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On partitions of finite vector spaces of low dimension over $GF(2)$

S.I. El-Zanati *, G.F. Seelinger, P.A. Sissokho, L.E. Spence, C. Vanden Eynden

4520 Mathematics Department, Illinois State University, Normal, IL 61790–4520, USA

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To Anthony Hilton on the occasion of his retirement

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ABSTRACT

Let $V_n(q)$ denote a vector space of dimension n over the field with q elements. A set $\mathcal{P}$ of subspaces of $V_n(q)$ is a *partition* of $V_n(q)$ if every nonzero vector in $V_n(q)$ is contained in exactly one subspace of $\mathcal{P}$. If there exists a partition of $V_n(q)$ containing a_i subspaces of dimension n_i for $1 \le i \le k$, then $(a_k, a_{k-1}, \ldots, a_1)$ must satisfy the Diophantine equation $\sum_{i=1}^{k} a_i(q^{n_i} - 1) = q^n - 1$. In general, however, not every solution of this Diophantine equation corresponds to a partition of $V_n(q)$. In this article, we determine all solutions of the Diophantine equation for which there is a corresponding partition of $V_n(2)$ for $n \le 7$ and provide a construction of each of the partitions that exist.

1. Introduction

Let $K = GF(q)$ be the finite field with q elements and $V = V_n(q)$ be a vector space of dimension n over K. We say that a set $\mathcal{P} = \{V_i\}_{i=1}^k$ of subspaces of V is a *partition* of V if and only if $V = \cup_{i=1}^k V_i$ and $V_i \cap V_j = \{0\}$ when $i \neq j$.

Suppose that $\mathcal{P} = \{V_i\}_{i=1}^k$ is a partition of V, and let $n_i = \dim V_i$. It is easy to see that the following conditions are satisfied:

$$q^n - 1 = \sum_{1}^{k} (q^{n_i} - 1) \tag{1}$$

and

$$n_i + n_j \le n \quad \text{when } i \neq j. \tag{2}$$

Condition (1) is true because every nonzero vector in $V_n(q)$ belongs to exactly one of the subspaces V_i, and (2) holds because $V_i \oplus V_j$ is a subspace of V with dimension $n_i + n_j$. Let W be a subspace of $V_n(q)$ having dimension $n - 1$. Then $\{V_i \cap W\}_{i=1}^k$ is a partition of W, which we call the *partition of W induced by* $\mathcal{P}$, and, for $i = 1, 2, \ldots, k$, we have $\dim(V_i \cap W) = \dim V_i$ if $V_i \subseteq W$, and $\dim(V_i \cap W) = \dim V_i - 1$ otherwise.

Bu [3] presents a number of sufficient conditions for the existence of certain partitions of $V_n(q)$. The first of these is a well-known result.

Lemma 1.1. *If m is a divisor of n and $k = (q^n - 1)/(q^m - 1)$, there exists a partition of $V_n(q)$ consisting of k subspaces of dimension m.*

Lemma 1.2. *For $1 \le d < \frac{1}{2}n$, there exists a partition of $V_n(q)$ consisting of 1 subspace of dimension $n - d$ and q^{n-d} subspaces of dimension d.*

* Corresponding author.
E-mail address: saad@ilstu.edu (S.I. El-Zanati).

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Lemma 1.3. *If $n = ks - 1$, where $s > 1$, there exists a partition of $V_n(q)$ consisting of $q^{(k-1)s}$ subspaces of dimension $s - 1$ and $(q^{(k-1)s} - 1)/(q^s - 1)$ subspaces of dimension s.*

Other authors have considered variations on this problem. Let $T = \{t_1, t_2, \dots, t_k\}$ be a set of positive integers with

$$2 \le t_1 < t_2 < \cdots < t_k = t. \tag{3}$$

A partition $\mathcal{P}$ of $V_n(q)$ is called a *T-partition* if (i) for every $\alpha \in T$, there is a $U \in \mathcal{P}$ with $\dim U = \alpha$; and (ii) $\dim U \in T$ for any $U \in \mathcal{P}$. Beutelspacher proved in [1] that if $t_1 = 2$, then $V_{2t}(q)$ has a T-partition. Heden shows in [6] that $V_{2t}(q)$ has a T-partition for any arbitrary set of integers satisfying (3).

In a later paper [7], Heden characterized the partitions of $V_n(2)$ for $n \ge 9$ that contain only subspaces of dimensions 1, 2, 3, and $n - 3$. He also presents there results about the existence and nonexistence of certain partitions of $V_6(2)$ and $V_7(2)$.

Before proceeding, we note that this partition problem also relates to the problem of finding optimal partial spreads and has applications in byte error control codes. For further information, we direct the interested reader to the article by Clark and Dunning [4], and to the references therein.

Suppose there exists a partition of $V_n(q)$ into a_i subspaces of dimension i for $1 \le i \le k$. By (1), $(a_1, a_2, \dots, a_k)$ must satisfy the Diophantine equation

$$\sum_{i=1}^{k} a_i(q^{n_i} - 1) = q^n - 1. \tag{4}$$

In general, however, not every solution of (4) corresponds to a partition of $V_n(q)$. For example, Corollary 2.2 shows that there is no partition of $V_5(2)$ into 10 subspaces of dimension 2 and 1 subspace of dimension 1. In this article, we classify all solutions of (4) for which there is a corresponding partition of $V_n(2)$ when $n \le 7$ and provide an explicit construction of those that exist. Some of these partitions were shown to exist in [7], but we include them here for completeness.

2. Feasible partition types

We begin by proving a third necessary condition for a solution to (4) to correspond to a partition of $V_n(2)$.

Lemma 2.1. *Let V be a finite dimensional vector space over a finite field F. Then $\sum_{v \in V} v = 0$ if $\dim V \ge 2$ or $\operatorname{char} F \ne 2$.*

Proof. If $\operatorname{char} F \ne 2$, then every nonzero vector $v \in V$ has a unique additive inverse $-v \ne v$. So, in this case, it is clear that $\sum_{v \in V} v = 0$.

If $\operatorname{char} F = 2$, then we can reduce to the case where F is the field with two elements. Let V be a finite-dimensional vector space over F of dimension $n \ge 2$, and let $\{u_1, u_2, \dots, u_n\}$ be a basis for V. Define $w = \sum_{v \in V} v$. Then $w = \sum_{i=1}^{n} \alpha_i u_i$ for some $\alpha_i \in F$.

