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Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads

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For Alex Heller on his 65th birthday

Abstract

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A right adjoint functor is said to be of *descent type* if the counit of the adjunction is pointwise a coequalizer. Building on the results of Tholen's doctoral thesis, we give necessary and sufficient conditions for a composite to be of descent type when each factor is so. We apply this to show that every finitary monad on a locally-finitely-presentable enriched category $\mathscr{A}$ admits a presentation in terms of basic operations and equations between derived operations, the arties here being the finitely-presentable objects of $\mathscr{A}$.

1. Introduction

Our primary goal is to show that—in the context of enriched category theory—every finitary monad on a locally finitely presentable category $\mathscr{A}$ admits a *presentation* in terms of $\mathscr{A}$-objects Bc of 'basic operations of arity c' (where c runs through the finitely-presentable objects of $\mathscr{A}$) and $\mathscr{A}$-objects Ec of 'equations of arity c' between derived operations. We shall explain in greater detail these ideas on monads in a paper in preparation to which the present paper is preliminary; here, in Sections 4 and 5, we say only enough about these ideas to exhibit our problem as that of showing the counit of a certain adjunction to be a coequalizer.

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We do show this in Section 5, using sufficient conditions given in Section 3 for the composite of two adjunctions to have a coequalizer for its counit when each factor does so. Even for ordinary (unenriched) categories $\mathscr{A}$, the result on presentations of monads would seem to be new except when $\mathscr{A} = \mathbf{Set}$.

The core of the present article, therefore, is a study—now in the classical context of ordinary category theory—of adjunctions whose counits are coequalizers. Certain facts about these are well known, apparently going back to unpublished work of Beck from some 25 years ago. Given an adjunction

$$\eta, \varepsilon : F \dashv U : \mathscr{A} \to \mathscr{B} , \tag{1.1}$$

write $T = (UF, \eta, U\varepsilon F)$ for the associated monad on $\mathscr{B}$, write $\mathscr{B}^T$ for the Eilenberg–Moore category of T-algebras, and write $K : \mathscr{A} \to \mathscr{B}^T$ for the comparison functor. Barr and Wells prove in [1, p. 111, Corollary 7 and Theorem 9] the equivalence of

(a) each component $\varepsilon A : FUA \to A$ of the counit ε is a coequalizer;

(b) for each A we have a coequalizer diagram

$$FUFUA \underset{}{\overset{\varepsilon FUA}{\rightrightarrows}} FUA \xrightarrow[\varepsilon A]{} A ; \tag{1.2}$$

(c) $K : \mathscr{A} \to \mathscr{B}^T$ is fully faithful;

they say (see [1, p. 102]) that a functor U is *of descent type* when it has a left adjoint and these equivalent conditions are satisfied. Observe that, in this case, each $A \in \mathscr{A}$ not only has *some* presentation as a coequalizer of two maps $f, g : FX \to FY$ between 'free' objects, but in fact has the *canonical* such presentation (1.2).

Much more about adjunctions with these properties is contained in Tholen's 1974 doctoral thesis [15] (wherein a U of descent type is said to be *premonadic*; see his p. 8). Tholen's Proposition 10.1 adds six more equivalent conditions to the three above; his thesis, although distributed, being unpublished, we take the liberty of re-proving in Section 2 below the equivalence of his nine conditions, augmenting them by yet another four (of lesser importance).

For the sake of expository tidiness we recall some elementary facts about adjunctions that are not all essential to our arguments. It is well known that, for an adjunction (1.1), U is faithful precisely when each εA is epimorphic, and is fully faithful precisely when each εA is invertible. It is perhaps less well known, but very easy to prove, that U is faithful and conservative—a *conservative* functor being one that reflects isomorphisms—precisely when each εA is an extremal epimorphism; that is, an epimorphism that factorizes through no proper subobject of its codomain. (Under mild conditions on $\mathscr{A}$—the existence of equalizers, or of both pullbacks and pushouts, or of coequalizers and arbitrary cointersections of epimorphisms—every morphism that factorizes through no proper subobject of its codomain is—see [7]—an epimorphism; in such cases U is automatically faithful if conservative.)

A coequalizer being clearly an extremal epimorphism, a functor U of descent type is necessarily faithful and conservative. To see that the converse is false, take $U : \mathbf{Cat} \to \mathbf{Set}$ to be the functor sending a small category to its set of morphisms; it is easily verified that εA is not a coequalizer when A is the monoid generated by e with $e^2 = e$. On the other hand, it follows from (c) above that a monadic functor U is of descent type; it is in fact classical that (1.2) is a coequalizer diagram when U is monadic. In particular, it is a coequalizer diagram when U is the forgetful functor from abelian groups to sets. Accordingly it remains a coequalizer diagram when U is the forgetful functor from torsion-free abelian groups to sets; this, therefore, is an example of a non-monadic U of descent type.

Clearly, if a composite $U = VW$ is faithful or conservative, so is W. In fact (see [15, Korollar 10.4] or Proposition 3.1 below), if VW is of descent type so is W, provided that it has a left adjoint. This 'cancellation' result is not matched by a 'composition' result: while VW is of course faithful or conservative when both V and W are so, it may fail to be of descent type even when both V and W are monadic; for, as MacDonald and Stone point out in [13, Section 3], the Applegate–Tierney tower decomposition of an adjunction can have any length. The only positive result we have seen in this direction is that of Tholen [15, Korollar 10.4]: to wit, $VW : \mathcal{A} \to \mathcal{B}$ is of descent type when V and W are so if every composite of coequalizers in $\mathcal{A}$ is again a coequalizer. This condition on $\mathcal{A}$, however—equivalent by [8] when $\mathcal{A}$ is finitely complete to the requirement that every extremal epimorphism in $\mathcal{A}$ be a coequalizer—is far too restrictive for our purposes, failing already when $\mathcal{A} = \mathbf{Cat}$.

Our contribution to this matter is the giving in Section 3 below of various sufficient conditions, based upon Tholen's results and our generalizations of them, for VW to be of descent type when V and W are so; the chief of these is Theorem 3.2, which we use in Section 5 to prove that monads do admit presentations of the desired kind; the others constitute a congeries of results, perhaps worth recording, that emerged during our investigations.

2. Conditions equivalent to being of descent type

We first recall some definitions and a few classical things that we need to refer to.

Because the results of this section involve no completeness hypotheses of any kind on $\mathcal{A}$, we use *regular epimorphism* not, as Barr and Wells do in [1], to mean a coequalizer (of some parallel pair), but in the weaker sense of [8]. Thus $f : A \to B$ is a regular epimorphism if it is the joint coequalizer of all those parallel pairs x, y with codomain A for which $fx = fy$; equivalently, if it is the joint coequalizer of *some* family (perhaps large) $x_i, y_i : C_i \to A$ of parallel pairs. Of course any coequalizer is a regular epimorphism, and the converse is true if $\mathcal{A}$ admits kernel-pairs.

