JOURNAL OF DIFFERENTIAL EQUATIONS 50, 234–259 (1983)

Analytic Resolvent Operators for Integral Equations in Banach Space

R. C. GRIMMER

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901

AND

A. J. PRITCHARD

Control Theory Centre, University of Warwick, Coventry CV47AL, England

Communicated by Jack K. Hale

Received January 4, 1982; revised April 19, 1982

1. INTRODUCTION

In this paper we study the integrodifferential equation

$$\dot{x}(t) = A_0 x(t) + \int_0^t B(t-s)\, x(s)\, ds + f(t), \qquad t \geqslant 0,$$

$$x(0) = x_0 \in D(A) \subset X \tag{VE}$$

and the integral equation

$$x(t) = \int_0^t a(t-s)\, x(s)\, ds + f(t) \tag{IE}$$

in a Banach space X. The operators A, A_0, $B(t)$, and $a(t)$ are closed operators on X with dense domain and frequently we will assume the domains $D(B(t))$, $D(a(t))$ are constant in t and equal to $D(A)$ while $D(A_0) \supset D(A)$. However, unlike a good deal of the work on this subject it will not be necessary to assume that for suitable x, $B(t)x$ or $a(t)x$ are continuously differentiable functions of t, or more generally, in a Sobolev space $W^{1,p}$.

In an earlier work [11], the first author showed that under fairly general circumstances (VE) has associated with it (even in the nonconvolution case) a resolvent operator $R(t)$ which is a bounded operator for $t \geqslant 0$. This

0022-0396/83 $3.00

operator satisfies a number of properties reminiscent of a semigroup. In particular $R(0)=I$ and for $x \in D(A)$

$$\dot{R}(t)x = A_0 R(t)x + \int_0^t B(t-s)\, R(s)\, x ds$$

and

$$\dot{R}(t)x = R(t)\, A_0 x + \int_0^t R(t-s)\, B(s)\, x ds.$$

Further, (IE) has a resolvent operator associated with it, $r(t)$, which satisfies for $x \in D(a(0)^2)$

$$r(t)x = -a(t)x + \int_0^t a(t-s)\, r(s)\, x ds$$

and

$$r(t)x = -a(t)x + \int_0^t r(t-s)\, a(s)\, x ds.$$

These equations naturally lead to variation of parameter formulas for (VE) and (IE) which yield solutions for (VE) and (IE) if the function f satisfies appropriate conditions and if $x_0 \in D(A)$. This indicates that the families $\{R(t), t \geqslant 0\}$, $\{r(t), t \geqslant 0\}$ are similar to semigroups. There is a rich theory for analytic semigroups and we wish to develop theories for (VE) and (IE) which yield analytic resolvents. This will enable us to extend the class of functions for which solutions can be obtained. Rather than requiring f to be continuously differentiable or f and Af to be continuous, or possibly $f \in W^{1,1}((0, T), X)$ or $f \in L^1((0, T), D(A))$ as in [1–3, 6, 7, 18] and in some sense in [21, 25], we are able to obtain solutions to the problem

$$\dot{x}(t) = A_0 x(t) + \int_0^t B(t-s)\, x(s)\, ds + f(t), \qquad t > 0,$$

$$x(0) = x_0 \in D((-A)^\alpha)$$

with $(-A)^\alpha f$ continuous. Further if $B(t)$ is of "lower order" than A, the initial value problem

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)\, x(s)\, ds, \qquad t > 0,$$

$$x(0) = x_0 \in X$$

has a solution.

In a similar manner it will be shown that if f and $(-A)^{1+\beta}f$ are continuous, $0<\beta<1$, then one obtains a solution of (IE) if $r(t)$ is an analytic resolvent. This contrasts with the usual requirement that A^2f is continuous or f satisfies one of the conditions $f\in L^p((0,T), D(A^2))$ or $f\in W^{1,1}((0,T), D(A))$ as in, for example, [4, 11–13].

Our technique involves showing that under certain assumptions on $A, A_0, B(t)$ and $a(t)$ that (VE) and (IE) have resolvent operators in the sense of [11]. Then we show that the variation of parameters formulas associated with $R(t)$ and $r(t)$ in fact yield solutions (rather than "mild" or "weak" solutions) if x_0 and f satisfy certain assumptions.

Our results for (VE) are similar to those obtained by Friedman and Shinbrot [10] and used in [9] where the equation

$$x(t)=\int_0^t h(t-s)\,Ax(s)\,ds+k(t) \tag{1.1}$$

was studied. In [10] it is assumed that $h(t)$ is a scalar function with $h(0)>0$, $h(t)$ absolutely continuous on $(0,T)$ with $\dot{h}\in L^1(0,T)$, $k\in C^1([0,T),X)$ with $k(0)\in D((-A)^\alpha)$, $\dot{k}(t)\in D((-A)^\alpha)$, and $(-A)^\alpha k\in L^1((0,T),X)$, $0<\alpha\leqslant 1$; the solution is then obtained via a resolvent operator (fundamental solution in [10]). Alternatively, this equation is also examined under the condition that $h(0)>0$, $h\in C^1([0,T))$, $\dot{h}$ is absolutely continuous with $\ddot{h}\in L^p((0,T))$, $p\geqslant 1$, and $k(0)\in D((-A)^\alpha)$, $0<\alpha\leqslant 1$, with $\dot{k}$ Holder continuous in X. Various other assumptions on h and k are also studied mostly in connection with obtaining results concerning asymptotic behavior of the solutions. Also required in [10] were conditions on the Laplace transform of h and the fact that A generated an analytic semigroup. More generally, they also considered the equation

$$x(t)=\int_0^t h(t-s)\,A(s)\,x(s)\,ds+k(t)$$

which will not be considered here.

The resolvent operator in [10] was obtained by taking the limit of contour integrals of the form

$$(2\pi i)^{-1}\int_{C_n}(\lambda I-A)^{-1}\,S_\lambda(t)\,d\lambda,$$

where $S_\lambda^*(s)=1/(s-\lambda g(s))$, $g(s)=h(0)+\dot{h}^*(s)=sh^*(s)$, where $*$ indicates Laplace transform. As they considered the differential version of (1)

$$\dot{x}(t)=h(0)\,Ax(t)+\int_0^t \dot{h}(t-s)\,Ax(s)\,ds+\dot{k}(t),$$

$$x(0)=k(0)$$

they obtained a resolvent for an equation of the form of (VE).

Also related to our work is the paper of Da Prato and Iannelli [7] which considers the equation

$$\dot{x}(t) = \int_0^t K(t-s)\,Ax(s)\,ds,$$

$$x(0) = x_0, \tag{1.2}$$

where K is a scalar function which is locally integrable on $\mathbb{R}^+$. An analytic resolvent is obtained for (2) in [7, Theorem 4] with $R(t)$ given by

$$R(t) = (2\pi i)^{-1} \int_\Gamma e^{\lambda t}(\lambda I - K^*(\lambda)A)^{-1}\,d\lambda,$$

where Γ is a contour of the type used to obtain an analytic semigroup. Our result concerning analytic resolvents for (VE) is a generalization of [7, Theorem 4]. However, no results were obtained in [7] concerning which types of functions f yielded solutions of (VE).

If $B(t) = h(t)A$ and $a(t) = k(t)A$, where h and k are scalar functions, resolvent operators for (VE) and (IE) have been obtained using an integral representation by Clement and Nohel [4] if the resolvents associated with $\lambda h(t)$ and $\lambda k(t)$ satisfied certain positivity conditions. It must be noted, however, that the resolvent operator and variation of parameters formula obtained in [4] corresponding to (IE) is different than that obtained here. In particular Clement and Nohel consider the resolvent equation

$$r(t)x = -k(t)x + \int_0^t k(t-u)\,Ar(u)\,x du,$$

whereas we consider

$$r(t)x = -k(t)\,Ax + \int_0^t k(t-u)\,Ar(u)\,x du$$

for (IE).

In the case where X is a Hilbert space and A is self-adjoint with spectral decomposition E_λ,

$$Ax = \int_{-\infty}^{\mu} \lambda dE_\lambda x, \qquad x \in D(A), \quad \mu < 0,$$

Carr and Hannsgen [1] and Hannsgen [15–19] obtain the resolvent operator

$$R(t) = \int_{-\infty}^{\mu} u(t, \lambda)\,dE_\lambda$$

for the equation

$$\dot{x}(t) + \int_0^t h(t-s)\,Ax(s)\,ds = f(t),$$

where $u(t, \lambda)$ is the resolvent associated with $\lambda h(t)$. That is,

$$\dot{u}(t, \lambda) + \lambda \int_0^t h(t-s)\,u(s, \lambda)\,ds = 0, \qquad u(0, \lambda) = 1.$$

Although there may be many ways to obtain a resolvent operator $R(t)$ for (VE) we note that it follows from [11, Theorem 2.3] that there exists at most one resolvent operator $R(t)$ in the sense that it is used here.

