Available online at www.sciencedirect.com
ScienceDirect
Historia Mathematica 37 (2010) 345–375
HISTORIA MATHEMATICA
www.elsevier.com/locate/yhmat

The scholar and the fencing master: The exchanges between Joseph Justus Scaliger and Ludolph van Ceulen on the circle quadrature (1594–1596)

Jan P. Hogendijk

Department of Mathematics, University of Utrecht, P.O. Box 80.010, 3508 TA Utrecht, Netherlands

Abstract

In Chapter 21 of Vanden Circkel (On the Circle) [Van Ceulen, 1596], the arithmetic teacher and fencing master Ludolph van Ceulen published his analysis of 16 propositions which had been submitted to him by an anonymous "highly learned man". In this paper, the author of the propositions will be identified as the classicist and humanist Joseph Justus Scaliger (1540–1609), who lived in the city of Leiden, just like Van Ceulen. The whole Chapter 21 of Van Ceulen's Vanden Circkel turns out to be a criticism of Scaliger's Cyclometrica (1594), a work which includes a false circle quadrature and many other incorrect theorems. The exchanges between Van Ceulen and Scaliger are analyzed in this paper and related to difference in social status and to different approaches to mathematics.

Samenvatting

In Hoofdstuk 21 van *Vanden Circkel* publiceerde de reken- en schermmeester Ludolph van Ceulen zijn analyses van 16 proposities die aan hem waren gestuurd door een anonieme "hooggeleerde." In dit artikel wordt de auteur ervan geïdentificeerd als de classicus en humanist Joseph Justus Scaliger (1540–1609), die net als Van Ceulen in Leiden woonde. Het blijkt dat het hele hoofdstuk 21 van *Vanden Circkel* een systematische kritiek is op de *Cyclometrica* van Scaliger van 1594, een werk waarin een foute cirkelkwadratuur en andere onjuiste meetkundige stellingen voorkomen. De uitwisseling tussen Van Ceulen en Scaliger worden in dit artikel geanalyseerd en in verband gebracht met verschil in maatschappelijke status en verschillende opvattingen over wiskunde.

MSC: 01A40

Keywords: Scaliger; Ludolph Van Ceulen; circle quadrature; Adriaan Van Roomen; Andreas Helmreich

E-mail address: J.P.Hogendijk@uu.nl

0315-0860/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.hm.2010.03.004

1. Introduction

Joseph Justus Scaliger (1540–1609) was a famous classicist and humanist. Although mathematics was not his main area of expertise, he announced around 1590 that he had solved the three famous problems of ancient Greek mathematics, namely the circle quadrature, the trisection of the angle, and the construction of two mean proportionals [Grafton, 1993, 380–384]. In the summer of 1593 he was hired by the University of Leiden, where his *Cyclometrica Elementa* [Elements of Circle Measurement] appeared in June 1594. The work was lavishly produced. The definitions and theorems are in ancient Greek with Latin translation, and the proofs are in Latin. The chapter titles, page headers, and geometrical figures are printed in red, as well as all letters in the Latin text indicating points in the figures. The *Cyclometrica Elementa* consists of two books, and will henceforth be abbreviated as *Cyclometrica*. In Book 1, Scaliger "shows" that the square of the circumference of the circle is ten times the square of the diameter (i.e., $\pi = \sqrt{10}$). In Book 2, he "proves" that the area of a circle is $1\frac{1}{5}$ times the area of the inscribed regular hexagon (i.e., $\pi = \frac{9}{5}\sqrt{3}$). Scaliger does not accept the relationship between the circumference and area of a circle, which had been proved by Archimedes, and which can be expressed in modern terms as follows: surface area (πr^2) is radius times half circumference ($r \cdot \pi r$). Archimedes proved this relationship by reductio ad absurdum, but Scaliger warns against the corrupting influence of this method of reasoning on young people [Scaliger, 1594b, 11; Grafton, 1993, 382]. It was not a problem for Scaliger that his circle rectification was inconsistent with the Archimedean limits $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Scaliger states that his own quadrature of the circle is done in a geometrical way and according to scientific method, and not done in a tyrannical way as was that by Archimedes [Scaliger, 1594b, 15; Goulding, 2005, 251].

