Mean-VAR model with stochastic volatility

Hanen Ould Ali\textsuperscript{a}\textsuperscript{*}, Faouzi Jilani\textsuperscript{b}

\textsuperscript{a,b} Faculty of Economics and Management of Tunis B.P 248 El Manar II 2092 Tunis, Tunisia

Abstract

The objective of this article is the research of optimal portfolio strategy under a probability constraint of type Value-at-Risk in setting of stochastic volatility model. Our decision problem is the optimal combination of risky asset and certain asset by maximizing the utility function under the VaR constraint that is limited by a loss proportional to current return. The empirical results show that the VaR constraint decreases the amount invested in risky asset gradually over time. We also note that volatility has a significant impact on the optimal solution. Furthermore, this approach is more appropriate when the investor sets a confidence level high enough and low holding period. In addition, this model allows us to better understand the form of the distribution and offers the advantage to the investor to take account of asymmetric and leptokurtic distributions of stock market returns.

Keywords: dynamic programming, leptokurtic, optimal portfolio, and stochastic volatility;

1. Introduction

Several studies have developed the Mean-Value-at-Risk model. It is, in particular, about Kast and al. (1999), Campbell and al. (2001) and Alexander and Baptista (2002) among others, who are interested in maximizing the expected utility under a VaR constraint in a static setting. However, Basak and Shapiro (2001) and KFC Yiu (2004) developed the study of the model to a dynamic setting. Thus, for Basak and Shapiro (2001), the VaR remains limited as a measure of risk, because it does not contributes to information on the level of losses when they really occur. KFC Yiu (2004) has reinforced this last by proving that the results of Basak and Shapiro (2001) are subject to assumptions that are inappropriate with regard to the experience and methods that determine the portfolio management. The author presents a setting of resolution described as coherent, in which the risk is estimated dynamically, but his model is questionable since the volatility is assumed constant. So, both models Mean-Risk which experienced most interest, within both the professional and academic framework are the MV model and the MVaR model. For these two models, the risk is defined by the variance or volatility. Therefore, the empirical or economic result of the asset allocation lies on the quality of the estimate of volatility. Many efforts have been made to develop financial models managing to take into account the statistical properties of stock market fluctuations. The financial literature has mainly concentrated on two approaches: the autoregressive models ARCH (Auto Regressive Conditional Heteroscedasticity) introduced by Engel (1982) and the non-deterministic volatility model, which is based on the work of Bachelier (1900). However, the objection could be formulated against ARCH models is her
deterministic setting, while the other approach leads to the volatility of a stochastic variable. Indeed, it is commonly accepted that volatility is not determined for the returns process and thus for the calculation of VaR. Our objective is to propose a general method to solve a stochastic optimization problem under probability constraint of type dynamic VaR in an incomplete market. This article is organized as follows: Section 2 is devoted to the formalization of the model. Section 3 determines the optimal strategy. Finally, a conclusion ends the article.

2. Formalization of the model

The model presented in this paper is an extension of the model of KFC Yiu (2004), this model has the distinction of introducing the dynamic VaR in the choice of the managers of asset allocation managers. We consider the situation of an investor seeking to determine the optimal share between a risk-free asset B at the deterministic rate of interest r, and an investment in a risky asset S whose price follows an Ito process, a management period equated to the interval [0, T]. In setting of risk management, volatility is an explicit measure of market fluctuations. In this work, we borrow the model of Harvey and al. (1994), where volatility \( \sigma(t) \) is governed by Ornstein-Uhlenbeck process in its logarithmic version (log-OU), This type of specification is characterized by volatility that moves around a reference level, noted \( \vartheta \) and the coefficient \( \kappa \) indicates the speed of recall. \( \gamma \) is the volatility of volatility and the Brownian movements \( dW_1(t) \) and \( dW_2(t) \) are potentially correlated, one has \( E(dW_1(t) dW_2(t)) = \rho \). The model without VaR constraint is:

\[
\max_{\omega(t)} E \left[ \int_0^T u(t,X(t)) \, dt \right]
\]

s.c \( dX(t) = [\omega(t) \mu - r + X(t)] \, dt + \sigma(t) \, dW_1(t) \) \hspace{5cm} (1)

\[
d \ln \sigma(t)^2 = \kappa [\vartheta - \ln \sigma(t)^2] \, dt + \gamma \, dW_2(t)
\]

With \( \mu \) is constant, \( X(t) \) is the wealth at time \( t \) and \( \omega(t) \) is the share of wealth invested in the risky asset.