We claim $\alpha_i = 0$ for all i. Indeed, for any $v \in V$, we can write $v = \sum_{i=1}^{n} \epsilon_{i,v} u_i$. Therefore $w = \sum_{v \in V} \sum_{i=1}^{n} \epsilon_{i,v} u_i = \sum_{i=1}^{n} \left(\sum_{v \in V} \epsilon_{i,v}\right) u_i$, so that $\alpha_i = \sum_{v \in V} \epsilon_{i,v}$. In particular, $\alpha_i = t(1_F)$, where t is the number of vectors such that $\epsilon_{i,v} = 1_F$. But the set of vectors whose ith component is 1_F is given by $\{u_i + \sum_{j \ne i} \epsilon_j u_j : \epsilon_j \in F\}$. Since the order of this set is even, $t(1_F) = 0_F$; so $\alpha_i = 0$ for all i. Therefore $\sum_{v \in V} v = 0$. $\square$

Corollary 2.2. *In any partition of $V_n(q)$ $(n \ge 2)$ into subspaces, the sum of the vectors in the 1-dimensional subspaces of the partition is 0.*

Proof. From Lemma 2.1, the sum of the vectors in $V_n(q)$ is 0, and the same is true for the sum of the vectors in the subspaces of dimension greater than 1. The result follows. $\square$

Note that Corollary 2.2 shows that there can be no partition of $V_5(2)$ containing exactly ten subspaces of dimension 2 and one subspace of dimension 1. However, such a partition satisfies both of the necessary conditions (1) and (2). Thus Corollary 2.2 provides a new necessary condition for the existence of a partition of $V_n(2)$. Because every vector in $V_n(2)$ is its own additive inverse, Corollary 2.2 implies that *if a partition of $V_n(2)$ contains subspaces of dimension 1, then there cannot be fewer than 3 subspaces of dimension 1.*

We call a sequence of nonnegative integers $\langle a_k, a_{k-1}, \dots, a_1 \rangle$ a *partition type* for $V_n(2)$ and say that it is *feasible* if the following four conditions hold:

(a) $\sum_{i=1}^{k} a_i(2^i - 1) = 2^n - 1$.
(b) If both a_i and a_j are nonzero for $i \ne j$, then $i + j \le n$.
(c) If $i > n/2$, then $a_i \le 1$.
(d) $a_1 \ne 1$ and $a_1 \ne 2$.

Thus a necessary condition for the existence of a partition of $V_n(2)$ containing exactly a_i subspaces of dimension i for $i = 1, 2, \dots, k$ and no subspaces of dimension greater than k is that the partition type $\langle a_k, a_{k-1}, \dots, a_1 \rangle$ be feasible.

If there is a partition of $V_n(2)$ containing exactly a_i subspaces of dimension i for $i = 1, 2, \ldots, k$ and no subspaces of dimension greater than k, then we say that the partition type $\langle a_k, a_{k-1}, \ldots, a_1\rangle$ is *realized*. In Sections 3, 5 and 6 below, we determine, for $n \leq 7$, all of the feasible partition types for $V_n(2)$ that are realized. Specifically, we prove the following result.

Main theorem. *For $n \leq 7$, all the feasible partition types of $V_n(2)$ are realized except $\langle 7, 3, 5\rangle$ when $n = 6$ and $\langle 1, 13, 7, 0\rangle$, $\langle 1, 13, 6, 3\rangle$, $\langle 1, 14, 3, 5\rangle$, and $\langle 17, 1, 5\rangle$ when $n = 7$.*

In [5], we use this result to provide the base of a proof by mathematical induction showing that, for every integer $n > 2$, a feasible partition type of $V_n(2)$ with the form $\langle a_3, a_2, 0\rangle$ is realized if and only if $a_2 \neq 1$. In another article [2], we show how partitions of $V_n(q)$ relate to uniformly resolvable designs.

Convention. Throughout the remainder of this paper, if S is a subset of $V_n(2)$ such that $S \cup \{0\}$ is a subspace, we will use the term *subspace* to refer to S.

3. Partitions of $V_n(2)$ for $n \leq 5$

Partitions of $V_3(2)$.

The 3 feasible partition types are $\langle 1, 0, 0\rangle$, $\langle 1, 4\rangle$, and $\langle 7\rangle$. Clearly all of these can be realized.

Partitions of $V_4(2)$.

The 8 feasible partition types are $\langle 1, 0, 0, 0\rangle$, $\langle 1, 0, 8\rangle$, and $\langle 5-i, 3i\rangle$ for $0 \leq i \leq 5$. Partitions of type $\langle 1, 0, 0, 0\rangle$ and $\langle 5, 0\rangle$ are realized by Lemma 1.1. A partition of type $\langle 1, 0, 8\rangle$ can be obtained by extracting a 3-dimensional subspace of $V_4(2)$ and regarding the remaining vectors as the nonzero vectors in subspaces of dimension 1. The other partitions are obtained by regarding the nonzero vectors in 2-dimensional subspaces in a partition of type $\langle 5, 0\rangle$ as the nonzero vectors in 3 subspaces of dimension 1.

Partitions of $V_5(2)$.

The 21 feasible partition types are $\langle 1, 0, 0, 0, 0\rangle$, $\langle 1, 0, 0, 16\rangle$, and $\langle 1-i, 8+i-j, 4i+3j\rangle$ for $0 \leq j \leq 8+i$ and $0 \leq i \leq 1$. Partitions of types $\langle 1, 0, 0, 0, 0\rangle$ and $\langle 1, 0, 0, 16\rangle$ clearly exist, and a partition of type $\langle 1, 8, 0\rangle$ can be realized by Lemma 1.2. The remaining partitions can be obtained from one of type $\langle 1, 8, 0\rangle$ by regarding the subspace of dimension 3 as 1 subspace of dimension 2 and 4 subspaces of dimension 1, or regarding some subspaces of dimension 2 as 3 subspaces of dimension 1.

4. Reconfiguring subspaces

In Section 3, subspaces in a partition were changed into new subspaces to create a new partition. We call this process *reconfiguring* the subspaces. Reconfiguring subspaces plays an essential role in our construction of partitions of $V_6(2)$ and $V_7(2)$.

We have already seen that a subspace of dimension m can be reconfigured into a subspace of dimension $k < m$ and subspaces of dimension 1. In addition, any subspace of dimension 4 can be reconfigured into 5 subspaces of dimension 2.

If $n = 2k$, then any three subspaces of dimension k in a partition of $V_n(2)$ can be reconfigured into $r = 2^k - 1$ subspaces of dimension 2. For suppose that V_1, V_2, and V_3 are any three k-dimensional subspaces in a partition of $V_n(2)$. Let the nonzero elements in these subspaces be

$$x_1, x_2, \ldots, x_r, \qquad y_1, y_2, \ldots, y_r, \quad \text{and} \quad z_1, z_2, \ldots, z_r,$$

respectively. Since $V_n(2) = V_1 \oplus V_2$, it is possible to number the elements of V_1, V_2, and V_3 so that $x_i + y_i = z_i$ for $i = 1, 2, \ldots, r$. Thus V_1, V_2, and V_3 can be reconfigured into the 2-dimensional subspaces

$$\{0, x_1, y_1, z_1\}, \{0, x_2, y_2, z_2\}, \ldots, \{0, x_r, y_r, z_r\}.$$

In particular, any 3 subspaces of dimension 3 in a partition of $V_6(2)$ can be reconfigured into 7 subspaces of dimension 2.