Scaliger's pompous *Cyclometrica* did not fail to make a big impression on many scholars, as appears from his extant correspondence between 1594 and 1596.[1] Only a few experts in mathematics realized that Scaliger's work was mathematically very incorrect. Three such experts were Francois Viète (1540–1603), Adriaan van Roomen (1561–1615), and Ludolph van Ceulen (1540–1610). At the time when Scaliger published his *Cyclometrica*, Viète worked as a diplomat at the court of King Henri IV in Tours [Hofmann, 1970, xvii–xiv], and Van Roomen was professor of medicine in Würzburg [Busard, 1975]. Van Ceulen, who did not have academic training, worked as an arithmetic teacher and fencing master in the city of Leiden since 1593 [Katscher, 1971]. All three of them had determined π to many decimal places: Viète printed 10 decimals in 1593 [Viète, 1646, 392]; Van Roomen gave 14 decimals, also in 1593 [Busard, 1975]; and Van Ceulen published 20 decimals in his work *Vanden Circkel* [Van Ceulen, 1596]. Van Roomen and Van Ceulen were close friends and maintained an intensive correspondence.

Later in 1594, Viète printed his *Defense against the New Cyclometrica, or Anti-axe* [Viète, 1594]. Viète did not mention Scaliger explicitly but the *Defense* was full of humoristic insults. The title is a pun on Scaliger's newly introduced technical term *axe*, that is, the area contained inside the concave arcs *GC* and *GD* and the convex arc *CD* in Fig. 1. Van Roomen corresponded with Scaliger about the errors in the *Cyclometrica*, but when Scaliger remained unconvinced, Van Roomen printed a refutation in the form of dialogues [Van Roomen, 1597]. Hitherto, historians have assumed that Van Ceulen could not afford to

[1] I thank Dr. Dirk van Miert for showing me his edition of the Scaliger correspondence, which he and Dr Paul Botley prepared at the Warburg Institute, London, and which will appear in 2011 as *The Correspondence of Joseph Justus Scaliger*, 7 vols., Droz, Geneva.

publish a refutation, because he lived in the same city as Scaliger and his social status was much lower [Bierens de Haan, 1876; Bierens de Haan, 1878, 152; Katscher, 1971, 111].

In this paper we will see that Van Ceulen did publish a refutation, and that there was also a mathematical exchange between him and Scaliger. The refutation and at least part of the exchange were printed in 1596 in Chapter 21 of *Vanden Circkel* [Van Ceulen, 1596, 61a–66b]; that Chapter will be analyzed in this paper. In Section 2, I discuss Propositions 2 and 17 of Book 2 of Scaliger's *Cyclometrica*, which involve his quadrature of the circle. Section 3 is about 16 questions and answers in Chapter 21 of *Vanden Circkel*. These questions were sent to Van Ceulen by an anonymous "highly learned man," whose name is not mentioned explicitly, and who has been wrongly identified as Adriaan van Roomen in [Bierens de Haan, 1878, 142]. The "highly learned man" turns out to be Scaliger, as must have been clear to many contemporary readers of Van Ceulen's work. In Section 4, I discuss the rest of Chapter 21, which at first sight appears to be a random collection of propositions, but which can now be interpreted as a systematic criticism of Scaliger's *Cyclometrica*. In Section 5 the exchanges between Van Ceulen and Scaliger will be analyzed. Because Van Ceulen had much lower social status, he treated Scaliger very courteously, although the courtesy can perhaps be interpreted as irony. The interactions between Scaliger and Van Ceulen can to some extent be explained in terms of their different approaches to mathematics. A Dutch transcription and English translation of the 16 questions by Scaliger and their answers by Van Ceulen can be found in the Appendix to this paper.

