Finite semigroups, feedback, and the Letichevsky criteria on non-empty words in finite automata

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Abstract

This paper relates classes of finite automata under various feedback products to some well-known pseudovarieties of finite semigroups via a study of their irreducible divisors (in the sense of Krohn–Rhodes). In particular, this serves to relate some classical results of Krohn, Rhodes, Stiffler, Eilenberg, Letichevsky, Gécseg, Ésik, and Horváth. We show that for a finite automaton satisfaction of (1) the Letichevsky criterion for non-empty words, (2) the semi-Letichevsky criterion for non-empty words, or (3) neither criterion, corresponds, respectively, to the following properties of the characteristic semigroup of the automaton: (1) non-constructability as a divisor of a cascade product of copies of the two-element monoid with zero U, (2) such constructability while having U but no other non-trivial irreducible semigroup as a divisor, or (3) having no non-trivial irreducible semigroup divisors at all. The latter two cases are exactly the cases in which the characteristic semigroup is $\mathscr{R}$-trivial.

This algebraic characterization supports the transfer of results about finite automata to results about finite semigroups (and vice versa), and yields insight into the lattice of pseudovarieties of finite semigroups—or, equivalently via the Eilenberg correspondence, the lattice of +-varieties of regular languages—and the operators on these lattices that are naturally associated to various automata products with bounded feedback. In particular, all operators with non-trivial feedback are shown to be equivalent, and we characterize all pseudovarieties of finite semigroups closed

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under each type of feedback product either explicitly or by reducing the question to closure under the cascade product.

1. Preliminaries and previous results

1.1. Automata

A finite automaton $\mathscr{A} = (A, X, \delta)$ is a finite set of states A, finite input alphabet X, and transition function $\delta : A \times X \to A$. Members of X are called the *input letters* of $\mathscr{A}$. X^* denotes the set of finite words over X. If $w \in X^*$, then the length $|w|$ of w is n if $w = x_1 \ldots x_n$ ($x_i \in X$, $1 \leqslant i \leqslant n$). The unique word of length zero in X^* is denoted λ. $X^+ = X^* \setminus \{\lambda\}$ denotes the words over X of positive length. We extend δ to words over X inductively by letting

$$\delta(a, \lambda) = a \quad \text{and} \quad \delta(a, wx) = \delta(\delta(a, w), x)$$

for all $a \in A$, $x \in X$, and $w \in X^*$. We write $a \cdot w$ for $\delta(a, w)$ if no confusion can result. Clearly $(a \cdot w) \cdot w' = a \cdot ww'$ for all $a \in A$, $w, w' \in X^*$. Note that we do *not* exclude the possibility that A or X or both may be empty.

Let $\mathscr{B} = (B, Y, \delta')$ also be a (finite) automaton. Then a homomorphism of automata $\varphi : \mathscr{A} \to \mathscr{B}$ is a pair of mappings $\varphi_1 : A \to B$ and $\varphi_2 : X \to Y$ such that $\varphi_1(a \cdot x) = \varphi_1(a) \cdot \varphi_2(x)$ holds for all $a \in A$, $x \in X$. If both φ_1 and φ_2 are surjective, then $\mathscr{B}$ is said to be a *homomorphic image* of $\mathscr{A}$. If on the other hand both φ_1 and φ_2 are injective, then $\mathscr{A}$ is said to be (isomorphic to) a *subautomaton* of $\mathscr{B}$. If φ_1 and φ_2 are both bijective, then we say φ is an *isomorphism* from $\mathscr{A}$ to $\mathscr{B}$. We shall generally not distinguish among isomorphic structures.

1.2. Products of automata with feedback

Let $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be finite automata and let X be a finite alphabet. Then a *general product* (with arbitrary feedback among factors) is an automaton with states $A_1 \times \cdots \times A_n$ and transition function of the form

$$(a_1, \ldots, a_n) \cdot x = (a'_1, \ldots, a'_n) \quad \text{with} \quad a'_i = a_i \cdot f_i(a_1, \ldots, a_n, x),$$

where $a_i, a'_i \in A_i$, $x \in X$, $1 \leqslant i \leqslant n$, for some $f_i : A_1 \times \cdots \times A_n \times X \to X_i$. For i from 1 to n, the function f_i is called the ith *feedback function* of the general product, and gives an input letter to $\mathscr{A}_i$ depending on the input letter x and the state components

$(a_1, \ldots, a_n)$.[1] Such a product is completely determined by its component automata, the input alphabet X, and feedback functions.[2]

In this paper, we shall study some products which restrict the length of feedback. If each f_i may depend only on x and the coordinates a_j with $j < i$, then we have a *cascade product*. For $k \geqslant 0$, if each f_i may depend only on x and a_j with $j - k < i$ then we have an α_k-*product*, that is, a *product with length of feedback bounded by k*. The cascade product is thus an α_0-product, and any general product is an α_k product for some k (e.g. for $k \geqslant n$, the number of factors).[3] We have a *quasi-direct product* or *q-product* if each f_i may depend only on x. Every α_k-product is obviously also an α_{k+n}-product for all $n \geqslant 0$. Given a class of finite automata $\mathcal{K}$ and a product π, let $\pi(\mathcal{K})$ denote all finite automata which can be constructed as π-products of members of $\mathcal{K}$. (In speaking of classes of automata, we shall assume they are closed under isomorphism.) We say a general product of automata has *non-trivial feedback* if it is an α_k-product for some $k > 0$ but is not an α_0-product. Thus we have a hierarchy

$$\mathcal{K} \subseteq q(\mathcal{K}) \subseteq \alpha_0(\mathcal{K}) \subseteq \alpha_1(\mathcal{K}) \subseteq \alpha_2(\mathcal{K}) \subseteq \cdots \subseteq \alpha_k(\mathcal{K}) \subseteq \alpha_{k+1}(\mathcal{K})$$
$$\subseteq \cdots \subseteq \alpha_\infty(\mathcal{K}),$$

where $\alpha_\infty(\mathcal{K}) = \bigcup_{k=0}^{\infty} \alpha_k(\mathcal{K})$ is of course the general product. It is easy to see [6]:

Lemma 1. *For all $0 \leqslant k, n \leqslant \infty$, and classes of finite automata $\mathcal{K}, \mathcal{K}'$:*
1. $\mathcal{K} \subseteq \alpha_k(\mathcal{K})$,
2. $\mathcal{K} \subseteq \mathcal{K}' \Rightarrow \alpha_k(\mathcal{K}) \subseteq \alpha_k(\mathcal{K}')$,
3. $\alpha_0(\mathcal{K}) = \alpha_0(\alpha_0(\mathcal{K}))$,
4. $\alpha_\infty(\mathcal{K}) = \alpha_\infty(\alpha_\infty(\mathcal{K}))$,
5. $\alpha_0(\alpha_k(\mathcal{K})) = \alpha_k(\mathcal{K})$,
6. $\alpha_k(\mathcal{K}) \subseteq \alpha_{k+n}(\mathcal{K})$. $\square$

In particular, α_0 and α_∞ are *closure operators* on classes of finite automata (since 1,2,3 resp. 1,2,4 of the lemma hold). It is certainly *not* true that $\alpha_k(\alpha_n(\mathcal{K})) = \alpha_{k+n}(\mathcal{K})$ for general k, n. It is also *not* true for general k that $\alpha_k(\alpha_n(\mathcal{K})) = \alpha_k(\mathcal{K})$.

An automaton $\mathcal{A}$ *homomorphically represents* an automaton $\mathcal{B}$ if $\mathcal{B}$ is a homomorphic image of a subautomation of $\mathcal{A}$. A class $\mathcal{K}$ of finite automata is said to be *homomorphically complete* if every finite automaton can be homomorphically represented by an automaton from $\mathcal{K}$.

If $\mathcal{K}$ is a class of finite automata, $H(\mathcal{K})$ denotes all homomorphic images of members of $\mathcal{K}$, and $S(\mathcal{K})$ denotes all subautomata of members of $\mathcal{K}$. We sometimes write $P_k(\mathcal{K})$ for $\alpha_k(\mathcal{K})$ and $P(\mathcal{K})$ for $q(\mathcal{K})$. We write $HSP_k(\mathcal{K})$ for $H(S(\alpha_k(\mathcal{K})))$ for $0 \leqslant k \leqslant \infty$. Thus, $HSP_k(\mathcal{K})$ is the class of automata which can be homomorphically

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[1] The general product is sometimes also called the *Gluškov product*.

[2] For $n = 0$, the empty product is an automaton with exactly one state—'the unique zero-tuple'—on which each input letter $x \in X$ acts in the only possible way.

[3] The cascade (or feedback-free) product has been studied since at least the early 1960s in computer science and electrical engineering. The α_k-products were introduced by F. Gécseg in 1975.

represented by α_k-products of members of $\mathcal{K}$, and we write $HSP(\mathcal{K})$ if the quasi-direct product is used. Hence we also have a hierarchy

$$\mathcal{K} \subseteq HSP(\mathcal{K}) \subseteq HSP_0(\mathcal{K}) \subseteq HSP_1(\mathcal{K}) \subseteq HSP_2(\mathcal{K})$$
$$\subseteq \cdots \subseteq HSP_k(\mathcal{K}) \subseteq HSP_{k+1}(\mathcal{K}) \subseteq \cdots \subseteq HSP_\infty(\mathcal{K}).$$

It is an elementary exercise to check the well-known fact that $HSP(\mathcal{K}) = HSP(HSP(\mathcal{K}))$ and, moreover, $HSP_i(\mathcal{K}) = HSP(HSP_i(K))$ for all $0 \leqslant i \leqslant \infty$ (e.g. [6]).
We recall

Theorem 2 (Letichevsky Decomposition Theorem [12]). *For every class $\mathcal{K}$ of finite automata, $\alpha_\infty(\mathcal{K})$ is homomorphically complete if and only if there exists an automaton $\mathcal{A} = (A, X, \delta)$ in $\mathcal{K}$ such that*

$$(Let) \quad \exists a_0 \in A, x, y \in X, p, q \in X^+, a_0 \cdot x \neq a_0 \cdot y, \text{ and } a_0 \cdot xp = a_0 \cdot yq = a_0.$$
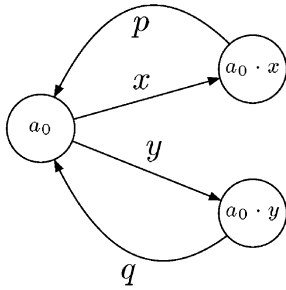
The Letichevsky Criterion

Remark. This formulation of the Letichevsky criterion is equivalent to the usual one which also allows p or q to be possibly empty: If the criterion holds with $p = \lambda$, then $a_0 = a_0 \cdot xp = a_0 \cdot x$, so we may replace p by the letter x. A similar observation holds for q (and for the semi-Letichevsky criterion, introduced in the sequel).