5. Partitions of $V_6(2)$

Clearly, only one feasible partition type for $V_6(2)$ contains a subspace of dimension 6, and, because of (b) in the definition, only one feasible partition type for $V_6(2)$ contains a subspace of dimension 5. A feasible partition type that contains a subspace of dimension 4 must be a nonnegative integer solution of the system

$$\begin{aligned} 15a_4 + 7a_3 + 3a_2 + a_1 &= 63 \\ a_4 \qquad\qquad\qquad &= 1 \\ a_3 \qquad\qquad &= 0, \end{aligned}$$

where $a_1 \neq 1$ and $a_1 \neq 2$, that is, a nonnegative integer solution of the equation $3a_2 + a_1 = 48$ such that $a_1 \neq 1$ and $a_1 \neq 2$. We can regard these feasible partition types as solutions of the inequality $3a_2 \leq 48$, and so there are 17 feasible partition types containing a subspace of dimension 4.

Table 1
Feasible partition types of $V_6(2)$

Category	Type	
1	$\langle 1, 0, 0, 0, 0, 0\rangle$	
2	$\langle 1, 0, 0, 0, 32\rangle$	
3	$\langle 1, 0, 16 - j, 3j\rangle$	$0 \le j \le 16$
4	$\langle 9 - 3i, 7i - j, 3j\rangle$	$0 \le j \le 7i$ and $0 \le i \le 3$
5	$\langle 8 - 3i, 1 + 7i - j, 4 + 3j\rangle$	$0 \le j \le 1 + 7i$ and $0 \le i \le 2$
6	$\langle 7 - 3i, 3 + 7i - j, 5 + 3j\rangle$	$0 \le j \le 3 + 7i$ and $0 \le i \le 2$

As above, we can regard a feasible partition type having no subspaces of dimension greater than 3 as a nonnegative integer solution of $7a_3 + 3a_2 \le 63$ such that $63 - (7a_3 + 3a_2) \ne 1$ and $63 - (7a_3 + 3a_2) \ne 2$. A straightforward count shows that there are 106 such solutions, and hence 106 feasible partition types having no subspaces of dimension greater than 3. Thus, in all, there are 125 feasible partition types for $V_6(2)$, which are shown in Table 1.

Each partition in categories 3, 4, 5, and 6 can be obtained by reconfiguration from the one in the same category with $i = j = 0$. For example, a partition of type $\langle 1, 0, 14, 6\rangle$ (category 3) can be obtained from one of type $\langle 1, 0, 16, 0\rangle$ by reconfiguring 2 subspaces of dimension 2 into 6 subspaces of dimension 1. Also, a partition of type $\langle 5, 4, 16\rangle$ (category 5) can be obtained from one of type $\langle 8, 1, 4\rangle$ by reconfiguring 3 subspaces of dimension 3 into 7 subspaces of dimension 2 and then reconfiguring 4 subspaces of dimension 2 into 12 subspaces of dimension 1. In fact, Table 1 is arranged so that if a partition in some category is obtained, then all other partition types in that category having larger values of i or j can be realized in this manner. Hence, to verify which feasible partition types of $V_6(2)$ can be realized, it is sufficient to show that one partition in each category exists and to prove that partition types in that category having smaller values of i or j cannot be realized.

Partitions in categories 1 and 2 clearly exist. A partition of type $\langle 1, 0, 16, 0\rangle$ (category 3) exists by Lemma 1.2. A partition of type $\langle 9, 0, 0\rangle$ (category 4) can be obtained from Lemma 1.1. A partition of type $\langle 8, 1, 4\rangle$ (category 5) can be obtained from a partition of type $\langle 9, 0, 0\rangle$ by reconfiguring 1 subspace of dimension 3 into 1 subspace of dimension 2 and 4 subspaces of dimension 1. Finally, Theorem 5 in [7] shows that a partition of type $\langle 7, 3, 5\rangle$ (category 6) does not exist. However, a partition of type $\langle 7, 2, 8\rangle$ can be obtained from a partition of type $\langle 9, 0, 0\rangle$ by reconfiguring 2 subspaces of dimension 3 into 2 subspaces of dimension 2 and 8 subspaces of dimension 1, and a partition of type $\langle 4, 10, 5\rangle$ is constructed in Example 2.

Although a partition of type $\langle 9, 0, 0\rangle$ exists by Lemma 1.1, we now describe an explicit construction of such a partition that is used in the rest of this paper.

Example 1. In what follows, we always identify a subspace by the set of its nonzero elements. Let V be the set of nonzero elements of a 3-dimensional subspace of $V_n(2)$ with basis $\{v_1, v_2, v_3\}$. Then

$$V = \{v_1, v_2, v_3, v_4 = v_1 + v_2, v_5 = v_2 + v_3, v_6 = v_1 + v_2 + v_3, v_7 = v_1 + v_3\}.$$

For any integer $1 \le k \le 7$, define V^k to be the 7-tuple generated by the elements of V with first entry v_k, that is,

$$V^k = (v_k, v_{k+1}, \dots, v_{k+6}),$$

where the addition of indices is done modulo 7. If we do not need to specify the first entry of this k-tuple, then we simply write $[V]$.

Consider the recurrence relation

$$v_{i+3} = v_i + v_{i+1},$$

with the addition of indices performed modulo 7. The sequence defined by this recurrence relation yields $[V]$.

Now let U and W be subspaces of $V_n(2)$ having dimension 3 such that $U \cap W = \{0\}$. As above, the nonzero elements $u_1, u_2, \dots, u_7$ and $w_1, w_2, \dots, w_7$ of U and W, respectively, can be chosen to satisfy the recurrence relations

$$u_{i+3} = u_i + u_{i+1} \quad \text{and} \quad w_{i+3} = w_i + w_{i+1},$$

where addition of indices is done modulo 7.