Our objective is not only the search of an optimal strategy of the investment but also for a better control of risk which will be translated by a constraint where the Value-at-Risk is limited by \( \text{VaR}(t) \), determined by: \( \text{VaR}(t) = \beta X(t) \), avec \( \beta \in [0,1] \). To formulate the VaR constraint, we follow the approach of KFC Yiu (2004), by writing equation (1) as an Ornstein-Uhlenbeck process, we obtain:

\[
\omega_1 \sigma(t)^2 \omega(t) + a_2 \omega(t) \leq \beta X(t)
\]

Where \( a_1 = - w_{\sigma} (e^{2rA} - 1)/(2r)^{1/2} \) ; \( a_2 = - \left( (\mu/r) - 1 \right) (e^{rA} - 1) \) and \( w_{\sigma} \) is the Cornish-Fisher expansion, taking into account the third and fourth moment of a distribution. The research of the optimal portfolio with a better control of risk is thus reduced to the resolution of the following stochastic control problem:

\[
\max_{\omega(t)} E \left[ \int_0^T u(t,X(t)) \, dt \right]
\]

s.c \( dX(t) = [\omega(t) \mu - r + X(t)] \, dt + \sigma(t) \, dW_1(t) \) \hspace{5cm} (2)

\[
d \ln \sigma(t)^2 = \kappa [\vartheta - \ln \sigma(t)^2] \, dt + \gamma \, dW_2(t)
\]

\[
(A1) \quad \sigma(t)^2 + a_2 \omega(t) \leq \beta X(t)
\]

The resolution of the optimal control problem passes by the application of the principle of the dynamic programming and leads under certain assumptions to the solution of a differential equation called Hamilton-Jacobi-Bellman. The equality between the solution of the optimal problem and the solution of the differential equation requires assumptions of sufficient regularity of the value function, which will be the subject of the next section.
3. OPTIMAL STRATEGIES

The state equations are (3) and (4), the criterion is (2) and the indirect utility function is:

\[
v(t, x, \ln(\sigma^2)) = \frac{\max}{\omega(t)} E \left[ \int_t^T u(s, x(s, \sigma^2)) \, ds \right]
\]

It is the utility that we hope to get if one holds the wealth \( x \) at this moment, the Hamilton-Jacobi-Bellman writes:

\[
\max_{\omega(t)} \left[ u(t, x(t, \sigma^2)) + V_x \left[ \omega(t, x, \sigma^2) (\mu - r) + r x \right] + V_{\ln(\sigma^2)}^2 \left[ \vartheta - \ln(\sigma(t)^2) \right] + \frac{1}{2} V_{xx}^2 \sigma^2 \omega(t, x, \sigma^2) + \frac{1}{2} \sigma^2 \omega(t, x, \sigma^2) + \frac{1}{2} \sigma^2 \omega(t, x, \sigma^2)^2 \right] = 0
\]

With the terminal condition: \( v(T, x, \ln(\sigma^2)) = 0 \)

The problem is constrained, then write the Lagrangian and the necessary conditions:

\[
\mathcal{L}(\omega(t, x, \sigma^2), \lambda(t, x, \sigma^2)) = u(t, x(t, \sigma^2)) + V_x \left[ \omega(t, x, \sigma^2) (\mu - r) + r x \right] + V_{\ln(\sigma^2)}^2 \left[ \vartheta - \ln(\sigma(t)^2) \right] + \frac{1}{2} V_{xx}^2 \sigma^2 \omega(t, x, \sigma^2) + \frac{1}{2} \sigma^2 \omega(t, x, \sigma^2) + \frac{1}{2} \sigma^2 \omega(t, x, \sigma^2)^2 \right] + \lambda(t, x, \sigma^2) (\beta x - (a_1 \sigma + a_2) \omega(t, x, \sigma^2)) = 0
\]

\[
\lambda(t, x, \sigma^2) \leq 0
\]