It is said that a finite automaton $\mathcal{A}$ *satisfies Letichevsky's criterion* if it has the above property (*Let*). We say a class of finite an automata $\mathcal{K}$ *satisfies Letichevsky's criterion* if $\mathcal{K}$ has a member that satisfies (*Let*). We then write $\mathcal{A} \models Let$ and $\mathcal{K} \models Let$, respectively. We write $\neg\mathscr{LET}$ for the class of finite automata that do not satisfy (*Let*), and $\mathscr{All}$ for the class of all finite automata.

Corollary 3. *For any class $\mathcal{K}$ of finite automata*:
1. $HSP_\infty(\mathcal{K}) = \mathscr{All} \Leftrightarrow \mathcal{K} \models Let$.
2. $\neg\mathscr{LET}$ *is closed under the general product, i.e.* $\alpha_\infty(\neg\mathscr{LET}) \subseteq \neg\mathscr{LET}$.
3. $HSP_\infty(\neg\mathscr{LET}) = \neg\mathscr{LET}$.

When is $\alpha_k(\mathcal{K})$ homomorphically complete? A strengthening of Letichevsky's Theorem gives a partial answer:

Theorem 4 (Ésik). $\mathscr{K}$ *is homomorphically complete for the* α_2-*product if and only if* $\mathscr{K}$ *satisfies the Letichevsky criterion.*

That is, $HSP_2(\mathscr{K}) = \mathscr{A}ll \Leftrightarrow \mathscr{K} \models Let.$

Proof. See [4] or [6, Theorem 4.10]. $\square$

This implies $HSP_\infty(\mathscr{K}) = HSP_2(\mathscr{K})$ holds if $\mathscr{K} \models Let$. But remarkably equality holds for any $\mathscr{K}$:
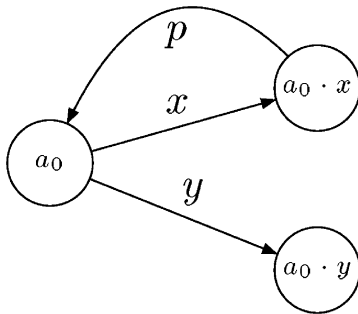
Theorem 5 (Ésik and Horváth [5]). *Let* $\mathscr{K}$ *be any class of finite automata. Then a finite automaton* $\mathscr{A} \in HSP_\infty(\mathscr{K})$ *if and only if* $\mathscr{A} \in HSP_2(\mathscr{K})$.

Proof. See [5] or [6, Theorem 5.4]. $\square$

Thus the HSP_k hierarchy collapses at $k = 2$ for every $\mathscr{K}$.

But in many cases it collapses for $k < 2$. If $\mathscr{A} = (A, X, \delta)$ does not satisfy Letichevsky's criterion but we have $a_0 \cdot x \neq a_0 \cdot y$, and $a_0 \cdot xp = a_0$ for some $a_0 \in A$, $x, y \in X$ and $p \in X^+$ then $\mathscr{A}$ satisfies the *semi-Letichevsky criterion* (*SL*):

(SL) $\neg Let$ and $\exists a_0 \in A, x, y \in X, p \in X^+, a_0 \cdot x \neq a_0 \cdot y$, and $a_0 \cdot xp = a_0$.

The Semi-Letichevsky Criterion

Examining the details of the proof of the Ésik–Horváth Theorem (as presented in [5] or [6, pp. 49–54]), one sees that it actually shows:

Corollary 6. *Let* $\mathscr{K}$ *be any class of finite automata. Then*

$$HSP_\infty(\mathscr{K}) = \begin{cases} HSP_2(\mathscr{K}) & \text{if } \mathscr{K} \text{ satisfies the Letichevsky criterion,} \\ HSP_1(\mathscr{K}) & \text{if } \mathscr{K} \text{ satisfies the semi-Letichevsky criterion,} \\ HSP_0(\mathscr{K}) & \text{otherwise.} \end{cases}$$

1.3. Semigroups, transformation semigroups, and pseudovarieties

A *semigroup* is a set S with an associative multiplication operation. That is, for all $x, y, z \in S$, $(xy)z = x(yz)$. A semigroup S is a *monoid* if it has an *identity element* $1 \in S$ such that $1s = s = s1$ for all $s \in S$. For any alphabet X, X^+ is a semigroup with

concatentation as the associative multiplication, and is called the *free semigroup* on X. Similarly, X^* is the *free monoid* on X, with identity element λ. A monoid is a *group* if in addition for each $s \in S$ there exists an *inverse* $s^{-1} \in S$ such that $s^{-1}s = 1 = ss^{-1}$. An *idempotent* in S is an element e such that $e^2 = e$. If S is a finite semigroup, it is easy to show that each element s of S has a unique idempotent power. Notation: we take $\omega(s)$ to be the least integer greater than 1, such that $s^{\omega(s)}s^{\omega(s)-1} = s^{\omega(s)-1}$. Note that $s^{\omega(s)}$ is the unique idempotent power of s. We write ω for $\omega(s)$, where there can be no confusion about s, and thus we shall write also $(s)^\omega$ or s^ω for this idempotent.

If S is a semigroup, then the *reverse semigroup* S^ρ has the same underlying set as S but multiplication $*$ with $x * y = yx$, where $x, y \in S^\rho$ and yx is their product in S.

If X and Y are subsets of S then $XY = \{xy \in S \mid x \in X, y \in Y\}$. (Of course XY is empty if either of X or Y is; and also $XY = X$ if $Y = \{1\}$ (and vice versa) if 1 is an identity element of S.) A subset $T \subseteq S$ is a *subsemigroup* of S if $T^2 \subseteq T$. A *homomorphism* $\varphi : S_1 \to S_2$ from a semigroup S_1 to a semigroup S_2 is a function such that $\varphi(s)\varphi(s') = \varphi(ss')$ for all $s, s' \in S$. If φ is surjective, then S_2 is a *homomorphic image* of S_1. A semigroup S *divides* (and is a *divisor* of) a semigroup T if S is a homomorphic image of a subsemigroup of T. We denote division by $S \prec T$. A non-trivial group G is *simple* if the only homomorphic images of G are G and the trivial group $\{1\}$ (up to isomorphism).

If S is a semigroup, let S^I denote the monoid $S^I = S \cup \{I\}$ with identity I, where I is a new symbol (i.e. $I \notin S$), and associative multiplication

$$ss' = \begin{cases} s & \text{if } s' = I, \\ s' & \text{if } s = I, \\ ss' & \text{if } s, s' \in S \end{cases}$$

for all $s, s' \in S^I$. Also, let

$$S^\bullet = \begin{cases} S & \text{if } S \text{ is a monoid,} \\ S^I & \text{otherwise.} \end{cases}$$

A *transformation semigroup* (A, S) is an automaton (A, S, δ) such that the set of inputs S is a semigroup and $(a \cdot s) \cdot s' = a \cdot ss'$ for all $a \in A$ and $s, s' \in S$. Here ss' is the product under the semigroup multiplication of S. Furthermore, the action of S is required to be *faithful*, i.e. if $a \cdot s = a \cdot s'$ holds for all $a \in A$, then $s = s'$.

Then the *right regular representation of S* is the transformation semigroup $(S^\bullet, S)$ with transition function given by the multiplication in $S^\bullet$. If S is a semigroup, sometimes we write "S" in a context where a transformation semigroup or an automaton is required; in this case, S denotes the right regular representation $(S^\bullet, S)$. For example, $\alpha_i(\{S\})$ denotes all α_i-products of factors $(S^\bullet, S)$, where the latter is viewed as an automaton.

Given any automaton $\mathscr{A} = (A, X, \delta)$, let $\mathscr{A}^+ = (A, S(\mathscr{A}), \delta^+)$ denote its *associated non-empty input word automaton* (also known as the *associated transformation semigroup*) whose states are the same as those of $\mathscr{A}$, and whose set of input letters $S(\mathscr{A})$ is the set of the transformations induced in $\mathscr{A}$ by words in X^+. That is, each non-empty word $w \in X^+$ represents an input letter of $\mathscr{A}^+$ with $a \cdot w = \delta(a, w)$ for all $a \in A$, and two

words $w, w' \in X^+$ represent the same input letter of $\mathscr{A}^+$ if and only if $\delta(a, w) = \delta(a, w')$ for all states $a \in A$. We then write $[w] = [w']$, and we have $S(\mathscr{A}) = \{[w] : w \in X^+\}$ with $\delta^+(a, [w]) = \delta(a, w)$. Of course, $S(\mathscr{A})$ is finite (since A is), and the map $w \mapsto [w]$ is a homomorphism of semigroups from the free semigroup X^+ onto $S(A)$. $S(\mathscr{A})$ is called the *characteristic semigroup* (or *transition semigroup*) of $\mathscr{A}$. If $\mathscr{A} = (A, X, \delta)$ is any automaton then of course $\mathscr{A}^+ = (A, S(\mathscr{A}), \delta^+)$ is a transformation semigroup. Obviously, by faithfulness, the characteristic semigroup of any transformation semigroup (A, S) is just the semigroup S. In particular, the characteristic semigroup of the right regular representation $(S^\bullet, S)$ is S, and moreover, the characteristic semigroup $S(\mathscr{A}^+)$ of $\mathscr{A}^+$ is just the characteristic semigroup $S(\mathscr{A})$ of $\mathscr{A}$.

An automaton $\mathscr{A}$ is a *group automaton* if each member of the input alphabet X acts as a permutation on the set A. If G is group, then its right regular representation $(G^\bullet, G) = (G, G)$ is the *group automaton corresponding to the group G*. It is easy to verify that a cascade product of group automata is itself a group automaton.