Define

$$A_{i-1} = \{u_1 + w_i, u_2 + w_{i+1}, \dots, u_7 + w_{i+6}\},$$

for $1 \le i \le 7$. Then A_{i-1} is the set of nonzero elements from a subspace of dimension 3 with basis

$$\{u_1 + w_i, u_2 + w_{i+1}, u_3 + w_{i+2}\}.$$

For $1 \le k \le 7$, let

$$U^k = (u_k, u_{k+1}, \dots, u_{k+6}) \quad \text{and} \quad W^k = (w_k, w_{k+1}, \dots, w_{k+6}),$$

and for $0 \le i \le 6$, let

$$A_i^k = (u_k + w_{i+k}, u_{k+1} + w_{i+k+1}, \ldots, u_{k+6} + w_{i+k+6}).$$

The subspaces U, W, and $A_i \cup \{0\}$ $(0 \le i \le 6)$ form a partition of

$$U \oplus W = \{u + w : u \in U \text{ and } w \in W\}$$

into 9 subspaces of dimension 3. Note that the 7-tuples U^k and W^k satisfy the same recurrence relation as the vectors u_k and w_k, that is,

$$U^k + U^{k+1} = U^{k+3} \quad \text{and} \quad W^k + W^{k+1} = W^{k+3}.$$

Moreover, we have $A_i^k = U^k + W^{i+k}$. □

Example 2. We prove the existence of a partition of $V_6(2)$ of type $\langle 4, 10, 5\rangle$ by showing that the 3-dimensional subspaces A_0, A_1, A_2, A_3, and A_4 in Example 1 can be reconfigured into 10 subspaces of dimension 2 and 5 subspaces of dimension 1. The nonzero vectors in the subspaces of dimension 2 are:

$$\begin{array}{lll}
\{u_1 + w_1, u_2 + w_3, u_4 + w_7\}, & & \{u_2 + w_2, u_6 + w_1, u_7 + w_4\}, \\
\{u_3 + w_3, u_4 + w_5, u_6 + w_2\}, & & \{u_4 + w_4, u_6 + w_7, u_3 + w_5\}, \\
\{u_5 + w_5, u_7 + w_1, u_4 + w_6\}, & & \{u_6 + w_6, u_2 + w_4, u_7 + w_3\}, \\
\{u_7 + w_7, u_1 + w_2, u_3 + w_6\}, & & \{u_3 + w_4, u_5 + w_7, u_2 + w_5\}, \\
\{u_5 + w_6, u_1 + w_4, u_6 + w_3\}, & \text{and} & \{u_1 + w_5, u_2 + w_6, u_4 + w_1\}.
\end{array}$$

Together with the 3-dimensional subspaces W, U, A_5, and A_6, these 10 subspaces of dimension 2 and the remaining 5 subspaces of dimension 1 form a partition of $V_6(2)$ of type $\langle 4, 10, 5\rangle$. □

The results obtained in this section show that every feasible partition type for $V_6(2)$ can be realized except $\langle 7, 3, 5\rangle$.

6. Partitions of $V_7(2)$

6.1. Feasible partition types of $V_7(2)$

By an analysis similar to that in Section 5, it can be shown that there are 758 feasible partition types for $V_7(2)$, which are shown in Table 2. Note that Table 2 is arranged so that if a partition in some category is obtained, then all other partition types in that category having larger values of j can be realized by reconfiguring 2-dimensional subspaces into subspaces of dimension 1. For example, the partition in category 3 of type $\langle 1, 0, 0, 32 - j, 3j\rangle$ can be obtained from one of type $\langle 1, 0, 0, 32, 0\rangle$ by regarding j subspaces of dimension 2 as $3j$ subspaces of dimension 1.

We will show that all of the partition types in Table 2 can be realized except for four: $\langle 1, 14, 3, 5\rangle$ (category 6), $\langle 1, 13, 7, 0\rangle$ (category 7), $\langle 1, 13, 6, 3\rangle$ (category 7), and $\langle 17, 1, 5\rangle$ (category 21).

Table 2
Feasible partition types of $V_7(2)$

Category	Partition type		Example	Category	Partition type		Example
1	$\langle 1, 0, 0, 0, 0, 0\rangle$						
2	$\langle 1, 0, 0, 0, 0, 64\rangle$						
3	$\langle 1, 0, 0, 32 - j, 3j\rangle$	$0 \le j \le 32$		21	$\langle 17, 1 - j, 5 + 3j\rangle$	$0 \le j \le 1$	
4	$\langle 1, 16, 0, 0\rangle$			22	$\langle 16, 5 - j, 3j\rangle$	$0 \le j \le 5$	
5	$\langle 1, 15, 1 - j, 4 + 3j\rangle$	$0 \le j \le 1$	5	23	$\langle 15, 6 - j, 4 + 3j\rangle$	$0 \le j \le 6$	
6	$\langle 1, 14, 3 - j, 5 + 3j\rangle$	$0 \le j \le 3$	5	24	$\langle 14, 8 - j, 5 + 3j\rangle$	$0 \le j \le 8$	8
7	$\langle 1, 13, 7 - j, 3j\rangle$	$0 \le j \le 7$	4	25	$\langle 13, 12 - j, 3j\rangle$	$0 \le j \le 12$	9
8	$\langle 1, 12, 8 - j, 4 + 3j\rangle$	$0 \le j \le 8$	5	26	$\langle 12, 13 - j, 4 + 3j\rangle$	$0 \le j \le 13$	
9	$\langle 1, 11, 10 - j, 5 + 3j\rangle$	$0 \le j \le 10$	6	27	$\langle 11, 15 - j, 5 + 3j\rangle$	$0 \le j \le 15$	
10	$\langle 1, 10, 14 - j, 3j\rangle$	$0 \le j \le 14$	3	28	$\langle 10, 19 - j, 3j\rangle$	$0 \le j \le 19$	
11	$\langle 1, 9, 15 - j, 4 + 3j\rangle$	$0 \le j \le 15$	5	29	$\langle 9, 20 - j, 4 + 3j\rangle$	$0 \le j \le 20$	
12	$\langle 1, 8, 17 - j, 5 + 3j\rangle$	$0 \le j \le 17$	7	30	$\langle 8, 22 - j, 5 + 3j\rangle$	$0 \le j \le 22$	
13	$\langle 1, 7, 21 - j, 3j\rangle$	$0 \le j \le 21$	3	31	$\langle 7, 26 - j, 3j\rangle$	$0 \le j \le 26$	
14	$\langle 1, 6, 22 - j, 4 + 3j\rangle$	$0 \le j \le 22$	5	32	$\langle 6, 27 - j, 4 + 3j\rangle$	$0 \le j \le 27$	
15	$\langle 1, 5, 24 - j, 5 + 3j\rangle$	$0 \le j \le 24$	6	33	$\langle 5, 29 - j, 5 + 3j\rangle$	$0 \le j \le 29$	
16	$\langle 1, 4, 28 - j, 3j\rangle$	$0 \le j \le 28$	3	34	$\langle 4, 33 - j, 3j\rangle$	$0 \le j \le 33$	
17	$\langle 1, 3, 29 - j, 4 + 3j\rangle$	$0 \le j \le 29$	5	35	$\langle 3, 34 - j, 4 + 3j\rangle$	$0 \le j \le 34$	
18	$\langle 1, 2, 31 - j, 5 + 3j\rangle$	$0 \le j \le 31$	6	36	$\langle 2, 36 - j, 5 + 3j\rangle$	$0 \le j \le 36$	
19	$\langle 1, 1, 35 - j, 3j\rangle$	$0 \le j \le 35$	3	37	$\langle 1, 40 - j, 3j\rangle$	$0 \le j \le 40$	
20	$\langle 1, 0, 36 - j, 4 + 3j\rangle$	$0 \le j \le 36$	5	38	$\langle 41 - j, 4 + 3j\rangle$	$0 \le j \le 41$	

6.2. Partition types of $V_7(2)$ that cannot be realized

In this subsection, we prove that 3 of the feasible partition types of $V_7(2)$ cannot be realized. A fourth feasible partition type, $\langle 1, 14, 3, 5\rangle$, cannot be realized by virtue of Theorem 5 in [7].