2. Preliminaries

In our work the operator A plays a dominant role. We shall assume that A generates an analytic semigroup and state our hypothesis in terms of A. To do this, it will be convenient to introduce some notation which we will use throughout this paper.

As A generates an analytic semigroup, for $\alpha > 0$, $(-A)^\alpha$ is a closed operator with dense domain in X. We denote this domain Y^α and endow it with the graph norm, which we denote $\|\cdot\|_\alpha$ so that $(Y^\alpha, \|\cdot\|_\alpha)$ is a Banach space. We shall write Y for Y^1 and it will often be convenient to let $Y^0 = X$ with its usual norm, $\|\cdot\|$. $\mathscr{L}(Y^\alpha, Y^\beta)$ will denote the space of bounded linear operators $Y^\alpha \to Y^\beta$ with norm $\|\cdot\|_{\alpha,\beta}$. If $\alpha = \beta$ we use the conventions $\mathscr{L}(Y^\alpha) = \mathscr{L}(Y^\alpha, Y^\alpha)$ and since there will be no confusion, $\|\cdot\|_\alpha = \|\cdot\|_{\alpha,\alpha}$, and for $Y^0 = X$, we write $L(X)$ with norm $\|\cdot\|$.

We shall also frequently use the Laplace transform of $f(t)$ which we shall denote $f^*(\lambda)$.

We consider the integrodifferential equation (VE) and the integral equation (IE) separately but in parallel. For the equation

$$\begin{aligned} &\dot{x}(t) = A_0 x(t) + \int_0^t B(t-s)\,x(s)\,ds + f(t), \\ &x(0) = x_0 \in D(A) \end{aligned} \tag{VE}$$

we shall in the future assume some or all of the following hypotheses:

(VI) A generates an analytic semigroup on X. $B(t)$ is a closed operator on X with domain at least $D(A)$ a.e. $t \geqslant 0$ with $B(t)x$ strongly measurable for each $x \in D(A)$ and $\|B(t)\|_{1,0} \leqslant b(t)$, $b \in LL^1(0, \infty)$ with $b^*(\lambda)$ absolutely convergent for $\operatorname{Re} \lambda > 0$. A_0 is closed with domain at least $D(A)$.

(V2) $\rho(\lambda) = (\lambda I - A_0 - B^*(\lambda))^{-1}$ exists as a bounded operator on X which is analytic for λ in the region $\Lambda = \{\lambda \in C: |\arg \lambda| < (\pi/2) + \delta\}$, where $0 < \delta < \pi/2$. In Λ if $|\lambda| \geqslant \varepsilon > 0$ there exists a constant $M = M(\varepsilon) > 0$ so that $\|\rho(\lambda)\| \leqslant M/|\lambda|$.

(V3) $A_0\rho(\lambda)$, $A\rho(\lambda) \in \mathscr{L}(X)$ for $\lambda \in \Lambda$ and are analytic on Λ into $\mathscr{L}(X)$. $B^*(\lambda) \in \mathscr{L}(Y, X)$ and $B^*(\lambda)\rho(\lambda) \in \mathscr{L}(Y, X)$ for $\lambda \in \Lambda$. Given $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ so that for $\lambda \in \Lambda$ with $|\lambda| \geqslant \varepsilon$, $\|A_0\rho(\lambda)\|_{1,0} + \|A\rho(\lambda)\|_{1,0} + \|B^*(\lambda)\rho(\lambda)\|_{1,0} \leqslant M/|\lambda|$, and $\|B^*(\lambda)\|_{1,0} \to 0$ as $|\lambda| \to \infty$ in Λ. In addition, $\|A\rho(\lambda)\| \leqslant M|\lambda|^n$ for some $n > 0$, $\lambda \in \Lambda$ with $|\lambda| \geqslant \varepsilon$. Further, there exists $D \subset D(A^2)$ which is dense in Y such that $A_0(D)$ and $B^*(\lambda)(D)$ are contained in Y and $\|B^*(\lambda)x\|_1$ is bounded for each $x \in D$, $\lambda \in \Lambda$, $|\lambda| \geqslant \varepsilon$.

(V4) $A_0 \equiv A$ and given $\varepsilon > 0$ and α, $0 \leqslant \alpha \leqslant 1$, there exists a constant $M = M(\varepsilon, \alpha)$ so that for $\lambda \in \Lambda$, $|\lambda| \geqslant \varepsilon$ $A\rho(\lambda)$, $B^*(\lambda)\rho(\lambda) \in \mathscr{L}(Y^\alpha, X)$ with $\|A\rho(\lambda)\|_{\alpha,0} + \|B^*(\lambda)\rho(\lambda)\|_{\alpha,0} \leqslant M/|\lambda|^\alpha$.

(V5) $A_0 \equiv 0$ and $A\rho(\lambda) \in \mathscr{L}(Y^\alpha, X)$ with $\|A\rho(\lambda)\|_{\alpha,0} \leqslant M/|\lambda|^{\beta(\alpha)}$, where M is a positive constant and $\beta(\alpha)$ is real.

We remark that the motivation for the set D is the possibility of considering (VE) when A is the Laplacian with Dirichlet boundary conditions and A_0 is a first order partial differential operator. In this case we might choose $D = C_0^\infty(\Omega)$.

For the integral equation

$$x(t) = \int_0^t a(t-s)\,x(s)\,ds + f(t) \tag{IE}$$

we shall require the hypotheses:

(I1) A generates an analytic semigroup on X. $a(t)$ is a closed operator on X with domain at least $D(A)$ a.e., $t \geqslant 0$. Also $a(t) \in \mathscr{L}(Y^{1+\beta}, Y^\beta)$, $0 \leqslant \beta \leqslant 1$ with $\|a(t)\|_{1+\beta,\beta} \leqslant b(t)$, $b \in L^1(0, \infty)$.

(I2) $(I - a^*(\lambda))^{-1}$ exists as a bounded operator on X which is analytic for λ in the region

$$\Lambda = \{\lambda \in C: |\arg \lambda| < (\pi/2) + \delta\},\ 0 < \delta < \pi/2.$$

Given $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ so that $\|(I - a^*(\lambda))^{-1}\| \leqslant M$ for $\lambda \in \Lambda$, $|\lambda| \geqslant \varepsilon$.

(I3) $a^*(\lambda) \in \mathscr{L}(Y^{1+\beta}, Y^\beta)$ $0 \leqslant \beta \leqslant 1$, for $\lambda \in \Lambda$. Further there is an α, $0 < \alpha \leqslant 1$, so that given $\varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ with the property $\lambda \in \Lambda$, $|\lambda| \geqslant \varepsilon$ implies $\|a^*(\lambda)\|_{1+\beta,\beta} < M/|\lambda|^\alpha$.

(I4) $\sigma(\lambda) = -a^*(\lambda)(I - a^*(\lambda))^{-1} \in \mathscr{L}(Y^\beta, X) \cap \mathscr{L}(Y^{1+\beta}, Y)$ for $\lambda \in \Lambda$. Also given $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ so that $\lambda \in \Lambda$, $|\lambda| \geqslant \varepsilon$ implies

$$\|\sigma(\lambda)\|_{\beta,0} + \|\sigma(\lambda)\|_{1+\beta,1} \leqslant M/|\lambda|^{\alpha\beta}, \qquad 0 \leqslant \beta \leqslant 1.$$

In formulating these hypotheses we have been guided by results concerning fractional powers of $(-A)$ and the special cases where $B(t)=h(t)A$ and $a(t)=h(t)A$ with $h(t)$ a scalar function. In particular (V4) is motivated by the consideration that $A(\lambda I-A)^{-1}\in\mathscr{L}(Y^\alpha, X)$. This follows since

$$\|A(\lambda I-A)^{-1}x\|\leqslant\|(-A)^{1-\alpha}(\lambda I-A)^{-1})\|\,\|x\|_\alpha, \qquad x\in y^\alpha$$

and the inequality [27, p. 42]

$$\|(-A)^\alpha x\|\leqslant C_{\gamma,\beta}\|(-A)^\beta x\|^{\gamma/\beta}\|x\|^{1-\gamma/\beta}, \qquad 0\leqslant\gamma<\beta\leqslant 1$$

with $\gamma=1-\alpha$, $\beta=1$ yields

$$\|(-A)^{1-\alpha}(\lambda I-A)^{-1}\|\leqslant C_{1-\alpha,1}M/|\lambda|^\alpha.$$

The motivation for (I3) is seen by considering the case when $a^*(\lambda)=h^*(\lambda)A$ while (I4) is similar to (V4).