For any functor $U : \mathcal{A} \to \mathcal{B}$, Tholen [15, p. 13] defines the notion of a *U-final*

morphism in $\mathscr{A}$; it is what many authors would call instead a *cocartesian arrow* for U. We need the notion only for a faithful U, and then the definition simplifies: a morphism $f : A \to B$ in $\mathscr{A}$ is said to be U-final if, whenever $t : UB \to UC$ is such that $t.Uf = Ug$ for some $g : A \to C$, we have $t = Us$ for some $s : B \to C$. As is shown by the example where U is the forgetful functor from topological spaces to sets, there is no need for Uf (much less f) to be epimorphic when f is U-final; however we have the following, which is part of [15, Lemma 9.5]:

Lemma 2.1. *For a faithful U, a regular epimorphism f is U-final if Uf is epimorphic.*

Proof. If $t.Uf = Ug$ as above, $fx = fy$ implies $Ug.Ux = Ug.Uy$ and hence $gx = gy$; because f is a regular epimorphism, this gives $g = sf$ for some s; thus $t.Uf = Ug = Us.Uf$, whence $t = Us$ since Uf is epimorphic. □

In the next section we use the following trivial results:

Lemma 2.2. *Let U be the composite VW where V and W are faithful. Then* (a) *if f is U-final it is W-final*; *and* (b) *if f is W-final and Wf is V-final then f is U-final.* □

We follow Mac Lane [14] in calling a diagram

$$C \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow[f]{} B \tag{2.1}$$

in $\mathscr{A}$ a *fork* if $fg = fh$. Applying a functor $U : \mathscr{A} \to \mathscr{B}$ to (2.1) gives a fork

$$UC \overset{Ug}{\underset{Uh}{\rightrightarrows}} UA \xrightarrow[Uf]{} UB \tag{2.2}$$

in $\mathscr{B}$. This fork is said to be *split* if we have $i : UB \to UA$ and $j : UA \to UC$ with

$$Uf.i = 1\ , \quad Ug.j = 1\ , \quad Uh.j = i.Uf\ ; \tag{2.3}$$

in which case (2.2) is a coequalizer diagram—in fact an *absolute coequalizer diagram*, since its image under any functor is again a split fork. The following is a strengthening of [15, Lemma 10.2]:

Lemma 2.3. *Given the adjunction* (1.1) *and any $f : A \to B$ for which Uf has a right inverse i, there is a fork* (2.1) *for which* (2.2) *is a split fork.*

Proof. Set $g = \varepsilon A : FUA \to A$ and set $h = \varepsilon A.Fi.FUf : FUA \to A$. By the naturality of ε we have $f.\varepsilon A = \varepsilon B.FUf$, so that

$$fh = \varepsilon B.FUf.Fi.FUf = \varepsilon B.FUf = f.\varepsilon A = fg\ ,$$

showing (2.1) to be a fork. A splitting of (2.2) is given by i and by $j = \eta UA : UA \to UFUA$; for $Ug.j = U\varepsilon A.\eta UA = 1$ by one of the triangular equations, while by the naturality of η we have

$$Uh.j = U\varepsilon A.UFi.UFUf.\eta UA = U\varepsilon A.\eta UA.i.UF = i.Uf\,,$$

as desired. □

For any adjunction (1.1) and any A in $\mathscr{A}$, the diagram (1.2) is a fork by the naturality of ε. Recall from [14, p. 139] that its image

$$UFUFUA \underset{UFU\varepsilon A}{\overset{U\varepsilon FUA}{\rightrightarrows}} UFUA \xrightarrow[U\varepsilon A]{} UA \tag{2.4}$$

is a split fork, a splitting being given by ηUA and $\eta UFUA$.

Of the thirteen conditions in the following theorem, the equivalence of all but (ii), (vii), (viii), and (ix) is given in Tholen's [15, Proposition 10.1]. The comparison functor mentioned in (i) is of course that of Section 1 above.

Theorem 2.4. *For an adjunction $\eta,\varepsilon : F \dashv U : \mathscr{A} \to \mathscr{B}$, the following conditions are equivalent:*

(i) *the comparison functor $K : \mathscr{A} \to \mathscr{B}^T$ is fully faithful;*

(ii) (2.1) *is a coequalizer diagram if* (2.2) *is a coequalizer diagram with $UFUf$ epimorphic;*

(iii) (2.1) *is a coequalizer diagram if* (2.2) *is a coequalizer diagram with Uf a retraction;*

(iv) *each εA is a coequalizer;*

(v) *each εA is a regular epimorphism;*

(vi) *each εA is the coequalizer of εFUA and $FU\varepsilon A$ in* (1.2);

(vii) *given any functor $P : \mathscr{J} \to \mathscr{A}$ and any inductive cone $\alpha = (\alpha J : PJ \to B)$, if the cone $U\alpha$ is an absolute colimit cone, then α is a colimit cone;*

(viii) *if* (2.1) *is a fork and* (2.2) *an absolute coequalizer diagram, then* (2.1) *is a coequalizer diagram;*

(ix) *if* (2.1) *is a fork and* (2.2) *is a split fork, then* (2.1) *is a coequalizer diagram;*

(x) *whenever Uf is a retraction, f is a coequalizer;*

(xi) *whenever Uf is a retraction, f is a regular epimorphism;*

(xii) *U is faithful, and whenever Uf is a retraction, f is U-final;*

(xiii) *U is faithful, and each εA is U-final.*

Proof. The implications (ii)$\Rightarrow$(iii), (iv)$\Rightarrow$(v), (vii)$\Rightarrow$(viii)$\Rightarrow$(ix), and (x)$\Rightarrow$(xi) are trivial. Because $U\varepsilon A$ in (2.4) is a retraction, the implications (iii)$\Rightarrow$(iv) and (xii)$\Rightarrow$(xiii) are immediate; as is the fact that (xi) gives εA epimorphic and hence U faithful—so that (xi)$\Rightarrow$(xii) by Lemma 2.1. Moreover, (ix)$\Rightarrow$(x) by Lemma 2.3. It only remains to prove (i)$\Rightarrow$(ii), (v)$\Rightarrow$(vi), (vi)$\Rightarrow$(vii), and (xiii)$\Rightarrow$(i).

