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Tabular degrees in α-recursion theory

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Abstract

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We introduce several generalizations of the truth-table and weak-truth-table reducibilities to α-recursion theory. A number of examples are given of theorems that lift from ω-recursion theory, and of theorems that do not. In particular it is shown that the regular sets theorem fails and that not all natural generalizations of wtt are the same.

0. Introduction

The study of strong reducibilities in ω-recursion theory began with the study of ω-recursion theory. However, α-recursion theory leapt straight into the study of Turing reducibility, bypassing all study of strong reducibilities in the generalized context. This paper is a first step to establish strong reducibilities in α-recursion theory, in that we develop definitions of α-truth-table, and several versions of α-weak-truth-table reducibilities.

These definitions do extend the relevant definitions on ω, and some of them behave in similar ways. But not always; for example, weak-truth-table generalizes in several distinct ways, but the useful property of ω-wtt—that permitting gives rise to it—does not apply to all of the generalizations.

Several problems new to these reducibilities quickly become apparent—the failure of a regular sets theorem; the fact that the power set function is rarely a total recursive function; and that the blocking machinery does not interact readily with the reducibilities (a search over a block is α-re, but not generally α-recursive).

We illustrate the above problems with examples that also give indications of how some theorems lift from ω to α. However, some very basic theorems do not, for instance Nerode's theorem that

$$A \leqslant_{\mathrm{wtt}} B \quad \text{iff} \quad \text{there is an } e \text{ such that } \forall X\, e^X \text{ is total and } e^B = A$$

makes heavy use of compactness, and indications are that it is necessary to use it. This means it must fail for many α.

This paper is primarily to establish definitions and show how they work. In later papers, we will show how the strong reducibilities may be used, as in ω-recursion theory, to give information about Turing degrees.

For the reader's convenience, we recall some basic definitions and notation of α-recursion theory.

Definition 0.1. α is *admissible* iff every total Σ_1-function f, with domain an element of L_α, has $\mathrm{rng}(f) \in L_\alpha$.

Definition 0.2.

ρ_α^n = the Σ_n-projectum

= the least $\delta \leqslant \alpha$ for which there is a total, injective, Σ_n-function from α into δ

= the greatest $\delta \leqslant \alpha$, such that every $A \subseteq \gamma < \delta$ which is Σ_n is an element of L_α.

Definition 0.3. The *Σ_n-cardinality* of a subclass of $\langle L_\alpha, D\rangle$ is

$|X|^{n,D} \stackrel{\text{def}}{=}$ the least $\delta \leqslant \alpha$ for which there is a total Σ_n^D-bijection from X into δ.

Definition 0.4.

κ_n^D = the Σ_n^D-cofinality

= the greatest $\delta \leqslant \alpha$, such that $\forall X \in L_\alpha\ |X|^{n,D} < \delta$, and $f: X \to \alpha$ is a total Σ_n^D-function, then $\mathrm{rng}(f) \in L_\alpha$.

If $D = \emptyset$ then it may be omitted or replaced by α.

These definitions may differ slightly from those usually used, but they are easily shown to be equivalent.

1. Definitions

Throughout the remainder of this paper we will assume α is any admissible ordinal.

In this section we will introduce a number of definitions, firstly for truth-table reducibility, and then for a variety of weak-truth-table reducibilities. Some results will be shown relating these definitions, and some basic facts about these definitions will be proven.

Definition 1.1. Let A, B be subsets of L_α. Then $A \leqslant_{\alpha\text{-tt}} B$ (A is α-truth-table reducible to B) iff $\exists e\, \forall K \in L_\alpha$

(1) $K \subseteq A$ iff $\exists M_1, M_2\ M_1 \subseteq B \wedge M_2 \cap B = \emptyset \wedge \langle\langle M_1, M_2\rangle, 0\rangle \in D_{\{e\}(K)}$,

(2) $K \cap A = \emptyset$ iff $\exists M_1, M_2\ M_1 \subseteq B \wedge M_2 \cap B = \emptyset \wedge \langle\langle M_1, M_2\rangle, 1\rangle \in D_{\{e\}(K)}$,

(3) $\bigcup \{\pi_0[D_{\{e\}(K')}] \mid K' \in L_\alpha \wedge K' \geqslant_{L_\alpha} K\} = L_\alpha \times L_\alpha$,

(4) $\forall M_1, M_2\ M_1 \cap M_2 = \emptyset \wedge \langle\langle M_1, M_2\rangle, 0\rangle \in D_{\{e\}(K)}$
$\Rightarrow \langle\langle M_1, M_2\rangle, 1\rangle \notin D_{\{e\}(K)}$.

This definition is meant to describe the situation in ω where elements of A are determined by Boolean polynomials with input information from B. In that case, we would naturally take $D_{\{e\}(K)}$ to be some set of possible inputs and results of a recursively determined polynomial. Since, on ω, $n \mapsto \wp(n)$ is recursive, this gives $D_{\{e\}(K)}$ recursively.

However, in the more general context where α may not be closed under powerset (or if it is, then $\beta \mapsto \wp(\beta)$ is a $0'$-function), we cannot do this, and so need to limit our attention to recursively describable information. Another property of ω that fails in general is König's lemma, and hence the usual proof of Nerode's theorem that

$$A \leqslant_{\omega\text{-tt}} B \quad \text{iff} \quad \exists e\, \forall X\, (\{e\}^X \text{ is total} \wedge \{e\}^B = \chi_A)$$

fails. In fact, we cannot even prove the left-to-right implication without additional assumptions.

Definition 1.2. α is *Σ_n-γ-admissible* iff $\alpha \leqslant \gamma$ and there is no $\lambda < \alpha$ for which there exists a function $f: \lambda \to \alpha$ cofinally in α, which is Σ_n^γ.

Hence "α is admissible" means the same as "α is Σ_1-α-admissible".

Proposition 1.3. *If $n \geqslant 1$ and α is Σ_{n+1}-γ-admissible, and $A \leqslant_{\alpha\text{-tt}} B$ then there is an e such that*

(i) *$\forall X \subseteq \alpha$ if X is Σ_n^γ then $\{e\}^X$ is total,*

(ii) *$\{e\}^B$ is total, and $\{e\}^B = \chi_{A^*}$ where $A^* = \{\langle K, 1\rangle \mid K \leqslant A\} \cup \{\langle K, 0\rangle \mid K \cap A = \emptyset\}$.*

Proof. Let $A \leqslant_{\alpha\text{-tt}} B$ via e. Define e' such that

$$\{e'\}^X(z, i) = \begin{cases} 0 & \text{iff } \exists M_1, M_2\ \langle\langle M_1, M_2\rangle, i\rangle \in D_{\{e\}(z)} \wedge M_1 \subseteq X \wedge M_2 \cap X = \emptyset, \\ 1 & \text{iff } \exists M_1, M_2\ M_1 \subseteq X \wedge M_2 \cap X = \emptyset \\ & \qquad \wedge \forall \langle \Lambda_1, \Lambda_2\rangle \in \pi_0[D_{\{e\}(z)}](M_1 \cap \Lambda_2 \neq \emptyset \vee M_2 \cap \Lambda_1 \neq \emptyset), \\ \uparrow & \text{otherwise.} \end{cases}$$

It is clear that $\{e'\}^B = A^*$, and $\{e'\}^B$ is total. Also, if $X \subseteq \alpha$ is Σ_n^γ, then we consider the function which searches through $\pi_1[D_{\{e\}(K)}]$ for a witness that either $M_1 \notin X$ or $M_2 \cap X \neq \emptyset$ is Δ_{n+1}^γ.

Since α is Σ_{n+1}-γ-admissible, this function has bounded range.

Now $\alpha \leq \rho_n^\gamma$ is a γ-cardinal, and hence the range of the function intersected with X is in L_α, and likewise the range of the function intersected with $\alpha \backslash X$ is in L_α.

This gives us witnesses to the case $\{e'\}^X(z, i) = 1$. □

Corollary 1.4. *Let α be an L-cardinal, and $A \leq_{\alpha\text{-tt}} B$. Then there is an e such that*

(1) $\forall X\ \{e\}^X$ *is total,*

(2) $\{e\}^B = A^*$. □

The converse of this theorem appears to require some degree of α-compactness, although how much is unclear.

There are several reducibilities which are weaker than α-tt. We first introduce the 'most' natural one of these, which is very readily seen to be a weakening of the tt-condition.

$$\langle\langle M_1, M_2\rangle, i\rangle \in D_{\{e\}(K)}.$$

Definition 1.5. Let A, B be subsets of L_α. Then $A \leq_{\alpha\text{-wtt}} B$ (A is α-weakly-truth-table reducible to B) iff $\exists e\ \forall K \in L_\alpha$

$$\begin{aligned}(1)\quad & K \subseteq A \quad \text{iff} \quad \exists M_1, M_2\ \ M_1 \subseteq B \wedge M_2 \cap B = \emptyset \\ & \qquad \wedge \{M_1, M_2\} \subseteq D_{\{e_0\}(K)} \wedge L_\alpha \models \Phi_{e_1}(K, M_1, M_2), \\ (2)\quad & K \cap A = \emptyset \quad \text{iff} \quad \exists M_1, M_2\ \ M_1 \subseteq B \wedge M_2 \cap B = \emptyset \\ & \qquad \wedge \{M_1, M_2\} \subseteq D_{\{e_0\}(K)} \wedge L_\alpha \models \Phi_{e_2}(K, M_1, M_2)\end{aligned}$$

where $e = \langle e_0, e_1, e_2\rangle$ and Φ_{e_i} denotes the e_ith Σ_1^α-formula.

$D_{\{e_0\}(K)}$ is meant to represent not only the fact that computations have recursive use, but also that we can collect together all the information that a computation might possibly use quickly (i.e., recursively).

We can further weaken this definition by only requiring that the use function be bounded, but the bounding function may be arbitrarily complex.

Definition 1.6. Let A, B be subsets of L_α. Then $A \leq_{\alpha\text{-mtt}} B$ (A is α-mildly-truth-table reducible to B) iff $\exists e = \langle e_0, e_1, e_2\rangle\ \forall K \in L_\alpha$

$$\begin{aligned}(1)\quad & K \subseteq A \quad \text{iff} \quad \exists M_1, M_2\ \ M_1 \subseteq B \wedge M_2 \cap B = \emptyset \\ & \qquad \wedge M_1, M_2 \subseteq L_{\{e_0\}(K)} \wedge L_\alpha \models \Phi_{e_1}(K, M_1, M_2), \\ (2)\quad & K \cap A = \emptyset \quad \text{iff} \quad \exists M_1, M_2\ \ M_1 \subseteq B \wedge M_2 \cap B = \emptyset \\ & \qquad \wedge M_1, M_2 \subseteq L_{\{e_0\}(K)} \wedge L_\alpha \models \Phi_{e_2}(K, M_1, M_2).\end{aligned}$$

We remark that for certain B it is possible to have $\{e_0\}$ be a trivial function and to have

$$A \leqslant_{\alpha\text{-mtt}} B \quad \text{iff} \quad A \leqslant_\alpha B.$$

For instance if $\rho_1^\alpha < \alpha$ and $B \subseteq \rho_1^\alpha$ just take $\{e_0\}(K) = \rho_1^\alpha$.

However, if B is regular, then the function $\{e_0\}(K)$ 'must' be unbounded in α (if A is non-recursive), and in fact we will only use this reduction in this context. This also means that this reducibility does not extend to inadmissible ordinals, whereas $\leqslant_{\alpha\text{-wtt}}$ is suitable for such extension.

An intermediate reducibility $\leqslant_{\alpha\text{-}\hat{w}}$ is of some interest—the improvement is to locally bound computations rather than globally as in the case of $\leqslant_{\alpha\text{-wtt}}$.

Definition 1.7. Let A, B be subsets of L_α. Then $A \leqslant_{\alpha\text{-}\hat{w}} B$ iff $\exists e\, \forall K \in L_\alpha$

(1) $K \subseteq A$ iff $\exists z, M_1, M_2\ M_1 \subseteq B \wedge M_2 \cap B = \emptyset$

$$\wedge\ M_1, M_2 \subseteq L_{\{e_0\}(K)} \wedge L_\alpha \models \Phi_{e_1}(z, K, M_1, M_2)$$

(2) $K \cap A = \emptyset$ iff $\exists z, M_1, M_2\ M_1 \subseteq B \wedge M_2 \cap B = \emptyset$

$$\wedge\ M_1, M_2 \subseteq L_{\{e_0\}(K)} \wedge L_\alpha \models \Phi_{e_2}(z, K, M_1, M_2)$$

(3) $\exists \sigma\, \forall M_1, M_2 \in L_{\{e\}(K)}\ (M_1 \cap M_2 = \emptyset \wedge \exists \beta\, L_\beta \models \exists z\ \Phi_{e_i}(z, K, M_1, M_2))$

$$\Rightarrow M_1, M_2 \in L_\sigma \wedge L_\sigma \models \exists z\ \Phi_{e_i}(z, K, M_1, M_2)$$

where Φ_{e_i} is the e_ith Δ_0-formula.

That this, and the other reducibilities, are transitive, is an easy exercise for the reader. There are several connections between reducibilities which are immediately apparent.

Proposition 1.8. *Let A, B be subsets of L_α. Then*

(1) $A \leqslant_{\alpha\text{-tt}} B \Rightarrow A \leqslant_{\alpha\text{-wtt}} B$ *and* $A \leqslant_{\alpha\text{-}\hat{w}} B$,

(2) $A \leqslant_{\alpha\text{-wtt}} B \Rightarrow A \leqslant_{\alpha\text{-mtt}} B$,

(3) $A \leqslant_{\alpha\text{-}\hat{w}} B \Rightarrow A \leqslant_{\alpha\text{-mtt}} B$.

Proof. (1) Let $A \leqslant_{\alpha\text{-tt}} B$ via e. Define $e' = \langle e_0', e_1', e_2' \rangle$ so that

$$D_{\{e_0'\}(K)} = \{M_1 \mid \exists i \in \mathbf{2}\, \exists M\ \langle\langle M_1, M\rangle, i\rangle \in D_{\{e\}(K)}\}$$
$$\cup \{M_2 \mid \exists i \in \mathbf{2}\, \exists M\ \langle\langle M, M_2\rangle, i\rangle \in D_{\{e\}(K)}\}$$

and

$$\Phi_{e_1'}(K, M_1, M_2) \Leftrightarrow \langle\langle M_1, M_2\rangle, 0\rangle \in D_{\{e\}(K)},$$
$$\Phi_{e_2'}(K, M_1, M_2) \Leftrightarrow \langle\langle M_1, M_2\rangle, 1\rangle \in D_{\{e\}(K)}.$$

Then $A \leqslant_{\alpha\text{-wtt}} B$ via e'.