For the equation

$$\dot{x}(t)=jAx(t)+\int_0^t h(t-s)\,Ax(s)\,ds+f(t), \qquad j=0,1, \qquad \text{(VE}')$$

where $h(t)$ is a scalar function defined on $(0,\infty)$ conditions (V1)–(V3) can be stated quite simply.

(V1$'$) A generates an analytic semigroup on X. In particular

$$\Lambda_1=\{\lambda\in C: |\arg\lambda|<(\pi/2)+\delta_1\},\ 0<\delta_1<\pi/2$$

is contained in the resolvent set of A and $\|(\lambda I-A)^{-1}\|\leqslant M/|\lambda|$ on Λ_1 for some constant $M>0$. The scalar function $h(\cdot)$ is in $L^1(0,\infty)$ with $\mathscr{L}^*(\lambda)$ absolutely convergent for $\operatorname{Re}\lambda>0$.

(V2$'$) There exists $\Lambda=\{\lambda\in C: |\arg\lambda|<(\pi/2)+\delta\}$, $0<\delta<\pi/2$, so that $\lambda\in\Lambda$ implies $g_j(\lambda)=j+h^*(\lambda)$ exists and is not zero. Further $\lambda g_j^{-1}(\lambda)\in\Lambda_1$ for $\lambda\in\Lambda$.

(V3$'$) In Λ, $h^*(\lambda)\to 0$ as $|\lambda|\to\infty$ and if $j=0$ $(h^*(\lambda))^{-1}=0(|\lambda|^n)$ as $|\lambda|\to\infty$ for some $n>0$.

It is clear that (V1) and (V1$'$) are equivalent and since $\rho(\lambda)=g_j^{-1}(\lambda)$ $(\lambda g_j^{-1}(\lambda)I-A)^{-1}$, (V2) follows immediately from (V2$'$). Now $A\rho(\lambda)=g_j^{-1}(\lambda)(\lambda g_j^{-1}(\lambda)(\lambda g_j^{-1}(\lambda)I-A)^{-1}-I)$ and since $s(sI-A)^{-1}$ is bounded for $s\in\Lambda$, we see that $\|A\rho(\lambda)\|$ has the growth properties of $g_j^{-1}(\lambda)$. If $j=1$, $g_j^{-1}(\lambda)\to 1$ as $|\lambda|\to\infty$ in Λ, while if $j=0$, $g_j^{-1}(\lambda)=(h^*(\lambda))^{-1}$ is $O(|\lambda|^n)$. The rest of (V3) is immediate.

Since $A\rho(\lambda) = g_j^{-1}(\lambda) A(\lambda g_j^{-1}(\lambda) I - A)^{-1}$, for $x \in Y^\alpha$

$$\begin{aligned}\|A\rho(\lambda)x\| &= \|(-A)^{1-\alpha} \rho(\lambda) (-A)^\alpha x\| \\ &\leqslant (|g_j^{-1}(\lambda)| C_{1-\alpha,1} M/|\lambda g_j^{-1}(\lambda)|^\alpha) \|x\|_\alpha \\ &\leqslant (C_{1-\alpha,1} M/|\lambda|^\alpha |g_j(\lambda)|^{1-\alpha}) \|x\|_\alpha .\end{aligned}$$

If $j = 1$, $g_j(\lambda) \to 1$ as $|\lambda| \to \infty$ so (V4) is immediate. Whereas if $j = 0$, $g_j(\lambda) = h^*(\lambda)$ so we require

(V5′) If $j = 0$, $|h^*(\lambda)|^{-1} \leqslant M|\lambda|^n$ for some positive constants M and n.

Then we may choose $\beta(\alpha) = \alpha + n(\alpha - 1)$ to obtain (V5).

For the equation

$$x(t) = \int_0^t h(t-s) A x(s)\, ds + f(t), \tag{IE′}$$

where again $h(t)$ is a scalar function defined on $(0, \infty)$ conditions (I1)–(I4) are also easily stated.

(I1′) A generates an analytic semigroup on X. In particular,

$$\Lambda_1 = \{\lambda \in C: |\arg \lambda| < (\pi/2) + \delta_1\}, 0 < \delta_1 < \pi/2$$

is contained in the resolvent set of A and $\|(\lambda I - A)^{-1}\| \leqslant M/|\lambda|$ on Λ_1 for some constant $M > 0$. The scalar function $h(\cdot) \in L^1(0, \infty)$.

(I2′) There exists $\Lambda = \{\lambda \in C: |\arg \lambda| < (\pi/2) + \delta\}$, $0 < \delta < \pi/2$ so that $\lambda \in \Lambda$ implies $h^*(\lambda)$ exists and is not zero, Further $|(h^*(\lambda))|^{-1} \in \Lambda_1$ for $\lambda \in \Lambda$.

(I3′) There is an α, $0 < \alpha \leqslant 1$, and constant $M > 0$ so that $|h^*(\lambda)| \leqslant M/|\lambda|^\alpha$, $\lambda \in \Lambda$.

Since

$$a^*(\lambda)(I - a^*(\lambda))^{-1} = A((h^*(\lambda))^{-1} I - A)^{-1}$$

and for $s \in \Lambda$, $x \in Y^\beta$,

$$\|A(sI - A)^{-1} x\| \leqslant (C_{1-\beta,1} M/|s|^\beta) \|x\|_\beta .$$

So that $\|a(\lambda)\|_{\beta,0} \leqslant k/|\lambda|^{\alpha\beta}$ for some constant k and hence (I4) follows from (I1′)–(I3′).

We remark also that given Eq. (VE) the transformation $y(t) = e^{-\gamma t} x(t)$, $\gamma > 0$, yields

$$\dot{y}(t) = (A_0 - \gamma I) y(t) + \int_0^t e^{-\gamma(t-s)} B(t-s) y(s)\, ds + e^{-\gamma t} f(t).$$

As $\rho_y(\lambda) = \rho(\lambda + \gamma)$, the effect of this transformation is to shift the regions Λ and Λ_1 to the left by γ and to enhance the integrability of the kernel $B(t)$. For Eq. (IE) this transformation yields

$$y(t) = \int_0^t e^{\nu(t-s)} a(t-s)\, y(s)\, ds + e^{-\nu t} f(t).$$

As this equation is exactly of the same form as (IE) we see that it is sufficient to assume $\|e^{-\nu t} a(t)\|_{1+\beta,\beta}$ is $L^1(0, \infty)$ for some $\gamma > 0$.

In the region Λ consider the contour Γ consisting of Γ_1, Γ_2, and Γ_3, where

$$\Gamma_1 = \{re^{i\phi} : r \geqslant 1\}\ \Gamma_3 = \{r^{-i\phi} : r \geqslant 1\}, \qquad \frac{\pi}{2} < \phi < \frac{\pi}{2} + \delta,$$

$$\Gamma_2 = \{e^{i\theta} : -\phi \leqslant \theta \leqslant \phi\},$$

oriented so that Im λ is increasing on Γ_1 and Γ_2. We define $R(t)$, $t \geqslant 0$, on X by $R(0) = I$ and

$$R(t)x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t} (\lambda I - A_0 - B^*(\lambda))^{-1}\, x d\lambda, \qquad t > 0,$$

or, equivalently, using the notation of (V2),

$$R(t)\, x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t} \rho(\lambda)\, x d\lambda, \qquad t > 0. \tag{R}$$

Similarly, we define $r(t)$, $t > 0$, on X by

$$r(t)\, x = -(2\pi i)^{-1} \int_\Gamma e^{\lambda t} a^*(\lambda)(I - a^*(\lambda))^{-1}\, x d\lambda$$

or

$$r(t)\, x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t} \sigma(\lambda)\, x d\lambda. \tag{r}$$

The purpose of this paper is to show that $R(t)$ and $r(t)$ are resolvents for (VE) and (IE), respectively, and to determine when the appropriate variation of parameter formulas yield solutions for (VE) and (IE). We define a resolvent operator for (VE) as in [11] but stated for convolution equations.

