Decomposition of Lie automorphisms of upper triangular matrix algebra over commutative rings

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Received 20 February 2006; accepted 18 May 2006
Available online 14 July 2006
Submitted by R.A. Brualdi

Abstract

Let $T_{n+1}(R)$ be the algebra of all upper triangular square matrices of order $n + 1$ over a commutative ring $R$ with the identity 1 and unit 2. For $n \geq 2$, we prove that any Lie automorphism of $T_{n+1}(R)$ can be uniquely written as a product of graph, central, inner and diagonal automorphisms.

AMS classification: 17B40; 17B45; 13C10

Keywords: Lie automorphism; Triangular matrix algebra; Commutative ring

1. Introduction

Let $M_{n+1}(R)$ be the $R$-algebra of all square matrices of order $n + 1$ over a commutative ring $R$ with the identity 1 and unit 2. $M_{n+1}(R)$ is a Lie algebra by the bracket operation $[x, y] = xy - yx$ for any $x, y \in M_{n+1}(R)$. Obviously if 2 is a unit in $R$, then $[x, y] = -[y, x]$. If an $R$-module automorphism $\varphi$ of $M_{n+1}(R)$ satisfies $\varphi([x, y]) = [\varphi(x), \varphi(y)]$, then $\varphi$ is called a Lie automorphism of $M_{n+1}(R)$. Let $n_f$ and $n_k$ be the subsets of $M_{n+1}(R)$. Lie bracket operation of $n_f$ and $n_k$ is denoted by $[n_f, n_k] = \{[x, y] | x \in n_f, y \in n_k\}$. We know that an $R$-algebra automorphism, which is a ring automorphism and also an $R$-module automorphism, of $M_{n+1}(R)$ must be a Lie automorphism. However, there are Lie automorphisms which are neither $R$-algebra automorphisms nor $R$-algebra
anti-automorphisms. The subalgebra \( T_{n+1}(R) \) of all triangular matrices over \( R \) has been previously investigated \([1–4]\). Doković [2] and Cao’s [1] papers described the automorphism groups of Lie algebra consisting of all upper triangular \( n \times n \) matrices of trace 0 over a connected commutative ring and a commutative ring with \( n \) invertible respectively. More generally over a commutative ring with unit 2 we consider the problem on decomposition of Lie automorphism of upper triangular matrix algebra into some standard automorphisms.

In this article, using the technique in [5], we prove that any Lie automorphism \( \varphi \) of \( T_{n+1}(R) \) can be uniquely expressed as \( \varphi = \omega_e \eta_c \theta \lambda_d \), where \( \omega_e, \eta_c, \theta \) and \( \lambda_d \) are graph, central, inner and diagonal automorphisms respectively for \( n \geq 2 \) and \( R \) is a commutative ring with the identity 1 and unit 2. In the remaining cases, we also show that any Lie automorphism of \( T_1(R) \) (\( n = 0 \)) is a central automorphism and that any Lie automorphism of \( T_2(R) \) (\( n = 1 \)) can be written as a product of central, inner and diagonal automorphisms.

2. Preliminaries

Let \( e_{ij} \) denote the matrix unit of \( M_{n+1}(R) \) and \( e \) the identity matrix of \( M_{n+1}(R) \). The matrix set \( \{e_{i,i+k} | i = 1, \ldots, n-k+1, k = 0, 1, \ldots, n\} \) is a basis of \( T_{n+1}(R) \). For any \( x \in T_{n+1}(R) \), \( x = \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} a_{i,i+k} e_{i,i+k} \) for some \( a_{i,i+k} \in R \). Let \( n_0 = T_{n+1}(R) \). Let \( n_1 = [n_0, n_0] \), \( n_2 = [n_1, n_1] \), \( n_j = [n_1, n_{j-1}] \), \( j = 3, \ldots, n \). It can be checked that \( n_j = \sum_{k=j}^{n} \sum_{i=1}^{n-k+1} Re_{i,i+k} \). Furthermore \( n_m n_l \subseteq n_{m+l} \) and \( [n_m, n_l] \subseteq n_{m+l} \) for \( m + l \leq n \) or are equal to zero for \( m + l \geq n + 1 \). For any \( \varphi \in \text{Aut}(n_0) \), we have \( \varphi(n_1) = [\varphi(n_0), \varphi(n_0)] = [\varphi(n_0), n_0] = n_1 \) and \( \varphi(n_j) = n_j, j = 2, \ldots, n \). Therefore \( \varphi(n_{j-1} \setminus n_j) = n_{j-1} n_j, j = 1, \ldots, n+1 \). Obviously \( n_1 \) is the subalgebra of all strictly upper triangular matrices of \( T_{n+1}(R) \). The matrix set \( \{e_{i,i+k} | i = 1, \ldots, n-k+1, k = 1, \ldots, n\} \) is a basis of \( n_1 \). The multiplicative group of all the invertible elements of \( R \) is denoted by \( R^* \). For any \( \varphi \in \text{Aut}(n_0) \), there exists \( b^* \in R^* \) such that \( \varphi(e_{1,n+1}) = b^* e_{1,n+1} \).

Lemma 2.1. The set \( \{e_{11}, e_{i+1,i+1}, e_{i,i+1} | i = 1, \ldots, n\} \) generates \( T_{n+1}(R) \) by Lie bracket operation.