(i)$\Rightarrow$(ii) We use the notation of Mac Lane in [14, Chapter 6]; U^T ; $\mathcal{B}^T \to \mathcal{B}$ is the forgetful functor from the category of algebras, and F^T is its standard left adjoint; as usual, the same letter denotes both an algebra in $\mathcal{B}^T$ and its underlying object in $\mathcal{B}$, and similarly for maps. Because the fully-faithful K reflects colimits, while $U^TK = U$ and $U^TF^TU^TK = TU^TK = UFU$, it suffices to verify that (ii) holds for the adjunction $F^T \dashv U^T : \mathcal{B}^T \to \mathcal{B}$. Suppose then that (2.1) is a fork in $\mathcal{B}^T$ which, seen as a fork in $\mathcal{B}$, is a coequalizer diagram with $U^TF^TU^Tf$ (or TU^Tf, which we write simply as Tf) an epimorphism in $\mathcal{B}$. Certainly any $k : A \to D$ in $\mathcal{B}^T$ with $kg = kh$ is, as a map in $\mathcal{B}$, of the form $k = tf$ for a unique $t : B \to D$ in $\mathcal{B}$; it remains only to show that t is a map of algebras. Writing a, b, d for the actions of T on A, B, D we have, since f and $k = tf$ are algebra maps, commutativity of the exterior and of the left square in the diagram

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[r, "Tt"] \arrow[d, "b"'] & TD \arrow[d, "d"] \\ A \arrow[r, "f"'] & B \arrow[r, "t"'] & D; \end{tikzcd}$$

the desired commutativity of the right square follows because Tf is epimorphic in $\mathcal{B}$.

(v)$\Rightarrow$(vi) The regular epimorphism εA is the joint coequalizer of some family $g_i, h_i : C_i \to FUA$ of parallel pairs. Suppose that $k : FUA \to D$ has $k.\varepsilon FUA = k.FU\varepsilon A$. This gives, since $\varepsilon A.g_i = \varepsilon A.h_i$ and hence $FU\varepsilon A.FUg_i = FU\varepsilon A.FUh_i$, the equation $k.\varepsilon FUA.FUg_i = k.\varepsilon FUA.FUh_i$, which by the naturality of ε is equally $kg_i.\varepsilon C_i = kh_i.\varepsilon C_i$. Since εC_i is epimorphic, we have $kg_i = kh_i$, so that $k = s.\varepsilon A$ for a unique s, as desired.

(vi)$\Rightarrow$(vii) Let $\beta = (\beta J : PJ \to D)$ be an inductive cone over P. Since $U\alpha$ is a colimit cone, we have $U\beta = t.U\alpha$ for some $t : UB \to UD$; and if t corresponds under the adjunction to $r : FUB \to D$, the equation $U\beta = t.U\alpha$ translates into $r.FU\alpha = \beta.\varepsilon P$. Using this twice and naturality three times, we have

$$\begin{aligned} r.\varepsilon FUB.FUFU\alpha &= r.FU\alpha.\varepsilon FUP = \beta.\varepsilon P.\varepsilon FUP = \beta.\varepsilon P.FU\varepsilon P \\ &= r.FU\alpha.FU\varepsilon P = r.FU\varepsilon B.FUFU\alpha\ . \end{aligned}$$

We deduce that $r.\varepsilon FUB = r.FU\varepsilon B$ since, $U\alpha$ being an absolute colimit cone, $FUFU\alpha$ is a colimit cone. By (vi), therefore, $r = s.\varepsilon B$ for some $s : B \to D$; whence $\beta.\varepsilon P = r.FU\alpha = s.\varepsilon B.FU\alpha = s\alpha.\varepsilon P$, giving $\beta = s\alpha$ since εP is (pointwise) epimorphic. As for the uniqueness of s with $\beta = s\alpha$, if $s\alpha = s'\alpha$ we have $Us.U\alpha = Us'.U\alpha$, giving $Us = Us'$ since $U\alpha$ is a colimit cone, and hence $s = s'$ since, each εA being epimorphic, U is faithful.

(xiii)$\Rightarrow$(i) K being faithful because $U = U^TK$ is so, it remains to prove K full. Since (see [14, p. 139]) KA is the object UA of $\mathcal{B}$ with the action $U\varepsilon A$, a map

$t : KA \to KB$ in $\mathscr{B}^T$ is just a map $t : UA \to UB$ in $\mathscr{B}$ satisfying $t.U\varepsilon A = U\varepsilon B.UFt$; because εA is U-final, any such t in $\mathscr{B}$ is Us for some $s : A \to B$ in $\mathscr{A}$; and then $Ks = t$ in $\mathscr{B}^T$. □

3. Cancellation and composition results

We consider a composite $U = VW : \mathscr{A} \to \mathscr{B}$ where $W : \mathscr{A} \to \mathscr{C}$ and $V : \mathscr{C} \to \mathscr{B}$. Condition (xi) of Theorem 2.4 gives an immediate cancellation result:

Proposition 3.1. *If $U = VW$ is of descent type so is W, provided that W has a left adjoint.* □

Note that, even if V has a left adjoint, this is no trivial consequence of the fact that adjunctions $\rho,\sigma : H \dashv W$ and $\alpha,\beta : G \dashv V$ give an adjunction $\eta,\varepsilon : HG \dashv VW$ with counit

$$HGVW \xrightarrow[H\beta W]{} HW \xrightarrow[\sigma]{} 1\,; \tag{3.1}$$

for (see [8]) f need not be a regular epimorphism when fg is so, unless g is known to be epimorphic.

The following is our chief composition result:

Theorem 3.2. *If $U = VW$ has a left adjoint as in* (1.1) *and V is of descent type, then U is of descent type if and only if W is faithful and each εA is W-final.*

Proof. If U is of descent type, W is faithful because $U = VW$ is so, and each εA is W-final by Lemma 2.2(a) and (xiii) of Theorem 2.4. For the converse, V being of descent type and $VW\varepsilon A = U\varepsilon A$ having the right inverse ηUA, the morphism $W\varepsilon A$ is V-final by (xii) of Theorem 2.4; thus εA is U-final by Lemma 2.2(b), and U is of descent type by (xiii) of Theorem 2.4. □

Observe that, in most practical applications of this result, W too is of descent type by Proposition 3.1; for W has a left adjoint, by [3] or [6], when VW does so and V is of descent type, provided that $\mathscr{A}$ admits coequalizers.