I conclude this Introduction by a quotation from [Van Roomen, 1597, 56], which describes the atmosphere of the exchange between Scaliger and Van Ceulen. Van Roomen says,

> The work of Scaliger [i.e., the *Cyclometrica*] had hardly been written, when the most excellent Mathematicus of our times, Ludolph van Ceulen, took it in his hands, and read and examined all of it. He found serious errors and informed him about them, through learned men who were familiar to Scaliger. At the same time he encouraged him to withdraw the work before it would reach others, because in this and no other way could his honour be preserved. But Scaliger laughed at the learned man [i.e., the messenger], and claimed that it would even be impossible for any learned Mathematicus, who used a long time, to examine, let alone to understand his writings. Thus, little importance should be attached to the judgement of some fighter (for this is how he called Ludolph, whom he considered to be unworthy of the name of a Mathematicus), who, because of his daily occupations, could not have examined them [i.e., the *Cyclometrica*] in ten or twelve days (for Ludolph had taken this much time). He said that he therefore wanted that Ludolph would publish his criticism. Although Ludolph accepted this as an answer, he did not stop to encourage that man [Scaliger] two or three more times to take better care of his honour. But in vain. ([Van Roomen, 1597, 56]; Latin text quoted in [Bierens de Haan, 1878, 153], German translation in [Katscher, 1971, 110])

2. Two propositions in Scaliger's *Cyclometrica*

In this section I summarize Propositions 2 and 27 of Book II of the *Cyclometrica*, which will be relevant in Section 3 below. For the sake of clarity, some of the (confusing) terminology of Scaliger will be printed in italics. For a general survey of the work, see [Bierens de Haan, 1877].

In Proposition 2 [Scaliger, 1594b, 72–78], Scaliger considers a circle with center G and inscribed regular hexagon $ABCDEF$, see Fig. 1. The diameters AD, BE, CF divide the hexagon into six equilateral triangles. Scaliger calls the figure bounded by the circular arc CD and the straight line CD a *segment of the hexagon* (segmentum hexagoni), or simply a *segment*. Note that the figure is a segment of the circle rather than the hexagon. Scaliger claims

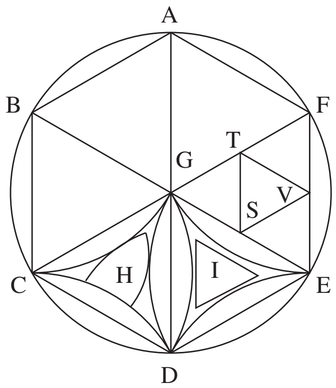

Fig. 1. Scaliger's circle quadrature.

that "the circle is equal in area to thirty-six segments of the hexagon inscribed in the circle" (*Circulus potest triginta sex segmenta Hexagoni ipsi circulo inscripti*). It follows that the inscribed hexagon itself is 30 such segments, so the area of the circle is $1\frac{1}{5}$ times that of the hexagon. In modern terms, this statement boils down to $\pi = \frac{9}{5}\sqrt{3}$. Scaliger's "proof" is as follows.

Inside the two equilateral triangles *CDG* and *DEG*, he constructs a total of six *segments of the hexagon* as in Fig. 1. He calls each of the two remaining figures, which are bounded by three concave arcs, a *complement* (*complementum*). From the *complement GCD* Scaliger subtracts a figure *H* equal in area to one *segment of the hexagon*, and he thus obtains a figure which he calls the *remainder of the segment* (*residuum segmenti*), which in Fig. 1 seems to consist of three parts. Then he considers an equilateral triangle *I*, whose area is equal to one-fifth of the area of equilateral triangle *GDE*. He calls *I* simply *the triangle*. From the *complement GDE* he subtracts *the triangle I* and he thus obtains a figure that he calls the *remainder of the triangle* (*residuum trianguli*).

He then deduces two true statements, namely (a) "thirty *segments* together with eight *remainders of the triangle* exceed the circle by two *triangles*," and (b) "thirty-six *triangles* together with thirty *segments* and six *remainders of the triangle* are equal in area to two circles." From statements (a) and (b) he draws the wrong conclusion (c): "Thirty-six *triangles* together with thirty *segments* and eight *remainders of the triangle* exceed two circles by two *triangles*."[2] He concludes that the *segment of the hexagon*, the *triangle*, the *remainder of the segment*, and the *remainder of the triangle* are all equal in area. Since the whole circle is equal to the inscribed hexagon (which is 30 *triangles*) plus six *segments*, the circle is equal in area to 36 *segments*. Q.E.D.