Proof. Let \( J_{n+1} \) be the subalgebra of \( T_{n+1}(R) \) generated by the set \( \{e_{11}, e_{i+1,i+1}, e_{i,i+1} | i = 1, \ldots, n\} \). For any \( e_{i,i+k} \in T_{n+1}(R) \), when \( k = 0 \), then \( e_{i,i+k} \in J_{n+1} \). When \( k = 2 \), we have \( e_{i,i+2} = [e_{i,i+1}, e_{i,i+1}] \in J_{n+1} \). Assume by induction that \( e_{i,i+k-1} \in J_{n+1} \), then \( e_{i,i+k} = [e_{i,i+k-1}, e_{i,i+k-1}] \in J_{n+1} \), that is, \( T_{n+1}(R) \subseteq J_{n+1} \). Clearly \( J_{n+1} \subseteq T_{n+1}(R) \).

Lemma 2.2. Let \( \varphi \) be in \( \text{Aut}(n_0) \). If \( \varphi(e_{jj}) \) and \( \varphi(e_{j,j+1}) \) are expressed, respectively, as

\[
\varphi(e_{jj}) = \sum_{i=1}^{n+1} a_{ii}^{(j)} e_{ii} \text{ mod } n_1, \quad j = 1, \ldots, n+1,
\]

\[
\varphi(e_{j,j+1}) = \sum_{i=1}^{n} b_{i,i+1}^{(j)} e_{i,i+1} \text{ mod } n_2, \quad j = 1, \ldots, n.
\]

Then the following matrices are invertible:

(i) \( A = (a_{mk})_{(n+1) \times (n+1)} \) where \( a_{mk} = a_{kk}^{(m)}, m = 1, \ldots, n+1, k = 1, \ldots, n+1; \)
(ii) \( B = (b_{mk})_{n \times n} \) where \( b_{mk} = b_{k,k+1}^{(m)}, m = 1, \ldots, n, k = 1, \ldots, n. \)
Proof. (i) It follows from the fact that the $R$-module $n_0/n_1$ is a free of rank $n + 1$ on the basis $\{e_{ii} + n_1 | i = 1, \ldots, n + 1\}$ and $\varphi$ induces an automorphism of that module. (ii) Note that the $R$-module $n_1/n_2$ is a free of rank $n$ on the basis $\{e_{i,i+1} + n_2 | i = 1, \ldots, n\}$.

Lemma 2.3. Let $\varphi$ be in $\text{Aut}(n_0)$. If $\varphi(e_{jj})$ and $\varphi(e_{j,j+1})$ are expressed, respectively, as the forms in Lemma 2.2, then the following conclusions hold.

(i) For $1 \leq m, k, l \leq n$, $a(l)^m_{i+i, l+i+1} = 0 (l \neq k)$.

(ii) For $1 \leq k, l \leq n$, $(a(k)^i_{i+i, l+i+1})$ $a(k)^i_{k+k, k+k+1} = 0 (l \neq k)$ where $i = 1, n$.

(iii) For $2 \leq m \leq n$ and $1 \leq i, k, l \leq n$, $(a(k)^{m}_{i+i, l+i+1})(a(k)^{m}_{m+1, m+1} - a(k)^{m}_{k+k, k+k+1}) = 0 (i \neq l, k \neq i, \text{here } n \geq 3)$.

Proof. When $j \neq m, m + 1, [\varphi(e_{jj}), \varphi(e_{m,m+1})] = 0$. So

$b(m)^{i}_{i+i, l+i+1} = (a(i)^{m}_{i+i, l+i+1}) = 0, \quad i = 1, \ldots, n$.

From $[\varphi(e_{mm}), \varphi(e_{m,m+1})] = \varphi(e_{m,m+1})$ and $[\varphi(e_{m+1,m+1}), \varphi(e_{m,m+1})] = -\varphi(e_{m,m+1})$, we have

$b(m)^{i}_{i+i, l+i+1}$ $a(m)^{0}_{m+1, m+1} = b(m)^{i}_{i+i, l+i+1}$, $i = 1, \ldots, n$.

Let $C = (c(m)^{n}_{i,i}) \in (n+1) \times (n+1)$ where $c(m)^{n}_{i,i} = a(m)^{0}_{k+k, k+k+1}$, $m = 1, \ldots, n+1$, $k = 1, \ldots, n$. $c(m)^{n}_{i,i} = a(m)^{0}_{k+k, k+k+1}$, $m = 1, \ldots, n+1$, $k = 1, \ldots, n$.