Since very morphism is trivially W-final when W is fully faithful, we have the following corollary:

Corollary 3.3. *If V is of descent type and W is fully faithful, VW is of descent type if it has a left adjoint.* □

For some further, less central, composition results we use the following lemma:

Lemma 3.4. *If the adjunction $\eta,\varepsilon : F \dashv U = VW$ is the composite of adjunctions $\rho,\sigma : H \dashv W$ and $\alpha,\beta : G \dashv V$, we have commutativity in*

$$\begin{tikzcd} WHGVW \arrow[r, "W\varepsilon"] & W \\ GVW \arrow[u, "\rho GVW"] \arrow[ur, "\beta W"'] & \end{tikzcd} \tag{3.2}$$

Proof. In the following diagram, the square commutes by naturality and the triangle by an adjunction equation, while the top edge is $W\varepsilon$ by (3.1):

$$\begin{tikzcd} WHGVW \arrow[r, "WH\beta W"] & WHW \arrow[r, "W\sigma"] \arrow[d, "\rho W"] & W \\ GVW \arrow[u, "\rho GVW"] \arrow[r, "\beta W"'] & W \arrow[ur, "1"'] & \end{tikzcd}$$ □

When V here is of descent type, so that each βWA is a coequalizer, it follows from (3.2) that each $W\varepsilon A$ is an epimorphism. Applying the left adjoint H we see that each $H\beta WA$ is a coequalizer and each $HW\varepsilon A$ is an epimorphism; compare these with conditions (ii) and (iii) of the following proposition:

Proposition 3.5. *When V and W are of descent type, each of the following—with the notation of Lemma 3.4—implies the next:*
(i) *W sends coequalizers to epimorphisms;*
(ii) *each $WH\beta WA$ is an epimorphism;*
(iii) *each $WHW\varepsilon A$ is an epimorphism;*
(iv) *$U = VW$ is of descent type.*

Proof. (i) implies (ii) by the remarks preceding the proposition, while (ii) implies (iii) by (3.2). Suppose now that (iii) holds. Recall that the image under $U = VW$ of (1.2) is the split coequalizer diagram (2.4). Because V is of descent type, the image under W of (1.2) is a coequalizer diagram by (iii) of Theorem 2.4. Because W is of descent type and $WHW\varepsilon A$ is epimorphic by hypothesis (iii) above, (2.4) is a coequalizer diagram by (ii) of Theorem 2.4. Thus U is of descent type by (vi) of Theorem 2.4. □

Our final composition result involves, not preservation properties of W as in Proposition 3.5, but reflexion properties. For comparison, recall from [7] that a right adjoint reflects strong epimorphisms (which coincide with the extremal ones in the presence of finite limits) if and only if its counit is (pointwise) a strong epimorphism; and observe that the condition below on W is a strengthening of (xi) of Theorem 2.4.

Proposition 3.6. *If $U = VW$ has a left adjoint and V is of descent type, then U is of descent type if U reflects regular epimorphisms—indeed, if f is a regular epimorphism whenever Wf is a coequalizer.*

Proof. Let $Uf = VWf$ be a retraction. Then, since V is of descent type, Wf is a coequalizer by (x) of Theorem 2.4, so that f is a regular epimorphism by hypothesis. Thus U is of descent type by (xi) of Theorem 2.4. □

4. Finitary enriched monads as algebras for a finitary monad

We now consider a symmetric monoidal closed category $\mathcal{V}$ that is, in the sense of Kelly [11], *locally finitely presentable as a closed category*; equivalently (see [11, Section 5]) the underlying ordinary category $\mathcal{V}_0$ of $\mathcal{V}$ is locally finitely presentable in the classical sense of Gabriel and Ulmer [5], the tensor product $x \otimes y$ of finitely presentable objects x and y of $\mathcal{V}_0$ is again finitely presentable, and the unit object I for the tensor product is finitely presentable in $\mathcal{V}_0$; examples from [11] of such closed categories are those of sets, pointed sets, abelian groups, R-modules for a commutative ring R, graded R-modules, differential graded R-modules, graphs, (small) categories, groupoids, preordered sets, and ordered sets—a non-example is the symmetric monoidal closed category of Banach spaces.

When we speak of a *functor* $T : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, we mean of course a $\mathcal{V}$-functor—otherwise we should have spoken of a functor $\mathcal{A}_0 \to \mathcal{B}_0$ between the underlying ordinary categories; similarly, by a *natural transformation* $\alpha : T \to S : \mathcal{A} \to \mathcal{B}$, we mean a $\mathcal{V}$-natural one. Such functors and natural transformations form an ordinary category $(\mathcal{A}, \mathcal{B})$; when $\mathcal{A}$ is small, this is (see [10, Section 2.2]) the underlying category $[\mathcal{A}, \mathcal{B}]_0$ of a $\mathcal{V}$-category $[\mathcal{A}, \mathcal{B}]$. Recall from [11, Section 1] that, when $\mathcal{A}$ admits filtered colimits, a functor $T : \mathcal{A} \to \mathcal{B}$ is said to be *finitary* when it (or equivalently its underlying ordinary functor $T_0 : \mathcal{A}_0 \to \mathcal{B}_0$) preserves these; we write $\mathrm{Fin}(\mathcal{A}, \mathcal{B})$ for the full subcategory of $(\mathcal{A}, \mathcal{B})$ determined by the finitary functors. Recall further from [11, Section 2] that the object c of $\mathcal{A}$ is said to be *finitely presentable* when the representable $\mathcal{A}(c, -) : \mathcal{A} \to \mathcal{V}$ is finitary, and that $\mathcal{A}_f$ denotes the full subcategory of $\mathcal{A}$ determined by the finitely presentable objects. Recall finally from [11, Section 3] that $\mathcal{A}$ is said to be *locally finitely presentable* (lfp) when it is cocomplete and has a small strongly-generating subcategory contained in $\mathcal{A}_f$; equivalently, by [11, Corollary 7.3], when $\mathcal{A}$ is cocomplete and $\mathcal{A}_f$ is small and dense in $\mathcal{A}$. This clearly agrees, in the case $\mathcal{V} = \mathbf{Set}$ of locally-small ordinary categories, with the classical Gabriel–Ulmer notion of lfp category; moreover, by [11, Proposition 7.5], $\mathcal{A}_0$ is lfp when $\mathcal{A}$ is so, and has the same finitely-presentable objects.

Examples from [11] of lfp $\mathcal{V}$-categories are $\mathcal{V}$ itself, the functor-$\mathcal{V}$-category

$[\mathcal{T}, \mathcal{V}]$ for any small $\mathcal{T}$, and the full subcategory $\mathrm{Lex}[\mathcal{T}, \mathcal{V}]$ of this when $\mathcal{T}$ is finitely complete in the appropriate sense; indeed every lfp $\mathcal{A}$ is (as in the classical set-based case) an instance of this last example, being equivalent by [11, Theorem 7.2] to $\mathrm{Lex}[\mathcal{A}_{\mathrm{f}}^{\mathrm{op}}, \mathcal{V}]$.