DEFINITION 2.1. $R(t)$ is said to be a resolvent operator for (VE) if $R(t) \in \mathscr{L}(X)$, $0 \leqslant t < \infty$, and if it satisfies:

(a) $R(t)$ is strongly continuous for $t \geqslant 0$ with $R(0) = I$ and $\|R(t)\| \leqslant Me^{\beta t}$ for some constants β and $M \geqslant 1$.

(b) $R(t) \in \mathscr{L}(Y)$ and $R(t)$ is strongly continuous, $t \geqslant 0$, on Y.

(c) For each $x \in Y$, $R(t)x$ is continuously differentiable, $t \geqslant 0$,

$$\dot{R}(t)x = A_0 R(t)x + \int_0^t B(t-u)R(u)x\,du$$

and

$$\dot{R}(t)x = R(t)A_0 x + \int_0^t R(t-u)B(u)x\,du.$$

To this point we have not placed any restrictions on the function $f(t)$ in (VE) or (IE). If we require $f(t)$ to be continuous we define a solution of (VE) as in [11].

DEFINITION 2.2. By a solution of (VE) is meant a function $x \in C([0, \infty), Y) \cap C^1([0, \infty), X)$ with $x(0) = x_0$ such that (VE) is satisfied for $t \geqslant 0$.

From the above definitions it follows immediately that if $R(t)$ is a resolvent operator and $x_0 \in Y$, then $x(t) = R(t)x_0$ is a solution of (VE) in the case $f = 0$. Actually more can be said.

THEOREM 2.3. *Suppose $R(t)$ is a resolvent operator for* (VE), $x_0 \in Y$ *and $f \in C([0, \infty), X)$. If $x(t)$ is a solution of* (VE), *then*

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)\,ds. \tag{VPR}$$

Proof. Consider

$$\begin{aligned} w(t) &= x(t) - R(t)x_0 - \int_0^t R(t-s)f(s)\,ds \\ &= \int_0^t \frac{\partial}{\partial s}(R(t-s)x(s))\,ds - \int_0^t R(t-s)f(s)\,ds. \end{aligned}$$

From Definition 2.1(c) it follows that for $x \in Y$,

$$\dot{R}(t-s)x = R(t-s)A_0 x + \int_s^t R(t-v)B(v-s)x\,dv.$$

Using the fact that $x(t)$ is a solution of (VE) we see that

$$w(t) = \int_0^t \int_0^s R(t-s)\, B(s-u)\, duds$$
$$- \int_0^t \int_s^t R(t-v)\, B(v-s)\, x(s)\, dvds$$
$$= 0$$

since $R(t-v)\, B(v-s)\, x(s)$ is integrable.

This result is the standard variation of parameters formula for (VE). Because our problem is infinite dimensional, however, (VPR) may not, in fact, be a solution. However, the following result is an easy consequence of the existence of a resolvent operator.

THEOREM 2.4. *Suppose* $R(t)$ *is a resolvent operator for* (VE). *If* (V1) *is valid,* $x_0 \in Y$ *and* $f \in C([0, \infty), Y)$, *then*

$$x(t) = R(t)\, x_0 + \int_0^t R(t-s)\, f(s)\, ds$$

is a solution of (VE).

Proof. As we have already noted, $R(t)\, x_0$ is a solution of (VE) with $f = 0$. We thus need only consider

$$w(t) = \int_0^t R(t-s)\, f(s)\, ds.$$

From Definition 2.1(c) we see that for $x \in Y$,

$$\dot{R}(t-s)\, x = A_0 R(t-s)\, x + \int_s^t B(t-c)\, R(v-s)\, x dv.$$

Hence,

$$\dot{w}(t) = f(t) + \int_0^t \dot{R}(t-s)\, f(s)\, ds$$
$$= f(t) + \int_0^t A_0 R(t-s) f(s)\, ds$$
$$+ \int_0^t \int_s^t B(t-v)\, R(v-s) f(s)\, dvds.$$

Thus since A_0 and $B(t)$ are closed operators and $R(t)$ is strongly continuous on Y,

$$\dot{w}(t) = f(t) + A_0 w(t) + \int_0^t B(t-v)\, w(v)\, dv.$$

In the next section we shall show that $R(t)$ is indeed a resolvent operator. Later we shall extend the class of functions $f(t)$ which yield solutions of (VE). We shall also extend the concept of solution so that solutions will be obtained for $x_0 \in Y^\alpha$, $\alpha > 0$.

Resolvent operators for integral equations are not as well behaved as resolvent operators for integrodifferential equations. In particular, they need not be bounded operators in general (cf. [11], for example).

DEFINITION 2.5. Operator $r(t)$ is said to be a resolvent operator for (IE) if $r(t)$ is locally Bochner integrable in X for each $x \in Y$ and locally Bochner integrable in Y for each $x \in Y^2$ and, in addition, for $x \in Y^2$, $r(t)$ satisfies

$$r(t)\, x = -a(t)\, x + \int_0^t r(t-u)\, a(u)\, x du, \qquad \text{a.e. } t \geqslant 0,$$

and

$$r(t)\, x = -a(t)\, x + \int_0^t a(t-u)\, r(u)\, x du, \qquad \text{a.e. } t \geqslant 0.$$

DEFINITION 2.6. By a solution of (IE) is meant a function $x(\cdot)$, $x \in L^1(0, T), Y)$ for every T, $0 < T < \infty$, which satisfies (IE) a.e. $t \geqslant 0$.

As a result of these definitions one has the following result:

THEOREM 2.7. *Suppose* $r(t)$ *is a resolvent for* (IE). *If* (I1) *is valid*, $f \in L^1([0, T], Y^2)$ *for every* T, $0 < T < \infty$, *then*

$$x(t) = f(t) - \int_0^t r(t-u)\, f(u)\, du \tag{VPr}$$

is a solution of (IE).

Proof. From Definition 2.5,

$$\begin{aligned} w(t) &= \int_0^t r(t-u)\, f(u)\, du \\ &= -\int_0^t a(t-u)\, f(u)\, du + \int_0^t \int_u^t a(t-v)\, r(v-u)\, f(u)\, dv du \\ &= -\int_0^t a(t-u)\, f(u)\, du + \int_0^t \int_0^v a(t-v)\, r(v-u)\, f(u)\, du dv \end{aligned}$$

and, since $a(t)$ is closed,

$$= -\int_0^t a(t-u)\,f(u)\,du + \int_0^t a(t-v)\,w(v)\,dv.$$

Letting $x(t) = f(t) - w(t)$, we see that $x(t)$ is in $L^1((O, T), Y)$ for every T, $0 < T < \infty$, and that

$$x(t) = f(t) + \int_0^t a(t-u)\,x(u)\,du.$$

As in the case of (VE) we shall extend the class of functions $f(t)$ which will yield a solution of (IE).

3. Analytic Resolvent Operators

In this section we shall determine conditions which imply that (R) and (r) define resolvent operators for (VE) and (IE), respectively. In addition, we shall show that $R(t)$ and $r(t)$ exhibit many of the properties of an analytic semigroup.

THEOREM 3.1. *Suppose* (V1)–(V3) *are valid, then* (R) *defines a resolvent operator* $R(t)$ *for* (VE). *In addition, there exists a constant* $N \geqslant 1$ *so that* $\|R(t)\| \leqslant N$. *Further,* $R(t)$ *has an analytic extension to the region* $\{t \in C: |\arg t| < \delta\}$.

Proof. First we note that for $t > 0$ and $x \in X$ it follows from (V2) and (V3) that the integrals

$$\int_\Gamma e^{\lambda t}\,\rho(\lambda)\,x d\lambda, \qquad \int_\Gamma e^{\lambda t}\,A\rho(\lambda)\,x d\lambda$$

both converge. Thus, for $t > 0$ and $x \in X$, $R(t)\,x \in Y$. To see that $R(t) \in \mathscr{L}(X)$ with $\|R(t)\|$ uniformly bounded for $t > 0$, let $\lambda t = \gamma$ and $J = t\Gamma$ to get

$$R(t)\,x = (2\pi i)^{-1} \int_J t^{-1}\,e^\gamma \rho(\gamma t^{-1})\,x d\gamma$$

and use Cauchy's theorem to obtain

$$R(t)\,x = (2\pi i)^{-1} \int_\Gamma t^{-1}\,e^\gamma \rho(\gamma t^{-1})\,x d\gamma.$$

It now follows from (V2) that

$$\|R(t)\,x\| \leqslant (2\pi)^{-1} \int_\Gamma |e^\gamma|\, M\, |\gamma|^{-1}\, |d\gamma|\, \|x\|$$

independently of $t > 0$. In exactly the same manner, for $x \in Y$, we use the estimate $\|A\rho(\lambda)\|_{1,0} \leqslant M/|\lambda|$ from (V3) to show

$$\|AR(t)\,x\| \leqslant (2\pi)^{-1} \int_\Gamma |e^\gamma|\, M\, |\gamma|^{-1}\, |d\gamma|\, \|x\|_1$$

so that $R(t) \in \mathscr{L}(Y)$ with $\|R(t)\|_1$ uniformly bounded for $t > 0$. As $R(0) = I$ by definition, $\|R(t)\|$ and $\|R(t)\|_1$ are uniformly bounded $t \geqslant 0$.

