective potential.

1. Introduction

Recent observations suggest that the universe is currently in a phase of accelerated expansion [1–5], that is assumed to be driven by an energy form with negative pressure called dark energy (DE). The most famous candidate for DE is a positive cosmological constant (CC), which is equivalent to a positive vacuum energy density. Although DE represents the dominant part (about 75%) of the total energy density of the universe, the observed value of the CC is only of the order $\rho_{\text{obs}} \simeq 10^{-47}$ GeV^4, which is extremely small compared to usual particle physics scales. So far, no generally accepted solution has been given to the problem of understanding such a tiny value of the CC, which is known as the CC problem [6].

It has been emphasized, that a non-zero CC arising from the Casimir effect [7,8] in Kaluza–Klein (KK) theories [9] might be relevant for the dynamical compactification of extra dimensions [10–12]. In this scenario, the Casimir energies produced by the fluctuations of gravitational and massless matter fields

E-mail addresses: fbauer@ph.tum.de (F. Bauer), gseidl@susygut.phy.okstate.edu (G. Seidl).

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propagating in the internal space, would yield a contribution to DE which depends on the size of the extra dimensions. DE could therefore provide via the Casimir effect a probe of the geometric infrared (IR) structure of the higher-dimensional manifold. It would now be interesting to see, whether the Casimir energies contributing to DE, might also be sensitive to the ultraviolet (UV) details of the theory. In fact, distinct higher-dimensional gauge theories that reproduce similar physics in the IR, can look drastically different in the UV. This may be best appreciated by the example of dimensional deconstruction [13,14], which yields a class of manifestly gauge-invariant and renormalizable effective Lagrangians for KK modes and thus represents a possible UV completion of higher-dimensional gauge theories.[1] In this type of models, one could only observe at high energies that the physics of extra dimensions actually emerges dynamically in a purely four-dimensional (4D) setting, which denotes a radical departure from the usual treatment of higher-dimensional theories near their UV cutoff. Recently, the idea of deconstruction has also been applied to an effective field theory for massive gravitons [16–18], which is defined in a "theory space" [19] containing "sites" and "links". This allows the construction of discrete gravitational extra dimensions, that show qualitatively new properties as compared to non-gravitational theory spaces [17,18]. A major feature of discrete gravitational extra dimensions is, that they exhibit a strong coupling scale Λ in the UV, which depends via an "UV/IR connection" on the size or IR length-scale of the compactified extra dimension [17]. We will therefore have to expect that a contribution to DE arising from the Casimir effect in discrete gravitational extra dimensions could be related to the UV structure of the theory in a non-trivial way.

In this Letter, we consider a vacuum energy contribution to DE, which is generated from the Casimir effect in a single discrete gravitational extra dimension. For this purpose, we treat the gravitational theory space as a flat background for quantum fields propagating in the latticized five-dimensional (5D) bulk. In determining the Casimir energy densities of the latticized bulk fields, we assume linearized gravity and truncate the theory at the 1-loop level. Since these energy densities contribute to the CC, they have to lie below the observed value $\rho_{\rm obs} \sim 10^{-47}$ GeV^4, associated with the accelerated expansion of the universe. For massless bulk fields,[2] the 4D Casimir energy density ρ scales with the size (circumference) R of the extra dimension as $|\rho| \sim R^{-4}$, which would lead to a lower bound $R \gtrsim (10^{-3}\ \text{eV})^{-1} \sim 0.1$ mm. A much smaller size R becomes possible, if the bulk fields have non-zero masses M_X, in which case the Casimir energies are exponentially suppressed for $M_X \gg R^{-1}$. In the discrete gravitational extra dimension, this suppression is only limited by the strong coupling scale Λ of the theory, since in a sensible effective field theory, M_X should be smaller than the UV cutoff Λ. By virtue of the UV/IR connection in minimal discretizations, however, the cutoff Λ depends on R and can be much lower than the usual 4D Planck scale $M_{\rm Pl} \simeq 10^{19}$ GeV. As a consequence, we expect from the Casimir effect a smallest possible value or lower limit on the size R, when M_X can at most be as large as the strong coupling scale Λ.

The Letter is organized as follows: in Section 2, we review the model for a single discrete gravitational extra dimension and briefly discuss the strong coupling behavior as the origin of the UV/IR connection. In Section 3, we include scalar and fermionic lattice fields in the gravitational theory space. Section 4 represents the main part of this work, where we first consider the vacuum energy of quantum fields on the transverse lattice and then determine the suppression of the Casimir energy density due to large bulk masses of the latticized matter fields. Then, we employ the UV/IR-connection and the observational constraints on the DE density to derive a lower limit on the size of the extra dimension. Finally, in Section 5, we present our summary and conclusions.