Central to our considerations are the results of [11, Proposition 7.6]: for lfp $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a functor $\mathcal{A} \to \mathcal{B}$ is finitary precisely when it is the left Kan extension of its restriction to $\mathcal{A}_{\mathrm{f}}$—indeed restriction along the inclusion $J : \mathcal{A}_{\mathrm{f}} \to \mathcal{A}$ induces an equivalence $\mathrm{Fin}(\mathcal{A}, \mathcal{B}) \to (\mathcal{A}_{\mathrm{f}}, \mathcal{B})$ of categories, an inverse of which sends $T : \mathcal{A}_{\mathrm{f}} \to \mathcal{B}$ to its left Kan extension $\mathrm{Lan}_J T$. (In fact, the proposition in question speaks of an equivalence $\mathrm{Fin}[\mathcal{A}, \mathcal{B}] \simeq [\mathcal{A}_{\mathrm{f}}, \mathcal{B}]$ of $\mathcal{V}$-categories; if we de-emphasize this higher level here, it is because we are going to consider shortly the monoids in $\mathrm{Fin}(\mathcal{A}, \mathcal{A})$, and the monoids in a $\mathcal{V}$-category do not form a $\mathcal{V}$-category—just as rings do not form an additive category.) At any rate, we have the conclusion that, since $[\mathcal{A}_{\mathrm{f}}, \mathcal{B}]$ is an lfp $\mathcal{V}$-category by (3.4) of [11], the ordinary category $\mathrm{Fin}(\mathcal{A}, \mathcal{B}) \simeq (\mathcal{A}_{\mathrm{f}}, \mathcal{B}) = [\mathcal{A}_{\mathrm{f}}, \mathcal{B}]_0$ is also lfp. If, for $T : \mathcal{A}_{\mathrm{f}} \to \mathcal{B}$ and $A \in \mathcal{A}$, we write $T \mathbin{\square} A$ for $(\mathrm{Lan}_J T)A$, the usual coend formula for the left Kan extension gives

$$T \mathbin{\square} A = (\mathrm{Lan}_J T)A = \int^{c \in \mathcal{A}_{\mathrm{f}}} \mathcal{A}(c, A) \otimes Tc\,, \tag{4.1}$$

the integrand here being the tensor product in $\mathcal{B}$ of $\mathcal{A}(c, A) \in \mathcal{V}$ and $Tc \in \mathcal{B}$. In fact, of course, we have here a $\mathcal{V}$-functor $\square : [\mathcal{A}_{\mathrm{f}}, \mathcal{B}] \otimes \mathcal{A} \to \mathcal{B}$; and a simple calculation shows $- \mathbin{\square} A$ to have a right adjoint, giving

$$\mathcal{B}(T \mathbin{\square} A, B) \cong [\mathcal{A}_{\mathrm{f}}, \mathcal{B}](T, \langle A, B\rangle)\,, \tag{4.2}$$

where $\langle A, B\rangle c$ is the cotensor product

$$\langle A, B\rangle c = \mathcal{A}(c, A) \pitchfork B\,. \tag{4.3}$$

We now fix on an lfp $\mathcal{V}$-category $\mathcal{A}$ and apply the last paragraph to the case $\mathcal{B} = \mathcal{A}$. The lfp category $\mathrm{Fin}(\mathcal{A}, \mathcal{A})$ of finitary endofunctors of $\mathcal{A}$ has a (non-symmetric) monoidal structure whose tensor product is composition; under the equivalence this translates into a monoidal structure on $(\mathcal{A}_{\mathrm{f}}, \mathcal{A})$ whose tensor product $T \circ S$ is $(\mathrm{Lan}_J T)(\mathrm{Lan}_J S)J = (\mathrm{Lan}_J T)S$, so that, using (4.1), we have

$$(T \circ S)d = T \mathbin{\square} Sd = \int^{c \in \mathcal{A}_{\mathrm{f}}} \mathcal{A}(c, Sd) \otimes Tc\,; \tag{4.4}$$

the unit object for this tensor product is of course the restriction $J \in (\mathcal{A}_{\mathrm{f}}, \mathcal{A})$ of $1 : \mathcal{A} \to \mathcal{A}$. Clearly $\square$, being itself the transform under the equivalence of 'evalua-

tion', is an *action* on $\mathscr{A}_0$ of monoidal $(\mathscr{A}_f, \mathscr{A})$—indeed an action on $\mathscr{A}$ of the monoidal $\mathscr{V}$-category $[\mathscr{A}_f, \mathscr{A}]$; that is, we have coherent $\mathscr{V}$-natural isomorphisms $(T \circ S) \square A \cong T \square (S \square A)$ and $J \square A \cong A$. The monoidal structure on $(\mathscr{A}_f, \mathscr{A})$ is closed on one side, (4.4) and (4.2) giving

$$[\mathscr{A}_f, \mathscr{A}](T \circ S, R) \cong [\mathscr{A}_f, \mathscr{A}](T, \{S, R\}) , \tag{4.5}$$

where, by (4.3),

$$\{S, R\}c = \int_{d \in \mathscr{A}_f} \mathscr{A}(c, Sd) \pitchfork Rd ; \tag{4.6}$$

we need below only the weaker version of (4.5) with $[\mathscr{A}_f, \mathscr{A}]$ replaced by $(\mathscr{A}_f, \mathscr{A})$. Unlike $- \circ S$, the endofunctor $T \circ -$ of $(\mathscr{A}_f, \mathscr{A})$ lacks a right adjoint in general; when $\mathscr{A} = \mathscr{V} = \mathbf{Set}$ and T is the functor constant at 1, it does not even preserve the initial object. Note, however, that $T \circ -$ is finitary—by (4.4), if you like, colimits, in $(\mathscr{A}_f, \mathscr{A})$ being formed pointwise; or equally by what $T \circ -$ means in terms of $\mathrm{Fin}(\mathscr{A}, \mathscr{A})$, in which colimits are again formed pointwise.

