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Darboux–Jouanolou–Ghys integrability for one-dimensional foliations on toric varieties

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Abstract

We use the existence of homogeneous coordinates for simplicial toric varieties to prove a result analogous to the Darboux–Jouanolou–Ghys integrability theorem for the existence of rational first integrals for one-dimensional foliations.

Keywords: Holomorphic foliations; Rational first integral; Toric varieties

1. Introduction

Darboux in his seminal work [8] of 1878 provided a theory on the existence of first integrals for polynomial differential equation based on the existence of sufficient invariant algebraic hypersurfaces. The Darboux's work has been inspired several researchers as for instance Poincaré who called it "admirable" and "oeuvre magistrale" [18].

The improvement and generalization of the Darboux theory of integrability was given by Jouanolou in [12] characterizing the existence of rational first integrals for Pfaff equations on $\mathbb{P}^n_k$, where k is an algebraically closed field of characteristic zero. Namely, let ω be a twisted 1-form $\omega \in \mathrm{H}^0(\mathbb{P}^n_k, \Omega^1_{\mathbb{P}^n_k} \otimes \mathcal{O}_{\mathbb{P}^n_k}(m+1))$, where m was called by Jouanolou the degree of ω. Then follows from [12, Theorem 3.3, p. 102] that ω admits a rational first integral if and only if possesses an infinite number of invariant irreducible hypersurfaces. More generally, Jouanolou proved in [13] that on a complex compact manifold X satisfying certain conditions on its Hodge-to-de Rham

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spectral sequence, a Pfaff equation $\omega \in \mathrm{H}^0(X, \Omega^1_X \otimes \mathcal{L})$, where $\mathcal{L}$ is a line bundle, admits a meromorphic first integral if and only if possesses an infinite number of invariant irreducible divisors. Moreover, if ω does not admit a meromorphic first integral, then the number of invariant irreducible divisors is at most

$$\dim_{\mathbb{C}}\bigl(\mathrm{H}^0\bigl(X, \Omega^2_X \otimes \mathcal{L}\bigr)/\omega \wedge \mathrm{H}^0\bigl(X, \Omega^1_X\bigr)\bigr) + \rho(X) + 1,$$

where $\rho(X)$ is the Picard number of X.

E. Ghys in [10] drops all hypotheses given by Jouanolou showing that this result is valid for all compact complex manifold. M. Brunella and M. Nicolau in [2] proved this same result for Pfaff equations in positive characteristic and for non-singular codimension one transversal holomorphic foliations on compact manifolds. A discrete dynamical version of Jouanolou's theorem was recently proved by S. Cantat. In [3] he proved that if there exist k invariant irreducible hypersurfaces by a holomorphic endomorphism $f : X \circlearrowleft$ with

$$k \geqslant \dim(X) + h^{1,1}(X)$$

then f preserves a nontrivial meromorphic fibration.

In this work we are interesting to proof a version of this results for vector fields on complete simplicial toric varieties. Let $\mathbb{P}_\Delta$ be a simplicial toric variety associated by a fan Δ and $\mathcal{T}\mathbb{P}_\Delta = \mathcal{H}om(\Omega^1_{\mathbb{P}_\Delta}, \mathcal{O}_{\mathbb{P}_\Delta})$ the Zarisk's sheaf of $\mathbb{P}_\Delta$. Since $\mathbb{P}_\Delta$ is a complex orbifold then $\mathcal{T}\mathbb{P}_\Delta \simeq i_*\Theta_{\mathbb{P}_{\Delta,reg}}$, where $i : \mathbb{P}_{\Delta,reg} \to \mathbb{P}_\Delta$ is the inclusion of regular part $\mathbb{P}_{\Delta,reg} := \mathbb{P}_\Delta - \mathrm{Sing}(\mathbb{P}_\Delta)$ and $\Theta_{\mathbb{P}_{\Delta,reg}}$ is the tangent sheaf $\mathbb{P}_{\Delta,reg}$, see [6, Appendix A.2]. A singular holomorphic foliation $\mathcal{F}$ on $\mathbb{P}_\Delta$ is a global section of $\mathcal{T}\mathbb{P}_\Delta \otimes K_{\mathcal{F}}$, where $K_{\mathcal{F}}$ is an invertible sheaf on $\mathbb{P}_\Delta$. We denote $\mathbb{T}^n$ the torus acting on $\mathbb{P}_\Delta$ and we call a $\mathbb{T}^n$-invariant Weil divisor as $\mathbb{T}^n$*-divisor*. We shall use the existence of homogeneous coordinate for simplicial toric varieties to prove the following result.

Theorem 1.1. *Let $\mathcal{F}$ be an one-dimensional foliation on a complete simplicial toric variety $\mathbb{P}_\Delta$ of dimension n and Picard number $\rho(\mathbb{P}_\Delta)$. If $\mathcal{F}$ admits*

$$\mathcal{N}(\mathbb{P}_\Delta, K_{\mathcal{F}}) = h^0\bigl(\mathbb{P}_\Delta, \mathcal{O}(K_{\mathcal{F}})\bigr) + \rho(\mathbb{P}_\Delta) + n$$

invariant irreducible $\mathbb{T}^n$-divisors, then $\mathcal{F}$ admit a rational first integral.

Observe that, in general $\mathbb{P}_\Delta$ is a singular variety with quotient singularities. Therefore, in two dimension this result show that the Darboux–Jouanolou–Ghys's theorem is valid for a class of singular toric variety.