The *flip-flop automaton* $\mathscr{F}$ has states $\{a, b\}$ and inputs $\{a, b, 1\} = F$ where the a and b act as constants and 1 acts as the identity. Its characteristic semigroup $F = S(\mathscr{F})$ is called the *flip-flop monoid*, and has multiplication table:

F	1	a	b
1	1	a	b
a	a	a	b
b	b	a	b

A semigroup S is *irreducible* if whenever $\mathscr{A}$ is an automaton with $S(\mathscr{A}) = S$ and $\mathscr{A} \in HSP_0(\{\mathscr{A}_1, \ldots, \mathscr{A}_n\})$ for some $\mathscr{A}_1, \ldots, \mathscr{A}_n \in \mathscr{A}ll$, then $S \prec S(\mathscr{A}_i)$ for some i ($1 \leqslant i \leqslant n$). If S is a finite semigroup, $IRRED(S)$ denotes the set of non-trivial irreducible divisors of S, i.e. those having at least two elements. If $\mathscr{A}$ is a finite automaton, $IRRED(\mathscr{A})$ denotes $IRRED(S(\mathscr{A}))$. If $\mathscr{K}$ is a class of finite automata, $IRRED(\mathscr{K})$ is the union of all $IRRED(\mathscr{A})$ for $\mathscr{A} \in \mathscr{K}$. $PRIMES(S)$ denotes the set of finite simple groups that divide S.

Theorem 7 (Krohn–Rhodes Theorem [10,11]). *Let $\mathscr{A}$ be a finite automaton. Then $\mathscr{A}$ can be homomorphically represented by a cascade of flip-flops $\mathscr{F}$ and group automata corresponding to $PRIMES(S(\mathscr{A}))$. That is, $\mathscr{A} \in HSP_0(\{\mathscr{F}\} \cup PRIMES(S(\mathscr{A})))$. Moreover, if $\mathscr{A}$ is a non-trivial group automaton, then the flip-flop $\mathscr{F}$ may be omitted.*

If $\mathscr{A}$ is homomorphically represented by a cascade of automata $\mathscr{A}_1, \ldots, \mathscr{A}_n$, then every irreducible semigroup that divides $S(\mathscr{A})$ divides $S(\mathscr{A}_i)$ for some i ($1 \leqslant i \leqslant n$). That is, $\mathscr{A} \in HSP_0(\mathscr{K}) \Rightarrow IRRED(\mathscr{A}) \subseteq IRRED(\mathscr{K})$. Moreover, a finite semigroup S is irreducible if and only if S is simple group or a divisor of the flip-flop monoid.

Corollary 8. *If every subgroup of $S(\mathscr{A})$ is trivial, then $\mathscr{A} \in HSP_0(\{\mathscr{F}\})$.*

Proof. Since every subgroup of $S(\mathscr{A})$ is trivial, $PRIMES(S(\mathscr{A})) = \emptyset$, so the conclusion follows from the first part of the Krohn–Rhodes Theorem. $\square$

The last part of the Krohn and Rhodes Theorem implies that the irreducible finite semigroups are exactly the finite simple groups and the subsemigroups of the flip-flop monoid F. These are the flip-flop monoid F itself, the two-element monoid U, the two-element right-zero semigroup 2^r, the one-element semigroup $\{1\}$, and the empty semigroup $\emptyset$.

U	1	0
1	1	0
0	0	0

2^r	a	b
a	a	b
b	a	b

Corollary 9. *If $\mathscr{K}$ is a class of finite automata such that $HSP_0(\mathscr{K}) = \mathscr{A}ll$, then $IRRED(\mathscr{K}) = all$ finite simple groups $\cup \{F, U, 2^r\}$.*

Moreover, suppose a semigroup S divides $S(\mathscr{A})$ for some finite automaton $\mathscr{A}$: If $S \prec F$ then S is (isomorphic to) a subsemigroup of $S(\mathscr{A})$; while, if S is a group, then S is the homomorphic image of a group G which is a subsemigroup of $S(\mathscr{A})$. (See e.g. [11] for proofs of the statements in this section).

A *pseudovariety* **S** of finite semigroups is a class of finite semigroups closed under division and finite direct products. That is, (1) if $S \prec T$ and $T \in \mathbf{S}$ then $S \in \mathbf{S}$, and (2) if $S_i \in \mathbf{S}$ for all $i \in I$, a finite index set, then $\prod_{i \in I} S_i \in \mathbf{S}$.

Taking $I = \emptyset$, the latter condition guarantees that the one-element semigroup is in **S**, so in particular **S** cannot be empty.

If $\mathscr{K}$ is a class of finite automata, then define $\mathscr{S}(\mathscr{K})$, *the semigroup pseudovariety corresponding to $\mathscr{K}$*, to be the smallest pseudovariety of finite semigroups containing the transition semigroup $S(\mathscr{A})$ for each automaton $\mathscr{A} \in \mathscr{K}$.

1.4. Eilenberg correspondences

Eilenberg's Theorem [3] states that pseudovarieties of finite semigroups are in a natural one to one correspondence with certain classes of recognizable languages, the varieties of languages. A *variety* $\mathscr{L}$ of languages assigns to each finite alphabet X a set $\mathscr{L}(X)$ of regular languages contained in X^+ such that (1) $\mathscr{L}(X)$ is closed under the Boolean operations of finite union, finite intersection, and complement within X^+, and (2) $\mathscr{L}(X)$ is closed under quotients: $L \in \mathscr{L}(X)$ and $x \in X$ implies Lx^{-1} and $x^{-1}L$ are in $\mathscr{L}(X)$, where $Lx^{-1} = \{w \in X^+ \mid wx \in L\}$ and $x^{-1}L = \{w \in X^+ \mid xw \in L\}$, and such that (3) $\mathscr{L}$ is closed under (non-erasing) inverse homomorphisms: $L \in \mathscr{L}(X)$ and $\varphi : Y^+ \to X^+$ is a homomorphism implies $\varphi^{-1}(L) \in \mathscr{L}(Y)$. [4]

If $L \subseteq X^+$ is a language over X, then the *syntactic semigroup* of L is the transition semigroup of its minimal automaton. $L \subseteq X^+$ is *recognized* by a finite semigroup S if $S = S(\mathscr{A})$ for some finite automaton $\mathscr{A} = (A, X, \delta)$ recognizing L. The reader is referred to [3] or [13] for full definitions and details, as well as relations to automata theory.

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[4] More exactly these are the $+$-varieties of languages. There is a related but somewhat different Eilenberg correspondence between $*$-varieties of regular languages (allowing the empty word) and pseudovarieties of monoids. (See [3] or [13] for precise details and differences between the two correspondences.)

Theorem 10 (Eilenberg [3, Theorem VII.3.2s]). *There is a one-to-one correspondence between pseudovarieties of semigroups and varieties of languages*: *The pseudovariety of finite semigroups* $\mathbf{V} \mapsto$ *the variety of languages* $\mathscr{L}_{\mathbf{V}}$ *where* $\mathscr{L}_{\mathbf{V}}(X)$ *is the set of the languages* $L \subseteq X^+$ *recognized by members of* $\mathbf{V}$. *The variety of languages* $\mathscr{L} \mapsto$ *the pseudovariety* $\mathbf{V}_{\mathscr{L}}$ *generated by syntactic semigroups of all the languages* $L \in \mathscr{L}(X)$ *with* X *some finite alphabet.*

The Eilenberg correspondence serves to systematize the study of regular languages algebraically. For instance, the pseudovariety **Sgp** of all finite semigroups corresponds to the variety of regular languages [7, Kleene's Theorem]. The pseudovariety **A** of aperiodic semigroups corresponds to the variety of star-free languages [14, Schützenberger's Theorem]. (A finite semigroup S is aperiodic if $s^\omega = s^{\omega+1}$ for all $s \in S$, i.e. every subgroup of S has only one element. Equivalently, by the first corollary of the Krohn–Rhodes Theorem, S is aperiodic if and only if S divides the transition semigroup of a cascade of flip-flops.)

Many instances of the Eilenberg correspondence between varieties of languages and pseudovarieties of finite semigroups have been studied (see [3,13] and subsequent publications by various researchers, including deep results of Knast, Simon, Brzozowski, and Straubing [2,8,9,15,17–19]). For purposes of this paper, we need only some relatively simple instances of this correspondence.

If $\mathbf{V}$ is a pseudovariety of finite semigroups then the *reverse pseudovariety* is $\mathbf{V}^\rho = \{S^\rho \mid S \in \mathbf{V}\}$, whose members are the reverse semigroups of members of S. The *reverse of a language* $L \subseteq X^+$ is the language $L^\rho = \{x_n \ldots x_1 \in X^+ \mid x_1 \ldots x_n \in L, n > 0\}$. Under the Eilenberg correspondence, reversing the languages in a variety corresponds to the reversing the semigroups in the corresponding pseudovariety (as is easy to see since the reverse language has the reverse syntactic semigroup) [3, Proposition VII.5.1]. Obviously, the Eilenberg correspondence preserves inclusion: $\mathbf{W} \subseteq \mathbf{V}$ if and only if $\mathscr{L}_{\mathbf{W}}(X) \subseteq \mathscr{L}_{\mathbf{V}}(X)$ for all finite alphabets X.

A semigroup S is *nilpotent of degree* n if and only if for all $x_1, \ldots, x_n \in S$, $x_1 \cdots x_n = 0$ holds, i.e. $x_1 \cdots x_n y = y x_1 \cdots x_n = x_1 \cdots x_n$ holds for all $y \in X$. $\mathbf{Nil}_n$ is the pseudovariety of finite semigroups that are nilpotent of degree n. In the corresponding language variety, $\mathscr{N}il_n(X)$ is the Boolean closure of the singleton languages $\{w\}$, $w \in X^+$, $|w| < n$. **Nil** is the pseudovariety which is the union of all the $\mathbf{Nil}_n$. **Nil** is also defined by $s^\omega = 0$. A language $L \subseteq X^+$ is *finite or cofinite* if either L or $X^+ \backslash L$ is finite, which is true if and only if its syntactic semigroup is nilpotent ([3, Proposition VIII.2.2] or [13, Chapter 2, Theorem 3.3]).