This "proof" can be clarified in modern notation, which was not available to Scaliger. Call h the area of Scaliger's *segment of the hexagon* and t the area of *the triangle I*. Then the area of the circle is $6h + 30t$. The area of each of the triangles *GCD* and *GDE* is $5t$, and the areas of the *complement*, the *remainder of the segment*, and the *remainder of the triangle* are $5t - 3h$, $5t - 4h$, and $4t - 3h$ respectively. In this notation, the three statements boil down to (a) $30h + 8(4t - 3h) = (30t + 6h) + 2t$ (true); (b) $36t + 30h + 6(4t - 3h) = 2(30t + 6h)$ (true), and (c) $36t + 30h + 8(4t - 3h) = 2(30t + 6h) + 2t$ (false). From (c), Scaliger concludes

[2] Latin text: "Ergo triginta segmenta cum octo residuis trianguli excedunt circulum duobus triangulis. Supra vero diximus triginta sex triangula cum triginta segmentis, & sex residuis trianguli esse aequalia duobus circulis. Ergo triginta sex triangula cum triginta segmentis, & octo residuis trianguli excedent duos circulos duobus triangulis" [Scaliger, 1594b, 73].

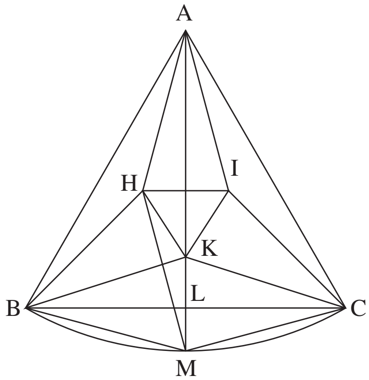

Fig. 2. Scaliger's study of the dodecagon.

that $t = h$, so the area of the circle is $30t + 6h = 36h$. The same notation may be used to clarify the six alternative "proofs" in [Scaliger, 1594b, 74–78], which will not be discussed here.

In *Cyclometrica* II:17 and its corollaries [Scaliger, 1594b, 107–112], Scaliger extends his investigations to the regular dodecagon. Fig. 2 is extracted from a larger figure in the *Cyclometrica*. Point A is the center of a circle, BC is a side of the inscribed equilateral hexagon, and M is the midpoint of the arc BC, so the line segments BM and MC are sides of the regular dodecagon inscribed in the circle. Scaliger calls the figure bounded by the line segment BM and the circular arc BM a *segment of the dodecagon* (*segmentum dodecagoni*). On the three sides of the equilateral triangle ABC Scaliger constructs three isosceles triangles ABH, BCK, CAI congruent to triangle BCM. Then he proves correctly that triangle HIK is equilateral, that triangle BMH is also equilateral, and that triangles BKH, AHI, and CIK are also equal in area to triangle BCM. In the proof and the corollary, Scaliger uses the terms *the isosceles* and *the equilateral* to indicate triangles BCM and HIK.

In the corollary he assumes the erroneous result of Proposition 2, and he proves, among other things, that "one *isosceles* and one *equilateral* are equal (in area) to ten *segments of the dodecagon*." (Ergo unum isosceles, & unum isopleuron sunt aequalia decem segmentis dodecagoni [Scaliger, 1594b, 111].) The proof is easy in modern notation: let t be Scaliger's *triangle* and h be his *segment of the hexagon* as above, and put d for his *segment of the dodecagon*. Then the area of the *isosceles* and the *equilateral* triangles are $h - 2d$ and $5t - 6(h - 2d) = 5t + 12d - 6h$ respectively, so the area of one *isosceles* plus one *equilateral* is $10d + 5(t - h)$. This is equal to $10d$ if and only if $t = h$, which was "proved" in *Cyclometrica* II:2.