It is convenient to regard the equivalence above as *identifying* $\mathrm{Fin}(\mathscr{A}, \mathscr{A})$ with $(\mathscr{A}_f, \mathscr{A})$. Then the category of *finitary monads* on $\mathscr{A}$—those monads (T, i, m) whose endofunctor-part T is finitary—as the category of monoids in the monoidal category $\mathrm{Fin}(\mathscr{A}, \mathscr{A})$, is identified with the category $\mathrm{Mon}(\mathscr{A}_f, \mathscr{A})$ of $\circ$-monoids in $(\mathscr{A}_f, \mathscr{A})$. Since each $- \circ S$ preserves all colimits, and since each $T \circ -$ is finitary, it follows from [9, Theorem 23.3] that the forgetful functor $W : \mathrm{Mon}(\mathscr{A}_f, \mathscr{A}) \to (\mathscr{A}_f, \mathscr{A})$ has a left adjoint H, which by (23.2) of [9] sends K to $HK = S$ given inductively by

$$S_0 = J , \quad S_{n+1} = J + K \circ S_n , \quad S = \operatorname*{colim}_{n<\omega} S_n , \tag{4.7}$$

with an evident monoid-structure. In fact, W is monadic: if $g,h : P \to Q$ in $\mathrm{Mon}(\mathscr{A}_f, \mathscr{A})$ have an absolute coequalizer $f : Q \to R$ in $(\mathscr{A}_f, \mathscr{A})$, we have the coequalizer $f \circ f : Q \circ Q \to R \circ R$ in $(\mathscr{A}_f, \mathscr{A})$ of $g \circ g$ and $h \circ h$, giving an induced $m : R \circ R \to R$; similarly we have an induced $i : J \to R$, turning R into a monoid which is the coequalizer in $\mathrm{Mon}(\mathscr{A}_f, \mathscr{A})$ of g and h; whence the monadicity of W follows by the Beck–Paré theorem given on p. 147 of [14]. Moreover, W is finitary; in fact it not only preserves, but creates, filtered colimits. To see this, let $\mathscr{K}$ be filtered and suppose a functor $\mathscr{K} \to \mathrm{Mon}(\mathscr{A}_f, \mathscr{A})$ sending α to T_α to be such that we have a colimit $T_\alpha \to T$ in $(\mathscr{A}_f, \mathscr{A})$; because each $P \circ -$ and each $- \circ Q$ preserves filtered colimits, the functor $\mathscr{K} \times \mathscr{K} \to (\mathscr{A}_f, \mathscr{A})$ sending (α, β) to $T_\alpha \circ T_\beta$ has the colimit $T_\alpha \circ T_\beta \to T \circ T$; the diagonal $\mathscr{K} \to \mathscr{K} \times \mathscr{K}$ being final because $\mathscr{K}$ is filtered, the functor $\mathscr{K} \to (\mathscr{A}_f, \mathscr{A})$ sending α to $T_\alpha \circ T_\alpha$ has the colimit $T_\alpha \circ T_\alpha \to T \circ T$; thus we get an induced $m : T \circ T \to T$, and similarly an induced $i : J \to T$, making T into a monoid with $T_\alpha \to T$ a colimit in $\mathrm{Mon}(\mathscr{A}_f, \mathscr{A})$. Accordingly

$\mathrm{Mon}(\mathscr{A}_f, \mathscr{A})$ is isomorphic to the Eilenberg–Moore category $(\mathscr{A}_f, \mathscr{A})^M$ of M-algebras, where M is the finitary monad WH on the lfp category $(\mathscr{A}_f, \mathscr{A})$; whence, by [5, Satz 10.3], $\mathrm{Mon}(\mathscr{A}_f, \mathscr{A})$ too is lfp.

5. Presentations of finitary monads

In the classical case $\mathscr{A} = \mathscr{V} = \mathbf{Set}$, wherein $\mathscr{A}_f$ is the category of finite sets, equivalent to the category $\mathbf{S}$ of finite cardinals, it is well known that to give a finitary monad on **Set** is equivalently to give a Lawvere theory $\mathscr{T}$ in the sense of [12], these two having the same algebras; in fact, identifying the monad with a monoid T in $(\mathbf{S}, \mathbf{Set})$, we obtain T from $\mathscr{T}$ via $Tn = \mathscr{T}(n, 1)$, and $\mathscr{T}$ from T via $\mathscr{T}(n, m) = (Tn)^m$; the multiplication on T corresponds to composition in $\mathscr{T}$, and so on. When one speaks in this context of a *free theory*, the forgetful functor one has in mind is not the above $W : \mathrm{Mon}(\mathbf{S}, \mathbf{Set}) \to (\mathbf{S}, \mathbf{Set})$, but instead its composite U with the $V : (\mathbf{S}, \mathbf{Set}) \to (\mathbf{N}, \mathbf{Set})$ induced by the inclusion $\mathbf{N} \to \mathbf{S}$, where $\mathbf{N}$ is the discrete category of natural numbers; U sends T to the mere sequence (Tn) of sets, Tn being called *the set of n-ary operations* of T (or of $\mathscr{T}$). So by a *free theory*—we would rather, to suit our more general context, speak of a *free monad*—what is commonly meant is one of the form FB, where F is the left adjoint of U and B is just a sequence (Bn) of sets, Bn being called *the set of basic n-ary operations* while $(FB)n$ is called *the set of derived n-ary operations*. As is observed in Bénabou's thesis [2], a Lawvere theory $\mathscr{T}$, or equally a finitary monad T on **Set**, is itself an algebra for an $\mathbf{N}$-sorted algebraic theory—see also Section 6. By the remarks on p. 140 of [5], therefore, the U above is monadic; it is moreover clearly finitary, since V preserves colimits.

We now imitate the above in the more general case of Section 4 . When we said there that, for an lfp $\mathscr{A}$, the category $\mathscr{A}_f$ is *small*, we did not mean that its set of objects is small—after all, in the classical case above, $\mathscr{A}_f$ is the category of finite sets—but rather that there is a small subset of its objects representing all the isomorphism classes; an examination of [11, Theorem 7.2] show that this is exactly what is proved there. Let us write N for such a small subset, seen (like the $\mathbf{N}$ of the classical case) as a discrete category; we may often speak for simplicity as if N and $\mathscr{A}_f$ have the same objects. We have the forgetful functor $V : (\mathscr{A}_f, \mathscr{A}) \to (N, \mathscr{A}_0)$ sending $K : \mathscr{A}_f \to \mathscr{A}$ to the mere family (Kc) of its objects and sending $\alpha : K \to L$ to the family $(\alpha c : Kc \to Lc)$ of its components; of course V is faithful and conservative. We can identify $(N, \mathscr{A}_0)$ with $(\mathscr{N}, \mathscr{A})$ where $\mathscr{N}$ denotes—see [10, Section 2.5]—the free $\mathscr{V}$-category on the ordinary category N; then V is in effect the functor $(\mathscr{A}_f, \mathscr{A}) \to (\mathscr{N}, \mathscr{A})$ induced by $\mathscr{N} \to \mathscr{A}_f$, and as such has a left adjoint G and a right adjoint as well, these being given by left and right Kan extensions. We need below the explicit form of GB; a simple calculation of the left Kan extension gives

$$(GB)c = \sum_{e \in N} \mathscr{A}_f(e, c) \otimes Be \,, \tag{5.1}$$

but it is simpler still to verify directly that the G so defined is indeed the left adjoint. We write U for the composite right adjoint $VW : \mathrm{Mon}(\mathscr{A}_\mathrm{f}, \mathscr{A}) \to (N, \mathscr{A}_\mathrm{o})$, and F for its left adjoint HG, with $\varepsilon : FU \to 1$ for the counit. Since V preserves colimits, U like W is finitary. In contrast, however, to the final sentence of the last paragraph concerning the classical case, we do not know—see Section 6 for further comments—whether, in this generality, U is monadic. But we do have the following:

Theorem 5.1. *$U : \mathrm{Mon}(\mathscr{A}_\mathrm{f}, \mathscr{A}) \to (N, \mathscr{A}_\mathrm{o})$ is of descent type.*

Proof. It suffices by Theorem 3.2 to prove that εT is W-final for each $T \in \mathrm{Mon}(\mathscr{A}_\mathrm{f}, \mathscr{A})$; that is to say, if also $S \in \mathrm{Mon}(\mathscr{A}_\mathrm{f}, \mathscr{A})$ and if $\alpha : T \to S$ is a morphism in $(\mathscr{A}_\mathrm{f}, \mathscr{A})$ for which $\alpha . \varepsilon T$ is a map of monoids, then α too is a map of monoids. Using m for the multiplication and i for the unit of each of the monoids involved, we have the commutativity of the exteriors and the left regions of the diagrams

$$\begin{tikzcd} FUT \circ FUT \arrow[r, "\varepsilon T \circ \varepsilon T"] \arrow[d, "m"'] & T \circ T \arrow[r, "\alpha \circ \alpha"] \arrow[d, "m"'] & S \circ S \arrow[d, "m"] \\ FUT \arrow[r, "\varepsilon T"'] & T \arrow[r, "\alpha"'] & S \end{tikzcd}$$

$$\begin{tikzcd} & J \arrow[dl, "i"'] \arrow[d, "i"'] \arrow[dr, "i"] & \\ FUT \arrow[r, "\varepsilon T"'] & T \arrow[r, "\alpha"'] & S , \end{tikzcd}$$

and we are to prove the right regions commutative; for which it suffices to prove $\varepsilon T \circ \varepsilon T$, more properly called $W\varepsilon T \circ W\varepsilon T$, to be epimorphic in $(\mathscr{A}_\mathrm{f}, \mathscr{A})$. Since $VW\varepsilon T = U\varepsilon T$ is a retraction and V is faithful, certainly $W\varepsilon T$ is epimorphic; and then, since $- \circ WFUT$ has a right adjoint by (4.5), the map $W\varepsilon T \circ WFUT : WFUT \circ WFUT \to WT \circ WFUT$ is epimorphic. It remains only to show that $WT \circ W\varepsilon T$ is epimorphic; and since $VW\varepsilon T$ is a retraction, this follows from the following lemma:

Lemma 5.2. *For any $P \in (\mathscr{A}_\mathrm{f}, \mathscr{A})$ and any $\beta : Q \to Q'$ in $(\mathscr{A}_\mathrm{f}, \mathscr{A})$, the map $P \circ Q \to P \circ Q'$ is epimorphic in $(\mathscr{A}_\mathrm{f}, \mathscr{A})$ if $V\beta$ is a retraction in $(N, \mathscr{A}_\mathrm{o})$.*

Proof. N being discrete, to say that $V\beta$ is a retraction is just to say that each of its components $(V\beta)d$ is a retraction; but $(V\beta)d$ is equally $\beta d : Qd \to Q'd$. By (4.4) we have

$$(P \circ \beta)d = \int^c \mathscr{A}(c, \beta d) \otimes Pc : \int^c \mathscr{A}(c, Qd) \otimes Pc \to \int^c \mathscr{A}(c, Q'd) \otimes Pc \,;$$

and this is a retraction since each βd is so. A fortiori each $(P \circ \beta)d$ is epimorphic in $\mathscr{A}$, so that $P \circ \beta$ is epimorphic in $(\mathscr{A}_{\mathrm{f}}, \mathscr{A})$. □□

Theorem 5.1 allows us to conclude, using Theorem 2.4, that every finitary monad on $\mathscr{A}$, here identified with a monoid T in $(\mathscr{A}_{\mathrm{f}}, \mathscr{A})$, admits a presentation as a coequalizer

$$FE \underset{\tau}{\overset{\sigma}{\rightrightarrows}} FB \xrightarrow[\rho]{} T \tag{5.2}$$

in $\mathrm{Mon}(\mathscr{A}_{\mathrm{f}}, \mathscr{A})$. We now examine what this means in more elementary terms.

First note a simple piece of general theory: since $\square$ is an action of $(\mathscr{A}_{\mathrm{f}}, \mathscr{A})$ on $\mathscr{A}_0$ for which each $- \square A$ has a right adjoint $\langle A, - \rangle$ as in (4.2), each $\langle A, A \rangle$ is a monoid in $(\mathscr{A}_{\mathrm{f}}, \mathscr{A})$, and to give an action $a : T \square A \to A$ turning A into a T-algebra is equally to give a monoid-map $\alpha : T \to \langle A, A \rangle$.

In particular, to give to $\mathscr{A}$ an FB-algebra structure is to give a map $\beta : FB \to \langle A, A \rangle$ in $\mathrm{Mon}(\mathscr{A}_{\mathrm{f}}, \mathscr{A})$, or equivalently a map $\beta' : B \to U\langle A, A \rangle$ in $(N, \mathscr{A}_0)$; that is, to give a family of maps $\beta' c : Bc \to (U\langle A, A \rangle)c = \langle A, A \rangle c = \mathscr{A}(c, A) \pitchfork A$, or again a family of maps $\bar{\beta} c : \mathscr{A}(c, A) \otimes Bc \to A$. In the classical case $\mathscr{A} = \mathscr{V} = \mathbf{Set}$ one recognizes here the concept of a free Lawvere theory, where to give to A an FB-algebra structure is to assign a map $A^n \to A$ to each basic n-ary operation $\omega \in Bn$. The less-classical case $\mathscr{V} = \mathbf{Set}$, $\mathscr{A} = \mathbf{Cat}$ was discussed in [4, Section 8], where Bc, for a finitely-presentable category c, was called *the category of basic c-ary operations*. In analogy with these concrete cases, Bc is aptly called in the general case *the $\mathscr{A}$-object of basic c-ary operations*.