To show $A \leqslant_{\alpha\text{-}\hat{w}} B$, take $e' = \langle e_0', e_1', e_2' \rangle$ where e_1', e_2' are as above. e_0' is such that $\{e'\}(K) = \text{L-rk}(D_{\{e\}(K)})$ and then $\sigma = \text{L-rk}(D_{\{e\}(K)}) + 1$ is an appropriate bound.

(2) Given $A \leqslant_{\alpha\text{-wtt}} B$ via $e = \langle e_0, e_1, e_2 \rangle$ to get $A \leqslant_{\alpha\text{-mtt}} B$ take $\{e_0'\}(K) = \text{L-rk}(D_{\{e\}(K)})$ and $e' = \langle e_0', e_1, e_2 \rangle$.

(3) Omit σ. □

Under certain additional conditions, there are other connections between these reducibilities.

Proposition 1.9. *Let $L_\alpha \models$ Power Set Axiom, and $A \leqslant_{\alpha\text{-mtt}} B$. Then $A \leqslant_{\alpha\text{-}\hat{w}} B$.*

Proof. First suppose $\alpha > \omega$. It suffices to find a suitable bound σ for all computations. Let $A \leqslant_{\alpha\text{-mtt}} B$ via e, and $K \in L_\alpha$, and let β be least such that $K, e \in L_\beta$.

Let $\kappa > \beta$ be the next α-cardinal greater than β and let γ be the least Σ_1-stable ordinal strictly greater than κ. Now, noting the e, $K \in L_\kappa$ and $L_\kappa <_1 L_\alpha$, we have $\{e_0\}(K) \in L_\kappa$, hence $\wp(\{e_0\}(K)) \subseteq L_\kappa$, and is an element of $L_{\kappa+1}$.

Thus if $L_\alpha \models \exists z\, (\Phi_{e_1}(z, K, M_1, M_2) \vee \Phi_{e_1}(z, K, M_1, M_2))$ and $M_1, M_2 \subseteq L_{\{e_0\}(K)}$ then $M_1, M_2 \in L_\gamma$, $K \in L_\gamma$, $e \in L_\gamma$ and so

$$L_\gamma \models \exists z\, (\Phi_{e_1}(z, K, M_1, M_2) \vee \Phi_{e_1}(z, K, M_1, M_2)).$$

This means γ is a suitable bound.

If $\alpha = \omega$, then since ω is closed under powerset, and $\rho_1^\omega = \omega$,

$$S \stackrel{\text{def}}{=} \{\langle M_1, M_2 \rangle \mid \exists z\, (\Phi_{e_1}(z, K, M_1, M_2) \vee \Phi_{e_2}(z, K, M_1, M_2)) \wedge (M_1 \cap M_2 = \emptyset)\}$$

is an α-finite subset of $\wp(\{e\}(K)^2)$.

By admissibility, the function $f : S \to \alpha$ given by

$$f(\langle M_1, M_2 \rangle) = \text{the least witness } z \text{ to } \langle M_1, M_2 \rangle \in S$$

has bounded range, say n is a bound. Then n is the bound required by our definition. □

Proposition 1.10. *Let $\rho_1^\alpha = \alpha$, and $A \leqslant_{\alpha\text{-wtt}} B$. Then $A \leqslant_{\alpha\text{-}\hat{w}} B$.*

Proof. Suppose $A \leqslant_{\alpha\text{-wtt}} B$ via e. Define $e' = \langle e_0', e_1', e_2' \rangle$ such that $\{e_0'\}(K) = \text{L-rk}(D_{\{e_0\}(K)})$ and for $i = 1, 2$

$$\Phi_{e_i'}(z, K, M_1, M_2) \iff (\langle M_1, M_2 \rangle \in D_{\{e_0\}(K)}) \vee (z \text{ is a witness to } \Phi_{e_i}(K, M_1, M_2)).$$

To obtain the required bound σ, note that $|D_{\{e_0\}(K)}|^{1,\alpha} < \rho_1^\alpha$, and so the set

$$S \stackrel{\text{def}}{=} \{\langle M_1, M_2 \rangle \mid \exists z\, (\Phi_{e_1'}(z, K, M_1, M_2) \vee \Phi_{e_2'}(z, K, M_1, M_2)) \wedge (M_1 \cap M_2 = \emptyset)\}$$

is a Σ_1-subset of a small set, and so is α-finite. Let $f : S \to \alpha$ be defined by

$$f(\langle M_1, M_2 \rangle) = \text{least witness } z \text{ to } \langle M_1, M_2 \rangle \in S.$$

Then, by admissibility, f has range bounded by L_σ say. σ is the required bound. $\square$

The converse of this theorem fails, as we will later show (see Section 4).

The relationship between $\leqslant_{\alpha\text{-wtt}}$ and $\leqslant_{\alpha\text{-}\hat{w}}$ when $\rho_1^\alpha < \alpha$ is unclear, since if $D_{\{e_0\}(K)}$ is a large set, the 'time' taken to find the appropriate witness could well be unbounded in α.

However, this proof does suggest a different restraint which we might impose upon reducibilities, and that is bounding the size of the set $D_{\{e\}(K)}$, i.e., if κ is either an α-cardinal or α, and r either tt or α-wtt, then we could define $\leqslant_{\kappa\text{-r}}$ as being the same as $\leqslant_\text{r}$ with the additional requirement that $|D_{\{e\}(K)}|^{0,\alpha} < \kappa$.

The following facts are now immediate.

(1) $\kappa_1 < \kappa_2$ and $A \leqslant_{\kappa_1\text{-r}} B \;\Rightarrow\; A \leqslant_{\kappa_2\text{-r}} B$

(2) $(A \leqslant_{\text{r}_1} B \;\Rightarrow\; A \leqslant_{\text{r}_2} B) \;\Rightarrow\; (A \leqslant_{\kappa\text{-r}_1} B \;\Rightarrow\; A \leqslant_{\kappa\text{-r}_2} B)$

(3) $A \leqslant_{\rho_1\text{-wtt}} B \;\Rightarrow\; A \leqslant_{\alpha\text{-}\hat{w}} B$ by the same proof as the last proposition.

We will not continue to investigate these reducibilities here.

A further alternative is relativizing the reducibilities. This is perhaps most interesting for $\leqslant_{\hat{w}}$.

Definition 1.11. Let $A, B, D \subseteq L_\alpha$. We say that $A \leqslant_{\alpha\text{-}\hat{w}\text{-}D} B$ iff
(1) $A \leqslant_{\alpha\text{-}\hat{w}} B$,
(2) the function $K \mapsto \sigma$ is total, and D-recursive.

This notion is closely related to the reducibilities so far introduced as the following propositions show.

Proposition 1.12. *Let $D_1 \leqslant_\alpha D_2$ and $A \leqslant_{\alpha\text{-}\hat{w}\text{-}D_1} B$. Then $A \leqslant_{\alpha\text{-}\hat{w}\text{-}D_2} B$.*

Proof. Immediate. $\square$

As a consequence of this proposition we will write $A \leqslant_{\alpha\text{-}\hat{w}\text{-}\mathbf{d}} B$ where $\mathbf{d}$ is an α-T-degree.

Proposition 1.13. *Let $A \leqslant_{\alpha\text{-}\hat{w}} B$. Then $A \leqslant_{\alpha\text{-}\hat{w}\text{-}\mathbf{0}'} B$.*

Proof. Let $A \leqslant_{\alpha\text{-}\hat{w}} B$ via e, and let $f: L_\alpha \to \alpha$ be the total function bounding computations. It suffices to show that some such $f' \leqslant_{w\alpha} \mathbf{0}'$.

Notice that σ is a bound on computations such that

$$\begin{aligned} K \in L_\alpha \quad \text{iff} \quad & \forall M_1, M_2 \subseteq L_{\{e\}(K)}\, \exists z\, (\Phi_{e_1}(z, K, M_1, M_2) \wedge \Phi_{e_2}(z, K, M_1, M_2)) \\ & \Rightarrow \exists z \in L_\sigma\, (\Phi_{e_1}(z, K, M_1, M_2) \wedge \Phi_{e_2}(z, K, M_1, M_2)). \end{aligned}$$

This is a Σ_2-relation R, given more explicitly by

$$\begin{aligned}\langle K_1, \sigma\rangle \in R \quad \text{iff} \quad & \exists x, y, m\ (\{e_0\}(K_1) = x \wedge y = L_x \wedge m = L_\sigma \\ & \wedge \forall M_1, M_2 \subseteq y\ \forall z\ (\neg\Phi_{e_1}(z, K, M_1, M_2) \wedge \neg\Phi_{e_2}(z, K, M_1, M_2)) \\ & \vee \exists z \in m\ (\Phi_{e_1}(z, K, M_1, M_2) \vee \Phi_{e_2}(z, K, M_1, M_2))).\end{aligned}$$

By uniformization there is a total Σ_2-function $f' : L_\alpha \to \alpha$ such that for all K $\langle K, f'(K)\rangle \in R$. Since f' is total, it is in fact Δ_2, and so $f' \leqslant_{\mathrm{w}\alpha} \mathbf{0}'$. □

Proposition 1.14. *$A \leqslant_{\alpha\text{-tt}} B$ iff $A \leqslant_{\alpha\text{-}\hat{\mathrm{w}}\text{-}\mathbf{0}} B$.*

Proof. Let $A \leqslant_{\alpha\text{-tt}} B$ via e. Define

$$D_{\{e'\}(K)} = \{L \mid \exists M, i \in \mathbf{2}\ \langle\langle L, M\rangle, i\rangle \in D_{\{e\}(K)} \text{ or } \langle\langle M, L\rangle, i\rangle \in D_{\{e\}(K)}\},$$

$f(K) = \text{L-rk}(D_{\{e\}(K)})$ and

$$\Phi_{e_1'}(z, K, M_1, M_2) \quad \text{iff} \quad \langle\langle M_1, M_2\rangle, 0\rangle \in D_{\{e\}(K)},$$

$$\Phi_{e_2'}(z, K, M_1, M_2) \quad \text{iff} \quad \langle\langle M_1, M_2\rangle, 1\rangle \in D_{\{e\}(K)}.$$

Then $f(K)$ is the desired bound, and f is Δ_1, hence $f \leqslant_{\mathrm{w}\alpha} \mathbf{0}$. This gives $A \leqslant_{\alpha\text{-}\hat{\mathrm{w}}\text{-}\mathbf{0}} B$.

Now let $A \leqslant_{\alpha\text{-}\hat{\mathrm{w}}\text{-}\mathbf{0}} B$ via e, and $f : L_\alpha \to \alpha$ be the recursive bounding function. Then define e' by

$$\begin{aligned}\langle\langle M_1, M_2\rangle, 0\rangle \in D_{\{e'\}(K)} \quad &\Leftrightarrow \quad M_1, M_2 \in D_{\{e_0\}(K)} \\ &\qquad \text{and } L_{f(K)} \models \exists z\ \Phi_{e_1}(z, K, M_1, M_2), \\ \langle\langle M_1, M_2\rangle, 1\rangle \in D_{\{e'\}(K)} \quad &\Leftrightarrow \quad M_1, M_2 \in D_{\{e_0\}(K)} \\ &\qquad \text{and } L_{f(K)} \models \exists z\ \Phi_{e_2}(z, K, M_1, M_2).\end{aligned}$$

This provides an α-tt reduction. □

For the readers' information, we note that if we relax the requirement that the bounding function be total, then

$$A \leqslant_{\alpha\text{-wtt}} B \;\Rightarrow\; A \leqslant_{\mathbf{0}\text{-r}} B \qquad (\mathbf{C}\text{-r is the } \Sigma_1^{\mathbf{C}}\text{-version})$$

and

$$A \leqslant_{\alpha\text{-mtt}} B \;\Rightarrow\; A \leqslant_{\mathbf{0}'\text{-r}} B.$$

With all of these reducibilities, it is natural to ask a reducibility version of Post's Problem, i.e. are there Σ_1-sets M, N such that

$$(\leqslant_{\alpha\text{-}\hat{\mathrm{w}}\text{-}\mathbf{0}}) \neq (\leqslant_{\alpha\text{-}\hat{\mathrm{w}}\text{-}M}) \neq (\leqslant_{\alpha\text{-}\hat{\mathrm{w}}\text{-}\mathbf{0}'})$$

and

$$(\leqslant_{\mathbf{0}\text{-r}}) \neq (\leqslant_{N\text{-r}}) \neq (\leqslant_{\mathbf{0}'\text{-r}}).$$

This problem will be left to the interested reader.

The next theorem is a technical result showing that we need only consider subsets of α rather than the apparently more general case of subsets of L_α.

Theorem 1.15. *Let $A \subseteq L_\alpha$. Then there is a set $B \subseteq \alpha$ such that A and B are many-one equivalent. Furthermore, if $A = W_e$ is α-re, then the function f such that $B = W_{f(e)}$ is total recursive.*

Proof. There is a parameter free, total Σ_1-function $h: L_\alpha \leftrightarrow \alpha$. Let $B = h[A]$. □

It remains to observe a connection between $\leqslant_{\alpha\text{-m}}$ and $\leqslant_{\alpha\text{-tt}}$.

Proposition 1.16. $A \leqslant_{\alpha\text{-m}} B \Rightarrow A \leqslant_{\alpha\text{-tt}} B$.

Proof. Let $A \leqslant_{\alpha\text{-m}} B$ via h, a total recursive function, with index e. Then define e' by

$$D_{\{e'\}(K)} = \{\langle\langle h[K], \emptyset\rangle, 1\rangle, \langle\langle \emptyset, h[K]\rangle, 0\rangle\}.$$

This gives the required reduction. □

As a consequence of the theorem, we may assume that all sets under consideration are subsets of the ordinals. The next question to address is that of complete sets.