The affine and non-singular version of this result was proved by J. Llibre and X. Zhang in [14], they showed that if the number of invariant algebraic hypersurfaces of a polynomial vector field X in $\mathbb{C}^n$ of degree d is at least

$$\mathcal{N}(n, d) = \binom{d+n-1}{n} + n$$

then X admits a rational first integral.

Remark 1. It follows from Hirzebruch–Riemann–Roch theorem for toric varieties (see [9]) that

$$h^0\bigl(\mathbb{P}_\Delta, \mathcal{O}(K_{\mathcal{F}})\bigr) = \sum_{k=0}^{n} \frac{1}{k!} \deg\bigl([K_{\mathcal{F}}]^k \cap Td_k(\mathbb{P}_\Delta)\bigr),$$

where $Td_k(\mathbb{P}_\Delta)$ is the k-th homology Todd class. Therefore, we have that

$$\mathcal{N}(\mathbb{P}_\Delta, K_\mathcal{F}) = \sum_{k=0}^{n} \frac{1}{k!} \deg\bigl([K_\mathcal{F}]^k \cap Td_k(\mathbb{P}_\Delta)\bigr) + rk\bigl(\mathcal{A}_{n-1}(\mathbb{P}_\Delta)\bigr) + n.$$

This paper is organized as follows. In Section 2 we recall some basic definitions and results about simplicial complete toric varieties emphasizing Cox's quotient construction and homogeneous coordinates. In Section 3 we use the generalized Euler exact sequence for simplicial toric varieties in order to consider a holomorphic foliation as a polynomial vector field in homogeneous coordinates. Finally, in Section 4 we proof Theorem 1.1.

2. Generalities on toric varieties

In this section we recall some basic definitions and results about simplicial complete toric varieties. For more details, we refer reader to the literature (e.g., see [7,5,9,17]).

Let N be a free $\mathbb{Z}$-module of rank n and $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual. A subset $\sigma \subset N \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^n$ is called a strongly convex rational polyhedral cone if there exists a finite number of elements $\vartheta_1, \ldots, \vartheta_k$ in the lattice N such that

$$\sigma = \{a_1\vartheta_1 + \cdots + a_k\vartheta_k;\ a_i \in \mathbb{R},\ a_i \geqslant 0\}.$$

We say that a subset τ of σ given by some a_i being equal to zero is a proper face of σ, and we write $\tau \prec \sigma$. A cone σ is called *simplicial* if its generators can be chosen to be linearly independent over $\mathbb{R}$. The dimension of a cone σ is, by definition, the dimension of a minimal subspace of $\mathbb{R}^n$ containing σ.

Definition 2.1. A non-empty collection $\Delta = \{\sigma_1, \ldots, \sigma_s\}$ of k-dimensional strongly convex rational polyhedral cones in $N \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^n$ is called a complete Fan if satisfy:

(i) if $\sigma \in \Delta$ and $\tau \prec \sigma$, then $\tau \in \Delta$;
(ii) if $\sigma_i, \sigma_j \in \Delta$, then $\sigma_i \cap \sigma_j \prec \sigma_i$ and $\sigma_i \cap \sigma_j \prec \sigma_j$;
(iii) $N \otimes_\mathbb{Z} \mathbb{R} = \sigma_1 \cup \cdots \cup \sigma_s$.

The dimension of a fan is the maximal dimension of its cones. An n-dimensional complete fan is simplicial if all its n-dimensional cones are simplicial.

Let Δ be a fan in $N \otimes_\mathbb{Z} \mathbb{R}$. It follows from Gordan's lemma (see [9]) that each k-dimensional cone σ^k in Δ (let say us generated by v_{ij}) defines a finitely generated semigroup $\sigma^k \cap N$. The dual $(n-k)$-dimensional cone

$$\check{\sigma} = \bigl\{m \in M \otimes_\mathbb{Z} \mathbb{R},\ \langle m, v_{ij}\rangle \geqslant 0\bigr\}$$

is then a rational polyhedral cone in $M \otimes_\mathbb{Z} \mathbb{R}$ and $\check{\sigma} \cap M$ is also a finitely generated semigroup. An affine n-dimensional toric variety corresponding to σ^k is the variety

$$\mathcal{U}_\sigma := \mathrm{Spec}\, \mathbb{C}[\check{\sigma} \cap M].$$

If a cone τ is a face of σ then $\check{\tau} \cap M$ is a subsemigroup of $\check{\sigma} \cap M$, hence $\mathcal{U}_\tau$ is embedded into $\mathcal{U}_\sigma$ as an open subset. The affine varieties corresponding to all cones of the fan Δ are glued together according to this rule into the toric variety $\mathbb{P}_\Delta$ associated with Δ. Is possible to show that a toric variety $\mathbb{P}_\Delta$ contain a complex torus $\mathbb{T}^n = (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of $\mathbb{T}^n$ on itself extends to an action of $\mathbb{T}^n$ on $\mathbb{P}_\Delta$. A toric variety $\mathbb{P}_\Delta$ determined by a simplicial complete fan Δ is projective and has quotient singularities, i.e., $\mathbb{P}_\Delta$ is a compact complex orbifold. For more details see [9].

Example 1. $\mathbb{T}^n$, $\mathbb{C}^n$ and $\mathbb{P}^n$ are toric varieties.