It is clear that $R(t)$ is strongly continuous for $t > 0$ in $\mathscr{L}(X)$ and in $\mathscr{L}(Y)$. Now, let $x \in Y$ and $t > 0$ so that

$$R(t)\,x - x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t} \{\rho(\lambda) - \lambda^{-1} I\}\, x d\lambda$$

$$= (2\pi i)^{-1} \int_\Gamma e^{\lambda t}\, \lambda^{-1} \rho(\lambda)(A_0 + B(\lambda))\, x d\lambda.$$

From (V3), $(A_0 + B^*(\lambda))\,x$ is bounded for $\lambda \in \Lambda, |\lambda| \geqslant 1$, so that

$$\int_\Gamma \lambda^{-1} \rho(\lambda)(A_0 + B^*(\lambda))\, x d\lambda = 0.$$

Thus, $\|R(t)\,x - x\| \to 0$ as $t \to 0^+$. Since $\|R(t)\|$ is uniformly bounded and Y is dense in X, $\|R(t)\,x - x\| \to 0$ as $t \to 0^+$ for all $x \in X$. From (V3) we see that $\|A\rho(\lambda)\|_{1,0} \leqslant M/|\lambda|$ and $\|A_0 x + B^*(\lambda)\,x\|_1$ is bounded for $x \in D$.

As D is dense in Y, we obtain $\|R(t)\,x - x\|_1 \to 0$ as $t \to 0^+$ for all $x \in Y$. Hence, $R(t)$ is strongly continuous for $t \geqslant 0$ in $\mathscr{L}(X)$ and $\mathscr{L}(Y)$. We have thus shown that $R(t)$ satisfies Definition 2.1(a) and (b).

Before proceeding to demonstrate that Definition 2.1(c) is satisfied by $R(t)$, let us note that $R^*(\lambda)\,x = \rho(\lambda)\,x$. To see this we compute for $\operatorname{Re} \lambda > 1$

$$R^*(\lambda)\,x = (2\pi i)^{-1} \int_0^\infty e^{-\lambda t} \int_\Gamma e^{\gamma t}\, \rho(\gamma)\, x d\gamma dt$$

$$= (2\pi i)^{-1} \int_\Gamma \int_0^\infty e^{-(\lambda - \gamma)t}\, \rho(\gamma)\, x dt d\gamma$$

$$= (2\pi i)^{-1} \int_\Gamma (\lambda - \gamma)^{-1}\, \rho(\gamma)\, x d\gamma.$$

If we define the new contour Γ_n to be that portion of Γ within distance n of the origin together with the arc of the circle $|\lambda| = n$, C_n, with $|\arg| \leqslant \phi$ and orient it so that the orientation along Γ is preserved, we see that

$$\rho(\lambda)\, x = (2\pi i)^{-1} \int_{\Gamma_n} (\lambda - \gamma)^{-1}\, \rho(\gamma)\, x d\gamma.$$

However, since $|\gamma - \lambda|/n \to 1$ on C_n as $n \to \infty$ and $\|\rho(\gamma)\, x\| \leqslant M/|\gamma|$ we have

$$\int_{C_n} (\gamma - \lambda)^{-1}\, \rho(\gamma)\, x d\gamma \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus, it follows that $R^*(\lambda)\, x = \rho(\lambda)\, x$.

Now, let $x \in X$ and consider for $t > 0$

$$\dot{R}(t)\, x = (2\pi i)^{-1} \int_{\Gamma} e^{\lambda t}\, \lambda \rho(\lambda)\, x d\lambda$$

which converges absolutely and uniformly by (V2) and

$$\dot{R}(t)\, x = (2\pi i)^{-1} \int_{\Gamma} e^{\lambda t} (A_0 + B^*(\lambda))\, \rho(\lambda)\, x d\lambda.$$

Now let $x \in Y$. Then

$$\begin{aligned} \dot{R}(t)\, x &= (2\pi i)^{-1} \int_{\Gamma} e^{\lambda t} \rho(\lambda)(A_0 + B^*(\lambda))\, x d\lambda \\ &= (2\pi i)^{-1} \int_{\Gamma} e^{\lambda t}\, \rho(\lambda)\, A_0 x d\lambda \\ &\quad + (2\pi i)^{-1} \int_{\Gamma} e^{\lambda t}\, \rho(\lambda)\, B^*(\lambda)\, x d\lambda. \end{aligned}$$

Note that

$$\int_{\Gamma} e^{\lambda t}\, \rho(\lambda)\, B^*(\lambda)\, x d\lambda = \int_{\Gamma} e^{\lambda t} R^*(\lambda)\, B^*(\lambda)\, x d\lambda$$

and define $F(t)$ by

$$F(t) = \int_0^t R(t-s)\, B(s)\, x ds.$$

Since $R(t)$ is bounded and strongly continuous for $t \geqslant 0$ and $\|B(t)\, x\| \leqslant b(t)\, \|x\|_1$, $F(t)$ exists and is bounded and continuous, $t \geqslant 0$.

It follows from Hille–Phillips [20, p. 219] that for $\gamma > 0$,

$$(2\pi i) \int_0^t F(s)\, ds = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \lambda^{-1} F^*(\lambda)\, d\lambda$$

$$= \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \lambda^{-1} R^*(\lambda)\, B^*(\lambda)\, x d\lambda$$

$$= \int_\Gamma e^{\lambda t} \lambda^{-1} \rho(\lambda)\, B^*(\lambda)\, x d\lambda.$$

Using this representation, differentiate to get

$$F(t)\, x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t} \rho(\lambda)\, B^*(\lambda)\, x d\lambda$$

so that

$$\dot{R}(t)\, x = R(t)\, A_0 x + \int_0^t R(t-s)\, B(s)\, x ds, \qquad t > 0. \tag{3.1}$$

To get the other resolvent equation for $x \in Y$, $t > 0$, we note that

$$\dot{R}(t)\, x = (2\pi i)^{-1} \int_\Gamma e^{\lambda t} (A_0 + B^*(\lambda))\, \rho(\lambda)\, x d\lambda$$

$$= A_0 R(t)\, x + (2\pi i)^{-1} \int_\Gamma e^{\lambda t}\, B^*(\lambda)\, \rho(\lambda)\, x d\lambda.$$

Now as we showed earlier $R(t)$ is strongly continuous and uniformly bounded in $\mathscr{L}(Y)$. Hence, arguing as above

$$(2\pi i) \int_0^t B(t-s)\, R(s)\, x ds = \int_\Gamma e^{\lambda t}\, B^*(\lambda)\, \rho(\lambda)\, x d\lambda$$

and so for $t > 0$ we obtain the resolvent equation,

$$\dot{R}(t)\, x = A_0 R(t)\, x + \int_0^t B(t-s)\, R(s)\, x ds, \qquad t > 0. \tag{3.2}$$

We now wish to show that for $x \in Y$, the resolvent equations are valid for $t \geqslant 0$. From (3.1) we see that for $x \in Y$,

$$\lim_{t \to 0^+} \dot{R}(t)\, x = A_0 x$$

as $R(t)$ is bounded in $\mathscr{L}(X)$ and $\|B(t)\|_{1,0} \leqslant b(t)$ while $R(t)$ is strongly continuous with $R(0)=I$. It now follows from [8, Problem 2, p. 158] that $R(t)x$ has a right derivative at $t=0$ equal to A_0x. Thus (3.1) and (3.2) are valid for $t \geqslant 0$.

The analyticity on $\{t \in C: |\arg t| < \delta\}$ is clear.

Remark 3.2. It follows from the proof of Theorem 3.1 that the key to obtaining a solution of

$$\dot{R}(t)x = A_0R(t)x + \int_0^t B(t-s)R(s)x\,ds, \qquad t>0, \tag{3.3}$$

is getting the convolution

$$\int_0^t B(t-s)R(s)x\,ds$$

to exist and be continuous for $t>0$ so that it can be shown that

$$\int_0^t B(t-s)R(s)x\,ds = (2\pi i)^{-1}\int_\Gamma e^{\lambda t}B^*(\lambda)\rho(\lambda)x\,d\lambda,\, t>0.$$

This observation leads us to consider the problem of determining the set of initial states x for which (3.3) is valid.