Now let us introduce standard Lie automorphisms of $T_{n+1}(R)$. (i) Let $\varepsilon$ be an idempotent of $R$. Then $\varepsilon, 1 - \varepsilon$ are orthogonal idempotents, that is, $(1 - \varepsilon) = 0$. Let $\varepsilon_0 = \sum_{i=1}^{n+1} e_{i,n-i+2}$. We define a map $\omega_\varepsilon: x \mapsto \varepsilon x - (1 - \varepsilon)(\varepsilon_0^\varepsilon x_0)^\varepsilon$ where $\tau$ denotes the transpose of a matrix. If both $\varepsilon$ and $\bar{\varepsilon}$ are idempotents of $R$, then $1 - (\varepsilon - \bar{\varepsilon})^2$ is also an idempotent of $R$ and $\omega_\varepsilon \omega\bar{\varepsilon} = \omega_1 - (\varepsilon - \bar{\varepsilon})^2$. This implies that $\omega_\varepsilon^2 = \omega_\varepsilon$ and $\omega_\varepsilon$ is an automorphism of $T_{n+1}(R)$. Obviously $\omega_1$ is the identity automorphism of $T_{n+1}(R)$ and $\omega_\varepsilon = \varepsilon \omega_1 + (1 - \varepsilon)\omega_0$. From $\omega_\varepsilon([x, y]) = [\omega_\varepsilon(x), \omega_\varepsilon(y)]$ for any $x, y \in T_{n+1}(R)$ we know that $\omega_\varepsilon$ is a Lie automorphism of $T_{n+1}(R)$. We call $\omega_\varepsilon$ a graph automorphism. If $\varepsilon$ is nontrivial, the graph automorphism $\omega_\varepsilon$ is neither an $R$-algebra automorphism nor an $R$-algebra anti-automorphism of $T_{n+1}(R)$ unless one of ideals $\varepsilon T_{n+1}(R)$ or $(1 - \varepsilon)T_{n+1}(R)$ of $T_{n+1}(R)$ is commutative. The graph automorphism $\omega_\varepsilon$ on the basis of $T_{n+1}(R)$ acts as $\omega_\varepsilon(e_{kj}) = \varepsilon(kj) - (1 - \varepsilon)e_{n-j+2,n-k+2} + k < j < k + 1, \omega_\varepsilon(e_{k,n-k+2}) = e_{k,n-k+2} - (1 - \varepsilon)1_{k} \leq k \leq \frac{n+1}{2}$ and $\omega_\varepsilon(e_{n-j+2,n-k+2}) = e_{n-j+2,n-k+2} - (1 - \varepsilon) e_{n-j+2,n-k+2} - (1 - \varepsilon)1_{k} \leq k \leq \frac{n+1}{2}$, $j \leq i < n - k + 1$, where $\frac{n+1}{2}$ is the integer part of $\frac{n+1}{2}$. The set of all graph automorphisms of $T_{n+1}(R)$ is a subgroup of $\text{Aut}(n_0)$, denoted by $\mathcal{G}$.

(ii) Regarding $R$ as an abelian Lie algebra, the map $\varphi: T_{n+1}(R) \rightarrow R$ such that $1 + \varphi(e) \in R^*$ is an endomorphism of $T_{n+1}(R)$. We define a map $\eta: x \mapsto x + f(x)e$ for any $x \in T_{n+1}(R)$.
Since $\eta_f \eta_g = \eta_{f+1+g}$, then $\eta_f^{-1} = \eta_{-(1+f(e))}^{-1}$ and $\eta_f$ is an automorphism of $T_{n+1}(R)$. From $\eta_f([x, y]) = [\eta_f(x), \eta_f(y)]$ for any $x, y \in T_{n+1}(R)$ we know that $\eta_f$ is a Lie automorphism of $T_{n+1}(R)$. We call $\eta_f$ a central automorphism. Obviously $\eta_f(y) = 0$ for any $y \in n_1$. Let $c_j = f(e_{jj})$, $j = 1, 2, \ldots, n + 1$. Then the central automorphism $\eta_f$ on the basis of $T_{n+1}(R)$ acts as $\eta_f(e_{jj}) = e_{jj} + c_j e (1 \leq j \leq n + 1)$, $\eta_f(e_{mj}) = e_{mj}$ ($m \neq j$). Let $c = (c_1, \ldots, c_{n+1})$. Then we denote $\eta_f$ by $\eta_c$. Since $f(e) = \sum_{j=1}^{n+1} c_j$, then $1 + \sum_{j=1}^{n+1} c_j \in R$. Therefore $\eta_c^{-1} = \eta_{-(1+\sum_{j=1}^{n+1} c_j)}^{-1} c$. The set of all central automorphisms of $T_{n+1}(R)$ is a subgroup of $\text{Aut}(n_0)$, denoted by $\mathcal{C}$.

(iii) For any $y \in n_1$, let $h = e + y$. The map $\theta_h: x \mapsto h x h^{-1}$ is called an inner automorphism which is an $R$-algebra automorphism of $T_{n+1}(R)$. If $h = h_{ij}(a) = e + a e_{ij}$ ($i < j$) with some $a \in R$, then $\theta_{h_{ij}(a)}$ is called the “simple” form. Using $[h_{ij}(a)]^{-1} = h_{ij}(-a)$ we know that $\theta_{h_{ij}(a)}(e_{ii}) = e_{ii} - a e_{ij}$, $\theta_{h_{ij}(a)}(e_{jj}) = e_{jj} + a e_{ij}$ for $i < j$ and $\theta_{h_{ij}(a)}(e_{kk}) = e_{kk}$ for $k \neq i, j$ and that $\theta_{h_{ij}(a)}(e_{i+1,j}) = e_{i,j+1} = a e_{i,j+1}$ and $\theta_{h_{ij}(a)}(e_{i,j+1}) = e_{i,j+1} - a e_{i,j}$ also $\theta_{h_{ij}(a)}(e_{k,k+1}) = e_{k,k+1}$ and $\theta_{h_{ij}(a)}(e_{k,k+1}) = e_{k,k+1}$ for $k \neq i, m, j$. It is easy to see that $\theta_{h_{ij}(a)}^{-1} = \theta_{h_{ij}(-a)}$. The set of all the “simple” inner automorphisms of $T_{n+1}(R)$ generate a subgroup of $\text{Aut}(n_0)$, denoted by $\mathcal{S}$.