Example 2 *(Weighted projective spaces).* Let $\varpi = \{\varpi_0, \dots, \varpi_n\}$ be the set of positive integers satisfying the condition $\gcd(\varpi_0, \dots, \varpi_n) = 1$. Choose $n + 1$ vectors $e_0, \dots, e_n$ in $\mathbb{R}^n$, such that $\mathbb{R}^n$ is spanned by $e_0, \dots, e_n$ and there exists the linear relation

$$\varpi_0 e_0 + \cdots + \varpi_n e_n = 0.$$

Define N to be the lattice in $\mathbb{R}^n$ consisting of all integral linear combinations of $e_0, \dots, e_n$. Let $\Delta(w)$ be the set of all possible simplicial cones in $\mathbb{R}^n$ generated by proper subsets of $\{e_0, \dots, e_n\}$. Then $\Delta(w)$ is a rational simplicial complete n-dimensional fan. The corresponding variety $\mathbb{P}_{\Delta(w)}$ is the n-dimensional weighted projective space $\mathbb{P}(\varpi_0, \dots, \varpi_n)$. We will see in the next section that $\mathbb{P}(\varpi_0, \dots, \varpi_n)$ is a quotient of $\mathbb{C}^{n+1} \backslash \{0\}$ by the diagonal action of the torus $\mathbb{C}^*$,

$$(z_0, \dots, z_n) \mapsto \left(\lambda^{\varpi_0} z_0, \dots, \lambda^{\varpi_n} z_n\right), \quad \lambda \in \mathbb{C}^*.$$

In particular, if $(\varpi_0, \dots, \varpi_n) = (1, \dots, 1)$, then $\mathbb{P}(1, \dots, 1) = \mathbb{P}^n$.

Example 3 *(Multiprojective spaces).* If X and Y are toric varieties then $X \times Y$ is so. Thus, the multiprojective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ are examples of toric varieties.

2.1. The toric homogeneous coordinates

Let $\mathbb{P}_\Delta$ be the toric variety determined by a fan Δ in $N \simeq \mathbb{Z}^n$. The one-dimensional cones of Δ form the set $\Delta(1) = \{\vartheta_1, \dots, \vartheta_{n+r}\}$, where ϑ_i denote the unique generator of one-dimensional cone. If σ is any cone in Δ, then $\sigma(1) = \{\vartheta_i \in \Delta(1); \rho \subset \sigma\}$ is the set of one-dimensional faces of σ. We will assume that $\Delta(1)$ spans $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$.

Each $\vartheta_i \in \Delta(1)$ corresponds to an irreducible $\mathbb{T}^n$-invariant Weil divisor D_i in $\mathbb{P}_\Delta$, where $\mathbb{T}^n = N \otimes_\mathbb{Z} \mathbb{C}^*$ is the torus acting on $\mathbb{P}_\Delta$. It is known that the of $\mathbb{T}^n$-invariant Weil divisors on $\mathbb{P}_\Delta$ is a free abelian group of rank $n + r$, given by $\mathbb{Z}^{n+r} = \bigoplus_{i=1}^{n+r} \mathbb{Z} \cdot D_i$. Thus an element $D \in \mathbb{Z}^{n+r}$ is a sum $\sum_{i=1}^{n+r} a_i D_i$. The $\mathbb{T}^n$-invariant Cartier divisors form a subgroup $\mathrm{Div}_{\mathbb{T}^n}(\mathbb{P}_\Delta) \subset \mathbb{Z}^{n+r}$.

Each $m \in M$ gives a character $\chi^m : \mathbb{T}^n \to \mathbb{T}$, and hence χ^m is a rational function on $\mathbb{P}_\Delta$. As is well known, χ^m gives the Cartier divisor

$$\mathrm{Div}\left(\chi^m\right) = \sum_{i=1}^{n+r} -\langle m, \vartheta_i \rangle D_i.$$

We will consider the map

$$M \to \mathbb{Z}^{n+r}, \quad m \mapsto \sum_{i=1}^{n+r} -\langle m, \vartheta_i \rangle D_i.$$

This map is injective since $\Delta(1)$ spans $N_{\mathbb{R}}$. By [9, §3.4], we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathrm{Div}_{\mathbb{T}}(\mathbb{P}_\Delta) & \longrightarrow & \mathrm{Pic}(\mathbb{P}_\Delta) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{n+r} & \longrightarrow & \mathcal{A}_{n-1}(\mathbb{P}_\Delta) & \longrightarrow & 0. \end{array} \tag{1}$$

In particular, the Picard number of $\mathbb{P}_\Delta$ is given by $\rho(\mathbb{P}_\Delta) = r$.

For each $\vartheta_i \in \Delta(1)$, we introduce a variable z_i, and consider the polynomial ring

$$\mathrm{S} = \mathbb{C}\big[z_i;\ \vartheta_i \in \Delta(1)\big].$$

Note that a monomial $\prod_{i=1}^{n+r} z_\rho^{a_\rho}$ determines a divisor $\sum_{i=1}^{n+r} a_i D_i$ and to emphasize this relationship, we will write the monomial as z^D. We will grade S as follows, the degree of a monomial z^D is $\deg(z^D) = [D] \in \mathcal{A}_{n-1}(\mathbb{P}_\Delta)$. Using the exact sequence (1), it follows that two monomials $\prod_{i=1}^{n+r} z_i^{a_i}$ and $\prod_{i=1}^{n+r} z_i^{b_i}$ in S have the degree if and only if there is some $m \in M$ such that $a_i = \langle m, \vartheta_i\rangle + b_i$ for each $i = 1, \dots, n+r$. Then

$$\mathrm{S} = \bigoplus_{\alpha \in \mathcal{A}_{n-1}(\mathbb{P}_\Delta)} \mathrm{S}_\alpha,$$

where $\mathrm{S}_\alpha = \bigoplus_{\deg(z^D)=\alpha} \mathbb{C} \cdot z^D$. Note that $\mathrm{S}_\alpha \cdot \mathrm{S}_\beta \subset \mathrm{S}_{\alpha+\beta}$. The ring S is called *homogeneous coordinates ring* of the toric variety $\mathbb{P}_\Delta$.