THEOREM 3.3. *Suppose* (V1)–(V4) *are valid and* $b(t)$ *is bounded on each interval of the form* $0<T_1 \leqslant t \leqslant T_2 < \infty$. *Then for* $0<\alpha \leqslant 1$ *and* $t>0$, $R(t) \in \mathscr{L}(Y^\alpha, Y)$ *with* $\|R(t)\|_{\alpha,1} \leqslant Kt^{\alpha-1}$, *K a positive constant. In addition, for* $x \in Y^\alpha$; $0<\alpha \leqslant 1$, *one has*

$$\dot{R}(t)x = AR(t)x + \int_0^t B(t-s)R(s)x\,ds, \qquad t>0.$$

Proof. Let $x \in Y^\alpha$ and consider for $t>0$,

$$AR(t)x = (2\pi i)^{-1}\int_\Gamma e^{\lambda t}A\rho(\lambda)x\,d\lambda = (2\pi i)^{-1}\int_\Gamma e^{\gamma}t^{-1}A\rho(\gamma t^{-1})x\,d\gamma,$$

where we have changed variable $\gamma = \lambda t$ and contour by Cauchy's theorem as in Theorem 3.1. Estimating, we obtain from (V4),

$$\|AR(t)x\| \leqslant (2\pi)^{-1}\int_\Gamma |e^\gamma|\, t^{-1}\, \|A\rho(\gamma t^{-1})\|_{\alpha,0}\, \|x\|_\alpha\, |d\gamma| \leqslant Kt^{\alpha-1}\|x\|_\alpha.$$

Now, for $x \in Y^\alpha$,

$$\int_0^t \|B(t-s)\,R(s)\,x\|\,ds \leqslant \int_0^t \|B(t-s)\|_{1,0}\,\|R(s)\|_{\alpha,1}\,\|x\|_\alpha\,ds$$
$$\leqslant \int_0^t b(t-s)\,Ks^{\alpha-1}\,ds\,\|x\|_\alpha$$

so that $\int_0^t B(t-s)\,R(s)\,xds$ exists. Also, as $R(t)\,x$ is continuous in Y, $t>0$, for $x \in Y^\alpha$ with $\|R(t)\,x\|_1 \leqslant Kt^{\alpha-1}\,\|x\|_\alpha$. It follows routinely from $\|B(t)\|_{1,0} \leqslant b(t)$ with $b(t)$ bounded on intervals of the form $0 < T_1 \leqslant t \leqslant T_2 < \infty$ that $\int_0^t B(t-s)\,R(s)\,xds$ is continuous for $t>0$. We thus see, as in Remark 3.2, that (3.3) is satisfied by $R(t)\,x$.

If $B(t)$ is bounded by a "lower order" operator than A we obtain a result reminiscent of the theory of analytic semi-groups.

THEOREM 3.4. *Suppose* (V1)–(V3) *are valid with* $A_0 = A$, *and in addition,* $B(t) \in \mathscr{L}(Y^\alpha, X)$ *for some* $\alpha, 0 \leqslant \alpha < 1$, *a.e.* $t \geqslant 0$, *with* $\|B(t)\|_{\alpha,0} \leqslant b_\alpha(t)$. *Suppose* $b_\alpha \in L^1(0,\infty)$ *with* $b_\alpha(t)$ *bounded on any interval of the form* $0 < T_1 \leqslant T_2 < \infty$. *Suppose also that there exists a constant* M *so that* $\|A\rho(\lambda)\| \leqslant M$ *for* $\lambda \in \Lambda$, $|\lambda| \geqslant 1$. *Then for any* $x \in X$.

$$\dot{R}(t)\,x = AR(t)\,x + \int_0^t B(t-s)\,R(s)\,xds, \; t > 0.$$

Proof. From the inequality (cf. [27, p. 38])

$$\|(-A)^\alpha x\| \leqslant C_{\alpha,1}\,\|Ax\|^\alpha\,\|x\|^{1-\alpha}$$

with $C_{\alpha,1}$ constant, $x \in Y$, we obtain for $x \in X$,

$$\|(-A)^\alpha\,\rho(\lambda)\,x\| \leqslant C_{\alpha,1}\,\|A\rho(\lambda)\,x\|^\alpha\,\|\rho(\lambda)\,x\|^{1-\alpha} \leqslant K_1\,|\lambda|^{\alpha-1}\,\|x\|,$$

where K_1 is constant, $\lambda \in \Lambda$ with $|\lambda| \geqslant 1$. Thus, changing variables, $\gamma = t\lambda$. and using Cauchy's theorem,

$$(-A)^\alpha R(t)\,x = (2\pi i)^{-1} \int_\Gamma e^\alpha t^{-1}(-A)^\alpha \rho(\gamma t^{-1})\,xd\gamma$$

and so

$$\|(-A)^\alpha R(t)\,x\| \leqslant (2\pi)^{-1} \int_\Gamma |e^\gamma|\,K\,|\gamma|^{\alpha-1} t^{-\alpha}\,\|x\|\,|d\gamma| \leqslant Kt^{-\alpha}\,\|x\|.$$

We see now that

$$\|B(t-s)\,R(s)\,x\| \leqslant b_\alpha(t-s)\,Ks^{-\alpha}\,\|x\|$$

for all $x \in X$. Hence as before, $\int_0^t B(t-s)\,R(s)\,xds$ is continuous for $t > 0$ and the result follows.

Related to Theorem 3.3 we have the following result when $A_0 = 0$.

THEOREM 3.5. *Suppose* (V1)–(V3) *and* (V5) *are valid with $b(t)$ bounded on each interval of the form $0 < T_1 \leqslant t \leqslant T_2 < \infty$. Then for $\alpha, 0 < \alpha < 1$, with $\beta(\alpha) > 0$, $R(t) \in \mathscr{L}(Y^\alpha, Y)$ and $\|R(t)\|_{\alpha,1} \leqslant K t^{\beta(\alpha)-1}$ for $t > 0$ with K a positive constant. In addition, for $x \in Y^\alpha$, we have*

$$\dot{R}(t)\,x = \int_0^t B(t-s)\,R(s)\,xds, \qquad t > 0.$$

Proof. The proof proceeds exactly as the proof of Theorem 3.3 and is omitted.

We now turn to the problem of obtaining a resolvent operator for (IE).

THEOREM 3.6. *Suppose* (I1)–(I4) *are valid. Then* (*r*) *defines a resolvent operator $r(t)$ for* (IE) *with*

$$r(t) \in \mathscr{L}(Y^\beta, X) \cap \mathscr{L}(Y^{1+\beta}, Y) \quad \text{and} \quad \|r(t)\|_{\beta,0} \quad \text{and} \quad \|r(t)\|_{1+\beta,1}$$

are bounded above by $Kt^{\alpha\beta-1}$, $0 < \beta \leqslant 1$, for $t > 0$, where K is some positive constant. In addition, for $x \in Y^{1+\beta}$

$$r(t)\,x = -a(t)\,x + \int_0^t a(t-u)\,r(u)\,xdu$$

and

$$r(t)\,x = -a(t)\,x + \int_0^t r(t-u)\,a(u)\,xdu$$

a.e. $t \geqslant 0$.

Proof. We first note that since $\sigma(\lambda)$ is uniformly bounded in $\mathscr{L}(X)$ and $\mathscr{L}(Y)$ that $r(t)\,x$ is continuous, $t > 0$, in X for $x \in X$ and in Y for $x \in Y$. Also, for $x \in Y^\beta$, $0 < \beta \leqslant 1$, changing variables, $\gamma = t\lambda$, and using Cauchy's theorem yields

$$r(t)\,x = (2\pi i)^{-1} \int_\Gamma e^\gamma t^{-1}\,\sigma(\gamma t^{-1})\,xd\gamma,$$

so

$$\|r(t)\,x\| \leqslant (2\pi)^{-1} \int_\Gamma |e^\gamma|\, t^{\alpha\beta-1} M\, |\gamma|^{\alpha\beta}\, |d\gamma|\, \|x\|_\beta \leqslant K t^{\alpha\beta-1}\, \|x\|_\beta.$$

In the same manner, for $x \in Y^{1+\beta}$,

$$\|r(t)\,x\|_1 \leqslant K t^{\alpha\beta-1}\, \|x\|_{1+\beta}.$$

Thus, for $x \in Y$, $r(t)\,x$ is integrable in X and for $x \in Y^2$, $r(t)\,x$ is integrable in Y. Further, for $0 < \beta \leqslant 1$, $r(t) \in \mathscr{L}(Y^\beta, X) \cap \mathscr{L}(Y^{1+\beta}, Y)$ with $\|r(t)\|_{\beta,0}$ and $\|r(t)\|_{1+\beta,1}$ both bounded by $Kt^{\alpha\beta-1}$.