(iv) Let $d = \sum_{i=1}^{n+1} d_i e_{ii}$. where $d_i \in R^*$, $i = 1, 2, \ldots, n + 1$. The map $\lambda_d: x \mapsto d x d^{-1}$ is called a diagonal automorphism which is an $R$-algebra automorphism of $T_{n+1}(R)$. It is obvious that $\lambda_d^{-1} = \lambda_d^{-1}$. A diagonal automorphism on the basis of $T_{n+1}(R)$ yields that $\lambda_d(e_{ii}) = e_{ii}$ and $\lambda_d(e_{i,i+k}) = \prod_{m=1}^{k} \tilde{d}_{i+m-1,i+m} e_{i+k}$ for $d_1 = 1$, $d_i = \prod_{m=2}^{i} \tilde{d}_{i-m+1,i-m+2} e_{i+k}$ for $i = 2, \ldots, n + 1$. The set of all diagonal automorphisms of $T_{n+1}(R)$ is a subgroup of $\text{Aut}(n_0)$, denoted by $\mathcal{D}$.

Lemma 2.4. Let $\varphi$ be in $\text{Aut}(n_0)$. If $\varphi(e_{12}) = \sum_{k=1}^{n} b_{k,k+1}^{(1)} e_{k,k+1}$ mod $n_2$, then there exists a graph automorphism $\omega_\varepsilon$ such that $\omega_\varepsilon \varphi(e_{12}) = \tilde{b}_{12}^{(1)} e_{12}$ mod $n_2$ where $\tilde{b}_{12}^{(1)} = (2\varepsilon - 1)(b_{12}^{(1)} + b_{n,n+1}^{(1)}) \in R^*$.

Proof. Since $e_{2,n+1} \in n_{n-1} \setminus n_2$, we have $\varphi(e_{2,n+1}) \in n_{n-1} \setminus n_2$. Assume that $\varphi(e_{1,n+1}) = b^* e_{1,n+1}$ where $b^* \in R^*$ and $\varphi(e_{2,n+1}) = a e_{1n} + b e_{2,n+1} + c e_{1,n+1}$. From

$$
\varphi(e_{1,n+1}) = [\varphi(e_{12}), \varphi(e_{2,n+1})] = (b_{12}^{(1)} - ab_{n,n+1}^{(1)}) e_{1,n+1}.
$$

We have $b_{12}^{(1)} b - ab_{n,n+1}^{(1)} = b^* \in R^*$. By Lemma 2.3, $b^* b_{k,k+1}^{(1)} = (b_{12}^{(1)} b - ab_{n,n+1}^{(1)}) b_{k,k+1}^{(1)} = 0$, $k = 2, \ldots, n - 1$, that is, $b_{k,k+1}^{(1)} = 0$, $k = 2, \ldots, n - 1$. So $\varphi(e_{12}) = b_{12}^{(1)} e_{12} + b_{n,n+1}^{(1)} e_{n,n+1}$ mod $n_2$. From

$$
(b_{12}^{(1)} + b_{n,n+1}^{(1)})[b_{12}^{(1)} (bb^* - 1) - b_{n,n+1}^{(1)} (ab^* - 1)^2] = (b_{12}^{(1)} b b - 1 - b_{n,n+1}^{(1)} ab - 1)^2 = (b^* b^* - 1)^2 = 1,
$$

we know $b_{12}^{(1)} + b_{n,n+1}^{(1)} \in R^*$. Take $\varepsilon = b_{12}^{(1)} b_{12}^{(1)} b_{n,n+1}^{(1)}$, then $1 - \varepsilon = b_{12}^{(1)} b_{12}^{(1)} b_{n,n+1}^{(1)}$. From

$$
\varepsilon^2 = (b_{12}^{(1)} b_{12}^{(1)} b_{n,n+1}^{(1)})^2 = b_{12}^{(1)} (b_{12}^{(1)} b_{n,n+1}^{(1)} b_{12}^{(1)} + b_{n,n+1}^{(1)} b_{12}^{(1)})^2 = \varepsilon
$$