Theorem 1.17. $\boldsymbol{K} \stackrel{\text{def}}{=} \{\langle x, y\rangle \mid x \in W_y\}$ *is an α-tt complete re set, i.e., if A is α-re, then $A \leqslant_{\alpha\text{-tt}} \boldsymbol{K}$.*

Proof. Let $A = W_e$ be α-re. Then, we have

$$M \subseteq A \quad \text{iff} \quad M \times \{e\} \subseteq \boldsymbol{K}$$

and

$$M \cap A = \emptyset \quad \text{iff} \quad M \times \{e\} \cap \boldsymbol{K} = \emptyset.$$

So define e' so that

$$D_{\{e'\}(M)} = \{\langle\langle M \times \{e\}, \emptyset\rangle, 1\rangle, \langle\langle \emptyset, M \times \{e\}\rangle, 0\rangle\}.$$

This gives us an α-tt reduction procedure. □

Some notation: Let $\hat{\boldsymbol{K}}$ denote the canonical subset of α which is α-m-equivalent to $\boldsymbol{K}$.

$\hat{\boldsymbol{K}}$ gives us the top of the re degrees for any of our reducibilities. The next result is that the Δ_1-sets give us the bottom of the re degrees.

Theorem 1.18. *Let A be Δ_1^α, and B be any subset of α. Then $A \leqslant_{\alpha\text{-tt}} B$.*

Proof. Let A be any Δ_1^α subset of L_α. Define e' so that

$$D_{\{e'\}(K)} = \begin{cases} \{\langle\langle\emptyset, \emptyset\rangle, 1\rangle\} & \text{if } K \subseteq A, \\ \{\langle\langle\emptyset, \emptyset\rangle, 0\rangle\} & \text{if } K \cap A = \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since A is Δ_1^α, and α is admissible, A^* is also Δ_1^α, and so $\{e'\}$ is a well-defined total recursive function such that $A \leqslant_{\alpha\text{-tt}} B$ via e'. □

The last of the basic facts about the use of degrees is that the usual join operation works.

Theorem 1.19. *Let B, C be subsets of L_α. Then $B \leqslant_{\alpha\text{-tt}} B \oplus C = (B \times \{0\}) \cup (C \times \{1\})$.*

Proof. This is clear as $B \leqslant_{\alpha\text{-m}} B \oplus C$. □

Associated with any reduction procedure $A \leqslant_{\alpha\text{-r}} B$ we have a pointwise functional, which we will denote by $\hat{\Phi}_e^{\mathrm{r}}$, or just $\hat{\Phi}_e$ when r is given by the context.

By way of example, we show how $\hat{\Phi}_e^{\mathrm{wtt}}$ is defined:

$$\hat{\Phi}_e^{\mathrm{wtt}}(B; x) = \begin{cases} 1 & \text{iff } \exists M_1, M_2\ \ M_1 \subseteq B \wedge M_2 \cap B = \emptyset \\ & \quad \wedge M_1, M_2 \in D_{\{e_0\}(\{x\})} \wedge \Phi_{e_1}(\{x\}, M_1, M_2), \\ 0 & \text{iff } \exists M_1, M_2\ \ M_1 \subseteq B \wedge M_2 \cap B = \emptyset \\ & \quad \wedge M_1, M_2 \in D_{\{e_0\}(\{x\})} \wedge \Phi_{e_2}(\{x\}, M_1, M_2), \\ \uparrow & \text{otherwise.} \end{cases}$$

Associated with this is its approximation at stage σ, where we use $\{e_0\}_\sigma(\{x\})$ and $L_\sigma \models \Phi_{e_1}$ or $L_\sigma \models \Phi_{e_2}$.

Also associated with $\hat{\Phi}_e$ is its use function, defined in the usual way, and usually denoted by ϕ_e, e.g., for α-wtt the use function ϕ_e for $\hat{\Phi}_e$ is:

$$\phi_e(x) = \bigcup D_{\{e_0\}(\{x\})}.$$

In the special case that $A \leqslant_{\alpha\text{-tt}} B$ via e we will also use the notation $B \models_e x \in A$ to mean $\hat{\Phi}_e^{\mathrm{tt}}(B; x) = 1$, and $B \models_e x \notin A$ to mean $\hat{\Phi}_e^{\mathrm{tt}}(B; x) = 0$.

2. Regularity

A fundamental property of subsets of α that is technically important in most arguments involving α-degrees (for $\leqslant_\alpha$) is regularity. We recall the definition.

Definition 2.1. $A \subseteq L_\alpha$ is *regular* iff for all $K \in L_\alpha$ $K \cap A \in L_\alpha$.

For $\leqslant_\alpha$ the Sack's regular sets theorem guarantees that every α-re degree contains an α-re regular set. However, this theorem fails for the stronger reducibilities we are considering, as the following theorem shows.

Theorem 2.2. *Let $A \leqslant_{\alpha\text{-mtt}} B$ and let B be regular. Then A is also regular.*

Proof. Let $A \subseteq L_\alpha$, and say $K \in L_\sigma$, and let $A \leqslant_{\alpha\text{-mtt}} B$ via e. We wish to show that $A \cap K \in L_\alpha$. Let $\tau = \bigcup \{\{e\}(x) \mid x \in K\}$.

Since $\{e\}$ is a total Σ_1-function, and $K \in L_\alpha$, we know that τ exists as an element of L_α. Also, as B is regular, we get $B \cap L_\tau \in L_\alpha$.

Also, for each $x \in K$ let

$$\begin{aligned}\sigma(x) = \mu\beta(\exists M_1, M_2, z \in L_\beta\ \ M_1 \subseteq B \cap L_\tau \wedge M_2 \cap L_\tau = \emptyset \\ \wedge M_1, M_2 \subseteq L_{\{e_0\}(\{x\})} \\ \wedge (L_\beta \vDash \Phi_{e_1}(\{x\}, M_1, M_2) \vee \Phi_{e_2}(\{x\}, M_1, M_2) \text{ with witness } z)).\end{aligned}$$

Notice that σ is total on K, since every element in K is either in or out of A. Also, σ is a Σ_1-function, hence rng σ is bounded in α, by γ say.

Now we obtain

$$\begin{aligned}x \in K \cap A \quad \text{iff} \quad L_\gamma \vDash (\exists M_1, M_2\ \ M_1 \subseteq (B \cap L_\tau) \wedge M_2 \cap (B \cap L_\tau) = \emptyset \\ \wedge M_1, M_2 \subseteq L_{\{e_0\}(\{x\})} \wedge \Phi_{e_1}(x, M_1, M_2)).\end{aligned}$$

Hence $K \cap A \in L_{\beta+1} \subseteq L_\alpha$, as required. $\square$

We notice that this proof does not require that A or B be α-re, only that B be regular, and furthermore it only requires a pointwise version of α-mtt reducibility, rather than the full reducibility.

It is an immediate consequence of this theorem that every α-mtt degree is completely regular or completely irregular, and hence the same is true for the stronger reducibilities we have defined.

Corollary 2.3. *Let $\rho_1^\alpha < \alpha$. Then there are 'natural' intermediate α-mtt-degrees.*

Proof. Let $\boldsymbol{K} = \{\langle x, y\rangle \mid x \in W_y\}$, and let $T \equiv_\alpha \boldsymbol{K}$ be regular and α-re. We claim T is the desired set.

(i) $T \leqslant_{\alpha\text{-tt}} \boldsymbol{K}$ by Theorem 2.2.

(ii) $\neg(T \geqslant_{\alpha\text{-tt}} \boldsymbol{K})$ since T is regular, but if $A_1 = W_a$ is a Σ_1-mastercode, then $A_1 \subseteq \rho_1^\alpha$ is not regular and $A_1 \times \{a\} \subseteq \boldsymbol{K}$. Therefore $\boldsymbol{K}$ cannot be regular.

(iii) $\neg(\emptyset \geqslant_{\alpha\text{-tt}} T)$ since if $\emptyset \geqslant_{\alpha\text{-tt}} T$ then we get $\emptyset \geqslant_{\alpha\text{-tt}} T \equiv_\alpha K$ whence $\emptyset \geqslant_\alpha \boldsymbol{K}$ an absurdity.

Hence $\emptyset <_{\alpha\text{-mtt}} T <_{\alpha\text{-mtt}} K$. $\square$

In light of this corollary, it is natural to ask if there is a greatest α-mtt regular α-re degree. This question is answered negatively by the following theorem.

Theorem 2.4. *Let $\rho_1^\alpha < \alpha$, and let $D \leqslant_{\alpha\text{-mtt}} \hat{\boldsymbol{K}}$, and suppose also that D is α-re and regular. Then there is a C such that $D <_{\alpha\text{-tt}} C$, $D <_{\alpha\text{-mtt}} C$ and C is α-re and regular.*

Proof. We construct C to satisfy the requirements

$$R_e: \quad \neg(C \leqslant_{\alpha\text{-mtt}} D \text{ via } e)$$

by finding arbitrarily large elements of $\hat{\boldsymbol{K}}$ to put into C to satisfy these requirements. By restraining C we ensure that it is regular. Then $D \oplus C$ provides the required set of higher degree.

Before continuing we need to verify that $\hat{\boldsymbol{K}}$ is sufficiently non-regular.

Definition 2.5. $A \subseteq L_\alpha$ is *strongly irregular* iff $\forall \delta,\ \gamma\ \exists \gamma' \geqslant \gamma + \delta\ [\gamma, \gamma'] \cap A$ is irregular.

We note that not every irregular set is strongly irregular, since, for instance, any Σ_1-mastercode is irregular, but, since it is bounded, it cannot be strongly irregular. We shall call such sets weakly irregular.

Lemma 2.6. *$\hat{\boldsymbol{K}}$ is strongly irregular.*

Proof. Let $A_1 = W_a$ be a Σ_1-mastercode, and $\gamma \in \alpha$ be any ordinal. Let $A' = \{\gamma + \delta \mid \delta \in W_a\} = W_{e(\gamma)}$. Then $W_{e(\gamma)} \subseteq [\gamma, \gamma + \rho_1]$ and is irregular.

Since $W_{e(\gamma)} \times \{e(\gamma)\} \subseteq \boldsymbol{K}$, we obtain the desired result, since m-reductions preserve the property of strong irregularity. □

Blocking Lemma 2.7. *Let $\kappa_2^\alpha \leqslant \rho_1^\alpha$ be the Σ_2-cofinality of α. Then there is a family $\langle B_\delta \mid \delta < \kappa_2^\alpha \rangle$ such that*

(1) $\quad \bigcup_\delta B_\delta = L_\alpha,$

(2) $\quad \delta_1 < \delta_2 \ \Rightarrow\ B_{\delta_1} \leqslant B_{\delta_2},$

(3) $\quad |B_\delta|^\alpha < \rho_1^\alpha.$

Proof. Let $q : \kappa_2 \to \rho_1$ be a total, Σ_2, cofinal function, and $f : \alpha \to \rho_1$ be a 1–1, Σ_1, total function. Note that q exists since the Σ_2^α-cofinality of $\rho_1 = \kappa_2$. Then define

$$B_\gamma = \{e \mid f(e) < q(\gamma)\}. \qquad \square$$

Since $\langle B_\gamma \mid \gamma < \kappa_2 \rangle$ is Σ_2, we require a recursive approximation to it, for which purpose we shall use

$$B_{\gamma,\sigma} = \{e \mid L_\sigma \models f(e) < q(\gamma)\}.$$

Note that

(1) $\delta_1 < \delta_2 \Rightarrow B_{\delta_1,\sigma} \subseteq B_{\delta_2,\sigma}$

and

(2) $\forall \gamma \, \exists \tau \, \forall \sigma \geq \tau \, \forall \delta \leq \gamma \; B_{\delta,\sigma} = B_\delta$.

We now define the priority function as

$$\gamma(e, s) \overset{\text{def}}{=} \text{least } \gamma \text{ such that } e \in B_{\gamma,\sigma}.$$

Now define the length of agreement function by

$$l(e, \sigma) = \sup\{x \mid \hat{\Phi}_{e,\sigma}(D_\sigma) \restriction x = C_\sigma \restriction x\}.$$

The restraint function $r(e, \sigma)$ will be defined inside the construction, and the block-restraint functions are given by

$$\bar{r}(\gamma, \sigma) = \sup\{r(e, \sigma) \mid e \in B_{\gamma,\sigma}\}$$

and

$$R(\gamma, \sigma) = \sup\{\bar{r}(\gamma', \sigma) \mid \gamma' < \gamma\}.$$

The construction at stage $\sigma + 1$

For each $e \in L_\sigma$, proceed as follows:

If σ is the first stage at which $l(e, \sigma)$ has attained its current value and there is a $z < l(e, \sigma)$ such that

(i) $z \in \hat{\boldsymbol{K}}_\sigma$,

(ii) $z > R(\gamma(e, \sigma), \sigma)$,

then put the least such z into $C_{\sigma+1}$, and set

$$r(e, \sigma + 1) = \max\{r(e, \sigma), z + 1, e\}.$$

Otherwise do nothing.

If σ is a limit stage, let

$$C_\sigma = \bigcup \{C_\tau \mid \tau < \sigma\} \quad \text{and} \quad r(e, \sigma) = \bigcup \{r(e, \tau) \mid \tau < \sigma\}.$$

In order to see that the construction succeeds, we show by induction that

(i) C is regular;

(ii) $\forall e \, l(e, \sigma)$ converges to a value $l(e) < \alpha$, and hence R_e is satisfied for all e;

(iii) $\forall \gamma \, R(\gamma, \sigma)$ converges to a value $\hat{R}(\gamma)$.

We illustrate the proof for $e \in B_0$, and indicate the changes for arbitrary B_γ.

We wish to show $l(e, \sigma)$ converges to a limit. Suppose not. Then $l(e, \sigma)$ is unbounded in α. Let γ be least such that $\hat{\boldsymbol{K}} \cap \gamma \notin L_\alpha$, and let σ be a stage such that

(i) $\forall\delta \leqslant \gamma\, \phi_{e,\sigma}(\delta) = \phi_e(\delta)$,

(ii) if $\gamma' = \bigcup \phi_e[\gamma + 1]$ then $D \cap \gamma' \subseteq D_\sigma$,

(iii) $\forall\tau \geqslant \sigma\, B_{0,\tau} = B_0$.