3. The 16 propositions and the identification of the "highly learned man"

In the middle of Chapter 21 of *Vanden Circkel* [Van Ceulen, 1596, 63a], the following subtitle appears in large print: "Here follow some pieces in the art [of mathematics], concerning the circle, proposed and found by a highly learned man, in which [pieces] his illustrious mind is evident. These pieces have been sent to me and he desires to have my opinion of them. Therefore I have answered them by investigating them, and I have found most of them to be correct by means of numbers." Pending the identification, we will call the highly learned man in the subtitle "the scholar." See the Appendix for a Dutch transcription and an English translation of the 16 "pieces" (i.e., propositions) and their answers.

Van Ceulen begins with a general introduction in which he computes the lengths of certain line segments and areas that he needs in his argument. He then presents each of the 16 propositions in what is apparently the wording of the scholar, followed by his

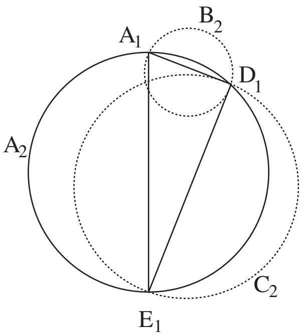

Fig. 3. The circle quadrature by the highly learned scholar.

own discussion. We first turn to proposition 16: "The circle is equal to 36 Segments of the hexagon inscribed in it." Van Ceulen quotes the confused proof that the scholar sent to him. In the following paraphrase, I have made slight changes in the order of the argument. I have added indices to the notation, and in Fig. 3 I have simplified the original figure, in order to (I hope) make the "proof" somewhat clearer. The original text can be consulted in the Appendix to this paper, where the original figure appears as Fig. 7.

The scholar considers a circle A_2 with diameter A_1E_1 (Fig. 3) and a point D_1 on the circle, which point will be defined below. Call B_2 the circle with diameter A_1D_1 and C_2 the circle with diameter D_1E_1. By the theorem of Pythagoras, we have $A_1E_1^2 = A_1D_1^2 + D_1E_1^2$. Since the areas of circles are proportional to their diameters, and the same is true for the areas of inscribed regular hexagons, it follows that the area of circle A_2 is the sum of the areas of circles B_2 and C_2, and that the area of the regular hexagon inscribed in A_2 is the sum of the areas of the regular hexagons inscribed in B_2 and C_2.

In his proof, the scholar uses two figures or quantities, which he calls *the isosceles* and *the irrational*, with reference to Fig. 3. To avoid a whole new notation, I will now explain these concepts with reference to Fig. 2. We assume that the circle in Fig. 2 is equal to circle A_2 in Fig. 3, that is to say, $AM = \frac{1}{2}A_1E_1$. Then *the isosceles* is triangle BMC, and *the irrational*[3] is $\frac{1}{10}(\Delta HIK + \Delta BMC)$, where ΔHIK is my notation for the area of triangle HIK. We have in Fig. 2 $\Delta ABC = 6\Delta BMC + \Delta HIK = 5\Delta BMC + (\Delta BMC + \Delta HIK)$. Multiplying by 6, we conclude that the regular hexagon in circle A_2 is equal in area to 30 *isosceles* plus 60 *irrationals*.

We now return to the main line of argument. The scholar chooses point D_1 in such a way that the area of C_2 is equal to 36 *isosceles* (Assumption 1). He also assumes that the hexagon inscribed in B_2 is equal in area to 60 *irrationals* (Assumption 2). Because the regular hexagon inscribed in circle A_2 is equal in area to the sum of the regular hexagons inscribed in circles B_2 and C_2, and also equal in area to 30 *isosceles* plus 60 *irrationals*, it follows that the hexagon inscribed in C_2 is equal in area to 30 *isosceles*. By Assumption 1, the area of the whole circle C_2 is 36 *isosceles*, so the difference is 6 *isosceles*. But the difference is also 6 *segments*, that is to say, segments cut off by the regular hexagon inscribed in circle C_2. Thus a *segment* (cut off by the inscribed hexagon) in circle C_2 is equal to an *isosceles*. Since the area of circle C_2 is 36 *isosceles*, the area of circle C_2 is also 36 *segments*. Q.E.D.