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we know \( \varepsilon \) and \( 1 - \varepsilon \) are orthogonal idempotents. From \((2\varepsilon - 1)^2 = 1\) we have \(2\varepsilon - 1 \in R^*\). Furthermore
\[
b_{12}^{(1)} \varepsilon = (b_{12}^{(1)})^2 (b_{12}^{(1)} + b_{n,n+1}^{(1)})^{-1} = b_{12}^{(1)} (b_{12}^{(1)} + b_{n,n+1}^{(1)}) (b_{12}^{(1)} + b_{n,n+1}^{(1)})^{-1} = b_{12}^{(1)}.
\]
Similarly \(b_{n,n+1}^{(1)} (1 - \varepsilon) = b_{n,n+1}^{(1)}\). Also \(b_{12}^{(1)} (1 - \varepsilon) = 0\) and \(b_{n,n+1}^{(1)} \varepsilon = 0\). Therefore
\[
\omega_\varepsilon \varphi(e_{12}) = b_{12}^{(1)} \omega_\varepsilon(e_{12}) + b_{n,n+1}^{(1)} \omega_\varepsilon(e_{n,n+1}) \mod n_2
\]
\[
= b_{12}^{(1)} [\varepsilon e_{11} - (1 - \varepsilon)e_{n,n+1}] + b_{n,n+1}^{(1)} [\varepsilon e_{n,n+1} - (1 - \varepsilon)e_{11}] \mod n_2
\]
\[
= (2\varepsilon - 1)(b_{12}^{(1)} + b_{n,n+1}^{(1)}) e_{12} \mod n_2.
\]
Let \(\tilde{b}_{12}^{(1)} = (2\varepsilon - 1)(b_{12}^{(1)} + b_{n,n+1}^{(1)})\). Then \(\omega_\varepsilon \varphi(e_{12}) = \tilde{b}_{12}^{(1)} e_{12} \mod n_2\).

\[\square\]

3. Lemmas for main results

In order to achieve our goal, we also need other lemmas.

**Lemma 3.1.** Let \( \varphi \) be in \( \text{Aut}(n_0) \). If \( \varphi(e_{12}) = b_{12}^{(1)} e_{12} \mod n_2 \) where \( b_{12}^{(1)} \in R^* \), then \( \varphi(e_{j,j+1}) = b_{i,j+1}^{(j)} e_{j,j+1} \mod n_2 \), where \( b_{i,j+1}^{(j)} \in R^* \), \( j = 1, \ldots, n \), and there exists a central automorphism \( \eta_c \) such that \( \eta_c \varphi(e_{jj}) = e_{jj} \mod n_1 \), \( j = 1, \ldots, n + 1 \).

**Proof.** Write \( \varphi(e_{j,j+1}) \) as
\[
\varphi(e_{j,j+1}) = \sum_{k=1}^{n} b_{k,k+1}^{(j)} e_{k,k+1} \mod n_2, \quad j = 1, \ldots, n + 1.
\]

From \( e_{13} = [e_{12}, e_{23}] \) we have
\[
\varphi(e_{13}) = [\varphi(e_{12}), \varphi(e_{23})] = b_{12}^{(1)} b_{23}^{(2)} e_{13} \mod n_2 \backslash n_3.
\]

By induction we assume that \( \varphi(e_{1,m}) = \prod_{k=1}^{m-1} b_{k,k+1}^{(k)} e_{1,m} \mod n_m \) holds. Further, since \( e_{1,m+1} \in n_m \backslash n_{m+1} \), we have
\[
\varphi(e_{1,m+1}) = [\varphi(e_{1,m}), \varphi(e_{m,m+1})] = \prod_{k=1}^{m-1} b_{k,k+1}^{(k)} b_{m,m+1}^{(m)} e_{1,m+1} \mod n_{m+1} \in n_m \backslash n_{m+1}.
\]

When \( m = n \), \( \varphi(e_{1,n+1}) = \prod_{k=1}^{n} b_{k,k+1}^{(k)} e_{1,n+1} = b^* e_{1,n+1} \). So \( \prod_{k=1}^{n} b_{k,k+1}^{(k)} = b^* \in R^* \), that is, \( b_{j,j+1}^{(j)} \in R^* \), \( j = 1, \ldots, n \). By Lemma 2.3 we have \( \varphi(e_{j,j+1}) = b_{i,j+1}^{(j)} e_{j,j+1} \mod n_2 \), \( j = 1, \ldots, n \). By the process of proving Lemma 2.3 we obtain that \( b_{12}^{(1)} (a_{i1}^{(1)} - a_{22}^{(1)}) = b_{12}^{(1)} \), \( b_{i,i+1}^{(i)} (a_{i+1}^{(i)} - a_{i+1,i+1}^{(i)}) = -b_{i,i+1}^{(i)} \), \( b_{i+1,i+2}^{(i)} (a_{i+1,i+1}^{(i)} - a_{i+2,i+2}^{(i)}) = b_{i+1,i+2}^{(i)} \), \( i = 1, \ldots, n \), and \( b_{n,n+1}^{(n)} (a_{n,n+1}^{(n)} - a_{n+1,n+1}^{(n)}) = -b_{n,n+1}^{(n)} \). Then \( a_{i1}^{(1)} - a_{22}^{(1)} = 1 \), \( a_{i,i+1}^{(i)} - a_{i+1,i+1}^{(i)} = -1 \) and \( a_{i+1,i+2}^{(i)} - a_{i+2,i+2}^{(i)} = 1 \), \( i = 1, \ldots, n \). By Lemma 2.3, we have that \( a_{kk}^{(1)} - a_{k,k+1,k+1}^{(1)} = 0 \) \( (2 \leq k \leq n) \), \( a_{kk}^{(1)} - a_{k,k+1,k+1}^{(1)} = 0 \) \( (1 \leq k < i, i + 1 < k \leq n - 1) \), \( a_{kk}^{(1)} - a_{k,k+1,k+1}^{(1)} = 0 \) \( (1 \leq k \leq n - 1) \). Let \( \tilde{c} = (\tilde{c}_{mk})_{(n+1) \times (n+1)} \) where \( \tilde{c}_{kk} = 1 \), \( k = 1, \ldots, n \), \( \tilde{c}_{m,n+1} = \sum_{j=1}^{m} a_{j+1,j+1}^{(j)}, m = 1, \ldots, n \), \( \tilde{c}_{n+1,n+1} = 1 + \sum_{j=1}^{n} a_{j+1,j+1}^{(j)} + a_{nn}^{(n)} \), otherwise \( \tilde{c}_{mk} = 0 \), for \( m \neq k \) and