Lemma 6.1. *Let $\mathcal{B}$ be a basis for an n-dimensional vector space V, and let $\mathcal{B}'$ be a proper subset of $\mathcal{B}$. Then V has an $(n-1)$-dimensional subspace W such that $W \cap \mathcal{B} = \mathcal{B}'$.*

Proof. We can assume that $\mathcal{B}$ is the basis $\{e_1, e_2, \ldots, e_n\}$ and $\mathcal{B}' = \{e_1, e_2, \ldots, e_k\}$ for some $k < n$, where e_i is the vector whose ith component is 1 and whose other components are 0. The desired subspace is $W = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in V : \alpha_{k+1} + \alpha_{k+2} + \cdots + \alpha_n = 0\}$. □

Proposition 6.2. *No partition of $V_7(2)$ of type $\langle 1, 13, 7, 0\rangle$ exists.*

Proof. Assume that $\mathcal{P}$ is a partition of type $\langle 1, 13, 7, 0\rangle$. By (2), the partition of a 6-dimensional subspace of $V_7(2)$ induced by $\mathcal{P}$ is of type $\langle 1, 0, a_2, a_1\rangle$, where a_2 and a_1 satisfy

$$\begin{aligned} a_2 + a_1 &= 20 \\ 3a_2 + a_1 &= 48, \end{aligned}$$

or of type $\langle a_3, a_2, a_1\rangle$, where a_3, a_2, and a_1 satisfy

$$\begin{aligned} a_3 + a_2 + a_1 &= 21 \\ 7a_3 + 3a_2 + a_1 &= 63. \end{aligned}$$

It is easily checked that there is no integral solution of either system in which $a_1 = 7$. Moreover, only the second system has an integral solution in which $a_1 = 0$, and it is $a_3 = 0$, $a_2 = 21$, $a_1 = 0$. However, there can be no induced partition of type $\langle 21, 0\rangle$ because there must be at least one subspace of dimension 3 in the induced partition. Thus no partition induced by $\mathcal{P}$ can contain 0 or 7 subspaces of dimension 1.

Let $V_1, V_2, \ldots, V_7$ denote the 2-dimensional subspaces in $\mathcal{P}$, and let $S = \{u_1, u_2, \ldots, u_k\}$ be a linearly independent subset of $V_1 \cup V_2 \cup \cdots \cup V_7$ that is maximal with respect to the property that each u_i is in a different 2-dimensional subspace in $\mathcal{P}$. By renumbering, if necessary, we may assume that $u_i \in V_i$ for $i = 1, 2, \ldots, k$.

Case 1: $k = 7$.

In this case, S is a basis for $V_7(2)$. Apply Lemma 6.1 to find a 6-dimensional subspace W of $V_7(2)$ that contains none of the vectors in S. Then W does not contain any V_i, and hence the partition of W induced by $\mathcal{P}$ must contain 7 subspaces of dimension 1, a contradiction.

Case 2: $k < 7$.

We claim that $V_i \subseteq \text{Span}\, S$ for $k < i \le 7$. For otherwise, there is a vector u_i in V_i that is not in Span S. But then the set $\{u_1, u_2, \ldots, u_k, u_i\}$ contradicts the maximality of S.

Let W be a 6-dimensional subspace of $V_7(2)$ containing S. Since $V_i \subseteq \text{Span}\, S$ for $k < i \le 7$, the 1-dimensional subspaces in the partition of W induced by $\mathcal{P}$ must be $V_i \cap W = \{0, u_i\}$, where $1 \le i \le k$. But the sum of the vectors in the 1-dimensional subspaces of this partition must be 0; so we have a contradiction of the linear independence of S.

Since each case leads to a contradiction, we conclude that there is no partition of type $\langle 1, 13, 7, 0\rangle$. □

Corollary 6.3. *No partition of $V_7(2)$ of type $\langle 1, 13, 6, 3\rangle$ exists.*

Proof. Suppose that $\mathcal{P}$ is a partition of $V_7(2)$ of type $\langle 1, 13, 6, 3\rangle$. Because the vectors in the subspaces of $\mathcal{P}$ of dimension 1 must sum to 0, the union of these subspaces must be a subspace of dimension 2. This yields a partition of type $\langle 1, 13, 7, 0\rangle$, in contradiction of Proposition 6.2. □

Proposition 6.4. *No partition of $V_7(2)$ of type $\langle 17, 1, 5\rangle$ exists.*

Proof. Assume that $\mathcal{P}$ is a partition of $V_7(2)$ of type $\langle 17, 1, 5\rangle$. Let $u_1, u_2, \ldots, u_5$ denote the nonzero vectors in the five subspaces of dimension 1. If these were all contained in a subspace S of dimension 3, then, by Lemma 2.1 and Corollary 2.2, both the sum of the vectors in S and the sum of $u_1, u_2, \ldots, u_5$ would be 0. But then the sum of the other two nonzero vectors in S would be 0, which is impossible. Thus we can assume that u_1, u_2, u_3, and u_4 are linearly independent and that u_5 is their sum.

Extend $\{u_1, u_2, u_3, u_4\}$ to a basis for $V_7(2)$, and apply Lemma 6.1 to obtain a 6-dimensional subspace W of $V_7(2)$ that contains none of u_1, u_2, u_3, and u_4. The complement of W in $V_7(2)$ can contain no 2-dimensional subspace, and because u_1 and u_2 are not in W, it follows that $u_1 + u_2$ is in W. Similarly, $u_3 + u_4$ is in W, and so $u_5 = (u_1 + u_2) + (u_3 + u_4)$ is in W. Therefore the partition of W induced by $\mathcal{P}$ contains either one or two subspaces of dimension 1, a contradiction of Corollary 2.2. Hence there is no partition of type $\langle 17, 1, 5\rangle$. □

Table 3
The subspaces S_i formed from Eq. (5)