Let $\sigma_0 \geqslant \sigma$ be a stage such that $l(e, \sigma_0) > \gamma$, and $l(e, \sigma_0)$ is taking this value for the first time, and there is a $z \in \hat{\boldsymbol{K}}_\sigma$ with $z < \gamma$. Then at stage σ_0 we put some $z < \gamma$ into C. Now the computation $\hat{\Phi}_{e,\sigma_0}(D_{\sigma_0}) \restriction \gamma$ is permanent by (ii) above, and so we have created a permanent disagreement below γ.

Furthermore by (ii), any later z to go into C below γ for sake of R_e has to be smaller, since the length of agreement is smaller, and so only finitely many more z go into C below γ for the sake of R_e. Notice also that for $\tau \geqslant \sigma_0$, $r(e, \tau) = r(e, \sigma_0)$.

Hence $l(e, \sigma)$ and $r(e, \sigma)$ both go to a limit.

Now, let

$$S_0 \stackrel{\text{def}}{=} \{e \in B_0 \mid \exists \text{ a stage } \sigma' \geqslant \sigma \text{ with } l(e, \sigma') > \gamma \text{ and } l(\epsilon, \sigma') \text{ takes this value for the first time and } \exists z \in \hat{\boldsymbol{K}}_\sigma \text{ with } z < \gamma\}.$$

S_0 is a Σ_1-subset of B_0, and since $|B_0|^\alpha < \rho_1^\alpha$, S_0 is α-finite.

The function $e \mapsto$ "*the least witness σ' to $e \in S_0$*" is total on S_0, and hence is bounded in L_α by admissibility. Let τ be an upper bound, then $\hat{R}(0) = R(0, \tau + 1)$ and so $R(0, -)$ goes to a limit.

Now $C \cap \hat{R}(0) \in L_\alpha$, as follows: Let β be such that $\beta \geqslant \tau$ and $D \cap (\bigcup \phi_e[\hat{R}(0)]) \subseteq D_\beta$. Consider $\hat{\boldsymbol{K}} \cap (C \backslash C_\beta) \cap \hat{R}(0)$. By the reasoning above, this has Σ_1^α-cardinality bounded by the Σ_1^α-cardinality of $B_0 \times \omega$, since after stage β only permanent additions for sake of $e \in B_0$ are made, and only finitely many of these. But $|B_0 \times \omega|^{1,\alpha} < \rho_1^\alpha$, hence the set is α-finite, and we have the result.

For the general case, suppose $e \in B_\gamma$, and we have $R(\delta)$ bounded for $\delta < \gamma < \kappa_2$. Let σ be a stage such that

(i) $\forall\delta \leqslant \gamma\, \forall\tau \geqslant \sigma\, B_{\delta,\tau} = B_\delta$,

(ii) $\forall\delta < \gamma\, \forall\tau \geqslant \sigma\, \hat{R}(\delta) = R(\delta, \tau)$.

We will assume that $\gamma = \gamma' + 1$. Let λ be such that $\hat{\boldsymbol{K}} \cap [R(\gamma'), \lambda]$ is irregular, and let $\sigma' \geqslant \sigma$ be a stage such that

(i) $\phi_{e,\sigma'} \restriction \lambda = \phi_e \restriction \lambda$,

(ii) if $\lambda' = \bigcup \operatorname{rng}(\phi_e \restriction \lambda)$ then $D \cap \lambda' \subseteq D_{\sigma'}$.

The remainder of the argument is exactly the same as the B_0 case, and so we obtain $\forall e \in B_\gamma\, R_e$ is satisfied, and $C \cap \hat{R}(\gamma) \in L_\alpha$.

Notice also that the definition of $\hat{R}$ is Δ_2—since it is total and Σ_2, hence if γ is a limit, then $\hat{R}(\gamma)$ is bounded, since $\gamma < \kappa_2^\alpha$.

The argument that $C \cap \hat{R}(\gamma) \in L_\alpha$ is similar to the successor case.

This proves the theorem. □ (Theorem 2.4)

It is an interesting question as to whether the above theorem holds below every irregular α-mtt degree. I.e., is it the case that if A is irregular with $D \leqslant_{\alpha\text{-mtt}} A$ then there is a C α-re and regular with $D \leqslant_{\alpha\text{-mtt}} C \leqslant_{\alpha\text{-mtt}} A$?

We remark that this theorem actually obtains a C such that $D \leqslant_{\alpha\text{-tt}} C$, and hence shows that for no given reducibility there is a maximum regular α-re degree.

The problem caused by the failure of the regular sets theorem is quite severe but will not be dealt with any further in this paper, and so we shall continue by generalizing theorems from ω-recursion theory to α-recursion theory only for regular sets, and leave open, for now, the extensions to irregular sets.

The first, and possibly simplest such example is the following theorem, due to Jockusch when $\alpha = \omega$, which shows that even when we have a regular degree, we need not have a simple set in that degree.

Theorem 2.8. *There is a regular α-re non-recursive set A whose α-tt degree contains no simple set.*

Proof. We wish to construct a set A which is α-re and non-recursive, and auxiliary sets V_e to satisfy the requirements

Q_e: $\bar{A} \neq W_e$ to make A nonrecursive,
R_e: $A \equiv_{\alpha\text{-tt}} W_e \Rightarrow V_e$ is α-infinite and $V_e \cap W_e$ is α-finite.

The form in which we will use R_e is

R_e: if $A \leqslant_{\alpha\text{-tt}} W_{e_0}$ via e_1 and $W_{e_0} \leqslant_{\alpha\text{-tt}} A$ via e_2
then V_e is α-infinite and $V_e \cap W_{e_0}$ is α-finite.

We assign priorities to the R_e by $<_{L_\alpha}$-order, and the priority of Q_e immediately follows that of R_e.

Let $h : \alpha \leftrightarrow \alpha \times \alpha$ be Δ_1^α, and define, for $\beta < \alpha$

$$Z_\beta = \{z \mid \exists y\, h(z) = \langle \beta, y \rangle\}.$$

Then

$$x \notin Z_\beta \quad \Leftrightarrow \quad \exists \gamma\, \exists y\, h(y) = \langle \gamma, y \rangle \wedge \gamma \neq \beta$$

and so Z_β is Δ_1^α. Let

$$Z_{<\beta} = \bigcup_{\gamma<\beta} Z_\gamma \quad \text{and} \quad Z_{\geqslant\beta} = \bigcup_{\gamma\geqslant\beta} Z_\gamma.$$

$Z_{\geqslant \mathrm{rk}(e)}$ will be used to provide witnesses for R_e.

At any stage σ, having built A_σ so far, and attacking requirement e we define $A_\sigma^+ = A_\sigma \cup Z_{\mathrm{rk}(e)}$, and use A_σ^+ instead of A_σ. This will ensure that lower priority requirements do not interfere with our work for R_e.

Q_e will be met by a Friedberg–Muchnik style argument—we will find the least element of $Z_{\geqslant \mathrm{rk}(e)} \cap W_e$ and put it into A.

Two strategies are followed in attempting to satisfy R_e. The first is to try to destroy $W_{e_0} \leqslant_{\alpha\text{-tt}} A$ via e_2 and the second is to try and destroy $A \leqslant_{\alpha\text{-tt}} W_{e_0}$ via e_1.

The first strategy has us build the auxiliary set V_e, such that

$$x \in V_{e,\sigma} \Leftrightarrow \exists M_1, M_2 \;\; M_1 \subseteq A_\sigma^+ \wedge M_2 \cap A_\sigma^+ = \emptyset \wedge \langle M_1, M_2, 1\rangle \in D_{\{e_2\}_\sigma(\{x\})}.$$

If $x \in V_{e,\sigma} \cap W_{e,\sigma}$ because of a pair $\langle M_1, M_2\rangle$ then put $M_1 \cap Z_{>\text{rk}(e)}$ into $A_{\sigma+1}$ (and say that R_e received attention under the first strategy). Give no further attention to R_e unless some higher priority requirement later receives attention. If this never occurs then A and A_σ^+ agree on M_1, and M_2 (they agree on M_1 since M_1 is put into $A_{\sigma+1}$, and they agree on M_2, as no lower priority requirement puts anything into M_2, and no higher priority requirement ever acts again). Thus we have

$$\exists M_1, M_2 \;\; M_1 \subseteq A \wedge M_2 \cap A = \emptyset \wedge \langle M_1, M_2, 1\rangle \in D_{\{e_2\}(\{x\})} \wedge x \in W_{e_0}$$

which gives us $W_{e_0} \nleqslant_{\alpha\text{-tt}} A$ via e_2.

Now suppose that R_e never receives attention under the first strategy after some stage σ_0, and no higher priority requirement receives attention either.

We now consider the second strategy. R_e will receive attention at most twice under this strategy, so there will be a stage $\sigma_1 \geqslant \sigma_0$ so that $A_{\sigma_1} \cap Z_{\leqslant\text{rk}(e)} = A \cap Z_{\leqslant\text{rk}(e)}$.

For $\sigma \geqslant \sigma_1$ no new elements of V_e will be in W_{e_0}, and so $V_e \cap W_{e_0}$ will be α-finite. Thus, if we ensure that V_e is α-infinite, then R_e will be satisfied.

So suppose V_e is α-finite, and $W_{e_0} \leqslant_{\alpha\text{-tt}} A$ via e_2, then the second strategy will ensure that $A \nleqslant_{\alpha\text{-tt}} W_{e_0}$ via e_1.

Let

$$P_{e,\sigma}(i) \stackrel{\text{def}}{=} \{z \mid \exists K\, \exists M_1, M_2 \;\; M_1 \subseteq A_\sigma^+ \wedge M_2 \cap A_\sigma^+ = \emptyset \wedge z \in K \wedge \langle M_1, M_2, i\rangle \in D_{\{e_2\}_\sigma(K)}\} \quad \text{for } i = 0 \text{ or } 1.$$

We will say that x is an eligible witness for R_e at stage σ iff

(i) $x \in Z_{\text{rk}(e)} \backslash A_\sigma$,

(ii) $\exists m_1, m_2\, \exists i \;\; m_1 \subseteq P_{e,\sigma}(0) \wedge m_2 \cap P_{e,\sigma}(0) = \emptyset \wedge \langle m_i, m_2, i\rangle \in D_{\{e_1\}_\sigma(\{x\})} \wedge (m_2 \subseteq P_{e,\sigma}(1)$ via information in $L_x)$.

At stage σ, if x is an eligible witness for R_e, we enumerate all elements of $Z_{>\text{rk}(e)}$ which are used positively in showing $m_1 \subseteq P_{e,\sigma}(0)$ and $m_2 \subseteq P_{e,\sigma}(1)$, into A.

Then do nothing for R_e until we come to a stage $\tau > \sigma$ such that $m_1 \subseteq W_{e_0,\tau}$ and

$$\exists K_1, K_2 \;\; K_1 \subseteq W_{e_0,\tau} \wedge K_2 \cap W_{e_0,\tau} = \emptyset \wedge \langle K_1, K_2, 1\rangle \in D_{\{e_1\}_\sigma(\{x\})}.$$

At the first such stage enumerate x into A, and give no further attention to R_e unless a higher priority requirement receives attention.

Lemma 2.9. $\lim_\gamma(\min Z_\gamma) = \alpha$.

Proof. The function $f(\gamma) = \min(Z_\gamma)$ is Δ_1 as $f(\gamma) = z$ iff $z \in Z_\gamma \wedge \forall y < z\ y \notin Z_\gamma$ is clearly total and Σ_1. Hence, for any $\gamma < \alpha$, $f[\gamma]$ is bounded in α.

Claim. rng(f) is Δ_1.

$$z \in \mathrm{rng}(f) \quad \text{iff} \quad \exists y\, f(y) = z,$$
$$z \notin \mathrm{rng}(f) \quad \text{iff} \quad \exists \gamma, x\ z \in Z_\gamma \wedge (x \in Z_\gamma \wedge x < z).$$

Now suppose rng(f) is bounded in α. Then by the claim rng(f) is α-finite, and so we have a 1–1, total, Δ_1-function from α to an α-finite set X. By Σ_1-uniformization f^{-1} is defined and Σ_1 on X, and is also onto (as f is 1–1). But, since α is admissible, $f^{-1}[X]$ is bounded in α, and so we have a contradiction. Hence rng(f) is unbounded in α.

Furthermore $\forall \gamma\, \exists \delta\, \forall \delta' \geqslant \delta\, f(\delta') > \gamma$. If not, say γ_0 is a counterexample. Then by the claim rng$(f) \cap \gamma_0$ is α-finite, and by the above argument $f \upharpoonright (\mathrm{rng}(f) \cap \gamma_0)$ is bounded in α, providing a contradiction. □

Lemma 2.10. *A is regular.*

Proof. Let $\beta < \alpha$. Let γ be such that $\forall \gamma' \geqslant \gamma\, \min(Z_{\gamma'}) > \beta$. Such a γ exists by Lemma 2.9. Each R_e acts at most α-finitely often, and so let σ be a stage by which all the R_i with $\mathrm{rk}(i) < \gamma'$ have finished acting.

Then all higher priority requirements only put in elements from Z'_γ, for $\gamma' > \gamma$, and so only put in elements above β. Hence $A \cap \beta = A_\sigma \cap \beta \in L_\alpha$. □

Lemma 2.11. (i) $\{\gamma \mid A \cap Z_\gamma \neq \emptyset\}$ *is unbounded in* α.

(ii) $\{\gamma \mid \bar{A} \cap Z_\gamma \neq \emptyset\}$ *is unbounded in* α.

Proof. (i) Let $W_{e_\gamma} = Z_{>\gamma}$. Then requirement Q_{e_γ} will be met since it contains arbitrarily large elements, and higher priority requirements eventually stop acting. This means that for all γ, $A \cap Z_{>\gamma} \neq \emptyset$.

(ii) By the construction $A \cap Z_\gamma$ is α-finite for all γ. But Z_γ is not α-finite, and so $Z_\gamma \backslash A$ is α-finite. □

Corollary 2.12. *A and $\bar{A}$ are not α-finite.* □

Lemma 2.13. *For all e, if $\{\gamma \mid W_e \cap Z_\gamma\}$ is unbounded in α, then $W_e \neq \bar{A}$.*

Proof. Given any stage, there is always a later one at which we will be able to satisfy Q_e. This will be done. □

Notice that acting on Q_e has no effect on higher priority requirements.