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Lemma 3.4. \( k \neq n + 1 \). Therefore \( \det \tilde{C} = \det C \in R^* \). So \( 1 + \sum_{j=1}^{n} a_{j+1,j+1}^{(j)} + a_{nn}^{(n+1)} \in R^* \). Take \( c = (c_1, \ldots, c_{n+1}) \) where \( c_j = -a_{j+1,j+1}^{(j)} (1 + \sum_{j=1}^{n} a_{j+1,j+1}^{(j)} + a_{nn}^{(n+1)})^{-1} \) and \( c_{n+1} = -a_{nn}^{(n+1)} (1 + \sum_{j=1}^{n} a_{j+1,j+1}^{(j)} + a_{nn}^{(n+1)})^{-1} \), \( j = 1, \ldots, n \), then \( \eta_c \varphi(e_{jj}) = e_{jj} \mod n_1, j = 1, \ldots, n + 1 \). □

Remark. Since the proofs of the following Lemmas 3.2–3.6 are similar to those in [5], we will omit the proofs.

Lemma 3.2. Let \( \varphi \) be in \( \text{Aut}(n_0) \). If \( \varphi(e_{jj}) = e_{jj} \mod n_1, j = 1, 2, \ldots, n + 1 \), then

\[
\begin{align*}
\varphi(e_{11}) &= e_{11} + a_{12}^{(1)} e_{12} \mod n_2, \\
\varphi(e_{jj}) &= e_{jj} + a_{j,j+1}^{(j)} e_{j,j+1} - a_{j-1,j}^{(j-1)} e_{j-1,j} \mod n_2, \quad j = 2, \ldots, n \ (n \geq 2), \\
\varphi(e_{n+1,n+1}) &= e_{n+1,n+1} - a_{n,n+1}^{(n)} e_{n,n+1} \mod n_2.
\end{align*}
\]

Lemma 3.3. Let \( \varphi \) be in \( \text{Aut}(n_0) \). If \( \varphi(e_{jj}) \) is expressed as the form in Lemma 3.2, we may find an inner automorphism

\[
\theta = \prod_{j=1}^{n} \theta_{h,j+1}^{(j)}(a_{j,j+1}^{(j)})
\]

such that

\[
\theta \varphi(e_{jj}) = e_{jj} \mod n_2, \quad j = 1, 2, \ldots, n + 1.
\]

Lemma 3.4. Let \( \varphi \) be in \( \text{Aut}(n_0) \). If \( \varphi(e_{jj}) = e_{jj} \mod n_{m-1}, j = 1, 2, \ldots, n + 1 \), then

\[
\begin{align*}
\varphi(e_{jj}) &= e_{jj} + a_{j,j+m-1}^{(j)} e_{j,j+m-1} \mod n_m, \quad 1 \leq j \leq \min\{m - 1, n - m + 2\}, \\
\varphi(e_{jj}) &= e_{jj} + a_{j,j+m-1}^{(j)} e_{j,j+m-1} - a_{j-m+1,j}^{(j-m+1)} e_{j-m+1,j} \mod n_m, \\
m &\leq j \leq n - m + 2 \left( m \leq \left\lceil \frac{n + 1}{2} \right\rceil \right), \\
\varphi(e_{jj}) &= e_{jj} \mod n_m, \\
n - m + 3 &\leq j \leq m \left( m \geq \left\lceil \frac{n + 1}{2} \right\rceil + 1 \text{ or when n is odd, } m > \left\lceil \frac{n + 1}{2} \right\rceil + 1 \right), \\
\varphi(e_{jj}) &= e_{jj} - a_{j-m+1,j}^{(j-m+1)} e_{j-m+1,j} \mod n_m, \quad \max\{n - m + 3, m\} \leq j \leq n + 1.
\end{align*}
\]

Lemma 3.5. Let \( \varphi \) be in \( \text{Aut}(n_0) \). If \( \varphi(e_{jj}) \) is expressed as the form in Lemma 3.4, we may find an inner automorphism

\[
\theta = \prod_{j=1}^{n-m+2} \theta_{h,j+m-1}^{(j)}(a_{j,j+m-1}^{(j)})
\]

such that

\[
\theta \varphi(e_{jj}) = e_{jj} \mod n_m, \quad j = 1, 2, \ldots, n + 1.
\]

When \( m = n + 1 \), \( \theta \varphi(e_{jj}) = e_{jj}, j = 1, 2, \ldots, n + 1. \)
Lemma 3.6. When \( n \geq 1 \), let \( \varphi \) be in \( \text{Aut}(\mathbf{n}_0) \). If \( \varphi(e_{jj}) = e_{jj} \), there exists a diagonal automorphism \( \lambda_d \) such that \( \lambda_d \varphi(e_{j,j+1}) = e_{j,j+1} \mod \mathbf{n}_2 \), \( j = 1, \ldots, n \).