Let $\mathcal{O}(D)$ be the coherent sheaf on $\mathbb{P}_\Delta$ determined by a Weil divisor D, then it follows from [4] that

$$\mathrm{S}_{\deg(D)} \simeq H^0\big(\mathbb{P}_\Delta, \mathcal{O}(D)\big),$$

moreover there is a commutative diagram

$$\begin{array}{ccc} \mathrm{S}_{\deg(D)} \otimes \mathrm{S}_{\deg(E)} & \longrightarrow & \mathrm{S}_{\deg(D+E)} \\ \downarrow & & \downarrow \\ \mathrm{H}^0(\mathbb{P}_\Delta, \mathcal{O}(D)) \otimes \mathrm{H}^0(\mathbb{P}_\Delta, \mathcal{O}(E)) & \longrightarrow & \mathrm{H}^0(\mathbb{P}_\Delta, \mathcal{O}(D+E)), \end{array}$$

where the top arrows is the polynomial multiplication. If $\mathbb{P}_\Delta$ is a complete toric variety, then:

(i) S_α is finite-dimensional for every α, and in particular, $\mathrm{S}_0 = \mathbb{C}$.
(ii) If $\alpha = [D]$ for an effective divisor $D = \sum_{i=1}^{n+r} a_i D_i$, it follows from [4] that $\dim_{\mathbb{C}} \mathrm{S}_\alpha = \#(\mathscr{P}_D \cap M)$, where

$$\mathscr{P}_D = \big\{m \in M_{\mathbb{R}};\ \langle m, \vartheta_i\rangle \geqslant -a_i \text{ for all } i = 1, \dots, n+r\big\}.$$

2.2. *Quotient construction*

We get the monomial

$$z^{\widehat{\sigma}} = \prod_{\vartheta_i \notin \sigma} z_i$$

which is the product of all variables not coming form edges of $\sigma \in \Delta$. Then define $\mathcal{Z}(\Delta) = V(z^{\widehat{\sigma}};\ \sigma \in \Delta) \subset \mathbb{C}^{n+r}$. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{T})$ to the exact sequence

$$0 \to M \to \mathbb{Z}^{n+r} \to \mathcal{A}_{n-1}(\mathbb{P}_\Delta) \to 0 \tag{2}$$

we get the sequence

$$1 \to \mathrm{Hom}_{\mathbb{Z}}\big(\mathcal{A}_{n-1}(\mathbb{P}_\Delta), \mathbb{T}\big) \to \mathbb{T}^{n+r} \to \mathbb{T}^n \to 1.$$

It is possible to show that the group $G(\Delta) := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{A}_{n-1}(\mathbb{P}_\Delta), \mathbb{T}) \subset \mathbb{T}^r$ given by

$$G(\Delta) = \left\{ (t_1, \dots, t_r) \in \mathbb{T}^r;\ \prod_{i=1}^{r} t_i^{\langle e_j, \vartheta_i \rangle} = 1,\ j = 1, \dots, r \right\}.$$

Define the action of $G(\Delta)$ on $\mathbb{C}^{n+r} - \mathcal{Z}(\Delta)$, given by

$$\begin{aligned} &G(\Delta) \times \big(\mathbb{C}^{n+r} - \mathcal{Z}(\Delta)\big) \to \mathbb{C}^{n+r} - \mathcal{Z}(\Delta), \\ &\big(g, (z_1, \dots, z_{n+r})\big) \mapsto \big(g(D_1)z_1, \dots, g(D_{n+r})z_{n+r}\big). \end{aligned}$$

Theorem 2.1. *(See D. Cox [4].) If $\mathbb{P}_\Delta$ is an n-dimensional toric variety where $\vartheta_1, \dots, \vartheta_{n+r}$ span $\mathbb{R}^n$, then*:

(i) *$\mathbb{P}_\Delta$ is a universal categorical quotient $(\mathbb{C}^{n+r} - \mathcal{Z}(\Delta))/G(\Delta)$;*
(ii) *$\mathbb{P}_\Delta$ is an orbifold $(\mathbb{C}^{n+r} - \mathcal{Z}(\Delta))/G(\Delta)$ if and only if $\mathbb{P}_\Delta$ is simplicial.*

To describe the action of $G(\Delta)$ when it has no torsion we consider the lattice of relations between generators of Δ, i.e. r linearly independent over $\mathbb{Z}$ relations between $\vartheta_1, \dots, \vartheta_{n+r}$,

$$\begin{cases} a_{11}\vartheta_1 + \cdots + a_{1(n+r)}\vartheta_{n+r} = 0, \\ \qquad\qquad \vdots \\ a_{r1}\vartheta_1 + \cdots + a_{r(n+r)}\vartheta_{n+r} = 0. \end{cases} \tag{3}$$