The condition (7) is the usual condition encountered in the optimal control problems, except it is applied to the Lagrangian and not to the Hamiltonian. The condition (8) is the relationship of exclusion from Kuhn-Tucker theorem. As for the condition (9), it accompanies any problem of maximization. And as it is about a problem of maximization under constraint, it is necessary that \( \mathcal{L}_{\omega} \leq 0 \). What will happen when \( V_{xx} \leq 0 \) and \( u(t, x(t)) \) is concave. Using the first order conditions of equation (6) gives us:

\[
\omega^*(t, x, \sigma^2) = \frac{-\left( \mu - r \right) V_x - V_{\ln(\sigma^2)} \left[ \vartheta - \lambda(t, x, \sigma^2) \right] (a_1 \sigma + a_2)}{V_{xx}^2} \]

To solve the problem completely it is necessary to bring back \( \omega^*(t, x, \sigma^2) \) in the equation (6), we obtain the partial differential equation PDE:

\[
u(t, x(t, \sigma^2)) + V_x \left[ \omega^*(t, x, \sigma^2) (\mu - r) + r x \right] + V_{\ln(\sigma^2)}^2 \left[ \vartheta - \ln(\sigma(t)^2) \right] + \frac{1}{2} V_{xx}^2 \sigma^2 \omega^*(t, x, \sigma^2) + \frac{1}{2} \sigma^2 \omega^*(t, x, \sigma^2)^2 \right] + \frac{1}{2} \sigma^2 \omega^*(t, x, \sigma^2) + \frac{1}{2} \sigma^2 \omega^*(t, x, \sigma^2)^2 \right] + \lambda(t, x, \sigma^2) (\beta x - (a_1 \sigma + a_2) \omega(t, x, \sigma^2)) = 0
\]

This equation allows finding the optimal Bellman function \( v^*(t, x, \ln(\sigma^2)) \), taking into account of the second order condition.

By reason of the non-linearity in \( \omega^*(t, x, \sigma^2) \), the first order conditions and the equation Hamilton-Jacobi-Bellman are a non-linear system therefore the numerical methods are required to solve \( \omega^*(t, x, \sigma^2), \lambda^*(t, x, \sigma^2) \) and \( v^*(t, x, \ln(\sigma^2)) \) iteratively. The expression \( \omega^* \) then gives the choice of the investor which is obtained explicitly by specifying the utility function of the economic agent.

It is assumed that the attitude of the investor toward a risk is represented by his following utility function:

\[
u(t, x(t, \sigma^2)) = \ln x
\]
This particular form of utility function makes the \( dI(t,x)/dx < 0 \), that is to say, the more rich the agent is important, the more his risk aversion is low. An infinitely rich agent is thus neutral-to-risk. So searching the solution of the Bellman’s PDE in the following form: \( v(t, x, \ln \sigma^2) = f(t, \ln \sigma^2) \ln x \)

With \( f(t, \ln \sigma^2) = H(t)(\ln \sigma^2) \)

The partial differential equation is written: \( \ln x + v_x [\omega(t, x, \sigma^2) (\mu - r) + r x] + v_{\ln \sigma^2} \kappa[\theta - \ln \sigma(t)^2] + 1/2 \sigma_{xx} \gamma \omega^*(t, x, \sigma^2) + v_{\ln \sigma^2} \gamma \omega^*(t, x, \sigma^2) \rho + v_t = 0 \)

Neglecting the derivative of \( H(t) \) with respect to \( x \), we obtain:

\[
v_x = H(t)(\ln \sigma^2) x^{-1}; \quad v_{xx} = (-1) H(t)(\ln \sigma^2) x^{-2}; \quad v_{\ln \sigma^2} = H(t) \ln x
\]

Substituting in equation (12), we obtain:

\[
H(t)(\ln \sigma^2) x^{-2} \sigma^2 \omega^*(t, x, \sigma^2) + H(t) x^{-1} \sigma \gamma \omega^*(t, x, \sigma^2) \rho + H'(t)(\ln \sigma^2) \ln x = 0
\]

With the terminal condition: \( H(T) = 0 \)