	$x+W^1$	$x+A_0^6$	A_1^1	A_6^6	$x+A_3^1$	$x+A_2^5$	A_5^5
S_1	$x+w_1$	$x+u_6+w_6$	u_1+w_2	u_6+w_5	$x+u_1+w_4$	$x+u_5+w_7$	u_5+w_3
S_2	$x+w_2$	$x+u_7+w_7$	u_2+w_3	u_7+w_6	$x+u_2+w_5$	$x+u_6+w_1$	u_6+w_4
S_3	$x+w_3$	$x+u_1+w_1$	u_3+w_4	u_1+w_7	$x+u_3+w_6$	$x+u_7+w_2$	u_7+w_5
S_4	$x+w_4$	$x+u_2+w_2$	u_4+w_5	u_2+w_1	$x+u_4+w_7$	$x+u_1+w_3$	u_1+w_6
S_5	$x+w_5$	$x+u_3+w_3$	u_5+w_6	u_3+w_2	$x+u_5+w_1$	$x+u_2+w_4$	u_2+w_7
S_6	$x+w_6$	$x+u_4+w_4$	u_6+w_7	u_4+w_3	$x+u_6+w_2$	$x+u_3+w_5$	u_3+w_1
S_7	$x+w_7$	$x+u_5+w_5$	u_7+w_1	u_5+w_4	$x+u_7+w_3$	$x+u_4+w_6$	u_4+w_2

6.3. Partitions of $V_7(2)$ containing a subspace of dimension 4 or more

Partition types of $V_7(2)$ that contain a subspace of dimension 5 or more are listed in categories 1, 2, and 3 of Table 2. Because a partition of type $\langle 1, 0, 0, 32, 0\rangle$ exists by Lemma 1.2, all the partition types in categories 1, 2, and 3 can be realized.

Although a partition of type $\langle 1, 16, 0, 0\rangle$ (category 4) is known to exist by Lemma 1.2, we now construct a particular partition of this type that is useful in our examples. This construction uses the notation of Example 1, where U and W denote 3-dimensional subspaces of $V_7(2)$ such that $U \cap W = \{0\}$. Choose an element x of $V_7(2)$ not in $U \oplus W$, and let W^* denote the set of nonzero elements in W. Then the sets $\{0, x\} \oplus U$, W^*, $x + W^*$, A_i, and $x + A_i$ $(0 \le i \le 6)$ form a partition of the set $V_7(2)$ into 17 subsets in which each set other than $\{0, x\} \oplus U$ contains precisely 7 elements.

For any 7-tuple $S = (s_1, s_2, \ldots, s_7)$, denote by $x + S$ the 7-tuple

$$x + S = (x + s_1, x + s_2, \ldots, x + s_7).$$

By using the ideas in Example 1, we see that the following equations are true.

$$(x + W^1) + (x + A_0^6) + A_1^1 + A_6^6 + (x + A_3^1) + (x + A_2^5) = A_5^5 \tag{5}$$

$$W^1 + A_0^2 + (x + A_1^6) + A_2^2 + (x + A_4^6) + (x + A_6^7) = (x + A_5^7). \tag{6}$$

We can use Eqs. (5) and (6) to create 16 subspaces of dimension 3 from the 7-tuples $[W]$, $x+[W]$, $[A_i]$, and $x+[A_i]$ $(0 \le i \le 6)$. In Table 3, the respective components of the 7-tuples in Eq. (5) are listed vertically. Using the recurrence relations defined in Example 1, we see that each set S_i consisting of the ith components of these 7-tuples is a 3-dimensional subspace of $V_7(2)$. Thus Eq. (5) yields 7 subspaces of $V_7(2)$ of dimension 3. Similarly, Eq. (6) yields 7 subspaces of $V_7(2)$ of dimension 3.

So we have a partition $\mathcal{P}^*$ of $V_7(2)$ of type $\langle 1, 16, 0, 0\rangle$ that consists of $\{0, x\} \oplus U$, A_3, A_4, and the 14 subspaces produced by Eqs. (5) and (6).

Note that the set of entries of W^j equals W, and the set of entries of A_i^j equals A_i. So from each of Eqs. (5) and (6), we can also obtain two sets of 7 subspaces of dimension 2 and 1 of the 3-dimensional subspaces A_i in a similar manner. These sets of 2-dimensional subspaces are described by the following equations.

$$(x + A_0^6) + A_1^1 = (x + A_2^5) \quad \text{and} \quad (x + W^3) + (x + A_3^1) = A_5^1 \tag{7}$$

$$A_0^2 + (x + A_1^6) = (x + A_6^7) \quad \text{and} \quad W^1 + (x + A_4^1) = (x + A_5^1). \tag{8}$$

In Examples 3–7, we use this partition $\mathcal{P}^*$ to construct all the desired partitions in categories 5–20 of Table 2. The relevant example number is indicated in Table 2.

Example 3. There is a partition of type $\langle 1, 10, 14, 0\rangle$ (category 10) that consists of $\{0, x\} \oplus U$, A_3, A_4, A_6, the 7 subspaces of dimension 3 produced by Eq. (6), and the 14 subspaces of dimension 2 produced by Eq. (7). In addition, there is a partition of type $\langle 1, 4, 28, 0\rangle$ (category 16) that consists of $\{0, x\} \oplus U$, A_2, A_3, A_4, A_6 and the 28 subspaces of dimension 2 produced by Eqs. (7) and (8). Moreover, because any three of the subspaces A_i are contained in the 6-dimensional space $U \oplus W$, they can be reconfigured into 7 subspaces of dimension 2. Thus these partitions yield partitions of types $\langle 1, 7, 21, 0\rangle$ (category 13) and $\langle 1, 1, 35, 0\rangle$ (category 19). Therefore all of the partitions in categories 4, 10, 13, 16, and 19 of Table 2 can be realized. □

Example 4. Proposition 6.2 and Corollary 6.3 show that partitions of types $\langle 1, 13, 7, 0\rangle$ and $\langle 1, 13, 6, 3\rangle$ do not exist. Thus, in category 7, we must construct a partition of type $\langle 1, 13, 5, 6\rangle$. Such a partition can be obtained from $\mathcal{P}^*$ by reconfiguring the 3-dimensional subspaces S_1, S_2, and S_3 into 5 subspaces of dimension 2 and 6 subspaces of dimension 1. The nonzero vectors in the subspaces of dimension 2 are:

$$\{x + u_6 + w_6, u_6 + w_4, x + w_3\}, \quad \{x + u_5 + w_7, x + u_2 + w_5, u_3 + w_4\},$$
$$\{x + w_1, u_1 + w_2, x + u_1 + w_4\}, \quad \{x + w_2, x + u_7 + w_7, u_7 + w_6\},$$

and

$$\{x + u_1 + w_1, x + u_3 + w_6, u_7 + w_5\}.$$

Applying this reconfiguration to $\mathcal{P}^*$, we obtain a partition of type $\langle 1, 13, 5, 6\rangle$ (category 7). □

Example 5. A partition of type ⟨1, 15, 1, 4⟩ (category 5) can be obtained from $\mathcal{P}^*$ by reconfiguring a subspace of dimension 3 into 1 subspace of dimension 2 and 4 subspaces of dimension 1. The same method can be used to construct a partition in categories 11, 14, 17, and 20 with $j = 0$ from the corresponding partition in the preceding category.