Lemma 3.7. When \( n \geq 1 \), let \( \varphi \) be in \( \text{Aut}(\mathbf{n}_0) \). If \( \varphi(e_{jj}) = e_{jj} \), \( j = 1, 2, \ldots, n+1 \) and \( \varphi(e_{j,j+1}) = e_{j,j+1} \mod \mathbf{n}_2 \), \( j = 1, \ldots, n \), then \( \varphi(e_{j,j+1}) = e_{j,j+1} \), \( j = 1, \ldots, n \).

Proof. We express \( \varphi(e_{j,j+1}) \) as

\[
\varphi(e_{j,j+1}) = e_{j,j+1} + \sum_{k=2}^{n} \sum_{m=1}^{n-k+1} b^{(j)}_{m,m+k} e_{m,m+k}, \quad j = 1, \ldots, n.
\]

Therefore

\[
\varphi(e_{12}) = [\varphi(e_{11}), \varphi(e_{12})] = e_{12} + \sum_{k=2}^{n} b^{(1)}_{1,1+k} e_{1,1+k} \quad (n \geq 2),
\]

\[
\varphi(e_{23}) = [\varphi(e_{22}), \varphi(e_{23})] = e_{23} \quad (n = 2),
\]

\[
\varphi(e_{23}) = [\varphi(e_{22}), \varphi(e_{23})] = e_{23} + \sum_{k=2}^{n-1} b^{(2)}_{2,2+k} e_{2,2+k} \quad (n \geq 3),
\]

\[
(3 \leq j \leq n-1, n \geq 4),
\]

\[
\varphi(e_{n,n+1}) = [\varphi(e_{nn}), \varphi(e_{n,n+1})] = e_{n,n+1} + \sum_{k=2}^{n-1} b^{(n)}_{n-k,n} e_{n-k,n} \quad (n \geq 3).
\]

So for \( j = 1, \ldots, n \)

\[
\varphi(e_{j,j+1}) = [\varphi(e_{j,j+1}), \varphi(e_{j+1,j+1})] = [\varphi(e_{j,j+1}), e_{j+1,j+1}] = e_{j,j+1}.
\]

In the case \( n = 1 \), \( \varphi(e_{12}) = e_{12} \).

4. Main results

Theorem 4.1. Let \( R \) be a commutative ring with the identity 1 and unit 2, and \( T_{n+1}(R) \), \( n \geq 2 \), the algebra consisting of all upper triangular \( (n+1) \times (n+1) \) matrices over \( R \). For any Lie automorphism \( \varphi \) of \( T_{n+1}(R) \) there exist graph, central, inner and diagonal automorphisms \( \omega_x, \eta_c, \theta \) and \( \lambda_d \), respectively, of \( T_{n+1}(R) \) such that \( \varphi = \omega_x \eta_c \theta \lambda_d \), and the decomposition \( \varphi = \omega_x \eta_c \theta \lambda_d \) is unique.

Proof. By Lemmas 2.4 and 3.1–3.7 there are \( \lambda_d^{-1}, \theta^{-1}, \eta_c^{-1} \) and \( \omega_x \) such that

\[
\lambda_d^{-1} \theta^{-1} \eta_c^{-1} \omega_x \varphi(e_{jj}) = e_{jj}, \quad j = 1, 2, \ldots, n+1.
\]

\[
\lambda_d^{-1} \theta^{-1} \eta_c^{-1} \omega_x \varphi(e_{j,j+1}) = e_{j,j+1}, \quad j = 1, \ldots, n.
\]

Since \( e_{11}, e_{j+1,j+1}, e_{j,j+1}, j = 1, \ldots, n \), generate \( T_{n+1}(R) \), then \( \varphi = \omega_x \eta_c \theta \lambda_d \).

The uniqueness of the decomposition follows from the following lemma. \( \square \)
Lemma 4.2. Let $\mathcal{G}$, $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{D}$ be the graph, central, inner and diagonal automorphism group of $T_{n+1}(R)$, respectively. When $n \geq 2$, then

$$\text{Aut}(n_0) = \mathcal{G} \times (\mathcal{C} \times (\mathcal{I} \times \mathcal{D})).$$