2. Review of discrete gravitational extra dimensions

Recently, Arkani-Hamed and Schwartz have applied general techniques for implementing gravity in

[1] For an early application of similar ideas, see Ref. [15].

[2] A scenario for obtaining the observed CC from a 5D Casimir effect of massless bulk matter fields with a sub-mm extra dimension has been proposed, e.g., in Ref. [20]. Current Cavendish-type experiments, however, put already very stringent upper bounds of the order $R \lesssim 0.1$ mm on the possible size R of extra dimensions [21].

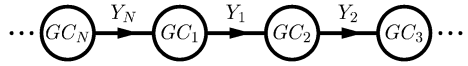

Fig. 1. Section of the gravitational theory space for a discrete fifth dimension compactified on the circle $\mathcal{S}^1$. Each site corresponds to one general coordinate invariance GC_i ($i = 1, 2, \ldots, N$), where two neighboring sites i and $i+1$ are connected by one link field Y_i and we identify $i + N = i$.

theory space [16] to a model for a single discrete gravitational extra dimension [17]. In this section, we briefly review this model for a discrete gravitational extra dimension, which describes pure gravity in the latticized bulk. In the next section, we then extend this setup to a model, that also includes matter fields.

Consider the minimal theory space for a single discrete gravitational extra dimension proposed in Ref. [17], which can be conveniently summarized by the "moose" [22] or "quiver" [23] type diagram shown in Fig. 1. Each circle or site i, where $i = 1, 2, \ldots, N$, corresponds to one general coordinate invariance (GC) symmetry GC_i and is equipped with a metric $g^i_{\mu\nu}$ for this site.[3] An arrow connecting two sites i and $i+1$ symbolizes a link field Y_i, which transforms as a vector under the two neighboring GCs. Since we suppose for the sites the identification $i + N = i$, the theory space is compactified on a circle. On each site i, we assume the usual Einstein–Hilbert action, i.e., the purely gravitational contribution from all sites to the total action is given by

$$\mathcal{S}^g_{\text{site}} = \sum_{i=1}^{N} \int \mathrm{d}^4x\, M^2 \sqrt{g^i}\, R\big(g^i\big), \tag{1}$$

where $R(g^i)$ is the Ricci scalar on the site i, while $M^2 = M_4^2/N$ and $M_4 = 1/\sqrt{16\pi G_{\text{N}}}$ with G_{N} as the 4D Newton's constant. We see in Eq. (1), that the action $\mathcal{S}^g_{\text{site}}$ is invariant under the large GC product group $\prod_{i=1}^N GC_i$. This N-fold product GC, however, is explicitly broken by the gravitational interactions $\mathcal{S}^g_{\text{link}}$ between the sites. In a minimal discretization with only nearest neighbor interactions, the action $\mathcal{S}^g_{\text{link}}$ is found to be on a Fierz–Pauli [26] form[4]

$$\mathcal{S}^g_{\text{link}} = \sum_{i=1}^{N} \int \mathrm{d}^4x \sqrt{g^i}\, M^2 m^2 \big(g^i_{\mu\nu} - g^{i+1}_{\mu\nu}\big) \times \big(g^i_{\alpha\beta} - g^{i+1}_{\alpha\beta}\big)\big(g^{i\mu\nu} g^{i\mu\nu} - g^{i\mu\alpha} g^{i\nu\beta}\big), \tag{2}$$

where the inverse mass m^{-1} of the heaviest graviton sets the lattice spacing $a = m^{-1}$, i.e., the discrete extra dimension has a size (circumference) $R = N/m$ such that the 5D Planck scale is given by $M_5 = (M_4^2/R)^{1/3}$, which defines the usual UV cutoff of the 5D theory. The product group $\prod_{i=1}^N GC_i$ is explicitly broken by the action in Eq. (2) to the diagonal GC. When we now expand in the weak field limit the metrics about flat space as $g^i_{\mu\nu} = \eta_{\mu\nu} + h^i_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski space metric, the mass-terms of the gravitons can be written as

$$\mathcal{S}^{\text{FP}}_{ij} = \int \mathrm{d}^4x\, M^2 m^2 (2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}) \times \big(h^i_{\mu\nu} h^{\mu\nu,j} - h^{\mu,i}_{\mu} h^{\nu,j}_{\nu}\big), \tag{3}$$

leading to a graviton spectrum with mass-squares

$$m_n^2 = 4m^2 \sin^2 \frac{\pi n}{N} \quad (n = 1, 2, \ldots, N). \tag{4}$$

The spectrum in Eq. (4) describes one diagonal zero-mode graviton which corresponds to the unbroken GC and a phonon-like spectrum of massive gravitons that matches in the IR, i.e., in the regime $n \ll N$, onto a linear KK tower. At this level, the phenomenology of the model appears to be very similar to that of a deconstructed gauge theory. An important qualitative difference to deconstruction, however, reveals itself in the peculiar strong coupling effects of the theory.