We now need to show that R_e is satisfied. Suppose that R_e is not satisfied, and so we have $A \leqslant_{\alpha\text{-tt}} W_{e_0}$ via e_1 and $W_{e_0} \leqslant_{\alpha\text{-tt}} A$ via e_2 and V_e is α-finite. We want to show that there is an abundance of eligible witnesses.

We note that $P_{e,\sigma}(1)=V_{e,\sigma}$, and that $P_{e,\sigma}(0)$ is a σ-stage approximation to W_{e_0} (since $W_{e_0} \leqslant_{\alpha\text{-tt}} A$ via e_2).

Also A is non-recursive, and so neither is W_{e_0} and, in particular, W_{e_0} is not α-finite. Because $Z_{\mathrm{rk}(e)} \cap A$ is α-finite, and $Z_{\mathrm{rk}(e)}$ is not α-finite and as $A \leqslant_{\alpha\text{-tt}} W_{e_0}$ via e_1, we will have many x's in $Z_{\mathrm{rk}(e)} \setminus A$ which bound V_e such that

$$\exists m_1, m_2 \ \ m_1 \subseteq W_{e_0} \wedge m_2 \cap W_{e_0} = \emptyset \wedge \langle m_1, m_2, 1\rangle \in D_{\{e_1\}(\{x\})}. \tag{*}$$

For each such pair m_1, m_2 we will eventually find σ, M_1^0, M_2^0, M_1^1, M_2^1 such that, for $i=0, 1$:

$$M_1^i \subseteq A \cap A_\sigma^+ \wedge M_2^i \cap A = M_2^i \cap A_\sigma^+ = \emptyset \wedge \langle M_1^i, M_2^i, i\rangle \in D_{\{e_2\}_\sigma(m_{i+1})}. \tag{**}$$

Let σ be a stage such that there is an x as above for which $m_1 \subseteq W_{e_0,\sigma}$ and $\{e_1\}_\sigma(\{x\})\downarrow$ (and hence (*) occurs), and at which (**) occurs. Then, at this stage $m_1 \subseteq P_{e,\alpha}(0)$ and $m_2 \subseteq P_{e,\sigma}(1)$ (by (**)). As $P_{e,\sigma}(1) \subseteq V_e$ which is α-finite, as x is a bound for V_e, then we have $m_2 \subseteq P_{e,\sigma}(1)$ via information in L_x. Thus x is an eligible witness that remains eligible at all later stages and so R_e will receive attention at some stage under the second strategy.

Now, $x \notin A$ and so, as $A \leqslant_{\alpha\text{-tt}} W_{e_0}$ via e_1, we must come to a stage τ as described above after which no elements conflicting with $m_1 \subseteq W_{e_0}$ via e_2 ever enter A, since we eventually get true computations. At the second step of this strategy, which must come, as we have 'correct' information, we put x into A. It remains to verify that the m_1, m_2 used by this x are true information about W_{e_0}.

By definition of τ, $m_1 \subseteq W_{e_0,\tau}$—since we put all information into A required by e_2 to verify this fact. Since we also put enough information into A to verify that $m_2 \subseteq P_{e,\tau}(1)$ and x is well above this, we will obtain $m_2 \cap W_{e_0} = \emptyset$ (since $A \leqslant_{\alpha\text{-tt}} W_{e_0}$ via e_2 and $A_{\tau+1}$ is correct on the use of the computation showing $m_2 \cap W_{e_0} = \emptyset$).

Hence, we get

$$m_1 \subseteq W_{e_0} \wedge m_2 \cap W_{e_0} = \emptyset \wedge \langle m_1, m_2, 1\rangle \in D_{\{e_1\}(\{x\})},$$

and so $W_{e_0} \not\leqslant_{\alpha\text{-tt}} A$ via e_1, a contradiction. □ (Theorem 2.8)

3. α-tt degrees

As an application of finite injury techniques to α-tt degrees, we include the following theorem relating α-m degrees, and α-tt degrees. Our presentation follows Downey [3]. We note that further work of Downey has shown that these singular degrees are in fact dense, but we shall not tackle this question for α-recursion theory within this paper.

Theorem 3.1. *Let α be Σ_1-admissible. Then there is an α-re regular non-recursive set A such that for all α-re sets B, if $B \equiv_{\alpha\text{-tt}} A$, then $B \equiv_{\alpha\text{-m}} A$.*

[**Remark.** The regularity of A is forced upon us by the method by which we construct A. It is therefore an interesting question as to whether such an A can be irregular, and incomplete.]

Proof. As usual we build A by stages, A_σ will denote the construction at the end of stage σ. Also, we use the Blocking Lemma to give us an appropriate blocking family $\langle B_\delta \mid \delta < \kappa_2^\alpha \rangle$ and a Δ_1-approximating family $\langle B_{\delta,\sigma} \mid \delta < \kappa_2^\alpha,\ \sigma < \alpha \rangle$.

The requirements we have to meet are:

P_e: $\bar{A} \neq W_e$,

N_e: $A \leqslant_{\alpha\text{-tt}} V_{e_0}$ via e_1, and $V_{e_0} \leqslant_{\alpha\text{-tt}} A$ via $e_2 \Rightarrow V_{e_0} \equiv_{\mathrm{m}} A$

(where V_{e_0} is an α-re set, in some previously fixed listing of the α-re sets).

P_e is met by finding $x \in W_e$ greater than the current e-restraint, at a stage σ where $W_{e,\sigma} \cap A_\sigma = \emptyset$. We then put x in A, and P_e is met once and for all.

Each block has a restraint associated with it:

$$r(\gamma, \sigma) = \text{restraint for } B_{\gamma,\sigma}.$$

Matters will be arranged as to ensure $\lim_\sigma r(\gamma, \sigma) = \hat{r}(\gamma)$ exists for all γ.

We will build $A \subseteq \alpha$, and assume every W_e and V_e is a subset of α. Our strategy for N_e will be to try and falsify the antecedent of the implication if at all possible, and if it's not possible the reason for this will provide the desired m-reductions. For more discussion of this, see Downey [3].

A will be built by 'dumping', i.e., at stage σ we let $\{a_{\beta,\sigma} \mid \beta < \lambda_\sigma \leqslant \sigma\}$ enumerate $\sigma \backslash A_\sigma$ monotonically. Then, if we put $a_{\beta,\sigma}$ into $A_{\sigma+1}$, we will also put $a_{\gamma,\sigma}$ into $A_{\sigma+1}$ for all γ such that $\beta \leqslant \gamma < \lambda_\sigma$.

Claim. If A is built by dumping, then A is regular.

Indeed, if $\beta \in \alpha$ is any ordinal, we wish to show that there is a stage σ with $A \cap \beta \subseteq A_\sigma$, from which we get that $A \cap \beta = A_\sigma \cap \beta \in L_\alpha$. Let $\sigma_0 = \beta$, and given σ_i, define, if possible, σ_{i+1} to be the least stage $\tau > \sigma_i$ such that $(A_\tau \backslash A_{\sigma_i}) \cap \beta \neq \emptyset$, i.e., σ_{i+1} is the next stage after σ_i at which an element below β enters A. Let x_{i+1} be the least element of $(A_{\sigma_{i+1}} \backslash A_{\sigma_i}) \cap \beta$. By the dumping property, we have $x_{i+1} < x_i$, and so, by well-foundedness, there is a greatest k for which σ_k is defined. It is then apparent that $A \cap \beta = A_{\sigma_k} \cap \beta$.

We now need to define our length of agreement functions. There will be auxiliary use functions which we also need to define, associated with each reduction:

$$\begin{aligned} L(e, \sigma) = \sup\{x \mid \forall y < x\ (&(y \in V_{e_0,\sigma} \wedge A_\sigma \vDash_{e_2} y \in V_{e_0,\sigma}) \\ &\vee (y \notin V_{e_0,\sigma} \wedge A_\sigma \vDash_{e_2} y \notin V_{e_0,\sigma}))\}, \\ \gamma_e(y) = \max(&\text{L-rk}(D_{\{e\}(\{y\})}), \text{L-rk}(y) + 1), \\ l(e, \sigma) = \sup\{x \mid \forall y < x\ (&(V_{e_0,\sigma} \vDash_{e_1} y \in A \wedge y \in A) \\ &\vee (V_{e_0,\sigma} \vDash_{e_1} y \notin A \wedge y \notin A)) \wedge L(e, \sigma) > \gamma_e(y)\}. \end{aligned}$$

l is the A-controllable length of the agreement function.

Definition 3.2. We say N_e *requires attention* via β at stage $\sigma+1$ iff

(i) N_e is currently unsatisfied, and $e \in B_{\gamma,\sigma}$

(ii) β is the least ordinal such that $a_{\beta,\sigma} > \max\{a_{|e|,\sigma}, \bigcup_{\delta<\gamma} r(\delta,\sigma)\}$ and one of options I or II hold:

Option I: $l(e,\sigma) > a_{\beta,\sigma}$ and if we set $A_\sigma^\beta = A_\sigma \cup \{a_{\delta,\sigma} \mid \beta \leqslant \delta < \lambda_\sigma\}$ then the following sentence fails:

$$\exists \text{ sequence } \langle\langle M_1^y, M_2^y\rangle, y\rangle_{y<l(e,\sigma)} \in L_{\sigma+1} \text{ such that for all } y \in l(e,\sigma)$$

(i) $\forall z \in M_1^y\ A_\sigma^\beta \vDash_{e_2} z \in V_{e_0}$ and $\forall z \in M_2^y\ A_\sigma^\beta \vDash_{e_2} z \notin V_{e_0}$,

(ii) $y \in A_\sigma^\beta \Leftrightarrow \langle\langle M_1^y, M_2^y\rangle, 1\rangle \in D_{\{e_1\}(\{y\})}$ and

$y \notin A_0^\beta \Leftrightarrow \langle\langle M_1^y, M_2^y\rangle, 0\rangle \in D_{\{e_1\}(\{y\})}$,

(iii) $V_{e_0,\sigma} \subseteq \bigcup_y M_1^y$.

Notice that by making $A_{\sigma+1} = A_\sigma^\beta$, and setting the restraint to be $a_{\beta,\sigma}$, then if this is never injured, and $A \geqslant_{\alpha\text{-tt}} V_{e_0}$ via e_2 and $V_{e_0} \geqslant_{\alpha\text{-tt}} A$ via e_1 then $A \cap \sigma = A_{\sigma+1}$, and we now have a contradiction. Indeed for $y < l(e,\sigma)$ we have $\gamma_e(y) < L(e,\sigma) \leqslant \sigma$ and hence there must be M_1, M_2 with

(i) $M_1 \subseteq V_{e_0} \wedge M_2 \cap V_{e_0} = \emptyset$,

(ii) $\langle M_1, M_2\rangle \in L_\sigma$ and $D_{\{e_1\}(\{y\})} \in L_\sigma$,

(iii) $\langle\langle M_1, M_2\rangle, i\rangle \in D_{\{e_1\}(\{y\})} \Leftrightarrow y \in A^i$ (where $A^0 = \bar{A}$ and $A^1 = A$).

Furthermore M_1, $M_2 \subseteq \gamma_e(y) < l(e,\sigma)$ and so for all $x \in M_1$ $A \vDash_{e_2} x \in V_{e_0}$, for all $x \in M_2$ $A \vDash_{e_2} x \notin V_{e_0}$, and this is known below σ, so is comprehended by L_σ. Thus in L_σ there is a sequence witnessing the truth of the sentence, which provides the contradiction. This, then, provides one way to destroy $A \equiv_{\alpha\text{-tt}} V_{e_0}$.

Option II: We have an $\alpha_{\beta+1,\sigma} < l(e,\sigma)$ such that if

$$A_\sigma^\gamma = A_\sigma \cup \{a_{\gamma',\sigma} \mid \gamma \leqslant \gamma' \leqslant \lambda_\sigma\} \quad \text{for } \gamma = \beta, \beta+1,$$

$$\hat{\gamma}(e,\sigma) \stackrel{\text{def}}{=} \max\{\gamma_e(y) \mid y < l(e,\sigma)\},$$

$$J_{e,\sigma} \stackrel{\text{def}}{=} \{x \mid A_\sigma^{\beta+1} \vDash_{e_2} x \in V_{e_0}\} \cap \hat{\gamma}(e,\sigma),$$

then there is a z such that

$$(z \notin A_\sigma^\beta \text{ and } J_{e,\sigma} \vDash_{e_1} z \in A) \quad \text{or} \quad (z \in A_\sigma^\beta \text{ and } J_{e,\sigma} \vDash_{e_1} z \notin A).$$

So, if option II obtains, we first set $A_{\sigma+1} = A_\sigma^{\beta+1}$, and set the restraint to $a_{\beta,\sigma+1}$. If this is permanent and $V_{e_0} \equiv_{\alpha\text{-tt}} A$ via e_1, e_2 then we eventually get to a stage τ where $V_{e_0,\tau} \cap \hat{\gamma}(e,\sigma) = J_{e,\sigma}$. At this stage we will be able to change A by adding $a_{\beta,\tau} = a_{\beta,\sigma}$ to A, and so obtain a permanent disagreement.

The construction at stage $\sigma+1$

Step 1. Find the least e such that N_e requires attention. If none exists go to Step 2, setting $\hat{A}_{\sigma+1} = A_\sigma$. Otherwise, find the least β such that N_e requires attention via β. Then for all $e' > e$ cancel $r(e',\sigma)$, and declare $N_{e'}$ to be unsatisfied.

If option I holds, set $\hat{A}_{\sigma+1} = A^{\beta}_{\sigma}$, declare N_e to be satisfied, and set $r(e, \sigma+1) = a_{\beta,\sigma}$.

If option I fails, but option II holds, set $\hat{A}_{\sigma+1} = A^{\beta+1}_{\sigma}$ and set $r(e, \sigma+1) = a_{\beta,\sigma}$.