In order to complete the proof we must first examine $r^*(\lambda)$. For $x \in Y^\beta$ and $\lambda \geqslant 0$,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\, r(t)\, x dt &= (2\pi i)^{-1} \int_0^\infty e^{-\lambda t} \int_\Gamma e^{\gamma t}\, \sigma(\gamma)\, x d\gamma dt \\ &= (2\pi i)^{-1} \int_\Gamma (\lambda - \gamma)^{-1}\, \sigma(\gamma)\, x d\gamma, \\ &= \sigma(\lambda)\, x\end{aligned}$$

(arguing as in the proof of Theorem 3.1). Also, if $x \in Y^{1+\beta}$, then $(r * a)(t)\, x = \int_0^t r(t-u)\, a(u)\, x du$ exists a.e. $t \geqslant 0$ and is in $L^1((0, \infty), X)$ since $\|a(t)\|_{1+\beta,\beta} \leqslant b(t)$ while $\|r(t)\|_{\beta,0} \leqslant K t^{\alpha\beta-1}$. Similarly, $(a * r)(t)\, x = \int_0^t a(t-u)\, r(u)\, x du$ exists a.e. $t \geqslant 0$ and is integrable since $\|r(t)\|_{1+\beta,1} \leqslant Kt^{\alpha\beta-1}$ while $\|a(t)\|_{1,0} \leqslant b(t)$. Taking Laplace transforms,

$$\begin{aligned}[-a(t)\, x + (r * a)(t)\, x]^* &= -a^*(\lambda)\, x + \sigma(\lambda)\, a^*(\lambda)\, x \\ &= -a^*(\lambda)(I - \sigma(\lambda))\, x \\ &= \sigma(\lambda)\, x,\end{aligned}$$

and similarly,

$$[-a(t)\, x + (a * r)(t)\, x]^* = \sigma(\lambda)\, x.$$

For $x \in Y^{1+\beta}$, $0 < \beta \leqslant 1$, we now obtain

$$\begin{aligned}r(t)\, x &= (2\pi i)^{-1} \int_\Gamma e^{\lambda t}\, \sigma(\lambda)\, x d\lambda \\ &= (2\pi i)^{-1} \int_\Gamma e^{\lambda t} [-a^*(\lambda)\, x + a^*(\lambda)\, r^*(\lambda)\, x]\, d\lambda\end{aligned}$$

or

$$r(t)\,x = -(2\pi i)^{-1}\int_{\Gamma} e^{\lambda t}\, a^*(\lambda)\, x d\lambda + (2\pi i)^{-1}$$
$$\times \int_{\Gamma} e^{\lambda t}\, a^*(\lambda)\, r^*(\lambda)\, x d\lambda.$$

From Hille–Phillips [20, p. 219], for $\gamma > 0$ and $t > 0$,

$$\int_0^t a(s)\, x ds = (2\pi i)^{-1}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\, \lambda^{-1}\, a^*(\lambda)\, x d\lambda.$$
$$= (2\pi i)^{-1}\int_{\Gamma} e^{\lambda t}\, \lambda^{-1}\, a^*(\lambda)\, x d\lambda,$$

and a differentiation yields,

$$a(t)\, x = (2\pi i)^{-1}\int_{\Gamma} e^{\lambda t}\, a^*(\lambda)\, x d\lambda. \tag{3.5}$$

Also, for $F(t) = (a * r)(t)\, x$ and $\gamma > 0$,

$$\int_0^t F(s)\, ds = (2\pi i)^{-1}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\, \lambda^{-1}\, F^*(\lambda)\, d\lambda.$$

Now $r^*(\lambda)\, x = -a^*(\lambda)(I - a^*(\lambda))^{-1} x$ so $\|r^*(\lambda)\|_{1+\beta,1} \leqslant M\,|\lambda|^{-\alpha\beta}$. Thus, $\|F^*(\lambda)\| = \|a^*(\lambda)\, r^*(\lambda)\, x\| \leqslant M^2\,|\lambda|^{-\alpha - \alpha\beta}\,\|x\|_{1+\beta}$ and

$$\int_0^t F(s)\, ds = (2\pi i)^{-1}\int_{\Gamma} e^{\lambda t}\lambda^{-1}\, a^*(\lambda)\, r^*(\lambda)\, x d\lambda.$$

A differentiation now yields

$$\int_0^t a(t-u)\, r(u)\, x du = (2\pi i)^{-1}\int_{\Gamma} e^{\lambda t} a^*(\lambda)\, r^*(\lambda)\, x d\lambda, \tag{3.6}$$

a.e. $t \geqslant 0$. Hence, from (3.4)–(3.6), we obtain

$$r(t)\, x = -a(t)\, x + \int_0^t a(t-u)\, r(u)\, x du,$$

a.e. $t \geqslant 0$ for $x \in Y^{1+\beta}$. In a completely analogous manner, we also obtain

$$r(t)\, x = -r(t)\, x + \int_0^t r(t-u)\, a(u)\, x du,$$

a.e. $t \geqslant 0$ for $x \in Y^{1+\beta}$.

4. Variation of Parameters Formulas

In this section we derive conditions on x_0 and $f(\cdot)$ so that the variation of parameter formulas

$$x(t) = R(t)\, x_0 + \int_0^t R(t-s) f(s)\, ds, \tag{VPR}$$

$$x(t) = f(t) - \int_0^t r(t-s)\, f(s)\, ds, \tag{VPr}$$

solve Eqs. (VE) and (IE), respectively. First we begin with (VE) which we write in the form

$$\dot{x}(t) = \mathscr{A}(t)\, x(t) + f(t), \qquad t > 0,$$

where

$$\mathscr{A}(t) \colon C([0,t]; X) \to X \text{ is defined by}$$

$$\mathscr{A}(t)\, x(t) = Ax(t) + \int_0^t B(t-s)\, x(s)\, ds.$$

Theorem 4.1. *Suppose the conditions of Theorem* 3.3 *hold and* $x_0 \in Y^\alpha$, $f \in C([0,T], Y^\alpha)$, *then* (VPR) *is the solution of* (VE) *in the sense that* $x \in C([0,T], X) \cap C^1((0,T], X)$ *and*

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)\, x(s)\, ds + f(t), \qquad t > 0$$

and $x(0) = x_0$.

Proof. Since $x_0 \in Y^\alpha$ by Theorem 3.3 we know that $R(t)\, x_0$ satisfies the homogeneous version of (VE) and hence we need only consider

$$v(t) = \int_0^t R(t-\rho)\, f(\rho)\, d\rho.$$

Formally

$$\mathscr{A}(t)\, v(t) = \int_0^t AR(t-\rho)\, f(\rho)\, d\rho$$
$$+ \int_0^t \int_0^\rho B(t-\rho)\, R(\rho - s)\, f(s)\, ds d\rho.$$

This expression makes sense since A and $B(t)$ are closed and we have the following estimate

$$\|\mathscr{A}(t)\, v(t)\| \leqslant K \int_0^t (t-\rho)^{\alpha-1} \|f(\rho)\|_\alpha \, d\rho$$

$$+ K \int_0^t \int_0^\rho \|B(t-\rho)\|_{1,0} (\rho - s)^{\alpha-1} \|f(s)\|_\alpha \, ds\, d\rho.$$

So $v(t) \in D(\mathscr{A}(t))$ and it is not difficult to show that $\mathscr{A}(\cdot)\, v(\cdot) \in C([0, T]; X)$. Also

$$\int_0^\alpha \mathscr{A}(t)\, v(t)\, dt = \int_0^\alpha \int_0^t A R(t-\rho) f(\rho)\, d\rho\, dt$$

$$+ \int_0^\alpha \int_0^t \int_0^\rho B(t-\rho) R(\rho - s) f(s)\, ds\, d\rho\, dt$$

$$= \int_0^\alpha \int_\rho^\alpha A R(t-\rho) f(\rho)\, dt\, d\rho$$

$$+ \int_0^\alpha \int_s^\alpha \int_s^t B(t-\rho) R(\rho - s) f(s)\, d\rho dt\, ds$$

by Fubini's theorem. But in establishing the resolvent equation (3.2) in Theorems 3.1 and 3.3 we in fact proved for $x \in Y^\alpha$

$$R(t-s)\, x - x = \int_s^t A R(a-s)\, s da$$

$$+ \int_s^t \int_s^a B(a-\rho) R(\rho - s)\, x d\rho da.$$

Hence

$$\int_0^\alpha \mathscr{A}(t)\, v(t)\, dt = v(\alpha) - \int_0^\alpha f(\rho)\, d\rho.$$

So

$$\dot{v}(t) = \mathscr{A}(t)\, v(t) + f(t), \qquad t > 0.$$

THEOREM 4.2. *Suppose the conditions of Theorem* 3.5 *hold and* $x_0 \in Y^\alpha$, $f \in C([0, T], Y^\alpha)$, *then* (VPR) *is the solution of* (VE) *in the sense that* $x \in C([0, T], X) \cap C^1([0, T], X)$ *and*

$$\dot{x}(t) = \int_0^t B(t-s)\, x(s)\, ds + f(t), \qquad t > 0$$

and $x(0) = x_0$.