It has been demonstrated in Ref. [16], that the strong coupling behavior of discrete gravitational extra dimensions is most conveniently exhibited by making use of the Callan–Coleman–Wess–Zumino formalism for effective field theories [28]. Following this lead, the product symmetry group $\prod_{i=1}^N GC_i$ can be formally restored in $\mathcal{S}^g_{\text{link}}$ by appropriately adding Goldstone bosons. To this end, one expands each link field around the identity as $Y^\mu_i = x^\mu + \pi^\mu_i$, where the Goldstone bosons π^μ_i transform non-linearly under GC_i

[3] Multi-graviton theories have been considered earlier, e.g., in Ref. [24] and in connection with discretized brane-worlds in Ref. [25].

[4] The Fierz–Pauli form for graviton mass terms ensures the absence of ghosts in the spectrum. For a recent discussion of ghosts in massive gravity, see Ref. [27].

and GC_{i+1}. The Goldstone bosons, which have three polarizations, are eaten by the massless gravitons, which have two polarizations, to generate the five polarizations of the massive gravitons with spectrum as given in Eq. (4). Now, the interactions of the lowest lying scalar longitudinal component ϕ of the Goldstone bosons allow to extract directly the scale of unitarity violation in the theory. It turns out that, for the model at hand, the amplitude $\mathcal{A}(\phi\phi \to \phi\phi)$ for ϕ–ϕ scattering is of the order $\mathcal{A} \sim E^{10}/\Lambda_4^{10}$, where E is the energy of ϕ and

$$\Lambda_4 = \left(\frac{M_4}{R^3}\right)^{1/4} \tag{5}$$

is the strong coupling scale of the theory that is set by the triple vertex of ϕ. From Eq. (5), it is seen that the UV cutoff scale Λ_4 of the effective theory depends on the IR length-scale R of the compactified extra dimension. This phenomenon has been called UV/IR connection [17]. Since in a sensible effective theory for massive gravitons the lattice spacing m^{-1} must always be larger than the minimal lattice spacing defined by $a_{\min} \sim \Lambda_4^{-1}$, this implies that the theory does not possess a naive continuum limit. In other words, for given radius R, the effective theory is characterized by a highest possible number of lattice sites $N_{\max} = R\Lambda_4$, which limits how fine grained the lattice can be made.

Besides the triple derivative coupling of ϕ, the Goldstone boson action contains other types of vertices, each of which can be associated with a characteristic strong coupling scale for that interaction [18]. As two such typical examples, we will consider the scales

$$\Lambda_3 = \left(\frac{M_4}{R^2}\right)^{1/3} \quad \text{and} \quad \Lambda_5 = \left(\frac{M_4}{R^4}\right)^{1/5}, \tag{6}$$

which we will later compare with Λ_4. It is important to note that the existence of the strong coupling scales in Eqs. (5) and (6) is qualitatively different from the UV cutoff in deconstructed gauge theories. In deconstruction, the strong coupling scale associated with the non-linear sigma model approximation is always by a factor $\sim 4\pi$ larger than the mass of the heaviest gauge boson, which is of the order the inverse lattice spacing. In this sense, deconstruction may provide, unlike the effective theory of massive gravitons discussed here, an UV completion of higher-dimensional gauge theories. It should be noted, however, that the emergence of the scales in Eqs. (5) and (6) is a result of choosing a minimal discretization with nearest-neighbor couplings and may be avoided in specific types of non-local theory spaces [18].

3. Incorporation of matter

Let us now extend the model in Section 2, which has been formulated for pure gravity, by adding on each site extra scalar and fermionic site variables. To illustrate the general idea, we shall restrict ourselves here, for simplicity, to the case where we have on each site i only one scalar Φ_i and one Dirac fermion Ψ_i. We suppose that the sets of scalar and fermionic site variables $\bigcup_{i=1}^N \Phi_i$ and $\bigcup_{i=1}^N \Psi_i$, respectively describe, in the sense of usual lattice gauge theory, a scalar Φ and a fermion Ψ propagating in the discretized fifth dimension discussed in Section 2. The total action $\mathcal{S}$ of our model can therefore be split into contributions from the sites and links as

$$\mathcal{S} = \sum_{X=g,\Phi,\Psi} \left(\mathcal{S}^X_{\text{site}} + \mathcal{S}^X_{\text{link}}\right), \tag{7}$$