Multiplying by \( [(\ln \sigma^2)(\ln x)]^{-1} \) we obtain: \( H'(t) + H(t) [(\ln x)^{-1} x^{-1} \omega^*(\mu - r) + r x] + (\ln \sigma^2)^{-1} \kappa[\theta - \ln \sigma(t)^2] + 1/2 (-1) (\ln x)^{-1} x^{-2} \sigma^2 \omega^* + (\ln \sigma^2)^{-1} (\ln x)^{-1} \sigma \gamma \omega^* \rho + (\ln \sigma^2)^{-1} = 0 \)

In the case \( \lambda = 0 \) the optimal solution is: \( \omega^*(t, x, \sigma^2) = (\mu - r)x \sigma^2 + x \gamma \rho \sigma \ln \sigma^2 \)

Substituting this solution into equation (13) we find:

\[
H'(t) + H(t) [(\ln x)^{-1} x^{-1} \left( \frac{(\mu - r)}{\sigma^2} + \frac{\gamma \rho}{\sigma \ln \sigma^2} \right) (\mu - r) + r] + (\ln \sigma^2)^{-1} \kappa[\theta - \ln \sigma^2] \quad i/2 (\ln x)^{-1} \sigma^2 \left( \frac{(\mu - r)}{\sigma^2} + \frac{\gamma \rho}{\sigma \ln \sigma^2} \right)^2 + (\ln \sigma^2)^{-1} (\ln x)^{-1} \sigma \gamma \left( \frac{(\mu - r)}{\sigma^2} + \frac{\gamma \rho}{\sigma \ln \sigma^2} \right) \rho + (\ln \sigma^2)^{-1} = 0
\]

In the unconstrained case the Hamilton Jacobi Bellman equation becomes an ordinary differential equation.

This unconstrained solution will be used as an initial hypothesis to the iterative algorithm proposed by KFC Yiu (2004). Dividing the computational domain into a grid of \( N_t \times N_x \times N_\sigma \) and omitting \( (t, x, \sigma^2) \) in all variables for the simplicity of notation, the final algorithm can be summarized as follows:

1. \( \lambda_0 = 0 \); \( \omega_0 = \left( \frac{(\mu - r)x}{\sigma^2} + \frac{x \gamma \rho}{\sigma \ln \sigma^2} \right) \); \( H_0(t) \) from equation (49) Set \( k = 0 \)

2. For \( t = [(N_t - 1)\Delta t, (N_t - 2)\Delta t, \ldots, 0] \); \( x = [\Delta x, 2 \Delta x, \ldots, N_x \Delta x] \) and \( \sigma = [\Delta \sigma, 2 \Delta \sigma, \ldots, N_\sigma \Delta \sigma] \). Calculate \( \lambda_{k+1} \) and \( \omega_{k+1} \) from

\[
\lambda_{k+1} = \beta x - (a_1 \sigma + a_2) \ln x;
\]

\[
\omega_{k+1} = \left( \frac{(\mu - r)H_k(t)(\ln \sigma^2)^{-1}}{\sigma^2 \left( -1 \right) H_k(t)(\ln \sigma^2)^{-1} x^{-2}} \right) - \left( H_k(t)(\ln x)^{-1} \sigma \gamma \rho - \lambda_{k+1} (a_1 \sigma + a_2) \right)
\]
For \( n = \left[ N, -1, N - 2, \ldots, 0 \right] \), \( x = \left[ \Delta X, 2 \Delta X, \ldots, N \Delta X \right] \) and \( \sigma = \left[ \Delta \sigma, 2 \Delta \sigma, \ldots, \Delta \sigma \right] \), solve

\[
H_{k+1} = H_{k+1}^n(t) + \Delta t \left[ \left( \ln x \right)^{-1} x^{-1} \omega_{k+1} \left( \mu - r \right) + r x \right] + \left( \ln \sigma^2 \right)^{-1} \kappa \left[ \theta - \ln \sigma^2 \right] - 1/2 \left( \ln x \right)^{-1} x^{-2} \sigma^2 \omega_{k+1}^2 + \left( \ln \sigma^2 \right)^{-1} \left( \ln x \right)^{-1} x^{-1} \omega_{k+1} \sigma \gamma \rho + \Delta t \left( \ln \sigma^2 \right)^{-1} Aavec \ H_{k+1}^N = 0
\]

(4) Return to (2) with \( k = k+1 \) until convergence.