Step 2. Find the least $\hat{e}$ such that $W_{\hat{e},\sigma} \cap \hat{A}_{\sigma+1} = \emptyset$ and

$$\exists x\ x \in W_{\hat{e},\sigma} \wedge x > a_{|\hat{e}|,\sigma} \wedge x > R(\hat{e}, \sigma) \stackrel{\text{def}}{=} \sup\{r(e', \tau) \mid e' \leq \hat{e},\ \tau \leq \sigma\}.$$

Choose the least such x, and let γ be such that $x = a_{\gamma,\alpha}$. Let $A_{\sigma+1} = A^{\gamma}_{\sigma}$. Cancel all $r(e', \sigma)$ for $e' > \hat{e}$, and declare $N_{e'}$ to be unsatisfied.

If there is no such $\hat{e}$, let $A_{\sigma+1} = \hat{A}_{\sigma+1}$.

By induction we will show that $\lim_{\sigma} r(e, \sigma)$ exists for all e and $\lim_{\sigma} r(\gamma, \sigma)$ exists for all $\gamma < \kappa_2$.

Suppose we know this for $\gamma' < \gamma < \kappa_2$. Let σ_0 be a stage such that for all $\gamma' < \gamma$ and all $\sigma \geq \sigma_0$, $r(\gamma', \sigma) = r(\gamma')$ is at its final value, and for $\gamma \leq \gamma$, $B_{\gamma',\sigma} = B_{\gamma'}$, and without loss of generality, for $\gamma' < \gamma$, $e \in B_{\gamma'} \Rightarrow P_e$ is satisfied.

Now, let $S = \{e \in B_{\gamma} \mid$ there exists a stage at which N_e requires attention after stage $\sigma_0\}$. This is a Σ_1-set, contained in B_{γ} and $|B_{\gamma}|^{\alpha} < \rho_1$. Hence S is α-finite.

We can let $S = S_1 \cup S_2$ where

$$S_1 = \{e \in S \mid N_e \text{ receives attention because of option I}\},$$
$$S_2 = \{e \in S \mid N_e \text{ receives attention because of option II}\}.$$

We now define a function f on S as follows:

$$f(\min S) = \begin{cases} \text{least } \tau > \sigma_0 \text{ at which } N_{\min S} \text{ receives attention} & \text{if } \min S \in S_1, \\ \text{second } \tau > \sigma_0 \text{ at which } N_{\min S} \text{ receives attention} & \text{if } \min S \in S_2, \end{cases}$$

$$f(e) = \begin{cases} \text{least } \tau > \bigcup f[\{e' \in S \mid e' < e\}] \text{ at which } N_e \text{ receives attention} & \text{if } e \in S_1, \\ \text{second } \tau > \bigcup f[\{e' \in S \mid e' < e\}] \text{ at which } N_e \text{ receives attention} & \text{if } e \in S_2. \end{cases}$$

f is Σ_1 and total on S, and so $\bigcup \operatorname{rng} f$ exists, call it σ_1.

By stage σ_1, every N_e ($e \in B_{\gamma}$) has received attention for the last time, and so we have $r(\gamma, \sigma_1) = r(\gamma)$ and in fact for all $e \in B_{\gamma}$, $r(e, \sigma_1) = r(e)$.

Now, let

$$T = \{e \in B_{\gamma} \mid P_e \text{ requires and receives attention after stage } \sigma_1\}.$$

Then T is a Σ_1-subset of B_{γ}, and so is α-finite, and the function $g: T \to \alpha$ given by

$$g(e) = \text{least stage } \sigma > \sigma_1 \text{ at which } P_e \text{ receives attention}$$

is Σ_1 and total on T. Therefore $\bigcup \operatorname{rng}(g) = \sigma_2$ exists below α. By stage σ_2 all requirements in B_{γ} are met forever.

The fact that the function $\gamma \mapsto r(\gamma)$ is Δ_2 enables us to get past limit points in κ_2.

The remainder of the argument follows Downey [2] and will be omitted. □ (Theorem 3.1)

This proof is not especially difficult, and we expect that the result of Downey [3] showing that the re T-degrees containing such tt-degrees are in fact dense will also succeed with minimal fuss. However this is beyond the scope of the current paper.

4. α-$\hat{\mathrm{w}}$ degrees

In this section we will generalize some results on ω-wtt degrees to α-$\hat{\mathrm{w}}$ degrees. It is unclear whether these generalizations extend to α-wtt degrees for arbitrary α, since the results use permitting.

In Section 2 we showed that if $\rho_1^\alpha < \alpha$ then there are many α-$\hat{\mathrm{w}}$ degrees. However that theorem says nothing about the case when $\rho_1^\alpha = \alpha$ nor about the 'spread' of the α-$\hat{\mathrm{w}}$ degrees, which this next theorem deals with, for the regular degrees.

Theorem 4.1. *Let α be admissible, and let D, C be regular α-re sets, such that $D <_\alpha C$. Then there are regular α-re sets A_1, A_2 such that*

$$D <_{\alpha\text{-tt}} A_1,\ A_2 <_{\alpha\text{-}\hat{\mathrm{w}}} C,$$

and

$$A_1 \cup A_2 = C \wedge A_1 \cap A_2 = \emptyset.$$

[**Note.** This proof actually needs $D <_\alpha C$ in general, but in special cases $D <_{\alpha\text{-}\hat{\mathrm{w}}} C$ will suffice. This will be discussed after the proof.]

Proof. We construct A_1, A_2 to satisfy requirements

$R_{e,i}$: $\neg(A_i \oplus D \geqslant_{\alpha\text{-}\hat{\mathrm{w}}} C \text{ via } e)$.

A_1 and A_2 will be $\leqslant_{\alpha\text{-}\hat{\mathrm{w}}} C$ from the construction. These requirements will be met by a finite injury, preservation of length of agreement strategy.

We recall from Shore [9], the existence of a blocking family $\langle B_\gamma \mid \gamma < \kappa_2^D\rangle$ and an approximating family $\langle B_{\gamma,\sigma} \mid \gamma < \kappa_2^D,\ \sigma < \alpha\rangle$ with $|B_\gamma|^{1,D} < \rho_1^D$.

Associated with $e = \langle e_0, e_1, e_2\rangle$ we will have an index $\hat{e}$ such that for all y, $\{\hat{e}\}(y) = \{e_0\}(\{y\})$, and we will define subblocks $B'_\gamma \subseteq B_\gamma$ to be the set of all $e \in B_\gamma$ such that $\{\hat{e}\}$ is increasing. This family also has an approximating family $B'_{\gamma,\sigma}$ which is defined so that

$$e \in B_{\gamma,\sigma} \backslash B'_{\gamma,\sigma} \quad \Leftrightarrow \quad e \in B_{\gamma,\sigma} \wedge \exists \tau \leqslant \sigma\ \exists y_1, y_2\ y_1 \leqslant y_2 \wedge \{\hat{e}\}_\tau(y_1) > \{\hat{e}\}_\tau(y_2).$$

Clearly, given an $e = \langle e_0, e_1, e_2\rangle$ there is an associated $e' = \langle e'_0, e'_1, e'_2\rangle$ where $e'_1 = e_1$, $e'_2 = e_2$ and $\{e'_0\}(K) = \bigcup \{D_{\{e_0\}(L)} \mid L \leqslant_{L_\alpha} K\}$ for which $\{\hat{e}'\}$ is increasing and $A \leqslant_{\alpha\text{-}\hat{\mathrm{w}}} B$ via $e \Rightarrow A \leqslant_{\alpha\text{-}\hat{\mathrm{w}}} B$ via e'. Hence, destroying all of these computations suffices for our argument.

We denote the eth α-$\hat{\mathrm{w}}$ reduction procedure applied to a set B by $\hat{\Phi}(e, B)$.

We now define the length of agreement and restraint functions:

$$l_i(e, \sigma) = \sup\{x \mid \hat{\Phi}(e, A_{i,\sigma} \oplus D_\sigma) \upharpoonright x = C_\sigma \upharpoonright x\}$$

and

$$\hat{l}_i(\gamma, \sigma) = \sup\{l(e, \sigma) \mid e \in B_{\gamma,\sigma}\} \quad \text{for } i = 1, 2.$$

Restraint is given by:

$$r_i(\gamma, \sigma) = \bigcup \{\hat{e}\}_\sigma[l_i(e, \sigma)]$$

and

$$R_i(\gamma, \sigma) = \bigcup \{r_i(e, \sigma) \mid e \in B'_{\gamma,\sigma}\}$$
$$= \bigcup \{\hat{e}\}_\sigma[\hat{l}_i(\gamma, \sigma)] \quad \text{for } i = 1, 2.$$

Without loss of generality, we will assume that for all σ, $|C_{\sigma+1} \setminus C_\sigma| \leq 1$.

We are now ready to describe the construction at stage $\sigma + 1$:

Let $x \in C_{\sigma+1} \setminus C_\sigma$ and let $\gamma \in \kappa_2^D$ be least such that $\exists e \in B'_{\gamma,\sigma}\ x < r_1(e, \sigma)$ or $x < r_2(e, \sigma)$. If $x < r_1(e, \sigma)$ put x into A_2 and set $r_2(e, \sigma + 1) = \max\{x, r_2(e, \sigma)\}$. Otherwise put x into A_1 and set $r_1(e, \sigma + 1) = \max\{x, r_1(e, \sigma)\}$.

At stage λ: Do nothing.

It is easy to see from this construction that $A_1 \cup A_2 = C$ and $A_1 \cap A_2 = \emptyset$. We now wish to show that

(i) $\lim_\sigma R_i(\gamma, \sigma)$ exists for all $\gamma < \kappa_2^D$,

(ii) $\lim_\sigma \hat{l}_i(\gamma, \sigma)$ exists for all $\gamma < \kappa_2^D$.

To prove this we require the following lemma.

Lemma 4.2. *For all $\gamma < \kappa_2^D$ and all x, $\{\{\hat{e}\}(y) \mid y < x$ and $e \in B'_\gamma\}$ is bounded in α.*

Proof. We recall that there is a total 1–1 function $f : \alpha \to \rho_1^D$ which is Δ_1^D, and a Σ_2^D cofinal function $q : \kappa_2^D \to \rho_1^D$ used in defining B_γ, and that $f[B_\gamma]$ is bounded in ρ_1^D, by $q(\gamma)$.

Since $\{\hat{e}\}$ is increasing, it suffices to show that $X = \{\{\hat{e}\}(x) \mid e \in B'_\gamma\}$ is bounded in α. But if $e \in B'_\gamma$, then $\{\hat{e}\}(x)$ is an element of the Σ_1^D-hull of $q(\gamma) \cup \{\langle x, p\rangle\}$ where p is the parameter for f. But this hull is already bounded in α, hence X is bounded in α. □

By this lemma, and as $R_i(\gamma, \sigma) = \bigcup \{\{\hat{e}\}[\hat{l}_i(\gamma, \sigma)] \mid e \in B'_\gamma\}$, in order to prove (i) and (ii), it suffices to prove (ii).

Suppose $\hat{l}_1(\gamma, \sigma)$ is unbounded in α, and $\gamma = \delta + 1$. Let $R_i(\delta) = \lim_\sigma R_i(\delta, \sigma)$, which exists by induction. Let σ be a stage such that for all $\tau \geq \sigma$

(a) $R_i(\delta, \tau) = R_i(\delta)$ for $i = 1, 2$,

(b) $C[R_i(\delta)] \subseteq C_\tau$ for $i = 1, 2$,

(c) $\forall \delta \leq \gamma\ B_{\delta,\tau} = B_\delta$ and $B'_{\delta,\tau} = B'_\delta$.

Notice that

$$S_\sigma \stackrel{\text{def}}{=} \{e \in B_\gamma \mid \exists \tau \geqslant \sigma\ \exists x\ \hat{\Phi}_{e,\tau}(e, A_{i,\tau} \oplus D_\tau)(x) = 0 \text{ using } D\text{-correct information and } C_\tau(x) = 1\}$$

is Σ_1^D, hence α-finite, and so the set of stages giving witness to $e \in S_\sigma$ is bounded, by σ' say. Notice also that, by the definition of σ, disagreements witnessing $e \in S_\sigma$ are permanent.

We will now indicate how D is able to compute C. For $\beta \in \alpha$, we wish to compute $C \restriction \beta$.

Let $\tau \geqslant \sigma$ be a stage such that

$$\forall \tau' \geqslant \tau\ \exists e \in B'_\gamma\ l_1(e, \tau') > \beta.$$

Now let $\tau' \geqslant \tau$, $e \in B'_\gamma \backslash S_\sigma$ be such that $l_1(e, \tau') > \beta$ using D-correct information. Then we have $\hat{\Phi}(e, A_{1,\tau} \oplus D_{\tau'}) \restriction \beta + 1 = C_{\tau'} \restriction \beta + 1$ with D-correct information.

Now suppose C changes below β. Then we claim that the left-hand side cannot change—so contradicting the fact that $e \notin S_\sigma$.

If any y goes into C below $R_1(\gamma, -)$ then it cannot be below any $R_1(\delta, -)$ for any $\delta < \gamma$ (since C through $R(\delta)$ is at its final value), hence that y goes into A_2. Therefore $A_{1,\tau'}$, through the use in the computation, is at its final value. Hence D computes $C \restriction \beta$.

Note that this is not a $\hat{w}$-reduction, since the choice of e is D-recursive, and not recursive.

This shows that $\limsup_\sigma \hat{l}_1(\gamma, \sigma) < \alpha$, and we will suppose it has value λ. Let

$$B_\gamma^\lambda = \{e \mid \{\hat{e}_0\} \text{ is total on } \lambda \wedge e \in B_\gamma\}.$$

This is Σ_1 (since α is admissible), and hence α-finite, and so we can find

$$\sup\{\{\hat{e}_0\}[\lambda] \mid e \in B_\gamma^\lambda\} = \mu < \alpha.$$

Let τ be a stage, $\tau \geqslant \sigma'$, such that $D_\tau \restriction \mu = D \restriction \mu$. Then after stage τ the argument proceeds as a standard finite injury argument with $D \restriction \mu$ as a parameter, which shows that $\lim_\sigma \hat{l}_1(\gamma, \sigma)$ exists.