Because there is no partition of type ⟨1, 14, 3, 5⟩, we can use the same technique to produce the necessary partition in category 6. If we reconfigure each of two 3-dimensional subspaces in $\mathcal{P}^*$ into a subspace of dimension 2 and 4 subspaces of dimension 1, we obtain a partition of type ⟨1, 14, 2, 8⟩ (category 6).

This technique cannot be used to produce a partition of type ⟨1, 12, 8, 4⟩ (category 8), however, because there is no partition of type ⟨1, 13, 7, 0⟩. Instead, we reconfigure the 3-dimensional subspaces S_1, S_2, S_3, and S_4 in $\mathcal{P}^*$ into 8 subspaces of dimension 2 and 4 subspaces of dimension 1. The nonzero vectors in the subspaces of dimension 2 are:

$$\begin{array}{lll}
\{x+w_1, u_1+w_7, x+u_1+w_3\}, & & \{u_6+w_5, u_7+w_6, u_2+w_1\},\\
\{x+u_7+w_7, u_7+w_5, x+w_4\}, & & \{x+u_6+w_1, x+u_3+w_6, u_4+w_5\},\\
\{x+u_6+w_6, u_5+w_3, x+u_1+w_4\}, & & \{u_2+w_3, x+u_2+w_5, x+w_2\},\\
\{x+u_1+w_1, u_3+w_4, x+u_7+w_2\}, & \text{and} & \{x+u_2+w_2, x+u_4+w_7, u_1+w_6\}.
\end{array}$$

Thus we obtain a partition of type ⟨1, 12, 8, 4⟩ (category 8). □

Example 6. It is possible to reconfigure the subspaces S_1, S_2, S_3, S_5, and S_6 into 10 subspaces of dimension 2 and 5 subspaces of dimension 1. The nonzero vectors in the subspaces of dimension 2 are:

$$\begin{array}{lll}
\{u_1+w_2, u_6+w_5, u_5+w_3\}, & & \{x+u_7+w_7, u_4+w_3, x+u_5+w_1\},\\
\{x+w_3, x+u_1+w_1, u_1+w_7\}, & & \{x+w_6, x+u_3+w_5, u_3+w_1\},\\
\{u_2+w_3, u_3+w_4, u_5+w_6\}, & & \{x+w_5, x+u_3+w_3, u_3+w_2\},\\
\{x+u_4+w_4, x+u_5+w_7, u_7+w_5\}, & & \{x+w_2, x+u_6+w_6, u_6+w_7\},\\
\{u_7+w_6, x+u_2+w_5, x+u_6+w_1\}, & \text{and} & \{x+w_1, x+u_6+w_2, u_6+w_4\}.
\end{array}$$

Applying this reconfiguration to $\mathcal{P}^*$, we obtain a partition of type ⟨1, 11, 10, 5⟩ (category 9).

In this partition, we can use Eq. (8) to reconfigure 7 subspaces of dimension 3 into 1 subspace of dimension 3 (which is A_2) and 14 subspaces of dimension 2. Applying this to the partition of type ⟨1, 11, 10, 5⟩ described in the preceding paragraph produces a partition of type ⟨1, 5, 24, 5⟩ (category 15). The subspaces of dimension 3 in this partition include A_2, A_3, and A_4. As in Example 3, they can be reconfigured into 7 subspaces of dimension 2 to yield a partition of type ⟨1, 2, 31, 5⟩ (category 18). □

Example 7. The 7 subspaces from Eq. (5) and A_3 can be reconfigured into 17 subspaces of dimension 2 and 5 subspaces of dimension 1. The nonzero vectors in the subspaces of dimension 2 are:

$$\begin{array}{ll}
\{u_4+w_3, x+w_1, x+u_4+w_7\}, & \{u_7+w_6, x+w_2, x+u_7+w_7\},\\
\{u_2+w_5, x+w_3, x+u_2+w_2\}, & \{u_1+w_2, x+w_4, x+u_1+w_1\},\\
\{u_3+w_2, x+w_5, x+u_3+w_3\}, & \{u_3+w_1, x+w_6, x+u_3+w_5\},\\
\{u_6+w_2, x+w_7, x+u_6+w_6\}, & \{u_5+w_1, x+u_4+w_4, x+u_7+w_2\},\\
\{u_7+w_1, x+u_4+w_6, x+u_5+w_5\}, & \{u_2+w_3, u_3+w_4, u_5+w_6\},\\
\{u_1+w_6, u_2+w_1, u_4+w_5\}, & \{u_6+w_7, x+u_1+w_3, x+u_5+w_1\},\\
\{u_4+w_7, u_7+w_5, u_5+w_4\}, & \{u_6+w_5, x+u_1+w_4, x+u_5+w_7\},\\
\{u_5+w_3, x+u_2+w_4, x+u_3+w_6\}, & \{u_2+w_7, x+u_7+w_3, x+u_6+w_1\},
\end{array}$$

and

$$\{u_1+w_4, u_3+w_6, u_7+w_3\}.$$

Applying this reconfiguration to $\mathcal{P}^*$ produces a partition of type ⟨1, 8, 17, 5⟩ (category 12). □

Thus Examples 3–7 produce all the desired partitions in categories 5–20.

6.4. Partitions of $V_7(2)$ containing no subspace of dimension greater than 3

The partitions of $V_7(2)$ that do not contain a subspace of dimension 4 or more are listed in categories 21–38 of Table 2.

By Proposition 6.4, the only realizable partition type in category 21 is ⟨17, 0, 8⟩. A partition of this type can be obtained from $\mathcal{P}^*$ by reconfiguring the subspace of dimension 4 into 1 subspace of dimension 3 and 8 subspaces of dimension 1.

With two exceptions, a partition type in category k ($22 \le k \le 38$) can be obtained from the corresponding partition in category $k - 18$ by reconfiguring the subspace of dimension 4 into 5 subspaces of dimension 2, and then reconfiguring 2-dimensional subspaces into subspaces of dimension 1 as necessary. For example, a partition of type ⟨16, 3, 6⟩ (category 22) can be obtained from $\mathcal{P}^*$ by reconfiguring the subspace of dimension 4 into 5 subspaces of dimension 2 and then reconfiguring 2 subspaces of dimension 2 into 6 subspaces of dimension 1.

The exceptions arise in categories 24 and 25 because there are no partitions of types ⟨1, 14, 3, 5⟩ (category 6) and ⟨1, 13, 7, 3⟩ (category 7). So, to complete our analysis, we must construct partitions of types ⟨14, 8, 5⟩ (category 24) and ⟨13, 12, 0⟩ (category 25).