where we have distinguished between the purely gravitational part ($X = g$), which is given in Eqs. (1) and (2), and the sum of contributions from the scalar ($X = \Phi$) and fermion ($X = \Psi$) species. Let us first specify in Eq. (7) the interactions $\mathcal{S}^X_{\text{site}}$ on the sites. For the lattice fields Φ and Ψ we take in Eq. (7) the matter actions

$$\mathcal{S}^\Phi_{\text{site}} = \sum_{i=1}^N \int \mathrm{d}^4x \sqrt{g^i} \left(-\frac{1}{2}\right) \times \left(g^i_{\mu\nu}\partial^\mu\Phi_i\partial^\nu\Phi_i + M^2_\Phi\Phi_i\Phi_i\right), \tag{8a}$$

$$\mathcal{S}^\Psi_{\text{site}} = \sum_{i=1}^N \int \mathrm{d}^4x \sqrt{g^i} \times \left[\mathrm{i}(\bar\Psi_i\gamma^\alpha V_\alpha{}^\mu(\partial_\mu + \Gamma_\mu)\Psi_i + M_\Psi\bar\Psi_i\Psi_i\right], \tag{8b}$$

where M_Φ and M_Ψ denote the bulk masses of the 5D scalar Φ and fermion Ψ, respectively. In Eq. (8b), we have written the fermion action using the vierbein formalism (see, e.g., Ref. [29]), where γ^α ($\alpha = 0, 1, 2, 3$) are the usual Dirac gamma matrices, while $V_\alpha{}^\mu$ is the vierbein and Γ_μ is the associated spin connection. It is

obvious, that the action $\sum_X \mathcal{S}^X_{\rm site}$, summarizing the interactions on the N sites, is invariant under N copies of GC. The N-fold product of GCs $\prod_{i=1}^N GC_i$, however, is explicitly broken in Eq. (7) by each term in the sum $\sum_X \mathcal{S}^X_{\rm link}$, which contains the interactions between the fields on the different sites. On the transverse lattice, we suppose that Φ and Ψ are coupled to their nearest neighbors via

$$\mathcal{S}^{\Phi}_{\rm link} = \sum_{i=1}^{N} \int \mathrm{d}^4x \sqrt{g^i} m^2 \Phi_i (\Phi_{i+1} - \Phi_i) + \text{h.c.}, \quad (9a)$$

$$\mathcal{S}^{\Psi}_{\rm link} = \sum_{i=1}^{N} \int \mathrm{d}^4x \sqrt{g^i} m \bar{\Psi}_{iL} (\Psi_{(i+1)R} - \Psi_{iR}) + \text{h.c.}, \quad (9b)$$

where $\Psi_{iL,R} = \frac{1}{2}(1 \mp \gamma_5)\Psi_i$, with $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, are the left- and right-handed components of the Dirac spinor Ψ_i. To arrive at Eq. (9b), we started with the Wilson–Dirac action [30]

$$\mathcal{S}_{\rm W} = \sum_{i=1}^{N} \int \mathrm{d}^4x \sqrt{g^i} m \times \left(\bar{\Psi}_i \frac{r+\gamma_5}{2} \Psi_{i+1} + \bar{\Psi}_i \frac{r-\gamma_5}{2} \Psi_{i-1} - r \bar{\Psi}_i \Psi_i \right), \quad (10)$$

where r is some arbitrary parameter. The action in Eq. (10) results from adding a Wilson term (which would vanish in the continuum limit $m \to \infty$) to the naive lattice action of fermions, thereby projecting out unwanted fermion doublers. We then obtain from $\mathcal{S}_{\rm W}$ the action $\mathcal{S}^{\Psi}_{\rm link}$ in Eq. (9b) by assuming for the parameter r Wilson's choice $r = 1$ [31]. As a consequence, we arrive at a common mass spectrum for scalars and fermions, which is given by

$$m_n^2 = 4m^2 \sin^2 \frac{\pi n}{N} + M_X^2 \quad (n = 1, 2, \ldots, N), \quad (11)$$

where $X = \Phi, \Psi$. The assumption of Wilson-fermions as in Eq. (9b) with $r = 1$ ensures for $M_\Psi = M_\Phi$ identical dispersion relations for the latticized fermions and bosons. Notice also, that Eq. (11) becomes for $X = g$ identical with the graviton spectrum in Eq. (4), when setting the bulk graviton mass to zero, i.e., $M_g = 0$. In the weak field limit, we observe that for even N, the action $\mathcal{S}^{\Psi}_{\rm link}$ in Eq. (9b) is characterized by $N/2$ global Z_2 symmetries[5]