A semigroup S is said to be *definite* if $se = e$ holds for all $e^2 = e, s \in S$. A semigroup S is said to be a *reverse definite*, if $es = e$ for all idempotents $e \in S$ and all $s \in S$. The pseudovariety of all definite semigroups is denoted **D**. The pseudovariety of all reverse definite semigroups is denoted $\mathbf{D}^\rho$. A language $L \subseteq X^+$ is *reverse definite* if L is of the form $Y X^* \cup Z$ where Y and Z are finite languages of X^+. The definite languages are the reverse of these. The definite languages are exactly those whose syntactic semigroups lie in **D**, while the reverse definite languages are exactly those whose syntactic semigroups lie in $\mathbf{D}^\rho$ ([3, Proposition VIII.4.1] or [13, Chapter 2, Theorem 3.4]). Inside **D** is a nested hierarchy of pseudovarieties $\mathbf{D}_n$ whose members

satisfy $x_1 \cdots x_n = y x_1 \cdots x_n$. **D** is the union of the $\mathbf{D}_n$. The pseudovarieties of left-zero semigroups $\mathbf{LZ} = \mathbf{D}_1^\rho$ and right-zero semigroups $\mathbf{RZ} = \mathbf{D}_1$ are the lowest levels of the two hierarchies. The characterization of the language variety recognized by $\mathbf{D}_n$ is the same as that for **D** except that the finite languages Y and Z may only contain words in X^+ of length not exceeding n and $n - 1$, respectively [13, p. 43]. Similar remarks characterize the language variety corresponding to each $\mathbf{D}_n^\rho$. An automaton $\mathscr{A} = (A, X, \delta)$ is called *reverse definite* if there is an $n > 0$, such for all $a \in A$, $x \in X$, $p \in X^+$, $a \cdot px = a \cdot p$ holds whenever $|p| \geqslant n$. It follows from the definition of the $\mathbf{D}_n^\rho$ that a language $L \subseteq X^+$ is reverse definite (i.e. recognized by a member of $\mathbf{D}^\rho$) if and only if L can be recognized by some reverse definite automaton.

In a semigroup S, x and y are *$\mathscr{R}$-related* (denoted: $x \mathscr{R} y$) if there exist $s, t \in S^\bullet$ such that $xs = y$ and $yt = x$. A semigroup S is *$\mathscr{R}$-trivial* if $x \mathscr{R} y$ always implies $x = y$. The finite $\mathscr{R}$-trivial semigroups comprise a pseudovariety **R**. A language $L \subseteq X^+$ is *extensive* if it can be written as the finite disjoint union of languages of the form Y^+, $Y \subseteq X$ and $X_0^* x_1 X_1^* x_2 \ldots x_n X_n^*$, where $n > 0$, $x_1, \ldots, x_n \in X$, $X_i \subseteq X \setminus \{x_{i+1}\}$ for $0 \leqslant i \leqslant n-1$ and $X_n \subseteq X$. These languages are exactly those whose syntactic semigroups lie in **R** as can be seen by using minor but straightforward modifications of the corresponding proof for *-varieties by Pin [13, Chapter 4, Theorem 3.3] (of the original result for $\mathscr{R}$-trivial monoids due to Eilenberg [3, Corollary X.3.3]). Another characterization of extensive languages is that they are exactly the languages which can be recognized by an *extensive finite automaton* $\mathscr{A} = (A, X, \delta)$, i.e. a finite automaton for which there is a partial ordering, or equivalently a total ordering, $\leqslant$ on A with $a \cdot x \leqslant a$ for all $a \in A, x \in X$ (cf. [13, Chapter 3.3;1]).

We record some of these correspondences between varieties of regular languages and pseudovarieties of finite semigroups:

$$\text{REGULAR} \Leftrightarrow \mathbf{Sgp}$$
$$\text{STAR-FREE} \Leftrightarrow \mathbf{A}$$
$$\text{EXTENSIVE} \Leftrightarrow \mathbf{R}$$
$$\text{DEFINITE} \Leftrightarrow \mathbf{D}$$
$$\text{REVERSE DEFINITE} \Leftrightarrow \mathbf{D}^\rho$$
$$\text{FINITE OR COFINITE} \Leftrightarrow \mathbf{Nil}$$

In the sequel the two pseudovarieties $\mathbf{D}^\rho$ and **R** will play a crucial role. We denote by **G** the pseudovariety consisting of all finite groups and the empty semigroup.

2. Algebratization

What is the relationship between homomorphic representation by the feedback products and pseudovarieties of finite semigroups or, equivalently, varieties of regular languages? To study this question, we examine the Letichevsky and semi-Letichevsky criteria algebraically. If we examine the transformation semigroups of automata satisfying the Letichevsky criterion, we are immediately confronted with the following fact.

Fact 11. *Let $\mathscr{A}$ be a finite automaton.*
1. $\mathscr{A} \models Let \Rightarrow \mathscr{A}^+ \models Let$.
2. $\mathscr{A}^+ \models Let \not\Rightarrow \mathscr{A} \models Let$.

Proof. (1) Since every letter of $\mathscr{A}$ yields a corresponding input symbol of $\mathscr{A}^+$, this is obvious. (2) Consider the 3-state counter automaton $\mathscr{C} = (\{0, 1, 2\}, \{x\}, \delta)$, $\delta(i, x) = i + 1 \,(\mathrm{mod}\,3)$ with a single input letter x. The non-empty input word automaton $\mathscr{C}^+$ associated to $\mathscr{C}$ has input letters corresponding to the transformations represented by the words x, xx and xxx (and no other transformations). $\mathscr{C}^+$ satisfies Letichevsky's criterion, but $\mathscr{C}$ does not.

The Three State Counter Automaton $\mathscr{C}$ and the Associated Non-Empty Input Word Automaton $\mathscr{C}^+$
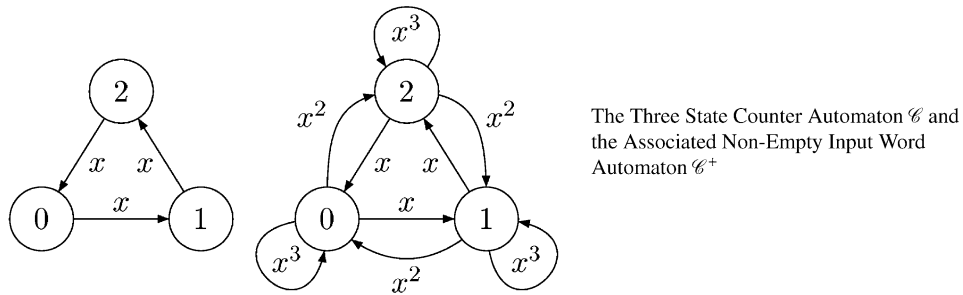
This completes the proof. □

Since the Eilenberg correspondence between varieties of languages and pseudovarieties of semigroups relies on the characteristic semigroups of the automata recognizing a language, the failure of the implication in Fact 11(2) suggests that, in order to develop an algebraic theory related to the Letichevsky criterion, it is desirable to study it for the corresponding transformation semigroup—i.e., the non-empty input word automaton $\mathscr{A}^+$ associated to a given automaton $\mathscr{A}$. Thus, if the transformation semigroup $\mathscr{A}^+$ associated with $\mathscr{A}$ satisfies the Letichevsky condition, let us write $\mathscr{A} \models Let^+$. By definition,

$$\mathscr{A}^+ \models Let \iff \mathscr{A} \models Let^+. \tag{1}$$

Now we say $\mathscr{A}$ satisfies the *Letichevsky criterion for non-empty words* (Let^+) if $\mathscr{A}^+$ satisfies the Letichevsky criterion. This is obviously equivalent to $\mathscr{A}$ satisfying the formula (Let^+):

$$(Let^+) \quad \exists a_0 \in A, x, y, p, q \in X^+, a_0 \cdot x \neq a_0 \cdot y, \quad \text{and} \quad a_0 \cdot xp = a_0 \cdot yq = a_0.$$

Thus we have the same condition as for (Let), except now for (Let^+), x and y need not be letters in the alphabet X of $\mathscr{A}$ but are allowed to be any non-empty words in X^+. In this notation, by Fact 11(1),

$$\mathscr{A} \models Let \Rightarrow \mathscr{A} \models Let^+ \tag{2}$$

but the reverse implication may fail to hold in general (Fact 11(2)).

Similarly, we say $\mathscr{A}$ satisfies *the semi-Letichevsky criterion on non-empty words*, and write $\mathscr{A} \models SL^+$, if $\mathscr{A}^+$ satisfies the semi-Letichevsky condition. By definition,

$$\mathscr{A}^+ \models SL \;\Leftrightarrow\; \mathscr{A} \models SL^+. \tag{3}$$

Thus, $\mathscr{A}$ satisfies the semi-Letichevsky criterion on non-empty words (SL^+) if $\mathscr{A}$ does not satisfy (Let^+) but the configuration of the semi-Letichevsky criterion occurs in $\mathscr{A}$ for some non-empty words $x, y, p \in X^+$. Precisely, $\mathscr{A} \models SL^+$ if $\mathscr{A} \not\models Let^+$ and there exist $x, y, p \in X^+$, $a_0 \in A$, such that $a_0 \cdot x \neq a_0 \cdot y$ and $a_0 \cdot xp = a_0$. This is equivalent to satisfaction by $\mathscr{A}$ of the formula (SL^+):

$$(SL^+) \quad \neg Let^+ \text{ and } \exists a_0 \in A, x, y, p \in X^+, a_0 \cdot x \neq a_0 \cdot y, \text{ and } a_0 \cdot xp = a_0.$$

If a class $\mathscr{K}$ of automata contains an automaton satisfying (Let^+), then we also say that $\mathscr{K}$ *satisfies Letichevsky's criterion on non-empty words* ($\mathscr{K} \models Let^+$). Otherwise, we say that $\mathscr{K}$ *does not satisfy* (Let^+), and write $\mathscr{K} \not\models Let^+$. Also, a class $\mathscr{K}$ of finite automata *satisfies the semi-Letichevsky criterion on non-empty words* ($\mathscr{K} \models SL^+$) if it does not satisfy the Letichevsky criterion on non-empty words and at least one member of $\mathscr{K}$ satisfies the semi-Letichevsky criterion on non-empty words.

Let us determine the remaining relations between the classical Letichevsky and semi-Letichevsky criteria and the corresponding criteria on non-empty words.

First we show that

$$\mathscr{A} \models SL^+ \Rightarrow \mathscr{A} \models SL \tag{4}$$

but the converse does not hold in general.