Thus by (1) the isomorphic to $\mathbb{T}^r$ factor of G defines an equivalence relation on $(\mathbb{C}^{n+r} - \mathcal{Z}(\Delta))/G(\Delta)$: let $u, v \in \mathbb{C}^{n+r} - \mathcal{Z}(\Delta)$ be, with $v = (v_1, \dots, v_{n+r})$, then $u \sim v$ if and only if

$$\exists (\lambda_1, \dots, \lambda_r) \in \mathbb{T}^r; \quad u = \big(\lambda_1^{a_{11}} \cdots \lambda_r^{a_{r1}} v_1, \dots, \lambda_1^{a_{1(n+r)}} \cdots \lambda_r^{a_{r(n+r)}} v_{n+r}\big). \tag{4}$$

Therefore, when $G(\Delta)$ has no torsion, the equivalence relation on $(\mathbb{C}^{n+r} - \mathcal{Z}(\Delta))/G(\Delta)$ is given by this formula. If $f \in S_\alpha$, it follows from [1, Lemma 3.8] the Euler's formula

$$i_{R_i}\, df = \theta_i(\alpha) f,$$

where $\theta_i \in \mathbb{C}$ and $R_i = \sum_{j=1}^{n+r} a_{ij} z_{ij} \frac{\partial}{\partial z_{ij}}$, $i = 1, \dots, r$. Moreover, $Lie(G) = \langle R_1, \dots, R_r \rangle$, see [4].

An element $\alpha \in \mathcal{A}_{n-1}(\mathbb{P}_\Delta)$ gives the character $\chi^\alpha : G(\Delta) \to \mathbb{T}$. The action of $G(\Delta)$ on $\mathbb{C}^{n+r}$ induces an action on S with the property that given $f \in S$, we have

$$f \in S_\alpha \quad \Leftrightarrow \quad f(g \cdot z) = \chi^\alpha(g) f(z), \quad \forall g \in G(\Delta),\ z \in \mathbb{C}^{n+r}. \tag{5}$$

The graded pieces of S are the eigenspaces of the action of $G(\Delta)$ on S. We say that $f \in S_\alpha$ is homogeneous of degree α. It follows that the set $\{f(z) = 0\}$ is well defined in $\mathbb{P}_\Delta$ and it defines a hypersurface.

We shall consider the following subfield of $\mathbb{C}(z_1, \dots, z_{n+r})$ given by

$$\widetilde{K}(\mathbb{P}_\Delta) = \left\{ \frac{P}{Q} \in \mathbb{C}(z_1, \dots, z_{n+r});\ P \in S_\alpha,\ Q \in S_\beta \right\}.$$

Thus, the field of rational functions on $\mathbb{P}_\Delta$, denoted by $K(\mathbb{P}_\Delta)$, is the subfield of $\widetilde{K}(\mathbb{P}_\Delta)$, such that $\deg(P) = \deg(Q)$.

3. One-dimensional foliations

Let $\mathbb{P}_\Delta$ be a complete simplicial toric variety of dimension n, and denote $\mathcal{O}_{\mathbb{P}_\Delta} := \mathcal{O}$. There exists an exact sequence known by *generalized Euler's sequence* [4]

$$0 \to \mathcal{O}^{\oplus r} \to \bigoplus_{i=1}^{n+r} \mathcal{O}(D_i) \to \mathcal{T}\mathbb{P}_\Delta \to 0,$$

where $\mathcal{T}\mathbb{P}_\Delta = \mathcal{H}om(\Omega^1_{\mathbb{P}_\Delta}, \mathcal{O})$ is the so-called Zarisk's sheaf of $\mathbb{P}_\Delta$. Let $i : X(\Delta) \to \mathbb{P}_\Delta$ be the inclusion of regular part $X(\Delta) := \mathbb{P}_\Delta - \mathrm{Sing}(\mathbb{P}_\Delta)$. Since $\mathbb{P}_\Delta$ is a complex orbifold then $\mathcal{T}\mathbb{P}_\Delta \simeq i_*\Theta_{X(\Delta)}$, see [6, Appendix A.2]. Let $\mathcal{O}(d_1, \dots, d_{n+r}) = \mathcal{O}(\sum_{i=1}^{n+r} d_i D_i)$ be, where $\sum_{i=1}^{n+r} d_i D_i$ is an effective divisor. Tensorizing the Euler's sequence by $\mathcal{O}(d_1, \dots, d_{n+r})$ we get

$$0 \to \mathcal{O}(d_1, \dots, d_{n+r})^{\oplus r} \to \bigoplus_{i=1}^{n+r} \mathcal{O}(d_1, \dots, d_i + 1, \dots, d_{n+r}) \to \mathcal{T}\mathbb{P}_\Delta(d_1, \dots, d_{n+r}) \to 0,$$

where $\mathcal{T}\mathbb{P}_\Delta(d_1, \dots, d_{n+r}) := \mathcal{T}\mathbb{P}_\Delta \otimes \mathcal{O}(d_1, \dots, d_{n+r})$.

Definition 3.1. A holomorphic foliation $\mathcal{F}$ on $\mathbb{P}_\Delta$ of degree $(d_1, \dots, d_{n+r})$ is a global section of $\mathcal{T}\mathbb{P}_\Delta \otimes \mathcal{O}(d_1, \dots, d_{n+r})$. We denote by $K_{\mathcal{F}} := \mathcal{O}(d_1, \dots, d_{n+r})$ the canonical sheaf associate to foliation $\mathcal{F}$.