In order to illustrate our approach, we composed a portfolio invested for its risky share in TUNINDEX index. The share invested in risk-free asset is placed at Assimilable Treasury Bond (ATB). Our resolution strategy was based on a Matlab program to implement the iterative algorithm above, with a final date \( T = 10 \) years, a number of period \( N_t = 500 \), and \( \Delta t = 1/50 \approx 7 \) days, which corresponds the period of time on which we want to measure the Value at Risk. Drift \( \mu = 12\% \), the long-term average \( \vartheta = -11.24 \), the speed of mean reversion \( \kappa = 21.78 \), volatility of volatility \( \gamma = 6.37 \) and the coefficient \( \rho = 0.6 \), these parameters are considered constant; they were estimated from the database of the return TUNINDEX index, using the kalman filter. The interest rate on long-term of the ATB is fixed at \( r = 4.95\% \). For the VaR constraint, the maximum loss is limited to \( \text{VaR}(t) = \beta X(t) \), with \( \beta = 0.05 \) and a probability \( \pi = 1\% \), that is to say we are certain to 99\% that losses will be less than \( \text{VaR}(t) \). Finally \( \Delta x = 501 \), \( \Delta X = 2 \) and \( \Delta \sigma = 11 \), \( \Delta \sigma = 8\% \) are respectively used for a partition of the computation domain of wealth \( x \) and volatility \( \sigma \). The variations of \( H(t) \) become negligible after six iterations. We will present later a graphic illustration of the numerical solutions, which will allow us to release specific results. Since our object is the optimal investment strategy while respecting the VaR constraint, we start with the representation of the optimal portfolio choice with and without constraint and to study through the curves the effect of time and volatility in the optimal allocation of the risky asset.

Figures 1, 2, 3, 4, 5 and 6 represent the amount invested in risky assets for different values of the portfolio. In Figure 1, where the volatility is very small, we note that the VaR constraint has an insignificant effect on the risky investment and the optimal solution suggests an important borrowing to invest in risky assets which is equal to \( \omega = 9.455 \). If the investor accepts the debt, he awaits a very important return, but it must be remembered that the leverage effect is to two sharp, if it can multiply gains, it can also multiply the losses. Also in the case of figure 1, getting the value of the portfolio \( x = 500 \), we find that optimal investment in risky asset equal to 4727 for the following two dates \( t = 0.1 \) and \( t = 9.96 \), which proves that the time factor has no influence on the decision of the investor when volatility is enough low. Contrary to figures 3, 4, 5 and 6, more we approach maturity more the amount invested in risky asset decreases.

Asset allocations with and without constraint

Fig 1: \( \sigma = 0.08 \) and \( t = 9.96 \)

Fig 3: \( \sigma = 0.48 \) and \( t = 9.96 \)

Fig 5: \( \sigma = 0.8 \) and \( t = 9.96 \)
These last four figures illustrate perfectly the modifications implied Value-at-Risk, we notice that the VaR constraint decreases the amount invested in risky asset. All these observations allows us to confirm the results obtained by Basak and Shapiro (2001), Cuoco and He (2001) and KFC Yiu (2004). We can also remark from Figures 1, 3 and 5 that volatility has a significant impact on the optimal solution. If we fixe volatility to 0.48 the investment equal to \( \omega = 0.264 x \), whereas for a volatility raised to 0.8 the risky investment becomes \( \omega = 0.097 x \). Thus, we can certify this result through the represent the allocation of the risky asset for different values of volatility.

The impact of volatility on the risky investment

Figures 7, 8, 9, 10, 11 and 12 represent the Lagrange multipliers. When the VaR constraint becomes active, the Lagrange multipliers are negative, which is totally coherent with the first-order conditions for an optimization problem. We also find that time has a remarkable effect on the Lagrange multiplier, when the decision date is near the final date we obtain a value of \( \lambda \) close to zero.