Proof of (4). We have $\mathscr{A}^+ \models SL$, so $\mathscr{A}^+ \not\models Let$, whence $\mathscr{A} \not\models Let$ by (2) and there exist non-empty words $x, y, p \in X^+$ and $a_0 \in A$, with $a_0 \cdot x \neq a_0 \cdot y$ but $a_0 \cdot xp = a_0$. Write $y = y_1 \cdots y_k$ with each y_j a letter in X. Clearly $a_0 \cdot y \neq a_0$ (lest $\mathscr{A}^+ \models Let$), so by removing some initial letters of y if necessary, we may suppose that $a_0 \cdot y_1 \ldots y_i \neq a_0$ for all i ($1 \leqslant i \leqslant k$). Let i be the greatest integer such that $0 \leqslant i \leqslant k$ and there exists $q \in X^+$ such that $a_0 \cdot y_1 \cdots y_i q = a_0$ (for $i = 0$, one may take $q = xp$).

Let $a_0' = a_0 \cdot y_1 \cdots y_i$. Then $a_0' \cdot q = a_0$. Write $q = q_1 \cdots q_\ell$, with letters $q_j \in X$. Since $\mathscr{A}^+ \not\models Let$, there can be no q with $a_0 \cdot yq = a_0$, so $i < k$, and we may set $y' = y_{i+1}$. Let $x' = q_1$ and $p' = q_2 \cdots q_\ell xp y_1 \cdots y_i$. Then $a_0' \cdot x' \neq a_0' \cdot y'$ (lest $a_0 \cdot y_1 \cdots y_i y_{i+1} q_2 \cdots q_\ell = a_0' \cdot y_{i+1} q_2 \cdots q_\ell = (a_0' \cdot y') \cdot q_2 \cdots q_\ell = (a_0' \cdot x') \cdot q_2 \cdots q_\ell = a_0' \cdot q_1 q_2 \cdots q_\ell = a_0' \cdot q = a_0$ contradicting the choice of i if $\ell > 1$ and contradicting $a_0 \cdot y_1 \cdots y_{i+1} \neq a_0$ if $\ell = 1$) and $a_0' \cdot x' p' = a_0' \cdot q_1 (q_2 \cdots q_\ell xp y_1 \cdots y_i) = a_0' \cdot qxp y_1 \cdots y_i = a_0 \cdot xp y_1 \cdots y_i = a_0 \cdot y_1 \cdots y_i = a_0'$. This proves $\mathscr{A} \models SL$.

To see that the converse may fail to hold, i.e. $\mathscr{A} \models SL \not\Rightarrow \mathscr{A} \models SL^+$, modify the counter automaton $\mathscr{C}$ of Fact 11(2) by adding a new input letter y which takes every state to a new "sink" state $*$ (i.e. $a \cdot y = *$ for all $a \in \{0, 1, 2\}$, and $* \cdot x = * \cdot y = *$). Denoting the modified automaton by $\mathscr{C}_*$, we have $\mathscr{C}_* \models SL$, but $\mathscr{C}_* \models Let^+$ (since $\mathscr{C} \models Let^+$), and so $\mathscr{C}_* \not\models SL^+$. $\square$

In the class of finite automata $\mathscr{All}$, let $\mathscr{LET}$ denote automata satisfying the (classical) Letichevsky criterion (Let), $\mathscr{LET}^+$ denote automata satisfying the Letichevsky
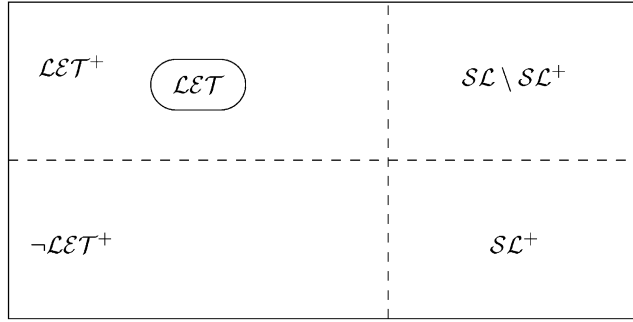
Fig. 1. The universe $\mathscr{A}ll$ of all finite automata: The dashed horizontal line separates the class $\mathscr{L}\mathscr{E}\mathscr{T}^+$ of automata satisfying the Letichevsky criterion on non-empty words (above) from the class $\neg\mathscr{L}\mathscr{E}\mathscr{T}^+$ of those not satisfying it (below). The dashed vertical line separates the class $\mathscr{S}\mathscr{L}$ of automata satisfying the classical semi-Letichevsky criterion (right) from those that do not (left). $\mathscr{S}\mathscr{L}$ is the disjoint union of $\mathscr{S}\mathscr{L}\cap\neg\mathscr{L}\mathscr{E}\mathscr{T}^+ = \mathscr{S}\mathscr{L}^+$, the automata satisfying the semi-Letichevksy criterion on non-empty words (below), and $\mathscr{S}\mathscr{L}\backslash\mathscr{S}\mathscr{L}^+ = \mathscr{S}\mathscr{L}\cap\mathscr{L}\mathscr{E}\mathscr{T}^+$, the automata satisfying both the classical semi-Letichevksy criterion and (Let^+). The class $\mathscr{L}\mathscr{E}\mathscr{T}$ of automata satisfying the classical Letichevsky criterion is a proper subclass of $\mathscr{L}\mathscr{E}\mathscr{T}^+$. Relations shown are established in the main text.

criterion on non-empty words (Let^+), $\neg\mathscr{L}\mathscr{E}\mathscr{T}^+$ denote automata not satisfying the Letichevsky criterion on non-empty words, $\mathscr{S}\mathscr{L}$ denote automata satisfying the (classical) semi-Letichevksy criterion (SL), and $\mathscr{S}\mathscr{L}^+$ denote automata satisfying the semi-Letichevsky criterion on non-empty words (SL^+). We have seen $\mathscr{S}\mathscr{L}^+ \subsetneq \mathscr{S}\mathscr{L}$ and $\mathscr{L}\mathscr{E}\mathscr{T} \subsetneq \mathscr{L}\mathscr{E}\mathscr{T}^+$. Observe that

$$\mathscr{S}\mathscr{L} \cap \neg\mathscr{L}\mathscr{E}\mathscr{T}^+ = \mathscr{S}\mathscr{L}^+. \tag{5}$$

Indeed, from the definition of SL^+, clearly $\neg Let^+$ and SL imply SL^+. Conversely, SL^+ requires that Let^+ be false, and we have already seen that SL^+ entails SL.

$$\mathscr{S}\mathscr{L} \cap \mathscr{L}\mathscr{E}\mathscr{T}^+ = \mathscr{S}\mathscr{L}\backslash\mathscr{S}\mathscr{L}^+. \tag{6}$$

Suppose SL holds. If Let^+ then SL^+ cannot hold by definition. If SL^+ is false, then either Let^+ holds or there are no a_0, x, y, p as in the SL condition. Since SL holds it can only be that Let^+ holds.

Obviously $\mathscr{L}\mathscr{E}\mathscr{T}^+ \cup \neg\mathscr{L}\mathscr{E}\mathscr{T}^+ = \mathscr{A}ll$ and $\mathscr{L}\mathscr{E}\mathscr{T} \cap \mathscr{S}\mathscr{L} = \emptyset$. The established relations between the various classes are shown in Fig. 1.

Now let us characterize these Letichevsky criteria on non-empty words algebraically.

Proposition 12. *Let $\mathscr{A}$ be any finite automaton. $\mathscr{A}$ satisfies the Letichevsky criterion on non-empty words (Let^+) if and only if the semigroup $S(\mathscr{A})$ of transformations of $\mathscr{A}$ is not $\mathscr{R}$-trivial.*

Proof. Suppose that the criterion (Let^+) is satisfied. Let state a_0 and non-empty $x, y, p, q \in X^+$ be as in the criterion. In particular, $a_0 \cdot x \neq a_0 \cdot y$. Let $e = ((yq)^\omega (xp)^\omega)^\omega$. Clearly $a_0 \cdot e = a_0$. We have $ex \mathscr{R} e$ since $exp(xp)^{\omega-1} = e(xp)^\omega = e$. Also $ey \mathscr{R} e$ since

$eyq(yp)^{\omega-1}(xp)^{\omega}((yq)^{\omega}(xp)^{\omega})^{\omega-1} = e^2 = e$. Therefore, $ex\mathscr{R}ey$, but $a_0 \cdot ex = a_0 \cdot x$ and $a_0 \cdot ey = a_0 \cdot y$. It follows from $a_0 \cdot x \neq a_0 \cdot y$ that $ex \neq ey$, whence $S(\mathscr{A})$ is not $\mathscr{R}$-trivial.

Conversely, let $S(\mathscr{A})$ be not $\mathscr{R}$-trivial. This means there are words $s, t \in X^+$ such that $s\mathscr{R}t$ but $s \neq t$ in $S(\mathscr{A})$. Then there exist $p, q \in X^*$ such $sp = t$ and $tq = s$ in $S(\mathscr{A})$. (Clearly, neither of p nor q is λ since $s \neq t$ in $S(\mathscr{A})$.) Since $s \neq t$, there is a state a_1 with $a_1 \cdot s \neq a_1 \cdot t$. Let $a_0 = a_1 \cdot s$. Then

$$a_0 \cdot pq = a_1 \cdot s\,pq = a_1 \cdot tq = a_1 \cdot s = a_0,$$

while

$$a_0 \cdot p = a_1 \cdot sp = a_1 \cdot t \neq a_1 \cdot s = a_0.$$

Then $x = pq$, $p' = pq$, $y = p$, and q are non-empty words such that $a_0 \cdot x \neq a_0 \cdot y$, and $a_0 \cdot xp' = a_0$, and $a_0 \cdot yq = a_0$. Thus $\mathscr{A}$ satisfies (Let^+). $\square$

Remark. The above proposition could also be proved via the fact that extensive automata correspond to $\mathscr{R}$-trivial semigroups.

Corollary 13. $\mathscr{A} \in \neg\mathscr{LET}^+ \iff S(\mathscr{A})$ *is* $\mathscr{R}$-*trivial* $\iff S(\mathscr{A}) \in \mathscr{S}(HSP_0(\{U\}))$.