From above exact sequence we conclude that a foliation on $\mathbb{P}_\Delta$ of degree $(d_1, \dots, d_r)$ is given by a polynomial vector in homogeneous coordinates of the form

$$X = \sum_{i=1}^{n+r} P_i \frac{\partial}{\partial z_i},$$

where P_i is a polynomial of degree $(d_1, \dots, d_i + 1, \dots, d_r)$ for all $i = 1, \dots, r$, modulo addition of a vector field of the form $\sum_{i=1}^{n+r} g_i R_i$. Therefore

$$\mathrm{Sing}(\mathcal{F}) = \pi\big(\{p \in \mathbb{C}^{n+r};\ R_1 \wedge \cdots \wedge R_{n+r} \wedge X(p) = 0\}\big),$$

where $\pi : (\mathbb{C}^{n+r} - \mathcal{Z}(\Delta))/G(\Delta) \to \mathbb{P}_\Delta$ is the canonical projection. Moreover, we have the following proposition.

Proposition 3.1. *Let $Fol((d_1, \dots, d_{n+r}), \mathbb{P}_\Delta)$ be the space of foliations of degree $(d_1, \dots, d_{n+r})$. Let $D^j = (d_j + 1)D_j + \sum_{\substack{i=1\\ i\neq j}}^{n+r} d_i D_i$ be and $D = \sum_{i=1}^{n+r} d_i D_i$. Then $Fol((d_1, \dots, d_{n+r}), \mathbb{P}_\Delta)$ is isomorphic to a complex projective space $\mathbb{P}^{N-1}$, where*

$$N = \sum_{j=1}^{n+r} \#(\mathscr{P}_{D^j} \cap M) - r \cdot \big[\#(\mathscr{P}_D \cap M)\big].$$

It is follows from (5) the polynomials $F, G \in S_\alpha$ define a rational function $\Theta : \mathbb{P}_\Delta \dashrightarrow \mathbb{P}^1$ by $\Theta = \frac{F}{G}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}_\Delta$ and $V = \{f = 0\}$ a hypersurface. We say that V is $\mathcal{F}$-invariant if and only if $X(f) = hf$, where X is a vector field which defines $\mathcal{F}$ in homogeneous coordinates. Also, a rational function Θ is a rational first integral for $\mathcal{F}$ if $X(\Theta) = 0$.

3.1. Examples

Rational scroll. Let $a_1, \dots, a_n$ be integers. Consider the $\mathbb{T}^2$-action on $(\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^n - \{0\})$ given as follows:

$$\mathbb{T}^2 \times \big(\mathbb{C}^2 - \{0\}\big) \times \big(\mathbb{C}^n - \{0\}\big) \to \big(\mathbb{C}^2 - \{0\}\big) \times \big(\mathbb{C}^n - \{0\}\big),$$
$$\big((\lambda, \mu), (x_1, x_2), (z_1, \dots, z_n)\big) \to \big((\lambda x_1, \lambda x_2), (\mu\lambda^{-a_1} z_1, \dots, \mu\lambda^{-a_n} z_n)\big).$$

The *rational scroll* $\mathbb{F}(a_1, \dots, a_n)$ is the quotient variety of $(\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^n - \{0\})$ by this action.

Let $E = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ be the vector bundle over $\mathbb{P}^1$. Write $\mathbb{P}(E)$ for the projectivized vector bundle

$$\mathbb{P}(E) \to \mathbb{P}^1$$

and let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological line bundle. It is possible show that $\mathbb{F}(a_1, \dots, a_n)$ is the image of $\mathbb{P}(E)$ by the embedding given by $\mathcal{O}_{\mathbb{P}(E)}(1)$, see [11].

Tow examples of this construction are:

(1) $\mathbb{F}(0, \dots, 0) \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$;
(2) $\mathbb{F}(a, 0)$ is a Hirzebruch surface; see [19].

We have that $\mathrm{Pic}(\mathbb{F}(a_1, \dots, a_n)) \simeq \mathbb{Z}L \oplus \mathbb{Z}M$, where L is the class of a fibre of π and M the class of any monomial $x_1^b x_2^c z_i$, with $b + c = a_i$. If all the $a_i > 0$, then M is the divisor class of the hyperplane section under the embedding $\mathbb{F}(a_1, \dots, a_n) \subset \mathbb{P}^{n+\sum_{i=1}^n a_i - 1}$. Let $\mathcal{O}(d_1, d_2) := \mathcal{O}(d_1 L + d_2 M)$. Thus, a foliation on $\mathbb{F}(a_1, \dots, a_n)$ is a global section of $T\mathbb{F}(a_1, \dots, a_n) \otimes \mathcal{O}(d_1, d_2)$ and has a bidegree (d_1, d_2). In this case Euler's sequence is given by

$$0 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1,0)^{\oplus 2} \oplus \bigoplus_{i=1}^n \mathcal{O}(-a_i, 1) \to \mathcal{T}\mathbb{F} \to 0,$$

and tensorizing by $\mathcal{O}(d_1, d_2)$ we get the sequence

$$0 \to \mathcal{O}(d_1, d_2)^{\oplus 2} \to \mathcal{O}(d_1 + 1, d_2)^{\oplus 2} \oplus \bigoplus_{i=1}^n \mathcal{O}(d_1 - a_i, d_2 + 1) \to \mathcal{T}\mathbb{F} \otimes \mathcal{O}(d_1, d_2) \to 0.$$

Therefore, a foliation on $\mathbb{F}(a_1, \dots, a_n)$ is given, in homogeneous coordinates, by a vector field

$$X = Q_1 \frac{\partial}{\partial x_1} + Q_2 \frac{\partial}{\partial x_2} + \sum_{i=0}^n P_i \frac{\partial}{\partial z_i},$$

where Q_i is bihomogeneous of bidegree $(d_1 + 1, d_2)$ and P_i bihomogeneous of bidegree $(d_1 - a_i, d_2 + 1)$, modulo $g_1 R_1 + g_2 R_2$, where

$$R_1 = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}, \qquad R_2 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \sum_{i=1}^n -a_i z_i \frac{\partial}{\partial z_i}$$

and g_i has bidegree (d_1, d_2).