It remains only to verify that $A_1, A_2 \leqslant_{\alpha\text{-mtt}} C$. Let $A_1 = W_a$ and $A_2 = W_b$. Then

$$K \subseteq A_1 \quad \text{iff} \quad \exists \sigma\ \forall x \in K\ L_\sigma \models x \in A_1 \qquad (\text{note this is } \Sigma_1)$$

and

$$K \cap A_1 = \emptyset \quad \text{iff} \quad \exists M_1, M_2\ \ M_1, M_2 \subseteq L_{\mathrm{rk}(K)} \wedge M_1 \subseteq C \wedge M_2 \cap C = \emptyset \wedge M_1 \cup M_2 = K \wedge M_1 \subseteq A_2.$$

Furthermore $A_1 \oplus A_2 \geqslant_{\alpha\text{-wtt}} C$ as $K \subseteq C$ is already Σ_1 and

$$K \cap C = \emptyset \quad \text{iff} \quad \exists M \in D_{\{e\}(K)} = \{K \times \{0, 1\}\} \wedge M \cap A_1 \oplus A_2 = \emptyset \wedge K \times \{0, 1\} = M.$$

Theorem 4.1 is proved. □

Now, having proven this when $D <_\alpha C$, the question arises about what happens if $D <_{\alpha\text{-}\hat{w}} C$. In one case we can readily answer this: if $\rho_2^\alpha \leq \kappa_2^\alpha$. If this happens, then block with $|B_\gamma|^\alpha < \rho_2^\alpha$ with κ_3^α many blocks—use the approximations given by Shore [9].

Now, to get the contradiction that shows length of agreement is bounded, consider $B_\gamma^* = \{e \in B_\gamma \mid \{\hat{e}_0\} \text{ is total}\}$.

This is a Σ_2-subset of B_γ and as $|B_\gamma|^\alpha < \rho_2^\alpha$ we have B_γ^* is α-finite. For $e \in B_\gamma \setminus B_\gamma^*$ there is a Σ_2-function f, such that $\{\hat{e}_0\}(f(e))$ is undefined, and as $\rho_2 \leq \kappa_2$, rng(f) is bounded, and we can work above this bound. So if $l_1(\gamma, \sigma)$ is unbounded, it is caused by $e \in B_\gamma^*$, but now we can choose any $e \in B_\gamma^*$ and demand that $D \restriction \{\hat{e}_0\}[\beta]$ be correct—hence getting an α-$\hat{w}$-computation of C in exactly the same way we previously got an α-computation.

The question of what happens if $\kappa_2^\alpha < \rho_2^\alpha$ remains, but some observations about $\lambda = \aleph_{\omega_1}$ may be relevant. Firstly the set $C = \{\beta \mid \beta < \lambda \text{ and } \beta \text{ is not a cardinal}\}$ is hypersimple, and by usual arguments, can be shown to be not α-$\hat{w}$-complete, i.e., $C <_{\alpha\text{-}\hat{w}} K$. Secondly, the arguments of Friedman [6] obtaining a negative solution to Post's Problem above $0'$ actually only use the set C. This suggests that a refinement of that analysis might give a failure of density in the λ-$\hat{w}$-re degrees.

The last theorem used a permitting argument. The next theorem shows that permitting does not always produce α-wtt reductions, although it does produce α-$\hat{w}$ reductions, by constructing an example of sets A, B, with $B \leq_{\alpha\text{-}\hat{w}} A$ by permitting, and $\neg(B \leq_{\alpha\text{-wtt}} A)$ by diagonalization. Essential use is made of the fact that the power set function on α is not recursive, and in our example, is not even total. It is possible to vary the argument to the case where the power set function is total, and this will be discussed following the proof.

Theorem 4.3. *Let $\alpha = \omega_1$. Then there exist Δ_2-sets A, B such that*

(a) $B \leq_{\alpha\text{-}\hat{w}} A$,

(b) $\neg(B \leq_{\alpha\text{-wtt}} A)$.

Proof. To obtain $B \leq_{\alpha\text{-}\hat{w}} A$ we use a permitting argument, i.e., we construct a recursive sequence $\langle x_\sigma \mid \sigma < \alpha\rangle$ with $\lim_\sigma x_\sigma = \alpha$, and

$$A_\sigma \restriction x_\sigma = A \restriction x_\sigma \quad \Rightarrow \quad B_\sigma \restriction x_\sigma = B \restriction x_\sigma.$$

This ensures $B \leq_{\alpha\text{-}\hat{w}} A$.

To obtain $B \not\leq_{\alpha\text{-wtt}} A$, we use requirements

R_e: $\neg(B \leq_{\alpha\text{-wtt}} A)$ via the eth reduction procedure.

Each of these is satisfied by finding witnesses x such that using e, A believes x is both in and out of B, or we create a disagreement.

The only problem is ensuring that we create a disagreement using a configuration of $A \restriction x$ never yet seen, and so we are able to make changes to B. This is possible because for any $\sigma < \alpha$, $\wp(\sigma)$ is unbounded in α.

For technical convenience we work, at stage σ, in $L_{b(\sigma)}$ where $a(\sigma)$ = the least admissible strictly larger than σ and $b(\sigma)$ = the least β which is p.r. closed such that $b > a(\sigma)$ and $L_\beta \models$ "ω is the greatest cardinal".

We define a restraint function $r(e, \sigma)$ for R_e.

At stage σ, for $e \leqslant \sigma$ we will define sets $A_{\sigma,e}$, $B_{\sigma,e}$, and elements $x_{\sigma,e}$.

$$A_{\sigma,0} = \emptyset = B_{\sigma,0}, \quad \text{and let} \quad x_{\sigma,0} = \omega.$$

Substage $\tau + 1$: We are given $A_{\alpha,\tau}$, $B_{\sigma,\tau}$ and $r(\tau', \sigma)$ for $\tau' < \tau$. Let

$$x_{\sigma,\tau} = \text{the least primitive recursively closed ordinal greater than or equal to}$$
$$\max\{r_1 + 1, \bigcup \{x_{\sigma'\tau'} \mid \langle \sigma', \tau' \rangle < \langle \sigma, \tau \rangle\}\}$$

(where $\langle \sigma', \tau' \rangle < \langle \sigma, \tau \rangle$ iff $\sigma' < \sigma$ or $\sigma' = \sigma$ and $\tau' < \tau$, and $r_1 = \bigcup_{\tau' < \tau} r(\tau', \sigma)$).

Notice that $\langle \sigma', \tau' \rangle \leqslant \langle \sigma, \tau \rangle \Rightarrow x_{\sigma',\tau'} \leqslant x_{\sigma,\tau}$, and that the only way $x_{\sigma,\tau}$ can change is because a higher priority requirement starts to act in a different (and hence earlier) case, causing the restraint function to increase.

Let $x = x_{\sigma,\tau}$, and $\tau = \langle e_0, e_1, e_2 \rangle$.

Definition 4.4. We say $\langle M_1, M_2 \rangle$ is *appropriate* if $\langle M_1, M_2 \rangle \in L_{b(\sigma)}$, and

$$L_{b(\sigma)} \models \langle M_1, M_2 \rangle \in D_{\{e_0\}(x)} \wedge M_1 \cap M_2 = \emptyset$$
$$\wedge M_1 \cap r_1 \subseteq A_{\sigma,\tau} \wedge M_2 \cap r_1 \cap A_{\sigma,\tau} = \emptyset.$$

We act according to the earliest of the cases below to occur:

Case 1: There exist $\langle L_1, L_2 \rangle$, $\langle M_1, M_2 \rangle$ which are appropirate with $L_1 \cap M_2 = \emptyset$, $L_2 \cap M_1 = \emptyset$ and $L_{b(\sigma)} \models \Phi_{e_1}(x, L_1, L_2) \wedge \Phi_{e_1}(x, M_1, M_2)$. Then take the least such pair $\langle \langle L_1, L_2 \rangle, \langle M_1, M_2 \rangle \rangle$ and let $A_{\sigma,\tau+1} = A_{\sigma,\tau} \cup L_1 \cup M_1$, and set

$$r(\tau, \sigma) = \max\{x + 1, \text{L-rk}(D_{\{e_0\}(x)})\}.$$

We now look for lengths of agreement previously established with $A_{\sigma,\tau+1}$, before defining $B_{\sigma,\tau+1}$.

Definition 4.5. (a) Let $w \leqslant x_{\sigma,\tau}$ be *agreeable by* σ' iff $\sigma' < \sigma$ and $A_{\sigma'} \cap (w + 1) = A_{\sigma,\tau+1} \cap (w + 1)$.

(b) w is *agreeable* iff it is agreeable by σ' for some $\sigma' < \sigma$.

Let

$$y = y_{\tau,\sigma} = \bigcup \{w \mid w \text{ is agreeable}\}.$$

If y is agreeable, let $\sigma' < \sigma$ be least such that y is agreeable by σ', and let $B_{\sigma,\tau+1} = B_{\sigma'} \cap (y + 1)$. Otherwise, let $\langle \sigma_i \mid i \in \omega \rangle$ be the L-least sequence for which there exists a sequence $\langle w_i \mid i \in \omega \rangle$ cofinal in y with w_i agreeable by σ_i. Then let

$$B_{\sigma,\tau+1} \cap (w_i + 1) = B_{\sigma_i} \cap (w_i + 1) \quad \text{for each } i < \omega.$$

Case 2: Case 1 fails, and for some appropriate $\langle M_1, M_2\rangle$ we have $L_{b(\sigma)} \models \Phi_{e_1}(x, M_1, M_2)$. Then pick the L-least such pair $\langle M_1, M_2\rangle$ and let $A_{\sigma,\tau+1} = A_{\sigma,\tau} \cup M_1$, and treat r and B as in Case 1 except that we work strictly below $x_{\sigma,\tau}$, instead of just below.

Case 3a: Case 1 and 2 both fail, and for some appropriate $\langle M_1, M_2\rangle$, $L_{b(\sigma)} \models \Phi_{e_2}(x, M_1, M_2)$ and this is the first time we have addressed this case for this value of $x_{\sigma,\tau}$, i.e., for $\sigma' < \sigma$: $x_{\sigma',\tau} \neq x_{\sigma,\tau}$ or if $x_{\sigma,\tau} = x_{\sigma',\tau}$ then we were in Case 4.

In this case, pick $S \subseteq x \setminus r_1$ to be $L_{b(\sigma)}$-generic ($S \in L_{b(\sigma)+\omega}$) and let

$$A_\sigma = A_{\sigma,\tau+1} = A_{\sigma,\tau} \cup S,$$
$$B_\sigma = B_{\sigma,\tau+1} = B_{\sigma,\tau} \cup \{x\},$$
$$r(\tau, \sigma) = x + 1$$

and go to stage $b(\sigma)$, i.e., for $\sigma < \tau < b(\sigma)$: $A_\tau = A_\sigma$ and $B_\tau = B_\sigma$.

Case 3b: Case 1, 2 and 3a all fail, and for some appropriate $\langle M_1, M_2\rangle$, $L_{b(\sigma)} \models \Phi_{e_2}(x, M_1, M_2)$. Then for some least $\sigma' < \sigma$ we have $x_{\sigma,\tau} = x_{\sigma'\tau}$ and at stage $\langle \sigma', \tau\rangle$ we were in Case 3.

Let $S = A_{\sigma',\tau} \cup (x \setminus r_1(\sigma', \tau))$ and

$$A_{\sigma,\tau+1} = A_{\sigma,\tau} \cup S,$$
$$B_{\sigma,\tau+1} = B_{\sigma,\tau} \cup \{x\},$$
$$r(\tau, \sigma) = x + 1.$$

Notice that $r_1(\sigma', \tau) = r_1(\sigma, \tau)$ since if not, we had to have $\tau = \delta + 1$ and $x_{\sigma',\delta} \neq x_{\sigma,\delta}$, as r_1 is defined using only $x_{\sigma',\delta}$. This means that $x_{\sigma,\delta} > x_{\sigma',\tau}$ which is impossible.

Case 4: All of the above cases fail in which case we do nothing, i.e., let $A_{\sigma,\tau+1} = A_{\sigma,\tau}$, $r(\tau, \sigma) = x + 1$ and $B_{\sigma,\tau+1} = B_{\sigma,\tau}$.

At limit τ, take unions.

Let $A_\sigma = A_{\sigma,\sigma}$ and $B_\sigma = B_{\sigma,\sigma}$. Then

$$x \in A \iff \exists\sigma\, \forall\tau \geq \sigma\; x \in A_\tau,$$
$$x \in B \iff \exists\sigma\, \forall\tau \geq \sigma\; x \in B_\tau.$$

Since both are built so that initial segments settle down (see Lemma 4.7),

$$x \notin A \iff \exists\sigma\, \forall\tau \geq \sigma\; x \notin A_\tau,$$
$$x \notin B \iff \exists\sigma\, \forall\tau \geq \sigma\; x \notin B_\tau,$$

and so both A and B are Δ_2.

Lemma 4.6. *Let $\langle \sigma, \tau\rangle$ be a stage at which Case 3 was addressed, and a generic S was used. Then for all $\delta < \sigma$ and $\rho \leq \delta$, $S \neq A_{\delta,\rho} \cap (x \setminus r)$ and for all $\langle M_1, M_2\rangle$, $\langle L_1, L_2\rangle \in D_{\{e_0\}(x)}$, $S \neq (A_{\sigma,\tau} \cup L_1 \cup M_1) \cap (x \setminus r)$ and $S \neq (A_{\sigma,\tau} \cup M_1) \cap (x \setminus r)$, where $x = x_{\sigma,\tau}$, $r = r(\sigma, \tau)$.*

Proof. By generality of S. $\square$

Lemma 4.7. *For all e,*
(a) $\lim_\sigma r(e, \sigma) = r(e)$ *exists,*
(b) $\lim_\sigma A_{\sigma,e} = A_e = A \cap r(e)$ *exists,*
(c) $\lim_\sigma x_{\sigma,e} = x_e$ *exists,*
(d) $\lim_\sigma B_{\sigma,e} = B_e = B \cap x_e$ *exists.*

Proof. By induction on e. The regularity of ω_1 will be used implicitly throughout this argument. Suppose that the result is true for $e' < e$ and let $r = \bigcup_{e'<e} r(e')$.

Let $\sigma > e$ be a stage such that for all $e' < e$ all of the above limits have been attained by stage σ. Then for all $\sigma' \geq \sigma$, $x_{\sigma',e} = x_e = x$, and the corresponding r_1 is r.