Proof. The first equivalence holds by the preceding proposition. The second equivalence holds, since by a theorem of Stiffler [16, Theorem 3.4(b)], a semigroup lies in **R** if and only if it divides the transition semigroup of a cascade of copies of U. $\square$

Corollary 14. $\mathscr{A} \in \neg\mathscr{LET}^+$ *implies* $IRRED(\mathscr{A}) \subseteq \{U\}$. *But the converse does not hold.*

Proof. By the preceding corollary, $S(\mathscr{A})$ divides the transition semigroup of a cascade of copies of $(U^{\bullet}, U) = (U, U)$. Since U is the only non-trivial irreducible divisor of U, the implication holds. To see that the converse may fail to hold consider the five element Brandt semigroup B_2 with elements $(1,1), (1,2), (2,1), (2,2)$, and 0 with multiplication $0 * (x,y) = (x,y) * 0 = 0 * 0 = 0$ for all $x, y \in \{1,2\}$, and for $x, y, x', y' \in \{1,2\}$,

$$(x,y) * (x', y') = \begin{cases} (x, y') & \text{if } y = x', \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that 2^r does not divide B_2, nor does any non-trivial group, but U does since it is isomorphic to the subsemigroup $\{(1,1), 0\}$. However, B_2 is not $\mathscr{R}$-trivial: $(1,1)\mathscr{R}(1,2)$ since $(1,1) * (1,2) = (1,2)$ and $(1,2) * (2,1) = (1,1)$. So $\mathscr{A} = (B_2^{\bullet}, B_2)$ satisfies the Letichevsky criterion by Proposition 12. $\square$

Corollary 15. $\mathscr{K} \not\models Let^+$ *implies* $IRRED(\mathscr{K}) \subseteq \{U\}$. *But the converse does not hold.*

Corollary 16. *If IRRED($\mathcal{K}$) contains 2^r, the flip-flop monoid F, or any non-trivial simple group, then $\mathcal{K} \models Let^+$.*

Proof. The condition $IRRED(\mathcal{K}) \not\subseteq \{U\}$ is equivalent to the presence of any non-trivial irreducible divisor other than U in $IRRED(\mathcal{K})$, i.e., one of 2^r, the flip-flop monoid F, or any simple group. It then follows from the contrapositive of the previous corollary that $\mathcal{K} \models Let^+$. $\square$

Proposition 17. *Let $\mathcal{A}$ be a finite automaton. $\mathcal{A}$ satisfies the semi-Letichevsky criterion on non-empty words if and only if $S(\mathcal{A})$ is $\mathcal{R}$-trivial but not reverse definite, i.e.*

$$\mathcal{A} \models SL^+ \quad \Leftrightarrow \quad S(\mathcal{A}) \in \mathbf{R} \backslash \mathbf{D}^\rho.$$

Proof. Given $\mathcal{A}$ satisfying SL^+, we have that $\mathcal{A}$ does not satisfy Let^+ (hence also not Let). From the previous proposition, we have that $S(\mathcal{A})$ is $\mathcal{R}$-trivial. We must show it is not reverse definite. If SL^+ holds, then we take $x, y, p \in X^+$ such that $a_0 \cdot x \neq a_0 \cdot y$ and $a_0 \cdot xp = a_0$. It follows that $(xp)^\omega x \neq (xp)^\omega y$ in $S(\mathcal{A})$. Therefore, for $e^2 = e = (xp)^\omega$, the equation $es = e$ does not hold in $S(\mathcal{A})$. Thus $S(\mathcal{A})$ is not reverse definite.

Conversely, suppose $S(\mathcal{A})$ is $\mathcal{R}$-trivial but not reverse definite. Again by the previous proposition, since it is $\mathcal{R}$-trivial it does not satisfy Let^+ (hence also does not satisfy Let). Since $S(\mathcal{A})$ is not reverse definite, there exist non-empty words $e, y \in X^+$ representing $[e], [y] \in S(\mathcal{A})$, with $[e]^2 = [e]$ and $[e][y] \neq [e]$. The latter means there is an $a_1 \in A$, with

$$a_1 \cdot ey \neq a_1 \cdot e.$$

Therefore, taking $a_0 = a_1 \cdot e$, we have

$$a_0 \cdot y = a_1 \cdot ey \neq a_1 \cdot e = a_0.$$

Moreover, since $[e]^2 = [e]$, we have

$$(a_0 \cdot e) \cdot e = a_0 \cdot ee = a_0 \cdot e = (a_1 \cdot e) \cdot e = a_1 \cdot ee = a_1 \cdot e = a_0.$$

Thus, taking x and p both equal to e, we have $(a_0 \cdot x) \cdot p = a_0$, and $a_0 = a_0 \cdot x \neq a_0 \cdot y$. This shows that $S(\mathcal{A})$ satisfies SL^+. $\square$

Corollary 18. *Let $\mathcal{A}$ be a finite automaton. $\mathcal{A}$ satisfies neither the semi-Letichevsky nor the Letichevsky criterion on non-empty words if and only if $S(\mathcal{A})$ is reverse definite. That is,*

$$\mathcal{A} \not\models Let^+ \quad and \quad \mathcal{A} \not\models SL^+ \quad \Leftrightarrow \quad S(\mathcal{A}) \in \mathbf{D}^\rho.$$

Corollary 19. *$\mathcal{A}$ satisfies neither the semi-Letichevsky nor the Letichevsky criterion on non-empty words if and only if $IRRED(\mathcal{A}) = \emptyset$.*

Proof. Stiffler [16, Fact 4.8(b)] proved that if a finite semigroup S has no non-trivial irreducible divisors then it is a nilpotent extension of a left-zero semigroup, or,

equivalently, idempotents in S are left-zeros ($e^2 = e$ implies $es = e$ for all $s \in S$), i.e. S is reverse definite. So the result follows from the corollary above. $\square$

We remark that Stiffler [16, Theorem 3.4(a)] also shows that $\mathscr{S}(HSP_0(\{2^r\})) = \mathbf{D}$. Therefore $\mathscr{S}(HSP_0(\{2^r\}))^\rho = \mathbf{D}^\rho$. Thus $\mathscr{A}$ satisfies neither of the Letichevsky criteria on non-empty words if and only if $\mathscr{S}(A)$ divides the reverse semigroup of a cascade of copies of $(2^{r\bullet}, 2^r)$.
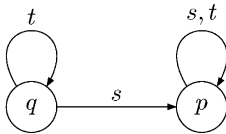
It is easy to check that:

Fact 20. *Let $\mathscr{A}$ be a finite automaton. Then*
1. *$\mathscr{A}$ satisfies (Let^+) if and only if the configuration*
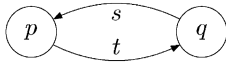
    occurs in $\mathscr{A}$. Precisely, for some states $p, q \in A$, $p \neq q$, and inputs $s, t \in X^+$, $q \cdot s = p$, $p \cdot t = q$.
2. *$\mathscr{A}$ satisfies (SL^+) if and only if $\mathscr{A}$ does not satisfies (Let^+) and the following configuration occurs in $\mathscr{A}$:*


    occurs in $\mathscr{A}$. Precisely, $\mathscr{A} \not\models Let^+$, and for some $p, q \in A$, $s, t \in X^+$, $p \neq q$, $q \cdot t = q$, $q \cdot s = p$, and $p \cdot s = p \cdot t = p$.

3. Summary of results

The results obtained so far easily entail the following series of theorems.

Theorem 21. *Let $\mathscr{K}$ be any class of finite automata. Then the following are equivalent.*
1. *$\mathscr{K}$ satisfies the Letichevsky criterion on non-empty words, i.e. $\mathscr{K} \models Let^+$.*
2. *$\mathscr{S}(\mathscr{K}) \not\subseteq \mathbf{R}$. That is, $S(\mathscr{A})$ is not $\mathscr{R}$-trivial for some automaton $\mathscr{A} \in \mathscr{K}$.*
3. *There is an automaton $\mathscr{A} \in \mathscr{K}$, such that $S(\mathscr{A})$ does not divide the semigroup of any cascade of copies of (U, U), where U is the two element monoid.*
4. *The configuration*
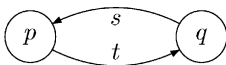
    occurs in some automata $\mathscr{A} \in \mathscr{K}$, for some states p, q, $p \neq q$, and inputs $s, t \in X^+$, $q \cdot s = p$, $p \cdot t = q$.

5. *There is a language L is recognized by an automaton from $\mathscr{K}$ such that L is not extensive.*

Theorem 22. *For any class $\mathscr{K}$ of finite automata, the following are equivalent, and imply that $\mathscr{K}$ satisfies the Letichevsky criterion on non-empty words*:

1. *There exists $\mathscr{A} \in \mathscr{K}$, such that the two-element reset semigroup 2^r divides $S(\mathscr{A})$ or a simple group G divides $S(\mathscr{A})$.*
2. *There exists $\mathscr{A} \in \mathscr{K}$, such that the two-element reset semigroup 2^r embeds in $S(\mathscr{A})$ or a non-trivial group G embeds in $S(\mathscr{A})$.*
3. *A non-trivial irreducible semigroup other than the two element monoid U divides $S(\mathscr{A})$ for some $\mathscr{A} \in \mathscr{K}$.*
4. *IRRED$(\mathscr{S}(\mathscr{K})) \not\subseteq \{U\}$.*

Proof. (1) implies (2): For any finite semigroup S, $2^r \prec S$ implies 2^r is a subsemigroup of S, and $G \prec S$ for a group G implies there is a group G' that is a subsemigroup of S mapping homomorphically onto G. (See e.g. [11].) The rest is now clear from Corollary 16 and the characterization of finite irreducible semigroups in the Krohn–Rhodes Theorem. $\quad\square$

Theorem 23. *For any class $\mathscr{K}$ of finite automata, the following are equivalent.*

1. *$\mathscr{K}$ satisfies the semi-Letichevsky criterion for non-empty words, i.e. $\mathscr{K} \models SL^+$ (thus, there exists an automaton $\mathscr{A} \in \mathscr{K}$ with $\mathscr{A} \models SL^+$, but no $\mathscr{A} \in \mathscr{K}$ satisfies Let^+).*
2. *The above configuration does not occur in any automaton $\mathscr{A} \in \mathscr{K}$, for any distinct states p, q and non-empty input words $s, t \in X^+$. But the configuration*
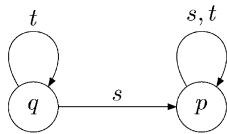
occurs in at least one $\mathscr{A}$ in $\mathscr{K}$ ($q, p \in A$, $p \neq q$, $s, t \in X^+$), $q \cdot t = q, q \cdot s = p, p \cdot s = p \cdot t = p$.
3. *For all $\mathscr{A} \in \mathscr{K}$, $S(\mathscr{A})$ divides the semigroup of a cascade of copies of (U, U), and moreover $IRRED(\mathscr{S}(\mathscr{K})) = \{U\}$.*
4. *For all $\mathscr{A} \in \mathscr{K}$, $S(\mathscr{A})$ lies in $\mathscr{S}(HSP_0(\{U\}))$ and U divides $S(\mathscr{A})$ for some $\mathscr{A} \in \mathscr{K}$, but no other non-trivial irreducible semigroup divides any $S(\mathscr{A})$ for $\mathscr{A} \in \mathscr{K}$.*
5. *$S(\mathscr{A})$ is $\mathscr{R}$-trivial for all $\mathscr{A} \in \mathscr{K}$, but there is an $\mathscr{A}$ with $S(\mathscr{A}) \notin \mathbf{D}^\rho$. That is, $\mathscr{S}(\mathscr{K}) \subseteq \mathbf{R}$ but $\mathscr{S}(\mathscr{K}) \not\subseteq \mathbf{D}^\rho$.*
6. *Every language recognized by automata from $\mathscr{K}$ is extensive, but there is at least one language recognized by some member of $\mathscr{K}$ which is not reverse definite.*

Remark. Considering the counterexample B_2, described in the proof of Corollary 14 above, which has $IRRED(B_2) = \{U\}$ and satisfies Let, one sees that the conditions in Theorem 22 only imply but are not equivalent to the Letichevsky criterion on

non-empty words. By the same counterexample, condition 3 of Theorem 23 cannot be weakened to $IRRED(\mathscr{S}(\mathscr{K})) = \{U\}$.