$$Z_2^{(i)}: \quad \Psi_{(i+k)L} \to -\Psi_{(i-k)L}, \quad \Psi_{(i+k)R} \leftrightarrow \Psi_{(i-k+1)R}, \quad h^{i+k}_{\mu\nu} \leftrightarrow h^{i-k}_{\mu\nu}, \quad (12)$$

where $i = 1, 2, \ldots, N/2$ is held fixed, while k runs over all the values $k = 0, \pm 1, \pm 2, \ldots, \pm N/2$. Starting with the Wilson–Dirac action $\mathcal{S}_{\rm W}$ in Eq. (10), the discrete symmetries $Z_2^{(i)}$ are only consistent with the form of the action $\mathcal{S}^{\Psi}_{\rm link}$ in Eq. (9b), which is obtained for the choice $r = 1$. We wish to point out, that the "locality" of the actions $\mathcal{S}^X_{\rm link}$ with nearest neighbor couplings might be understood in terms of scale-invariant renormalization group transformations acting in theory space [33].

4. Casimir energies

In this section, we investigate the Casimir energies of matter fields propagating in the discrete extra dimension introduced in Sections 2 and 3. For a continuous 5D space–time manifold, the Casimir energy densities of free massless scalars and fermions have been computed in Ref. [11], whereas the Casimir contribution of a massless graviton in the same background, using the standard effective action theory, can be found in Ref. [10]. In our model with a discrete fifth dimension, one can summarize in the 4D low-energy theory the vacuum energy contributions of the massive modes to the 1-loop effective potential as

$$V_{\rm eff} = (s - 4f + 5g) \sum_{n=1}^{N} V_0(m_n), \quad (13)$$

where s, f, and g, respectively, denote the number of real scalar, fermionic, and gravitational fields propagating in the latticized bulk. In Eq. (13), we have summed for each latticized field over the vacuum energy densities $V_0(m_n)$ of all the modes with masses m_n belonging to the phonon-like spectrum in Eq. (11). Notice in Eq. (13), that the factors -4 and 5 reflect the spin-degrees of freedom that contribute to each massive fermion and graviton loop. In continuum KK

[5] Discrete non-Abelian flavor symmetries from deconstruction have recently been analyzed in Ref. [32].

theories, a gauge-independent gravitational quantum-effective action can be consistently formulated by employing the Vilkovisky–De Witt effective action [34], for which, however, only a few explicit examples in special topologies are known [35]. For our model with a discrete extra dimension, the contribution $V_0(m_n)$ to the effective potential from a single real scalar degree of freedom with $M_X = 0$ has been calculated in Refs. [36,37], where

$$V_0(m_n) = \frac{m_n^4}{64\pi^2}\left(\ln\frac{m_n^2}{\mu^2} - \frac{3}{2}\right)$$

has been obtained by a zeta-function regularization technique [38]. In our theory space, the purely gravitational contribution to the effective potential which includes only the tower of massive gravitons [i.e., $s = f = 0$ and $g = 1$ in Eq. (13)], for example, was then found to be

$$V_{\text{eff}}|_{s,f=0} = \frac{15Nm^4}{32\pi^2}\left(\ln\frac{4m^2}{\mu^2} - \frac{3}{2}\right) + \frac{5m^4}{2\pi^2}\sum_{n=1}^{N-1}\sin^4\left(\frac{\pi n}{N}\right)\ln\sin\left(\frac{\pi n}{N}\right), \tag{14}$$

where, from Eq. (4), $m_n^2 = 4m^2\sin^2\frac{\pi n}{N}$. For a related discussion in a supersymmetric context see also Refs. [39,40]. Note that Eq. (14) contains also terms that are not due to the Casimir effect or terms that depend on an arbitrary renormalization scale μ originating from the regularization process.[6] Since we wish to consider only the 4D Casimir energy density, we will, in the following, eliminate the unwanted parts in the effective potential. This can be realized by subtracting off the vacuum energy density that corresponds to an uncompactified (unbounded) extra dimension as explained in Ref. [41]. As a nice advantage of this renormalization procedure we obtain that the transverse lattice result converges in the limit $N \to \infty$ *exactly* to the value expected from the continuum theory.