The Lagrange multipliers \( \lambda \)
Figures 13, 14 and 15 correspond to the maximum loss incurred, with a probability of 99%, and for a horizon period of 7 days. For each of the Figures 16 and 17, we notice that the integration of constraint allows to reduce the potential loss of the portfolio. On the contrary in Figure 15, we notice that the constraint doesn’t have any significant effect on the potential loss of the portfolio, which has marked a relatively high loss to others, this is due to leverage effect. Comparing these graphs, it is clear that volatility affects the value of VaR which means that the more the portfolio is volatile the less the VaR is. This description agrees perfectly with the analytical expression of VAR. To look for the effect of the variation of the confidence level and the horizon period on the Value-at-Risk, firstly we changed the confidence level, secondly we have expanded the horizon of management to a one-month period. Consequently, this enables us to note that whenever the confidence level decreases, the VaR drops. For example, for a wealth $x = 758$ MUs the VaR (99%) equivalent to 30 MUs, however, when the confidence level becomes 95% the VaR decreases to 28.79 MUs, and to 27.18 MUs for a confidence level equal to 90% we can explain this result as follows: the more the investor is risk-averse the more the VaR increases. The repercussions of a variation on the horizon period enables us to conclude that the higher the horizon period is the more the portfolio chosen by this approach is risky. For example, for a wealth of 600 MUs, the VaR is 27.42 MUs with a $\Delta t = 7$ days.

On the other hand, increasing $\Delta t$ to one month the maximum loss of the portfolio increases to 35.78 MUs.

Figures 16, 17 and 18 illustrate the optimal function of Bellman at the initial date, with and without constraint. The results seem to confirm the assumption of concavity of the optimal utility function. The latter showed a small change after the introduction of the VaR constraint. Also, we note that these three graphs are almost identical, which explains that volatility did not significantly affect the expected utility function.

The optimal expected utility at $t = 0$

In order to demonstrate the independence assumption of the $H(t)$ function and the portfolio value $x$, we have presented in figures 21, 22 and 23 the $H(t)$ function for a portfolio of low value equal to 90 MUs and for a relatively large value equal to 900 MUs.

The evolution of function H
Clearly, from the figures 19, 20 and 21, we can affirm that the $H(t)$ function depends only on time. Consequently, the final results are not affected by the negligence of the derivative of the $H(t)$ function with respect to $x$. Besides, we can evaluate the exactitude of the solution by calculating the residual value of the Hamilton-Jacobi-Bellman equation:

$$
\varepsilon_{i,j,k} = u(t, x(t, \sigma^2)) + V_x \omega(t, x, \sigma^2) (\mu - r) + r x + V_{\ln \sigma^2} \kappa \left[ \theta - \ln \sigma(t) \right] + \frac{1}{2} V_{xx} \sigma^2 \omega^2(t, x, \sigma^2) + \frac{1}{2} V_{\ln \sigma^2 \ln \sigma^2} \gamma^2 + V_{\ln \sigma^2} \sigma \gamma \omega(t, x, \sigma^2) \rho + V_t
$$

Finally, the discrete error can be measured by the norm:

$$
\| \varepsilon \| = \sqrt{\sum_{j=1}^{N_j} \sum_{i=1}^{N_i} \sum_{k=1}^{N_k} \varepsilon_{i,j,k}^2} \left( N_i \times N_j \times N_k \right)
$$

In our study applies this norm is worth $1.99 \times 10^{-9}$, it is a very low value which proves that the results are accurate and the iterative algorithm of KFC Yiu, (2004) is considered reliable. Nevertheless, this new approach to portfolio optimization, although more complex to implement, is an important advance in view of the formidable progression of the Value-at-Risk over the past decade.

4. Conclusion

Following this research, it should be noted that our model does not miss insufficiencies too. Note especially that on the empirical level, this model called stochastic volatility does not take into account the rare and extreme variations. Among the future ways of research, it seems interesting to extend our framework to more realistic strategic asset allocation models. This could be achieved by using jump models originally proposed by Merton (1976). The researches of Bates (1996), Duffie and al. (2000) and Pan (2002) conclude on the need to incorporate jumps directly in the process of volatility. It would also be interesting to generalize our approach by combining jump models with the risk measure Value-at-Risk. A first approach was carried out by Bakshi and Panayotov (2010) who established the sufficient conditions for combining the VaR model with jumps model of Merton (1976).

References


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