Theorem 24. *Let $\mathscr{K}$ be any class of finite automata. Then the following are equivalent.*
1. *$\mathscr{K}$ satisfies neither Let^+ nor SL^+.*
2. *Neither of the configurations above occurs in any automaton in $\mathscr{K}$.*
3. *No non-trivial irreducible semigroup divides $S(\mathscr{A})$ for any $\mathscr{A} \in \mathscr{K}$. That is, $IRRED(\mathscr{S}(\mathscr{K})) = \emptyset$.*
4. *$S(\mathscr{A})$ is a reverse definite for all $\mathscr{A} \in \mathscr{K}$. That is, $\mathscr{S}(\mathscr{K}) \subseteq \mathbf{D}^\rho$. In other words, $S(\mathscr{A})$ is a nilpotent extension of a left-zero semigroup; that is, $S(\mathscr{A})$ satisfies $x^\omega y = x^\omega$.*
5. *$S(\mathscr{A})$ divides the reverse of the transition semigroup of a cascade of copies of $(2^{r\bullet}, 2^r)$, for all $\mathscr{A} \in \mathscr{K}$.*
6. *Every language is recognized by an automaton from $\mathscr{K}$ is reverse definite.*

4. Feedback operators for pseudovarieties of finite semigroups

Now we return to the question at the beginning of Section 2 on the relationship between pseudovarieties and feedback products. The pseudovariety characterizations obtained above will allow us to relate the action of α_i operators on classes of automata with their action on pseudovarieties for $i = 0, 1, 2, \ldots, \infty$.

Define for each $i = 0, \ldots, \infty$, an operator $\hat{\alpha}_i : PV \to PV$ on the lattice PV of pseudovarieties of finite semigroups:

$$\hat{\alpha}_i(\mathbf{V}) := \mathscr{S}(HSP_i(\{(S^\bullet, S) \mid S \in \mathbf{V}\})).$$

Clearly, $\mathscr{S}(HSP_i(\mathscr{K})) = \mathscr{S}(P_i(\mathscr{K})))$ always holds for every class $\mathscr{K}$ of finite automata.

We write $\mathbf{V} \models Let^+$, if $(S^\bullet, S) \models Let^+$ for some $S \in \mathbf{V}$, and write $\mathbf{V} \models SL^+$ if $\mathbf{V} \not\models Let^+$ but for some $S \in \mathbf{V}$, $(S^\bullet, S) \models SL^+$. Then by the corollary of Ésik–Horváth Theorem, the following is immediate:

Lemma 25. *Let $\mathbf{V}$ be a pseudovariety of finite semigroups. Then*
1. *$\mathbf{V} \models Let^+ \Rightarrow \hat{\alpha}_2(\mathbf{V}) = \mathbf{Sgp}$.*
2. *$\mathbf{V} \models SL^+ \Rightarrow \hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V})$.*
3. *$\mathbf{V} \not\models SL^+$ and $\mathbf{V} \not\models Let^+ \Rightarrow \hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V})$.*

Lemma 26. *Let $\mathbf{V}$ be a pseudovariety of finite semigroups. Then $\mathbf{V} \models SL^+$ implies $\hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) = \mathbf{R}$.*

Proof. The hypothesis implies $U \in \mathbf{V}$. Since $\mathscr{S}(HSP_0(\{U\})) = \mathbf{R}$ by Stiffler's results [16], it follows that $\mathbf{R} \subseteq \hat{\alpha}_0(\mathbf{V})$. Since $\mathbf{V} \not\models Let$, we have $\hat{\alpha}_\infty(\mathbf{V}) \subseteq \mathbf{R}$ by Corollary 3(2) and Proposition 12. Therefore, $\mathbf{R} \subseteq \hat{\alpha}_0(\mathbf{V}) \subseteq \hat{\alpha}_1(\mathbf{V}) \subseteq \hat{\alpha}_\infty(\mathbf{V}) \subseteq \mathbf{R}$. □

Lemma 27. *If $\mathscr{A} = (A, X, \delta) \models Let^+$, then all finite automata are isomorphic to sub-automata of members of $\alpha_1(P(\mathscr{A}^+))$.*

Proof. Given any finite automaton $\mathscr{B} = (B, Y, \delta')$, we show $\mathscr{B}$ embeds in a single-factor α_1-product of a direct product of copies of $\mathscr{A}^+$. In $\mathscr{A}$, we have states $p \neq q$, and non-empty words $s, t \in X$ with $p \cdot s = q$ and $q \cdot t = p$, whence $q \cdot ts = q$ and $p \cdot st = p$. We map a state $b \in B$ to the state "b" of the $|B|$-fold direct product of copies of $\mathscr{A}^+$, where "b" $= (p, \ldots, p, q, p, \ldots, p)$, such that q occurs in the bth position and p occurs in all other positions. Now define the feedback function $f : A^{|B|} \times Y \to (X^+)^{|B|}$ to have value in its bth-component:

$$
(f(a, y))_b = \begin{cases}
ts & \text{if } a = \text{``}b\text{''} \text{ and } b \cdot y = b, \\
t & \text{if } a = \text{``}b\text{''} \text{ and } b \cdot y \neq b, \\
s & \text{if } a = \text{``}c\text{''}, b \neq c, \text{ and } c \cdot y = b, \\
st & \text{if } a = \text{``}c\text{''}, b \neq c, \text{ and } c \cdot y \neq b, \\
s & \text{otherwise,}
\end{cases}
$$

where $a \in A^{|B|}$, $y \in Y$, $b, c \in B$. (Note the value of f in the fifth case is arbitrary.)

It is straightforward to check that "b"$\cdot y =$ "$b \cdot y$" holds for all $b \in B$ and $y \in Y$, so $\mathscr{B}$ is isomorphic to a subautomaton of the α_1-product. $\square$

Lemma 28. *If $\mathbf{V}$ is a pseudovariety of finite semigroups, and $\mathbf{V} \models Let^+$, then $\hat{\alpha}_1(\mathbf{V}) = \mathbf{Sgp}$.*

Proof. Take $S \in \mathbf{V}$ with $(S^\bullet, S) \models Let^+$. Since S is not $\mathscr{R}$-trivial, choose distinct $p, q \in S$ such that there exist $s, t \in S$ with $ps = q$ and $qt = p$. By the construction of the previous lemma, any $(T^\bullet, T)$ embeds in an α_1-product of the $|T^\bullet|$-fold direct product of copies of $(S^\bullet, S)$. The image of $T^\bullet$ has all components in $\{p, q\}$, thus the image of $(T^\bullet, T)$ is actually isomorphic to a single-factor α_1-product of $(Q^\bullet, Q)$, where Q is the $|T^\bullet|$-fold direct product of copies of S. Since $Q \in \mathbf{V}$, we have $T \in \hat{\alpha}_1(\mathbf{V})$. $\square$

Theorem 29. *Let $\mathbf{V}$ be a pseudovariety of finite semigroups. Then*
1. $\mathbf{V} \models Let^+ \Rightarrow \hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) = \mathbf{Sgp}$.
2. $\mathbf{V} \models SL^+ \Rightarrow \hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) = \mathbf{R}$.
3. $\mathbf{V} \not\models SL^+$ *and* $\mathbf{V} \not\models Let^+ \Rightarrow \hat{\alpha}_\infty(\mathbf{V}) = \hat{\alpha}_0(\mathbf{V}) \subseteq \mathbf{D}^\rho$.

Moreover, the converses hold.