If the bulk masses M_X of the fields in Eq. (13) are all set to zero, the resulting 4D Casimir energy density of each latticized bulk field would be of the order $\sim R^{-4}$. As already mentioned in the introduction, this would lead to the bound $R \gtrsim 0.1$ mm. Let us therefore now consider latticized matter fields with non-vanishing bulk masses $M_X \neq 0$. In the extra dimension, the boundary conditions for the quantum fields can be periodic or anti-periodic, and the corresponding fields are called untwisted and twisted, respectively. The Casimir energy densities of these field configurations differ by a factor of order one and have opposite sign. Following Ref. [41], the 4D Casimir energy density of a single untwisted real scalar field in the latticized fifth dimension can be written as

$$\rho_{\text{untwisted}} = \frac{1}{2(2\pi)^3} \times \frac{4\pi}{8}\left[\sum_{n=1}^{N} m_n^4 \ln m_n - N\int_0^1 \mathrm{d}s\, m_s^4 \ln m_s\right], \tag{15}$$

where, from Eq. (11), $m_n^2 = 4m^2\sin^2(\pi n/N) + M_X^2$ and s is treated in the integral as a continuous parameter which replaces n/N in the sine function. As long as the number of lattice sites is $N \gtrsim \mathcal{O}(10)$, the Casimir energy density on the transverse lattice in Eq. (15) differs less than $\lesssim 1\%$ from the value in the naive continuum limit $N \to \infty$. In the remainder of this section, we will therefore employ the expressions for the Casimir energy densities of quantum fields in the continuum theory. In this approximation, the vacuum energy density of a real (un)twisted scalar field reads [41]

$$\rho_{\text{(un)twisted}} = \frac{\pm 1}{8(2\pi)^2}\frac{(2\pi)^5}{R^4}\int_x^\infty \mathrm{d}n \frac{(n^2 - x^2)^2}{\exp(2\pi n) \pm 1}, \tag{16}$$

where the "+" and "−" signs belong to twisted and untwisted fields, respectively, and $x = M_X R/(2\pi)$, in which M_X denotes the bulk mass of the scalar field. The integral in Eq. (16) can be performed exactly after neglecting the term ± 1 in the denominator, i.e., both densities differ only in an overall sign:

$$\rho_{\text{(un)twisted}} = \pm\frac{(M_X R)^2 + 3M_X R + 3}{(2\pi)^2 R^4} e^{-M_X R}. \tag{17}$$

When taking the sum of contributions for twisted and untwisted fields, the integrals must be added before

[6] The dependence on the renormalization scale μ leads, in a cosmological setup, to a running CC. Some recent work on such renormalization group motivated DE models and their cosmological implications can be found in Ref. [42] and references therein.

carrying out the approximation, which gives

$$\rho_{\text{sum}} = -\frac{4(M_X R)^2 + 6M_X R + 3}{16(2\pi)^2 R^4} e^{-2M_X R}. \tag{18}$$

The corresponding energy densities of Dirac fermions are obtained by simply multiplying the scalar densities $\rho_{\text{(un)twisted}}$ by -4. Note that the applied approximation works fine even in the limit of vanishing bulk masses $M_X \to 0$. The basic feature expressed in Eqs. (17) and (18) is that for large bulk masses $M_X \gg R^{-1}$, the energy density of massive matter fields becomes exponentially suppressed, which could compensate for the possibly large factor $\sim R^{-4}$, even when R is comparatively small.

Now, we are in a position to calculate the Casimir energy densities with the bulk masses M_X set equal to the strong coupling scales Λ_3, Λ_4, and Λ_5 given in Eqs. (5) and (6). The effective field theory description suggests that these are the largest possible values that M_X can take in the gravitational theory space. If the UV cutoff Λ is much larger than $\sim R^{-1}$, the expressions in Eqs. (17) and (18) are dominated by the exponential damping factors, such that the Casimir energy densities are most strongly suppressed when M_X becomes of the order the strong coupling scale Λ, with $\Lambda = \Lambda_3, \Lambda_4, \Lambda_5$. Moreover, this suppression is most effective, when the number of lattice sites N is maximized by choosing the inverse lattice spacing $m = N/R$ to be also of the order Λ. The lower limit R_{min} on the size R of the extra dimension emerges from requiring that the Casimir energy densities remain below the observed value $\rho_{\text{obs}} \simeq 10^{-47}$ GeV^4 of the DE density. The results for an untwisted scalar field and the sum of twisted and untwisted fields are plotted in Fig. 2. Since the smallest value R_{min} that R can take is, due to the UV/IR connection, a function of Λ, we have considered $R_{\text{min}}(\Lambda)$ for all three scales $\Lambda = \Lambda_3, \Lambda_4, \Lambda_5$. These values together with the corresponding maximum number of lattice sites $N = R_{\text{min}} \cdot \Lambda(R_{\text{min}})$, where $\Lambda(R_{\text{min}})$ is the strong coupling scale associated with R_{min}, are summarized in Table 1. Note that we can apply here the relations from the continuum theory, since (i) the number of lattice sites N is of the order $\sim 10^2$ and (ii) the lattice calculation leads to energy densities (drawn in Fig. 2 as circles), that agree very well with the values