[3] According to Van Ceulen's introduction to the 16 propositions, *the irrational* is defined as $\frac{1}{5}(LM^2 + \Delta HIK)$. From Propositions 5 and 6 (see Appendix), it follows that one *irrational* is also equal to $\frac{1}{5}(\Delta BMC - LM^2)$. Hence two *irrationals* are equal to $\frac{1}{5}(\Delta HIK + \Delta BMC)$.

This "proof" is incorrect because Assumptions 1 and 2 define different points D_1. Assumption 2 is equivalent to saying that the inscribed hexagon in circle C_2 is equal to 30 *isosceles*. If we assume that Assumptions 1 and 2 define the same point D_1, we might as well assume that the area of circle C_2 is $\frac{36}{30}$ times the area of its inscribed hexagon, which is what the scholar would like to prove. So the "proof" rests on a circular argument. Nevertheless the scholar concludes: "There is no one who is to some extent *Mathematicus* and who does not understand this. . . ."

I will now identify the scholar by means of the information in Section 2. Proposition 16 is equivalent to the false circle quadrature in *Cyclometrica* II:2. The terms *segment* and *isosceles* have the same meaning as in *Cyclometrica* II:2 and 17. *The irrational* in the proof of Proposition 16 also occurs in the following sense in the corollary to *Cyclometrica* II:17. In the notation of Section 2, one irrational is equal to $d + \frac{1}{2}(t - h)$, and if Scaliger's circle quadrature is accepted, we have $t = h$, in which case *the irrational* is equal to Scaliger's *segment of the dodecahedron d*.

Thus the scholar must have been someone of limited mathematical competence, who nevertheless considered himself to be a brilliant *Mathematicus*. He used the odd mathematical terminology of the *Cyclometrica* and introduced the term *the irrational*, which is related to *Cyclometrica* II:17. He firmly believed in the circle quadrature in *Cyclometrica* II:2. I conclude that the scholar was Scaliger himself.

Additional evidence for this identification can be found in the end of Section 4 below, and also in Propositions 1 through 15, which can be found in the appendix to this paper. These propositions begin with a large figure (Fig. 4), which resembles Scaliger's Fig. 1, and which includes some elements of Fig. 2. In *Cyclometrica* II:2, Scaliger introduced a special term *the triangle* for triangle *I*, which he believed to be equal to his *segment of the hexagon*. This triangle appears as triangle *AIH* in Fig. 4. Some of the terminology in the Propositions 1 through 15 is characteristic of Scaliger: *segment of the dodecagon* in Propositions 1, 8, 9, 10; *segment of the hexagon* in Propositions 6, 10; and *remainder of the segment* in Proposition 9. In the refutation of Proposition 16, Van Ceulen uses the Dutch equivalents of Scaliger's terms *isosceles* and *equilateral* in *Cyclometrica* II:7. Propositions 1 and 4 concern triangle *BMH* in Fig. 2, which I have derived from the *Cyclometrica*. I leave it to the reader to uncover further similarities between *Cyclometrica* II: 2, 17 and Propositions 1–16.

I will not discuss all propositions in detail. Some of them can easily be verified by means of the notation in the end of Section 2, such as, for example, the ninth proposition (see Fig. 4): "The triangle *AHI* together with 8 Irrationals amount to the same as [a] the remainder if 1 *Segment* of the hexagon is taken away from the curvilinear triangle *H*, which (remainder) is called the remainder of the *Segment*, together with [b] 8 *Segments* of the dodecagon." This proposition boils down to the true statement $t + 8\left(d + \frac{1}{2}(t - h)\right) = (5t - 4h) + 8d$.

Propositions 1 through 15 occur neither in the *Cyclometrica* nor in the *Appendix ad Cyclometricam* [Scaliger, 1594a], which appeared in December 1594, so they constitute some hitherto unknown mathematical work of Scaliger, which probably dates back to the period between 1594 and 1596. The first 13 propositions are correct, and they are witnesses of Scaliger's limited mathematical competence.