Multiprojective foliations. The Euler's sequence over the multiprojective space $\mathbf{P}^{(n_1,\dots,n_r)}$ is given by

$$0 \to \mathcal{O}^{\oplus r} \to \bigoplus_{j=1}^{r} \mathcal{O}(0,\dots,\underbrace{1}_{j},\dots,0)^{\oplus n_j+1} \to T\mathbf{P}^{(n_1,\dots,n_r)} \to 0.$$

Tensorizing this sequence by $\mathcal{O}(d_1-1,\dots,d_r-1)$ we get the exact sequence

$$0 \to \mathcal{O}(d_1-1,\dots,d_r-1)^{\oplus r} \to \bigoplus_{i=1}^{r} \mathcal{O}(d_1-1,\dots,d_i,\dots,d_r-1)^{\oplus n_i+1}$$
$$\to T\mathbf{P}^{(n_1,\dots,n_r)}(d_1-1,\dots,d_r-1) \to 0.$$

We conclude that a foliation on $\mathbf{P}^{(n_1,\dots,n_r)}$ of multidegree $(d_1-1,\dots,d_r-1)$ can be represented in multihomogeneous coordinates of $\mathbb{C}^{\sum_{i=1}^{r}(n_i+1)}$ for a polynomial vector field of the form

$$X = \sum_{i=1}^{r} X_i$$

with $X_i = \sum_{j=0}^{n_i} P_{ij} \frac{\partial}{\partial Z_{ij}}$, and P_{ij} is a multihomogeneous polynomial of multidegree $(d_1-1, \dots, d_i, \dots, d_r-1)$ modulo

$$\sum_{i=1}^{r} g_i R_i,$$

where g_i has multidegree $d = (d_1-1,\dots,d_r-1)$ and $R_i = \sum_{j=0}^{n_i} x_{ij} \frac{\partial}{\partial x_{ij}}$.

Weighted projective foliations. The Euler's sequence on $\mathbb{P}(\varpi)$ can be see since an exact sequence of orbibundle

$$0 \to \underline{\mathbb{C}} \to \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}(\varpi)}(\varpi_i) \to T\mathbb{P}(\varpi) \to 0$$

where $\underline{\mathbb{C}}$ is the trivial line orbibundle on $\mathbb{P}(\varpi)$, see [15]. It follows that a quasi-homogeneous vector field X induces a foliation $\mathcal{F}$ of $\mathbb{P}(\varpi)$ and that $gR_w + X$ define the same foliation as X, where R_w is the adapted radial vector field $R_\varpi = \sum_{i=0}^{n} \varpi_i z_i \frac{\partial}{\partial z_i}$, with g a quasi-homogeneous polynomial of type $(\varpi_0,\dots,\varpi_n)$ and degree $d-1$.

4. Proof of Theorem 1.1

Proof. Let $f_1, f_2, \dots, f_{N+n+r}$ be the $\mathcal{F}$-invariant irreducible hypersurfaces, where $N = h^0(\mathbb{P}_\Delta, \mathcal{O}(K_\mathcal{F}))$. Let $X = \sum_{i=1}^{n+r} P_i \frac{\partial}{\partial z_i}$ be a polynomial vector fields that defines $\mathcal{F}$ in homogeneous coordinates. It is follows from invarianceness that

$$\frac{X(f_j)}{f_j} = h_j \in \mathrm{S}_{[K_\mathcal{F}]}, \quad j = 1, 2, \dots, N+n+r.$$

We get the following relations

$$\begin{aligned}
&\lambda_{11}h_1+\lambda_{12}h_2+\lambda_{13}h_3+\cdots+\lambda_{1(N+1)}h_{N+1}=0,\\
&\lambda_{22}h_2+\lambda_{23}h_3+\cdots+\lambda_{2(N+1)}h_{N+1}+\lambda_{2(N+2)}h_{N+2}=0,\\
&\lambda_{33}h_3+\lambda_{34}h_4+\cdots+\lambda_{3(N+2)}h_{N+2}+\lambda_{3(N+3)}h_{N+3}=0,\\
&\quad\vdots\\
&\lambda_{jj}h_j+\lambda_{j(j+1)}h_{j+1}+\cdots+\lambda_{j(N+j)}h_{N+j}=0,
\end{aligned}$$

where $j=n+r$. We can suppose that $\lambda_{ii}\neq 0$, for all $i=1,\dots,n$. Define the rational 1-form on $\mathbb{C}^{n+r}$ given by

$$\eta_k=\sum_{j=k}^{N+k}\lambda_{kj}\frac{df_j}{f_j},\quad k=1,\dots,n+r.$$

Observe that by construction $|\eta_i|_\infty\neq|\eta_j|_\infty$ for all $i\neq j$, where $|\cdot|_\infty$ denote the set of poles. Contracting by X we get

$$i_X\eta_k=\sum_{j=k}^{N+k}\lambda_{kj}\frac{X(f_j)}{f_j}=\sum_{j=k}^{N+k}\lambda_{kj}h_j=0,$$

for all $k=1,\dots,n+r$. We affirm that $\eta_1,\dots,\eta_{n+r}$ are linearly dependent over the rational functions field $\widetilde{K}(\mathbb{P}_\Delta)$. Otherwise, there exists a rational function $R\neq 0$ such that