Now, either for some $\sigma' \geq \sigma$, $L_{b(\sigma')} \models \exists z\ \{e_0\}(x) = z$ or not. If not, nothing changes for e after stage σ. So assume $\sigma_1 \geq \sigma$ is the least stage for which $L_{b(\sigma_1)} \models \exists z\ \{e_0\}(x) = z$, and let $\sigma_2 \geq \sigma_1$ be the least stage such that for all $\langle M_1, M_2\rangle$ which are appropriate at stage σ_2, if $L_{\omega_1} \models \Phi_{e_1}(x, M_1, M_2)$ then $L_{b(\sigma_2)} \models \Phi_{e_1}(x, M_1, M_2)$.

Then at stage σ_2, $A_{\sigma_2,e}$, $B_{\sigma_2,e}$, $r(e, \sigma_2)$ and $x_{\sigma_2,e}$ reach their final values. By construction $A_e = A \cap r(e)$ and $B_e = B \cap x_e$. □

Lemma 4.8. *Let $\sigma' < \sigma$ and $A_{\sigma,\tau+1} \restriction x_{\sigma,\delta} + 1 = A_{\sigma'} \restriction x_{\sigma',\delta} + 1$ for any $\tau \leq \sigma$. Then*
$$B_{\sigma,\tau+1} \restriction x_{\sigma',\delta} + 1 = B_{\sigma'} \restriction x_{\sigma,\delta} + 1.$$

Proof. Let σ be the least stage for which the lemma fails, for some τ, σ', and let τ be least for which it fails for some σ', so that
$$A_{\sigma,\tau+1} \restriction x_{\sigma,\delta} + 1 = A_{\sigma'} \restriction x_{\sigma'\delta} + 1 \quad \text{but} \quad \neg(B_{\sigma,\tau+1} \restriction x_{\sigma,\delta} + 1 = B_{\sigma'} \restriction x_{\sigma',\delta} + 1).$$
Then at stage σ, $x_{\sigma,\delta} \leq y(\sigma, \tau) = y$, and $x_{\sigma,\delta}$ is agreeable—if $\langle \sigma, \tau\rangle$ is in Case 1 or 2. We first suppose $\langle \sigma, \tau\rangle$ is in Case 1 or 2.

If we need to use a cofinal sequence to reach y, i.e., $\langle w_i \mid i \in \omega\rangle$ by $\langle \sigma_i \mid i \in \omega\rangle$, then for some i, we have $w_i > x_{\sigma',\delta}$ and so $A_{\sigma'} \restriction x_{\sigma',\delta} + 1 = A_{\sigma_i} \restriction x_{\sigma,\delta} + 1$ and as $B_{\sigma,\tau+1} \restriction x_{\sigma',\delta} + 1 = B_{\sigma_i} \restriction x_{\sigma',\delta} + 1$ we also have $B_{\sigma'} \restriction x_{\sigma',\delta} + 1 \neq B_{\sigma_i} \restriction x_{\sigma',\delta} + 1$. This contradicts our choice of σ, τ and σ'. Likewise, we cannot have any agreeable w with $w > x_{\sigma',\delta}$, and w agreeable by $\sigma'' \neq \sigma'$ without achieving a similar contradiction to minimality. Hence, we may assume y is agreeable by σ'. But, in this case, we set $B_{\sigma,\tau+1} = B_{\sigma'} \cap (y + 1)$ (notice, it is equal to $B_{\sigma'} \cap (y + 1)$, since, if not, we would get another contradiction to minimality). But $y + 1 \geq x_{\sigma',\delta} + 1$ contradicting our choice of $\langle \sigma, \tau, \sigma'\rangle$. Thus we cannot be in Case 1 or 2 at stage $\langle \sigma, \tau\rangle$.

Suppose we were in Case 3 at stage $\langle \sigma, \tau\rangle$. If $\delta < \tau$, then $r(\sigma, \tau) > x_{\sigma',\delta}$ and as nothing happens to either A or B below r_1, σ cannot be the least stage of the contradiction. Hence $\delta \geq \tau$.

If $x_{\sigma',\delta} \neq x_{\sigma,\tau}$, then we have to have $x_{\sigma,\tau} > x_{\sigma',\delta}$—by choice of $x_{\sigma,\tau}$ and, in fact $x_{\sigma,\tau-1} \geq x_{\sigma',\delta}$, and thus $r(\sigma, \tau) \geq x_{\sigma',\delta}$, as it is $\geq x_{\sigma,\tau-1}$.

This means that $x_{\sigma',\delta} = x_{\sigma,\tau}$, and this, necessarily, implies $\delta = \tau$, and so
$$A_{\sigma'} \restriction x_{\sigma',\delta} + 1 = A_{\sigma',\delta} \restriction x_{\sigma',\tau} + 1.$$

Now, since we are in Case 3 at $\langle\sigma, \tau\rangle$, we cannot be in Case 3a, since otherwise we create a difference in A. Hence we are in Case 3b, and so we make A, and therefore B, look like earlier values of A, B. This will contradict the minimal choice of σ, τ and σ'.

If we are in Case 4 at $\langle\sigma, \tau\rangle$, the above analysis gives again that $x_{\sigma',\delta} = x_{\sigma,\tau}$ and $\delta = \tau$, and so we had to have been in Case 4 at $\langle\sigma', \tau\rangle$, and since both give no change in A and B, we get a contradiction to the choice of τ. $\square$

Lemma 4.9. *Let $A_\sigma \restriction x_{\sigma,\tau} = A \restriction x_{\sigma'\tau}$ for any σ and τ. Then $B_\sigma \restriction x_{\sigma,\tau} = B \restriction x_{\sigma,\tau}$.*

Proof. This follows immediately from Lemmas 4.7 and 4.8. $\square$

Corollary 4.10. $B \leqslant_{\alpha\text{-}\hat{\mathrm{w}}} A$.

Proof. We show how to determine $K \subseteq B$. The case $K \cap B = \emptyset$ is similar.

Let $K \subseteq x_{\sigma',\tau}$, and let $\sigma \geqslant \sigma'$ be any stage such that $A_\sigma \restriction x_{\sigma,\tau} = A \restriction x_{\sigma,\tau}$. Then $K \subseteq B \Leftrightarrow K \subseteq B_\sigma$ by Lemma 4.9.

σ is a bound suitable for all subsets of $x_{\sigma',\tau}$, and indeed for subsets of $x_{\sigma,\tau}$. This gives us an α-$\hat{\mathrm{w}}$-reduction. $\square$

Lemma 4.11 *For all e, $\neg(B \leqslant_{\alpha\text{-wtt}} A$ via $e)$.*

Proof. Suppose not, and e is least such that $B \leqslant_{\alpha\text{-wtt}} A$ via e.

Let σ be a stage such that for $e' < e$ and all $\tau \geqslant \sigma$, $A \cap r(e') = A_\tau \cap r(e')$ and $B \cap r(e') = B_\tau \cap r(e')$ and all relevant information for e has been established correctly in $L_{b(\sigma)}$— as in the proof of Lemma 4.6. As in that proof, we create a permanent computation and A and B take their limiting values through $r(e)$, at stage σ. We claim, that at this stage, we created a disagreement.

If $\langle\sigma, e\rangle$ is a Case 1 stage, this is clear, as A says that $x_{\sigma,e}$ is both in and out of B. Likewise in Cases 3a, and 4.

In Case 2, A says that $x_{\sigma,e}$ is in B, but our construction of B omits $x_{\sigma,e}$—as we do all our work strictly below $x_{\sigma,e}$, and restrain above it.

In Case 3b, A can only say, if it does, that $x_{\sigma,e}$ is out of B, but we put $x_{\sigma,e}$ into B, so creating a disagreement.

Since this a permanent computation, and all cases give disagreement, we are unable to choose e as in the hypothesis. $\square$

This completes the proof of Theorem 4.3. $\square$

To adapt this to the case when the power set function is total (and $\alpha > \omega$), it suffices to notice that limits are reached by the next stable ordinal after e is first seen. Thus the same argument using $b(\sigma)$ can be used, except we need to have

$b(\sigma)$ = least p.r. closed $\beta > a(\sigma)$ such that

$$L_\beta \models (|a(\sigma)| < |\sigma| \text{ and there is a greatest cardinal}).$$

Now two cases occur, either $\wp(x)$ is unbounded in $b(\sigma)$ or it is bounded. However, any lub is a limit of stables, and so the construction before the eth requirement takes place well below the bound. Thus the desired new sets can still be seen. The only problem is at σ itself, and in this case, the choice of $b(\sigma)$ ensures that $\wp^{L_{b(\sigma)}}(\sigma)$ is unbounded in $L_{b(\sigma)}$.

The construction can be adapted in fact, to any nonprojectible ordinal, using blocking. Nonprojectibility appears to be required in order to see that $\{\langle M_1, M_2\rangle \in D_{\{e_0\}(K)} \mid \langle M_1, M_2\rangle$ is appropriate at some stage and $(\Phi_{e_1}(x, M_1, M_2) \vee \Phi_{e_2}(x, M_1, M_2))\}$ is α-finite.

We do not know if the result can be improved to make A and B to be α-re. This particular argument makes strong use of the ability to go back and make 'corrections' at some x which we have chosen to be our witness for R_e, i.e., in the second part of Case 2.

However, our next theorem points out further difficulties.

Theorem 4.12. *Let $B \leq_{\alpha\text{-}\hat{\mathrm{w}}} A$ and B, A be α-re. Then there exists $A^* \equiv_{\alpha\text{-}\hat{\mathrm{w}}} A$ with $B \leq_{\alpha\text{-wtt}} A^*$. A^* depends on the reduction procedure.*

Proof. Let $B \leq_{\alpha\text{-}\hat{\mathrm{w}}} A$ via $e = \langle e_0, e_1\rangle$—the function $\{e_1\}$ bounds the use of the T-reduction given by e_0. Let $A^+ = A \times \{0\} \cup (L_\alpha \times L_\alpha \setminus \{0\})$.

We now define A^* as a subset of A^+. We devote the Kth column up to $\{e_1\}(K)$ of A^+ (for $K \neq \emptyset$) to testing $K \subseteq B$ *or* $K \cap B = \emptyset$.

Let σ_K be the least stage σ at which we see A_σ correctly giving us information about K as a 'subset' of B_σ.

Then for each x that later goes into A below $\{e_1\}(K)$ we put a new element $(\langle x, K\rangle)$ into $L_\alpha \times \{K\} \cap A^*$—in order of first appearance, and if this is simultaneous, then in order of L-rank.

Now define

$$D_{\{e_0'\}(K)} = \{\langle M_1, M_2\rangle \mid M_1 \cup M_2 = \{e_1\}(K) \times \{K\} \wedge M_1 \cap M_2 = \emptyset\}.$$

We now show that there is an α-wtt reduction computing B from A^*.

$$\begin{aligned} K \subseteq B \iff\ & \exists \sigma_1 < \sigma_2\, \exists \langle M_1, M_2\rangle \in L_{\sigma_2}\, \exists \langle m_1, m_2\rangle \in D_{\{e_0'\}}(K) \\ & \sigma_1 \text{ is least such that} \\ & (A_{\sigma_1} \text{ correctly gives information about } K \text{ relative to } B_{\sigma_1}) \\ & \wedge\, m_1 \subseteq A^* \wedge m_2 \cap A^* = \emptyset \\ & \wedge\, \exists S \in L_{\sigma_2+1}\ S \subseteq \sigma_2 \\ & \wedge\, \exists f \in L_{\sigma_2+1}\, \forall \tau \in S\ A_{\tau+1} \setminus A_\tau \neq \emptyset \end{aligned}$$

$$\wedge f \text{ is an order preserving bijection of } m_1 \text{ into } S$$
$$\wedge\, M_1 \subseteq A_{\sigma_2} \wedge M_2 \cap A_{\sigma_2} = \emptyset$$
$$\wedge\, \Phi_{e_1}(K, M_1, M_2).$$

σ_2 is a stage by which everything in A below the use of a computation determining $K \subseteq B$ is in A, and so A^* on the Kth column is finished, and so we can use it to get true answers about $K \subseteq B$.

$K \cap B = \emptyset$ is similar, except we change Φ_{e_1} to Φ_{e_2}.

This gives us e_1', and e_2' to get $e' = \langle e_0', e_1', e_2' \rangle$ and so $B \leqslant_{\alpha\text{-wtt}} A^*$ via e', as required. Notice, in this proof, if we use L_α^3 and form A^{**} by using column $\langle e, K \rangle$ instead of just K, we can make the set independent of the reduction procedure.

It remains to check that $A^* \equiv_{\alpha\text{-}\hat{\text{w}}} A$.

Clearly

$$K \subseteq A \quad \Leftrightarrow \quad K \times \{0\} \subseteq A^*$$

and

$$K \cap A = \emptyset \quad \Leftrightarrow \quad K \times \{0\} \cap A^* = \emptyset.$$

Thus $A \leqslant_{\alpha\text{-m}} A^*$.

Conversely

$$K \subseteq A^* \quad \Leftrightarrow \quad \pi_0[K] \subseteq A \text{ and for } \tau \in \pi_L[K] \;\; A_{\tau+1} \backslash A_\tau \neq \emptyset \text{ and } \tau > \sigma_L$$

and

$$K \cap A^* = \emptyset \quad \Leftrightarrow \quad \pi_0[K] \cap A = \emptyset \text{ and for } \tau \in \pi_L[K] \; (\tau < \sigma_L \text{ or } A_{\tau+1} \backslash A_\tau = \emptyset).$$

This can easily be modified to an α-$\hat{\text{w}}$ reduction—by obtaining the following bound on the use: Let $\hat{K} = \{M \mid \{M\} \times L_\alpha \cap K \neq \emptyset\}$ and $\sigma = \bigcup \{\pi_L[K] \mid L \in \hat{K}\}$. σ is a recursive function of K, and bounds all searches for the ordinals required above. Thus $A^* \leqslant_{\alpha\text{-}\hat{\text{w}}} A$. □

In a similar way we can see that $A^{**} \equiv_{\alpha\text{-}\hat{\text{w}}} A$—but note that in the construction of A^{**} we cannot use σ_K, so we just use 0 instead. Since A^{**} is α-re in A, the problem of obtaining A and B re with $B \leqslant_{\alpha\text{-}\hat{\text{w}}} A$ and $B \nleqslant_{\alpha\text{-wtt}} A$ is 'reduced' to finding an A with $A^{**} \nleqslant_{\alpha\text{-wtt}} A$.

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