Example 8. We construct a partition of type $\langle 14, 8, 5\rangle$ (category 24) by reconfiguring the subspaces $\{0, x\} \oplus U$, S_1, and S_2 into 8 subspaces of dimension 2 and 5 subspaces of dimension 1. The nonzero vectors in the subspaces of dimension 2 are:

$$\begin{array}{lll} \{u_3, u_5+w_3, u_2+w_3\}, & & \{x+u_7+w_7, x+u_2+w_5, u_6+w_4\}, \\ \{x+u_1, u_1+w_2, x+w_2\}, & & \{u_6+w_5, x+u_1+w_4, x+u_5+w_7\}, \\ \{u_6, x+w_1, x+u_6+w_1\}, & & \{x+u_2, x+u_6+w_6, u_7+w_6\}, \\ \{x+u_3, x+u_5, u_2\}, & \text{and} & \{x+u_4, x+u_7, u_5\}. \quad \square \end{array}$$

Example 9. We construct a partition of type $\langle 13, 12, 0\rangle$ (category 25) by reconfiguring the subspaces $\{0, x\} \oplus U$, S_1, S_2, S_3, and S_4 into 1 subspace of dimension 3 and 12 subspaces of dimension 2. The nonzero vectors in the subspace of dimension 3 are:

$$\{u_4, u_3, x, u_6, x+u_4, x+u_3, x+u_6\},$$

and the nonzero vectors in the subspaces of dimension 2 are:

$$\begin{array}{lll} \{u_1, x+u_1+w_4, x+w_4\}, & & \{u_1+w_2, u_5+w_3, u_6+w_5\}, \\ \{u_1+w_6, x+u_5, x+u_6+w_6\}, & & \{u_1+w_7, x+u_1+w_3, x+w_1\}, \\ \{u_2, x+u_2+w_2, x+w_2\}, & & \{u_2+w_1, x+u_6+w_1, x+u_7\}, \\ \{u_2+w_3, x+u_2, x+w_3\}, & & \{u_4+w_5, u_5, u_7+w_5\}, \\ \{u_3+w_4, x+u_1+w_1, x+u_7+w_2\}, & & \{u_7, x+u_4+w_7, x+u_5+w_7\}, \\ \{u_7+w_6, x+u_1, x+u_3+w_6\}, & \text{and} & \{u_6+w_4, x+u_2+w_5, x+u_7+w_7\}. \quad \square \end{array}$$

Thus all the partition types in categories 21–38 can be realized except the partition of type $\langle 17, 1, 5\rangle$.

The results in Sections 3, 5 and 6 establish our main theorem.

7. Equivalence of partitions

In this paper, we have concentrated on the type of a partition. Let us call partitions of $V_n(2)$ of the same type, $\mathcal{P} = \{V_1, V_2, \dots, V_k\}$ and $\mathcal{P}'$, *equivalent* if there is a change of basis $T: V_n(2) \to V_n(2)$ such that $\mathcal{P}' = \{T(V_1), T(V_2), \dots, T(V_k)\}$. In conclusion, we note that partitions of the same type need not be equivalent. Using the notation of Section 5, we define partitions $\mathcal{P}$ and $\mathcal{P}'$ of $V_6(2)$ as follows. The nonzero vectors in the subspaces of $\mathcal{P}$ are

$$\begin{array}{lll} V_1 = \{u_1, w_3, u_1+w_3\}, & V_2 = \{u_2, u_3, u_5\}, & V_3 = \{u_4, u_7+w_3, u_5+w_3\}, \\ V_4 = \{u_6, u_3+w_3, u_4+w_3\}, & V_5 = \{u_7, u_2+w_3, u_6+w_3\}, & \{w_1, w_2, w_4\}, \\ \{w_5, u_6+w_6, u_6+w_1\}, & \{w_6, u_3+w_7, u_3+w_2\}, & \{w_7, u_1+w_4, u_1+w_5\}, \\ \{u_1+w_1, u_2+w_4, u_4+w_2\}, & \{u_2+w_2, u_5+w_7, u_3+w_6\}, & \{u_4+w_4, u_2+w_5, u_1+w_7\}, \\ \{u_5+w_5, u_7+w_1, u_4+w_6\}, & \{u_7+w_7, u_6+w_2, u_2+w_6\}, & \{u_1+w_2, u_5+w_6, u_6+w_7\}, \\ \{u_3+w_4, u_5+w_2, u_2+w_1\}, & \{u_4+w_5, u_5+w_1, u_7+w_6\}, & \{u_7+w_2, u_4+w_1, u_5+w_4\}, \\ \{u_3+w_5, u_4+w_7, u_6+w_4\}, & \{u_7+w_4, u_2+w_7, u_6+w_5\}, & \{u_1+w_6, u_3+w_1, u_7+w_5\}, \end{array}$$

and the nonzero vectors in the subspaces of $\mathcal{P}'$ are

$$\begin{array}{lll} \{u_1, w_1, u_1+w_1\}, & \{u_2, w_2, u_2+w_2\}, & \{u_3, w_3, u_3+w_3\}, \\ \{u_4, w_4, u_4+w_4\}, & \{u_5, w_5, u_5+w_5\}, & \{u_6, w_6, u_6+w_6\}, \\ \{u_7, w_7, u_7+w_7\}, & \{u_1+w_2, u_4+w_6, u_2+w_7\}, & \{u_2+w_3, u_6+w_7, u_7+w_1\}, \\ \{u_3+w_4, u_2+w_6, u_5+w_3\}, & \{u_4+w_5, u_7+w_3, u_5+w_2\}, & \{u_5+w_6, u_1+w_4, u_6+w_3\}, \\ \{u_1+w_3, u_5+w_7, u_6+w_1\}, & \{u_2+w_4, u_4+w_7, u_1+w_5\}, & \{u_3+w_5, u_6+w_2, u_4+w_3\}, \\ \{u_7+w_2, u_6+w_4, u_2+w_1\}, & \{u_2+w_5, u_3+w_7, u_5+w_4\}, & \{u_3+w_6, u_4+w_1, u_6+w_5\}, \\ \{u_5+w_1, u_7+w_4, u_4+w_2\}, & \{u_1+w_6, u_3+w_1, u_7+w_5\}, & \{u_1+w_7, u_3+w_2, u_7+w_6\}. \end{array}$$

Both $\mathcal{P}$ and $\mathcal{P}'$ are partitions of type $\langle 21, 0\rangle$. It is easily checked that the subspaces $V_1, V_2, \dots, V_5$ in $\mathcal{P}$ can be reconfigured into a 4-dimensional subspace. A more tedious calculation shows that the span of no 2 subspaces in $\mathcal{P}'$ contains 5 subspaces in $\mathcal{P}'$, and so no 5 subspaces in $\mathcal{P}'$ can be reconfigured into a 4-dimensional subspace. Thus $\mathcal{P}$ and $\mathcal{P}'$ are not equivalent.

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