Proof. (1) follows from Lemma 28. (2) is just Lemma 26. (3) By Corollary 18, $\mathbf{V} \subseteq \mathbf{D}^\rho$. $\mathbf{D}^\rho$ is closed under α_0-product, so $\hat{\alpha}_0(\mathbf{V}) \subseteq \mathbf{D}^\rho$. But by Lemma 25(3), $\hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V})$. The converses follow from what we have seen before. $\square$

Corollary 30. *For each pseudovariety $\mathbf{V}$ of finite semigroups, $\hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V})$.*

Corollary 31. *For $0 \leqslant i \leqslant \infty$, the operator $\hat{\alpha}_i : PV \to PV$ is a closure operator.*

Proof. We already noted, when they were introduced, that α_∞ and α_0 are closure operators for classes of finite automata, so it follows that the corresponding operators are closure operators on PV. We have, for any $i > 0$, $\hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V})$, so $\hat{\alpha}_i(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V})$. $\square$

Theorem 32.

$$\mathbf{V} \not\models Let^+ \;\;\Leftrightarrow\;\; \hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) \subseteq \mathbf{R},$$

where equality with $\mathbf{R}$ *holds if and only if* $\mathbf{V} \models SL^+$ *if and only if* $IRRED(\mathbf{V}) = \{U\}$.

$$\mathbf{V} \models Let^+ \;\;\Leftrightarrow\;\; \hat{\alpha}_0(\mathbf{V}) \subseteq \hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) = \mathbf{Sgp},$$

where $\hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_1(\mathbf{V})$ *if and only if* $IRRED(\mathbf{V})$ *contains the flip-flop monoid and all finite simple groups.*

Proof. By the Krohn–Rhodes Theorem, equality of $\hat{\alpha}_0(\mathbf{V})$ and $\mathbf{Sgp}$ holds if and only if $IRRED(\mathbf{V})$ includes all finite simple groups and the flip-flop monoid. Everything else is clear for what we have already established. $\square$

Let us record the effect of the feedback operators on the lattice of pseudovarieties, which now follows directly:

Theorem 33 (Action of feedback operators on pseudovarieties). *Let* $\mathbf{V}$ *be a pseudovariety of finite semigroups. Then we have three cases determining the action of the* $\hat{\alpha}_k$ *operators on* $\mathbf{V}$ $(0 \leqslant k \leqslant \infty)$:
1. *If* $\mathbf{V} \subseteq \mathbf{D}^\rho$, *then* $\hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) \subseteq \mathbf{D}^\rho$.
2. *If* $\mathbf{V} \subseteq \mathbf{R}$ *but* $\mathbf{V} \not\subseteq \mathbf{D}^\rho$, *then* $\hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) = \mathbf{R}$.
3. *Otherwise,* $\mathbf{V} \not\subseteq \mathbf{R}$, *and then* $\hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) = \mathbf{Sgp}$, *and* $\hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_1(\mathbf{V})$ *if and only if* $\mathbf{V}$ *has all irreducibles.*

For case $\mathbf{V} \subseteq \mathbf{D}^\rho$: For which $\mathbf{V}$ does $\mathbf{V} = \hat{\alpha}_0(\mathbf{V})$? Some examples of such closed pseudovarieties include:

$\mathbf{D}^\rho$ satisfying $e^2 = e$ implies $es = e$; $(\mathbf{D}_n)^\rho$ satisfying $x_1 \cdots x_n = x_1 \cdots x_n y$; $\mathbf{Nil}_n$ satisfying $x_1 \cdots x_n = 0$; $\mathbf{Nil} = \bigcup_n \mathbf{Nil}_n$; $\mathbf{LZ} = \mathbf{D}_1^\rho$, satisfying $xy = x$. Their closure under $\hat{\alpha}_0$-product is easy to check directly. All these pseudovarieties are contained in $\mathbf{D}^\rho$ and closed for $\hat{\alpha}_0 = \hat{\alpha}_\infty$.

The smallest example of a pseudovariety $\mathbf{V} \subseteq \mathbf{R}$ but $\mathbf{V} \not\subseteq \mathbf{D}^\rho$ is $\mathbf{SL}$, the variety of semilattices, defined by equations $x^2 = x$ and $xy = yx$, since this is the smallest pseudovariety containing U. We have $\mathbf{SL} \subsetneq \hat{\alpha}_\infty(\mathbf{SL}) = \hat{\alpha}_0(\mathbf{SL}) = \mathbf{R} = \mathscr{S}(HSP_0(\{U\}))$.

There are many examples of $\mathbf{V} \not\subseteq \mathbf{R}$ closed under $\hat{\alpha}_0$: groups $\mathbf{G}$, solvable groups, p-groups, and many other pseudovarieties of groups,[5] whose closure under the $\hat{\alpha}_0$-operator is evident from considering irreducible divisors and using the fact that

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[5] Note that since we are considering semigroup varieties, the empty semigroup must be admitted as a member of any pseudovariety of groups, including $\mathbf{G}$, etc.
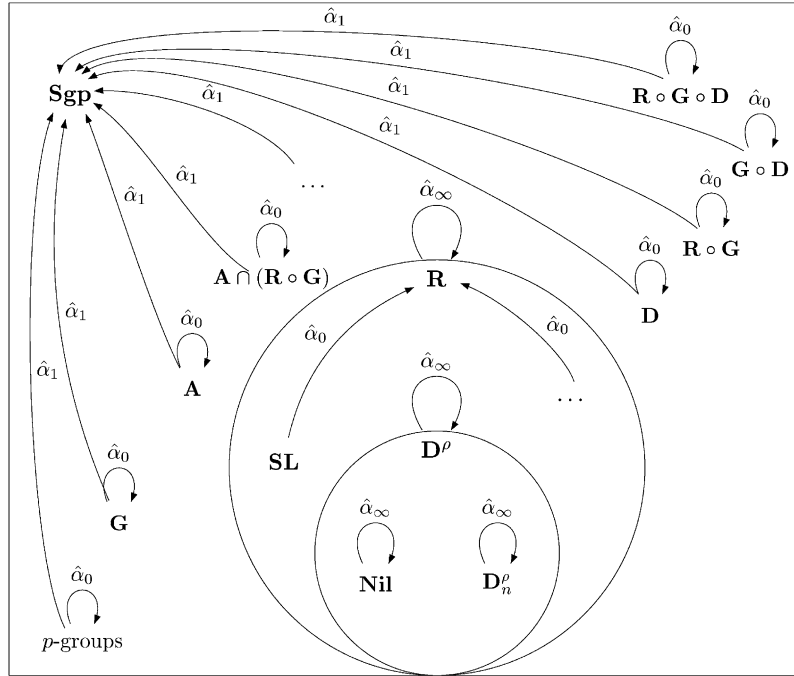
Fig. 2. For every pseudovariety $\mathbf{V}$ of finite semigroups: $\hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V})$. Within $\mathbf{R}$, $\hat{\alpha}_0 = \hat{\alpha}_1$ and there are many closed classes, i.e. $\mathbf{V} = \hat{\alpha}_0(\mathbf{V})$, within $\mathbf{D}^\rho$. However $\mathbf{R}$ is the only closed class within $\mathbf{R}$ not contained in $\mathbf{D}^\rho$. Outside of $\mathbf{R}$, $\hat{\alpha}_1(\mathbf{V}) = \hat{\alpha}_\infty(\mathbf{V}) = \mathbf{Sgp}$ (all finite semigroups); $\hat{\alpha}_0(\mathbf{V}) = \hat{\alpha}_1(\mathbf{V})$ if and only if $\mathbf{V}$ contains all irreducible finite semigroups; and there are many examples of closed classes $\mathbf{V} = \hat{\alpha}_0(\mathbf{V})$.

a cascade of group automata is a group automaton. Other $\hat{\alpha}_0$-closed pseudovarieties include $\mathbf{D}$ (satisfying $e^2 = e$ implies $se = e$) since $\mathbf{D} = \mathscr{S}(HSP_0(\{2^r\}))$; as well as $\mathbf{R} \circ \mathbf{G}$, $\mathbf{G} \circ \mathbf{D}$, and $\mathbf{R} \circ \mathbf{G} \circ \mathbf{D}$, which are defined by the exclusion of 2^r, U, and F, respectively, by results of Stiffler [16, Fact 4.16]); [6] and the exclusion varieties of finite semigroups not divided by any other particular irreducible or set of irreducibles (see examples in [16,3]). The pseudovariety of the finite aperiodic semigroups $\mathbf{A}$, satisfying $x^{\omega+1} = x^\omega$, is the class that excludes all finite simple group divisors and so is $\hat{\alpha}_0$-closed. Let $\mathbf{W}$ be the pseudovariety of aperiodic semigroups not divided by 2^r, then $IRRED(\mathbf{W}) = \{U\}$ and $\mathbf{W} \models Let^+$ by the counterexample in the proof of Corollary 14, so $\mathbf{R} \subsetneq \mathbf{W}$. Since $\mathbf{W}$ is defined by the exclusion of irreducibles (finite simple groups and 2^r), we have $\hat{\alpha}_0(\mathbf{W}) = \mathbf{W}$, but $\hat{\alpha}_1(\mathbf{W}) = \mathbf{Sgp}$ by Theorem 33(3). In fact, $\mathbf{W} = \mathbf{A} \cap (\mathbf{R} \circ \mathbf{G})$ by [16, Fact 4.16(a)].

This situation is schematized in Fig. 2 showing major divisions in the lattice of pseudovarieties of finite semigroups that characterize the effect of the various feedback operators.

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[6] Here $\mathbf{V}_n \circ \mathbf{V}_{n-1} \circ \cdots \circ \mathbf{V}_1$ denotes the pseudovariety generated by characteristic semigroups of α_0-products whose i^{th} factor automaton $\mathscr{A}_i = (S_i^\bullet, S_i)$ for some $S_i \in \mathbf{V}_{i\cdot}$.

Corollary 34. *Let* $\mathbf{V}$ *be a pseudovariety of finite semigroups such that* $\hat{\alpha}_i(\mathbf{V}) = \mathbf{V}$ *for a certain* i $(0 \leqslant i \leqslant \infty)$. *If either* $i \geqslant 1$ *or* $\mathbf{V} \not\models Let^+$, *then we have*

$$\mathbf{V} = \mathbf{Sgp} \Leftrightarrow \mathbf{V} \models Let^+$$
$$\mathbf{V} = \mathbf{R} \Leftrightarrow \mathbf{V} \models SL^+$$
$$\mathbf{V} \subseteq \mathbf{D}^\rho \ and \Leftrightarrow otherwise.$$

$$\hat{\alpha}_0(\mathbf{V}) = \mathbf{V}$$

While for $i = 0$ *and* $\mathbf{V} \models Let^+$, *we have* $\hat{\alpha}_0(\mathbf{V}) = \mathbf{V}$ *and*

$$\mathbf{V} = \{S \in \mathbf{Sgp} \mid PRIMES(S) \subseteq \mathbf{V}\} \Leftrightarrow F \in \mathbf{V}$$
$$\mathbf{V} \subseteq \mathbf{R} \circ \mathbf{G} \circ \mathbf{D} \Leftrightarrow otherwise,$$

where F *is the flip-flop monoid.*

Proof. The first part follows from Theorem 29. For the second part with $i = 0$ and $\mathbf{V} \models Let^+$, the Krohn–Rhodes Theorem yields the case when $F \in \mathbf{V}$. On the other hand, $F \notin \mathbf{V}$ if and only if $\mathbf{V} \subseteq \mathbf{R} \circ \mathbf{G} \circ \mathbf{D}$, by the result of Stiffler [16, Fact 4.16(c)]. □

Despite what one might have expected, in every case, the study of the feedback operators $\hat{\alpha}_i$ on the lattice of pseudovarieties of finite semigroups is completely solved or reduced to the study of $\hat{\alpha}_0$, the cascade closure.

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