Using (4.7) we can give an explicit inductive description of FB. First, by (4.4) and (5.1) along with the commutativity of colimits with colimits and the Yoneda isomorphism we have

$$\begin{aligned}
(GB \circ Q)d &= \int^{c \in \mathscr{A}_{\mathrm{f}}} \mathscr{A}(c, Qd) \otimes \Big[\sum_{e \in N} \mathscr{A}_{\mathrm{f}}(e, c) \otimes Be \Big] \\
&\cong \sum_e \Big[\int^{c \in \mathscr{A}_{\mathrm{f}}} \mathscr{A}(c, Qd) \otimes \mathscr{A}_{\mathrm{f}}(e, c) \Big] \otimes Be \\
&\cong \sum_e \mathscr{A}(e, Qd) \otimes Be\,.
\end{aligned} \tag{5.3}$$

By (4.7), therefore, we have $FB = \mathrm{colim}_{n<\omega} S_n$, where $S_0 = J$ and

$$S_{n+1}c = c + \sum_e \mathscr{A}(e, S_n c) \otimes Be\,. \tag{5.4}$$

This formula contains of course the usual construction of derived operations in the classical case. There, $S_n c$ is what may be called the set of derived operations

constructed at the nth stage; and by (5.4) an element of $S_{n+1}c$ is either an element of c, corresponding to a projection, or an element of some $\mathscr{A}(e, S_n c) \times Be$, given by an e-ary basic operation and an e-ad of already-constructed c-ary operations. In that classical case, the maps $S_n \to S_{n+1}$ and hence $S_n \to FB$ are monomorphic; so too when $\mathscr{V}$ is **Set** or **Cat** and $\mathscr{A}$ is **Cat**, and in many other important cases, but not in all. At any rate, we think of $(FB)c$ as *the $\mathscr{A}$-object of derived c-ary operations*.

Using (5.4), it is easy to reconstruct the monoid-map $\beta : FB \to \langle A, A \rangle$ of the penultimate paragraph from the family $\bar{\beta}c : \mathscr{A}(c, A) \otimes Bc \to A$. We get β by passage to the colimit from maps $\gamma_n c : S_n c \to \langle A, A \rangle c = \mathscr{A}(c, A) \pitchfork A$, which correspond to maps $\bar{\gamma}_n c : \mathscr{A}(c, A) \otimes S_n c \to A$; these are constructed inductively, it being clear how to define $\bar{\gamma}_{n+1} c$ on the summands of its domain

$$\mathscr{A}(c, A) \otimes c + \sum_e \mathscr{A}(c, A) \otimes \mathscr{A}(e, S_n c) \otimes Be\ ;$$

it is evaluation on the first summand, while on the eth summand of the second summand we use the canonical comparison map $\mathscr{A}(c, A) \otimes \mathscr{A}(e, S_n c) \otimes Be \to \mathscr{A}(e, \mathscr{A}(c, A) \otimes S_n c) \otimes Be$, whence $\bar{\gamma}_n c$ gives a map into $\mathscr{A}(e, A) \otimes Bc$, which we follow by $\bar{\beta}e$.

Finally, to give an action on A of the monoid T presented by (5.2), or a monoid-map $\alpha : T \to \langle A, A \rangle$, is to give a monoid-map $\beta : FB \to \langle A, A \rangle$ as above for which we have $\beta\sigma = \beta\tau$. This is equally to require that $U\beta.\sigma' = U\beta.\tau'$, where $\sigma', \tau' : E \to UFB$ correspond to σ and τ under the adjunction. By taking the image, one can suppose that $\sigma' c$ and $\tau' c$ exhibit Ec as a subobject of $(FB)c \times (FB)c$, called of course *the $\mathscr{A}$-object of equations of arity c*.

6. Final comments

As we said in Section 5, we have not been able to decide whether $U : \mathrm{Mon}(\mathscr{A}_{\mathrm{f}}, \mathscr{A}) \to (N, \mathscr{A}_0)$ is monadic in general; that is why we settled for Theorem 5.1, and why we developed the results in Section 3. We now indicate briefly why the proof in the classical case does not immediately extend.

Since U is finitary the monad we seek would be a finitary one on the lfp ordinary category $(N, \mathscr{A}_0)$, to which the theory above applies. In particular, the monad would have a presentation in terms of basic operations and equations. Let Bk be the object of basic operations of arity k; it is a family $(Bk)c$ of objects of $\mathscr{A}$. Here k is a finitely-presentable object of $(N, \mathscr{A}_0)$; which is to say that it is a family kc of finitely-presentable objects of $\mathscr{A}$, all but a finite number of which are the initial object of $\mathscr{A}$; let the non-zero ones be $kc_1, \ldots, kc_n$. To give the k-component of an action of the basic operations on an object $T = (Td)$ of $(N, \mathscr{A}_0)$ is, by Section 5, to give for each $d \in N$ a map

$$(\mathscr{A}_0(kc_1, Tc_1) \times \cdots \times \mathscr{A}_0(kc_n, Tc_n)) \cdot (Bk)d \rightarrow Td\ , \tag{6.1}$$

the tensor product in this case of ordinary categories being just the copower indicated by a dot. Of course the map (6.1) is trivial for those d for which $(Bk)d$ is the initial object.

Now compare this with what we must do to give to $T \in (N, \mathscr{A}_0)$ the structure of a monoid in $(\mathscr{A}_f, \mathscr{A})$. First we must make it a functor from $\mathscr{A}_f$ to $\mathscr{A}$, giving maps $\mathscr{A}_f(c, d) \rightarrow \mathscr{A}(Tc, Td)$, or

$$\mathscr{A}_f(c, d) \otimes Tc \rightarrow Td\ , \tag{6.2}$$

subject to suitable equational axioms. Then we must give the unit $i : J \rightarrow T$ and the multiplication $m : T \circ T \rightarrow T$, so that by (4.4) we are to give maps

$$d \rightarrow Td\ , \qquad \mathscr{A}(c, Td) \otimes Tc \rightarrow Td\ , \tag{6.3}$$

satisfying the equational axioms that make these maps $\mathscr{V}$-natural in d and in c, and the further equational axioms expressing the unit and associative laws for i and m.

Now in the classical case $\mathscr{A} = \mathscr{V} = \mathbf{Set}$, (6.2) and (6.3) have the form of (6.1); since the equational axioms are then indeed equations between derived operations, we conclude that U is monadic. In general, however, (6.2) and (6.3) do not have the form (6.1), even if $\mathscr{V} = \mathbf{Set}$. They are of that form in such important cases as $\mathscr{V} = \mathbf{Set}$, $\mathscr{A} = \mathbf{Cat}$; but already fail to be so in the equally important case $\mathscr{A} = \mathscr{V} = \mathbf{Cat}$. We have not pursued the matter further than this.

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