Table 1
Lower bound R_{min} on the size R of the extra dimension for an untwisted real scalar field and the sum of a twisted and an untwisted scalar. Additionally, the values of the strong coupling scale Λ and the number of lattice sites N are given when R is equal to R_{min}. For the scale Λ, we considered each of the three choices $\Lambda = \Lambda_3, \Lambda_4, \Lambda_5$ from Eqs. (5) and (6). The lower bound R_{min} emerges from the requirement that the absolute Casimir energy density lies below the observed value ρ_{obs} of the DE density, when the bulk field mass M_X in Eq. (11) takes the largest possible value $M_X \simeq \Lambda$

Untwisted	R_{min} [GeV^{-1}]	$\Lambda(R_{\text{min}})$ [GeV]	$N = R_{\text{min}} \cdot \Lambda(R_{\text{min}})$
Λ_3	6.1×10^{-12}	3.6×10^{13}	219
Λ_4	9.0×10^{-10}	2.2×10^{11}	198
Λ_5	1.1×10^{-7}	1.7×10^{9}	179
Sum	R_{min} [GeV^{-1}]	$\Lambda(R_{\text{min}})$ [GeV]	$N = R_{\text{min}} \cdot \Lambda(R_{\text{min}})$
Λ_3	8.2×10^{-13}	1.4×10^{14}	112
Λ_4	6.6×10^{-11}	1.6×10^{12}	103
Λ_5	4.4×10^{-9}	2.1×10^{10}	95

in the continuum theory.[7] For a mix of a twisted and an untwisted field, we observe that the Casimir energy density of massive bulk fields exhibits a stronger suppression due to the different signs of both components. From Fig. 2, we read off that the minimal radius R_{min} of the discrete gravitational extra dimension lies in the range

$$\left(10^{12}\ \text{GeV}\right)^{-1} \lesssim R_{\text{min}} \lesssim \left(10^{7}\ \text{GeV}\right)^{-1}, \tag{19}$$

where, typically, $\Lambda(R_{\text{min}}) \sim 10^2 \times R_{\text{min}}^{-1}$. For a radius R which is much smaller than the range given in Eq. (19), the Casimir energy densities of the bulk matter fields would significantly exceed ρ_{obs} and thus run into conflict with observation. Of course, there may be other possible sources of DE which might be responsible for the accelerated expansion of the universe, but it seems unlikely that they could exactly cancel the potentially large contributions from the Casimir effect in extra dimensions.

Let us now briefly comment on the gravitational contribution to the 1-loop quantum effective action V_{eff} in Eq. (13). For zero bulk mass $M_g = 0$, the gravi-

[7] For R_{min}, the values of the continuum and lattice formulas differ by about 15%, which is negligible, since the strong coupling scales $\Lambda_{3,4,5}$ are order of magnitude estimates. For instance, the lattice calculation for an untwisted scalar field and $\Lambda = \Lambda_3$ gives $R_{\text{min}} = 6.8 \times 10^{-12}$ GeV^{-1}, whereas the continuum approximation yields $R_{\text{min}} = 6.1 \times 10^{-12}$ GeV^{-1}.