$$\eta=\eta_1\wedge\cdots\wedge\eta_{n+r}=R\,dz_1\wedge\cdots\wedge dz_{n+r}.$$

Contracting η by $X=\sum_{i=1}^{n+r}P_i\frac{\partial}{\partial z_i}$ we have $R\imath_X(dz_1\wedge\cdots\wedge dz_{n+r})=0$, since $i_X\eta_k=0$, for all $k=1,\dots,n+r$. But $R\neq 0$, thus

$$0=\imath_X(dz_1\wedge\cdots\wedge dz_n)=\sum_{i=1}^{n+r}(-1)^{i+1}P_i\,dz_1\wedge\cdots\wedge\widehat{dz_i}\wedge\cdots\wedge dz_{n+r}.$$

This implies that $P_1=\cdots=P_{n+r}=0$, i.e., $X\equiv 0$, a contradiction. Let V be the $\widetilde{K}(\mathbb{P}_\Delta)$-linear space generated by $\{\eta_1,\dots,\eta_{n+r}\}$, suppose that $\dim_{\widetilde{K}(\mathbb{P}_\Delta)}V=k$ and

$$V=\langle\eta_1,\dots,\eta_k\rangle_{\widetilde{K}(\mathbb{P}_\Delta)},$$

for some $1\leqslant k<n+r$. There exist rational functions $R_1,\dots,R_k,R_{k+1}\in\widetilde{K}(\mathbb{P}_\Delta)$, with $R_{k+1}\neq 0$, such that

$$R_1\eta_1+\cdots+R_k\eta_k+R_{k+1}\eta_{k+1}=0,$$

multiplying this equation by $\operatorname{lcm}(R_1,\dots,R_{k+1})$ we obtain

$$Q_1\eta_1+\cdots+Q_k\eta_k+Q_{k+1}\eta_{k+1}=0,$$

where each Q_i is a homogeneous polynomial in the Cox ring of $\mathbb{P}_\Delta$. Now, we multiply this equation by $F=\prod_{i=1}^{N+n+r}f_i$

$$Q_1\widetilde{\eta_1}+\cdots+Q_k\widetilde{\eta_k}+Q_{k+1}\widetilde{\eta_{k+1}}=0,\tag{6}$$

where $\widetilde{\eta_i} = F\eta_i$. Since $\widetilde{\eta_i}$ are all homogeneous of the same degree, we can extract from relation (6) a relation

$$Q_{i_1}\widetilde{\eta_{i_1}} + \cdots + Q_{i_\ell}\widetilde{\eta_{i_\ell}} + Q_{k+1}\widetilde{\eta_{k+1}} = 0,$$

where $\deg(Q_{i_j}) = \deg(Q_{k+1})$, $i_j \in \{1,\dots,k\}$ and $j = 1,\dots,\ell \leqslant k$. Hence, we get

$$F\eta_{k+1} = R_{i_1}F\eta_{i_1} + \cdots + R_{i_\ell}F\eta_{i_\ell}, \tag{7}$$

where $R_{i_j} = -\frac{Q_{i_j}}{Q_{k+1}} \in K(\mathbb{P}_\Delta)$. Dividing by F and differentiating

$$0 = dR_{i_1} \wedge \eta_{i_j} + \cdots + dR_{i_\ell} \wedge \eta_{i_\ell}.$$

Now, contracting by X results

$$0 = X(R_{i_1}) \cdot \eta_{i_1} + \cdots + X(R_{i_\ell}) \cdot \eta_{i_\ell}.$$

Since $\ell \leqslant k$ then $X(R_{i_1}) = \cdots = X(R_{i_\ell}) = 0$. That is, the rational function R_{i_j}, $j = 1,\dots,\ell$, is either a first integral for the foliation $\mathcal{F}$ induced by the vector field X or it is constant. It remains to observe that at least one rational function R_{i_j} is not constant. Indeed, this follows from relation (7) and the fact that the set of poles $|\eta_{i_j}|_\infty \neq |\eta_{i_r}|_\infty$, for all $j \neq r$. $\square$

Example 4. It follows from [16] that

$$h^0\big(\mathbb{F}(a_1,\dots,a_n), \mathcal{O}(d_1,d_2)\big) = \left(\sum_{i=1}^n a_i\right)\binom{d_1+n-1}{n} + (d_2+1)\binom{d_1+n-1}{n-1}.$$

Let $\mathcal{F}$ be a foliation on $\mathbb{F}(a_1,\dots,a_n)$ of bidegree (d_1,d_2). If $\mathcal{F}$ admits

$$\mathcal{N}(a_1,\dots,a_n,d_1,d_2,n) = \left(\sum_{i=0}^n a_i\right)\binom{d_1+n-1}{n} + (d_2+1)\binom{d_1+n-1}{n-1} + n + 2$$

invariant irreducible algebraic hypersurfaces, then $\mathcal{F}$ admits a rational first integral.

Example 5. Let $\mathcal{F}$ be a foliation on $\mathbf{P}^{(n_1,\dots,n_r)}$ of degree $(d_1-1,\dots,d_r-1)$. If $\mathcal{F}$ admits

$$\mathcal{N}(n_1,\dots,n_r,d_1,\dots,d_r) = \prod_{i=1}^r \binom{d_i+n_i-1}{n_i} + \sum_{i=1}^r n_i + r$$

invariant irreducible algebraic hypersurfaces, then $\mathcal{F}$ admit a rational first integral.

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