Proof. The proof follows the same arguments as that of Theorem 4.1.

We now consider (IE) where the extra smoothness properties of the resolvent $r(t)$ enable us to extend Theorem 2.7 as follows:

THEOREM 4.3. *Suppose* (I1)–(I4) *are valid and that* $f \in L^1([0, T], Y)$. *If* $x(t)$ *is a solution of* (IE) *with* $x \in L^1([0, T], Y^{1+\beta})$, $0 < \beta \leqslant 1$, *then on* $[0, T]$ $x(t)$ *satisfies*

$$x(t) = f(t) - \int_0^t r(t-u) f(u)\, du, \tag{VPr}$$

where $r(t)$ *is defined by* (r).

Proof. We note that if $r(t)$ is defined by (r), then $r(t) \in \mathscr{L}(X)$ for $t > 0$. Further, we recall that $\|r(t)\|_{1,0} \leqslant Kt^{\alpha-1}$ so that $r(t-u) f(u)$, $r(t-u) x(u)$ and $a(t-u) x(u)$ are integrable on $0 \leqslant u \leqslant t \leqslant T$. From (IE) we see that

$$\begin{aligned} \int_0^t r(t-u)\, x(u)\, du - \int_0^t r(t-u) f(u)\, du \\ = \int_0^t r(t-u) \int_0^u a(u-s)\, x(s)\, ds du. \end{aligned}$$

Now as $r(t) \in \mathscr{L}(X)$ for $t > 0$,

$$\begin{aligned} &\int_0^t r(t-u) \int_0^u a(u-s)\, x(s)\, ds du \\ &\quad = \int_0^t \int_0^u r(t-u)\, a(u-s)\, x(s)\, ds du \\ &\quad = \int_0^t \int_s^t r(t-u)\, a(u-s)\, x(s)\, du ds \\ &\quad = \int_0^t r(t-s)\, x(s)\, ds + \int_0^t a(t-s)\, x(s)\, ds. \end{aligned}$$

Thus

$$-\int_0^t r(t-u) f(u)\, du = \int_0^t a(t-u)\, x(u)\, du$$

and $x(t)$ must satisfy (VPr).

On the other hand we have

THEOREM 4.4. *Suppose* (I1)–(I4) *are valid with* $f \in L^1([0, T], Y^{1+\beta})$, *then* (VPr) *is the solution of* (IE).

Proof. Since $f \in L^1([0, T], Y^{1+\beta})$, $\int_0^t r(t-s) f(s)\, ds \in L^1([0, T], Y)$ so $\int_0^t a(t-u) \int_0^u r(u-s) f(s)\, ds\, du$ is well defined in $L^1([0, T], X)$ and

$$\begin{aligned}
\int_0^t a(t-u) \int_0^u r(u-s) f(s)\, ds du \\
&= \int_0^t \int_s^t a(t-u)\, r(u-s) f(s)\, du ds \\
&= \int_0^t [r(t-s) + a(t-s)]\, f(s)\, ds.
\end{aligned}$$

So

$$\begin{aligned}
\int_0^t a(t-u)\, x(u)\, du &= \int_0^t a(t-u) f(u)\, du - \int_0^t r(t-s) f(s)\, ds \\
&= \int_0^t a(t-u) f(u)\, du \\
&= x(t) - f(t).
\end{aligned}$$

REFERENCES

1. R. W. CARR AND K. B. HANNSGEN, A non homogeneous integrodifferential equation in Hilbert Space, *SIAM J. Math. Anal.* **10** (1979), 961–984.
2. G. CHEN AND R. GRIMMER, Semigroups and integral equations, *J. Integral Equations* **2** (1980), 133–154.
3. G. CHEN AND R. GRIMMER, Integral equations as evolution equations, *J. Differential Equations*, **45** (1982), 53–74.
4. P. CLEMENT AND J. A. NOHEL, Abstract linear and nonlinear Volterra equations preserving positivity, *SIAM J. Math. Anal.* **10** (1979), 365–388.
5. R. F. CURTAIN AND A. J. PRITCHARD, "Infinite Dimensional Linear Systems Theory," Springer-Verlag, Berlin 1978.
6. G. DAPRATO AND M. IANNELLI, Linear abstract integro-differential equations of hyperbolic type in Hilbert spaces, *Rend. Sem. Mat. Univ. Padova* **62** (1980), 191–206.
7. G. DAPRATO AND M. IANNELLI, Linear integro-differential equations in Banach spaces, *Rend. Sem. Mat. Univ. Padova* **62** (1980), 207–219.
8. J. DIEUDONNÉ, "Foundations of Modern Analysis," Academic Press, New York, 1960.
9. A. FRIEDMAN, Monotonicity of solutions of Volterra integral equations in Banach space, *Trans. Amer. Math. Soc.* **138** (1969), 129–148.
10. A. FRIEDMAN AND M. SHINBROT, Volterra integral equations in Banach space, *Trans. Amer. Math. Soc.* **126** (1967), 131–179.
11. R. C. GRIMMER, Resolvent Operators for integral equations in a Banach space, *Trans. Amer. Math. Soc.* **273** (1982), 333–349.

12. R. C. Grimmer and R. K. Miller, Existence, uniqueness and continuity for integral equations in a Banach space, *J. Math. Anal. Appl.* **57** (1977), 429–447.
13. R. C. Grimmer and R. K. Miller, Well posedness of Volterra integral equations in Hilbert space, *J. Integral Equations* **1** (1979), 201–216.
14. S. I. Grossman and R. K. Miller, Perturbation theory for Volterra integrodifferential systems, *J. Differential Equations* **8** (1970), 457–474.
15. K. B. Hannsgen. A Volterra equation with parameter, *SIAM J. Anal.* **4** (1973), 22–30.
16. K. B. Hannsgen, A Volterra equation in Hilbert space, *SIAM J. Anal.* **5** (1974), 412–416.
17. K. B. Hannsgen, Uniform boundedness in a class of Volterra equations, *SIAM J. Anal.* **6** (1975), 689–697.
18. K. B. Hannsgen, The resolvent kernel of an integrodifferential equation in Hilbert space, *SIAM J. Anal.* **7** (1976), 481–490.
19. K. B. Hannsgen, Uniform L^1 behavior for an integrodifferential equation with parameter, *SIAM J. Anal.* **8** (1977), 626–639.
20. E. Hille and R. S. Phillips, "Functional Analysis and Semi-groups," Amer. Math. Soc., Providence, R.I., 1957.
21. R. K. Miller, Volterra integral equations in a Banach space, *Funkcial. Ekvac.* **18** (1975), 163–193.
22. R. K. Miller, An integrodifferential equation for rigid heat conductors with memory, *J. Math. Anal. Appl.* **66** (1978), 313–332.
23. R. K. Miller, "Nonlinear Volterra Integral Equations," Benjamin, Menlo Park, Calif., 1971.
24. R. K. Miller and R. L. Wheeler. Asymptotic behavior for a linear Volterra integral equation in Hilbert space, *J. Differential Equations* **23** (1977), 270–284.
25. R. K. Miller and R. L. Wheeler, Well-posedness and stability of linear Volterra integrodifferential equations in abstract spaces, *Funkcial. Ekvac.*, **21** (1978), 279–305.
26. A. Pazy, "Semigroups of Linear Operators and Applications to Parrtial Differential Equations," Dept. Math. Lecture Note, No. 10, University of Maryland, 1974.
27. H. Tanabe, "Equations of Evolution," Pittman, London, 1979.