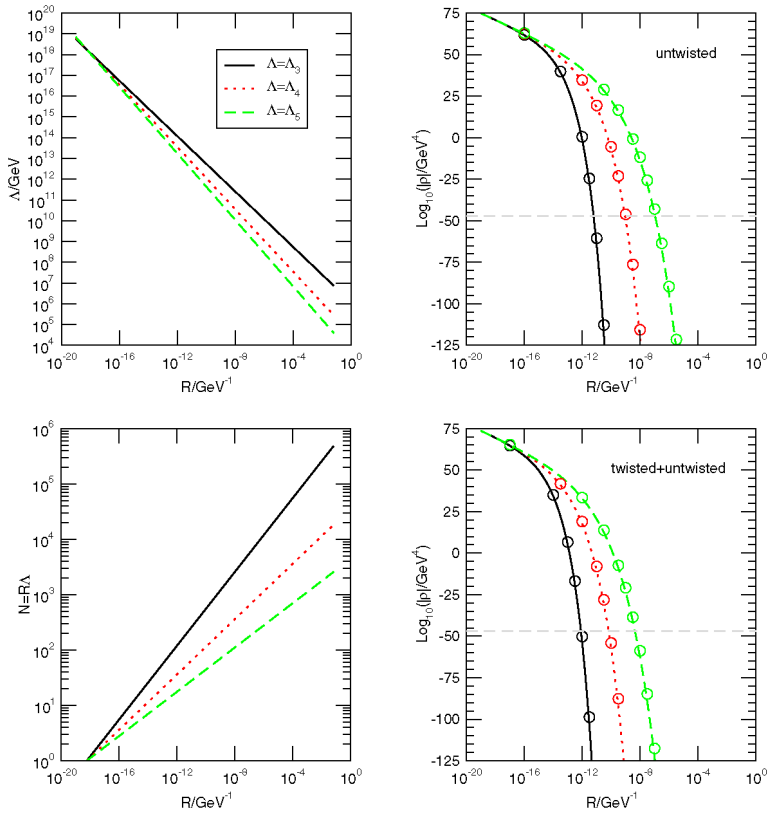

Fig. 2. For the three choices $\Lambda = \Lambda_3, \Lambda_4, \Lambda_5$ of the strong coupling scale Λ from Eqs. (5) and (6), we plotted the values of Λ, the Casimir energy densities ρ, and the corresponding number $N = R\Lambda$ of lattice sites as functions of the size R of the fifth dimension. The energy densities ρ are given for the untwisted scalar field [cf. Eq. (17)] and the sum of one untwisted and one twisted scalar field [cf. Eq. (18)]. Note, that ρ is negative in both cases, and the bulk masses of the fields have their maximal values, given by Λ, according to Section 4. In the plots of ρ, the horizontal dashed line marks the observed value $\rho_{\text{obs}} \sim 10^{-47}$ GeV^4 of the DE density and the circles represent exact lattice values from Eq. (15).

tational effective potential given in Eq. (14) would lead to a contribution to V_{eff} of the order $\sim m^4$. The gravitational vacuum energy, however, can be canceled in our model at the linear level, when we assume the presence of a suitable number of latticized matter fields with actions as given in Eqs. (8) and (9), which have vanishing bulk masses $M_X = 0$. For instance, choosing $b = 3$ massless scalars and $f = 2$ massless fermions, we find from Eq. (13) that in this case $V_{\text{eff}} = 0$, which holds in linearized gravity at the 1-loop level for an arbitrary number N of lattice sites. In this approximation, the cancelation of bosonic and fermionic degrees of freedom would actually be approached in the limit $N \to \infty$ for any value of the parameter r in the Wilson–Dirac action in Eq. (10). The requirement that this cancelation holds for arbitrary, i.e., also for small N, however, uniquely singles out Wilsons's choice $r = 1$. It is interesting to consider a possible origin of free massless scalars in effective field theories for KK modes. In a $(4 + d)$-dimensional KK theory with $d = 4$ compactified extra dimensions, e.g., we would have in the 4D low-energy theory one tower of massive spin-2 states, three towers of massive spin-1 states and six towers of massive spin-0 states with degenerate masses (see, e.g., Ref. [43]). The effective potential of these fields could, in a similar way as mentioned above, be canceled at the 1-loop level by adding only free Dirac fermions with zero bulk masses. Notice that, since the massless fields couple only gravitationally to the visible sector, a sufficiently low temperature of the massless states would allow to retain the predictions of standard big-bang nucleosyn-

thesis [44]. Finally, we note that the cancelation of vacuum energies in a supersymmetric multi-graviton theory on space–times with non-trivial topology was also considered very recently in Ref. [40], where bulk masses and different boundary conditions were taken into account.

5. Summary and conclusions

In this Letter, we have analyzed the Casimir effect of matter fields in the background of an effective 5D space–time. The underlying model of a discrete gravitational extra dimension exhibits a strong coupling behavior at an energy scale Λ, which depends via an UV/IR connection non-trivially on the size R of the extra dimension. For a small compactified extra dimension, massless quantum fields usually lead, due to the Casimir effect, to large vacuum energy contributions, which are in stark contrast to current observations. To circumvent this problem, we have assumed for the matter fields large bulk masses M_X to suppress the Casimir energy density exponentially, even for a tiny extra dimension. However, the strong coupling scale sets an upper bound on the values of the bulk masses $M_X \lesssim \Lambda$, and therefore limits the suppression effect. This yields a lower bound on the size of the fifth dimension, when the bulk masses take the maximal possible value $M_X \simeq \Lambda$. Here, we found that the minimal size $R_{\min}$ of the extra dimension lies in the range $R_{\min} \sim (10^{12}\ \text{GeV})^{-1} \cdots (10^{7}\ \text{GeV})^{-1}$ and that the corresponding maximum number of lattice sites is of the order $\sim 10^2$. Furthermore, we discussed the possibility of canceling the contribution of massless bulk fields to the quantum effective potential. Generally, it would be interesting to explore a possible relation of our model to holography, as suggested by the UV/IR connection [45], and analyze also supersymmetric realizations [39,40], e.g., in the framework of sequestered sector models of anomaly mediation [46].

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