**Proof.** By the first part of Theorem 4.1 we have $\text{Aut}(n_0) = \mathcal{G} \mathcal{C} \mathcal{I} \mathcal{D}$. For any $x \in n_0$ we have $\theta_{\lambda}(x) = h(dx^{-1})x^{-1} = \lambda x^{-1} \theta_{d^{-1}} x^{-1}$. Thus $\theta_{\lambda} = \lambda \theta_{d^{-1}}$. So $\mathcal{I} \triangleleft \mathcal{D}$. Obviously $\mathcal{I} \cap \mathcal{D} = 1$, then $\mathcal{D} = \mathcal{I} \times \mathcal{D}$. From

$$\eta_c \lambda(d(ej)) = \eta_c(ej) = ej + c je = \lambda d(ej + c je) = \lambda d \eta_c(ej),$$

$$\eta_c \lambda(d(ej, j+1)) = \eta_c(d j d_{j+1} e_{j, j+1}) = d j d_{j+1} e_{j, j+1} = \lambda d \eta_c(e_{j, j+1})$$

we have $\eta_c \lambda = \lambda d \eta_c$. Similarly $\eta_c \theta_{3n^2}(a) = \theta_{3n^2}(a) \eta_c$. Obviously $\mathcal{G} \cap \mathcal{D} = 1$, then $\mathcal{G} \mathcal{D} = \mathcal{G} \times (\mathcal{I} \times \mathcal{D})$. From

$$\omega_0 \theta_h(x) = -(e_0(hx^{-1})e_0)^T = \theta_{\omega_0(h^{-1})} \omega_0(x)$$

we have $\omega_0 \theta_h \omega_0 = \theta_{\omega_0(h^{-1})}$. Then

$$\omega_e \theta_h \omega_e(x) = (\varepsilon \omega_1 + (1 - \varepsilon) \omega_0) \theta_h (\varepsilon \omega_1 + (1 - \varepsilon) \omega_0)(x) = (\varepsilon \theta_h + (1 - \varepsilon) \omega_0 \theta_h \omega_0)(x) = \varepsilon \theta_h(x) + (1 - \varepsilon) \theta_{\omega_0(h^{-1})}(x) = \varepsilon^2 h x h^{-1} + (1 - \varepsilon)^2 \omega_0(h^{-1})x(\omega_0(h^{-1}))^{-1} = (\varepsilon h + (1 - \varepsilon) \omega_0(h^{-1}))x(\varepsilon h^{-1} + (1 - \varepsilon) \omega_0(h^{-1}))^{-1} = \omega_h(x) + (1 - \varepsilon) \omega_0(h^{-1})x(\varepsilon h^{-1} + (1 - \varepsilon) \omega_0(h^{-1}))^{-1} = \theta_{\varepsilon h + (1 - \varepsilon) \omega_0(h^{-1})}(x).$$

So $\omega_e \theta_h \omega_e(x) = \theta_{\varepsilon h + (1 - \varepsilon) \omega_0(h^{-1})}$. Similarly $\omega_e \lambda d \omega_e = \lambda \varepsilon d + (1 - \varepsilon) \omega_0(d^{-1})$. And also

$$\eta_c \omega_e(e_{j, j}) = \eta_c(\varepsilon e_{j, j} - (1 - \varepsilon)e_{n-j+2, n-j+2}) = (\varepsilon e_{j, j} + (1 - \varepsilon)(e_{n-j+2, n-j+2} + e_{n-j+2} e) = (\varepsilon e_{j, j} - (1 - \varepsilon)e_{n-j+2, n-j+2}) + (\varepsilon e_{j, j} - (1 - \varepsilon)e_{n-j+2, n-j+2}) = \omega_e(e_{j, j} + \varepsilon e_{j, j}) = \omega_e(e_{j, j})$$

where $\hat{c} = (\hat{c}, \ldots, \hat{c}^n + 1)$, $\hat{c}^j = \varepsilon e_{j, j} - (1 - \varepsilon)c_{n-j+2, j} = 1, \ldots, n + 1$. Thus $\eta_c \omega_e = \omega_e \eta_c$. Therefore $\mathcal{G} \triangleleft \mathcal{C} \mathcal{D}$. Clearly $\mathcal{G} \cap \mathcal{C} \mathcal{D} = 1$. Then $\text{Aut}(n_0) = \mathcal{G} \times (\mathcal{C} \times (\mathcal{I} \times \mathcal{D})).$ \qed

5. Discussion for $n = 0, 1$

When $n = 0$, it is obvious that $\varphi(e_{11}) = ae_{11}$, $a \in R^*$. Then $\eta_{a^{-1}(1-a)} \varphi(e_{11}) = e_{11}$ so that $\varphi = \eta_{a^{-1}}$. When $n = 1$, by Lemmas 3.1 and 3.6 we have $\lambda d \eta_c \varphi(e_{11}) = e_{11} + a_{12}^{(1)} e_{12}$, $\lambda d \eta_c \varphi(e_{22}) = e_{22} - a_{12}^{(1)} e_{12}$ and $\lambda d \eta_c \varphi(e_{12}) = e_{12}$. Furthermore $\theta_{a_{12}^{(1)}} \lambda d \eta_c \varphi(e_{11}) = e_{11}$, $\theta_{a_{12}^{(1)}} \lambda d \eta_c \varphi(e_{22}) = e_{22}$ and $\theta_{a_{12}^{(1)}} \lambda d \eta_c \varphi(e_{12}) = e_{12}$. Hence $\varphi = \eta_{a^{-1}} \lambda d^{-1} \theta_{a_{12}^{(1)}}$. 


Note. Comparing the decomposition of Lie automorphisms of \( n_0 \) with that of Jordan automorphisms of \( n_1 \) (see [5]), under the more general condition on a commutative ring \( R \) with unit 2 than that on a 2-torsionfree local ring \( R \), we may find that the central automorphism is needed here.

Acknowledgments

We are grateful to the referee for helpful comments.

References