There are two reasons that the "highly learned man" in the subtitle cannot have been Adriaan van Roomen. First, [van Roomen, 1597] includes a refutation of Scaliger's circle quadrature, which the scholar "proved" in Proposition 16. Second, Van Roomen can be excluded by looking at the table of contents in the beginning of *Vanden Circkel*. The title of Chapter 14 begins as follows: "[I]n this [chapter] you will find many mathematical problems . . ., sent to me by the highly learned Adriaen van Roomen . . ." But Chapter 21 is entitled: "Here the circle is discussed, and some mathematical questions concerning the circle, proposed and sent to be answered by me, by an illustrious highly learned Lord. …" If the Lord had been Van Roomen, Van Ceulen would have mentioned him by name.

Chapter 21 of *Vanden Circkel* may be connected to a letter that Van Roomen sent to Clavius on October 3, 1595. Van Roomen cites a passage from an otherwise lost letter by Van Ceulen that had recently arrived. According to Van Roomen, Van Ceulen said:

> Scaliger has recently presented [to me, i.e., Van Ceulen] a letter in which he again tried to confirm [the validity of] his work. I have again refuted the whole in a letter: I have shown that not Archimedes, but he erred, etc. This letter I have sent with some boy who is daily in his presence, and who is my student in arithmetic, and who even translated my letter to him; but he obstinately persists in his error. Thereafter (as the boy has told me) he has diligently started to work through the tenth Book of Euclid and arithmetic, which is a good thing: but it would have been better if he had done this before he had published his own writings. Perhaps he will in this way get back to normal. Here "Ludolph".
>
> I [says Van Roomen] guess that Ludolph has answered him by means of numbers which are roots, or incommensurables or irrationals, as they say. So it was necessary for him [i.e., Scaliger] to consult the tenth Book of Euclid and arithmetic. [Bockstaele, 1976, 120–121]

[De Wreede, 2007, 199] suggests that the young boy may have been the young Willebrord Snellius (1580–1626), who after Van Ceulen's death translated some of his work into Latin [Van Ceulen, 1615b].

In [Bockstaele, 1976, 121] it was suggested that the letter that Scaliger sent to Van Ceulen was the printed work *Appendix ad Cyclometricam* [Scaliger, 1594a], which appeared in December 1594. To me it seems more likely that the letter by Scaliger contained the 16 propositions that Van Ceulen received, and that Van Ceulen included most of Scaliger's letter and his own answer in what is now Chapter 21 of *Vanden Circkel*. The first part of Chapter 21 of *Vanden Circkel*, which we will discuss in the next section, also agrees with Van Ceulen's statement, "I have shown that not Archimedes, but he erred, etc."

4. Van Ceulen's refutation of the *Cyclometrica*

We now turn to the rest of Chapter 21 of *Vanden Circkel*. At first sight, this chapter seems to consist of a random collection of propositions. Some of them are redundant repetitions of material in earlier Chapters, and others have little or no connection to the rest of *Vanden Circkel*. All of them can now be interpreted as criticisms of Scaliger's *Cyclometrica* [Scaliger, 1594b] and his *Mesolabium* [Scaliger, 1594c].

The title of the chapter is as follows: "The XXI. Chapter. In which is proved that the square of the circumference of a circle is less than ten times the square of the diameter. Also that 36 triangles, 30 of which amount to the equiangular hexagon, are less, and 36 *Segments* of the hexagon are greater than the area of the circle, and that 35 of the segments are greater (than the area of the circle)" [Van Ceulen, 1596, 61a]. Thus Scaliger's circle rectification and circle quadrature are incorrect.

Van Ceulen begins Chapter 21 with an intuitive geometric argument about an inscribed polygon with an infinitely large number of angles, in order to show that the area of the circle is the product of the radius and half the circumference. He then shows by several arguments that the circumference of the circle is less than $\sqrt{10}$ times the diameter. In some of these arguments, he uses a circle with diameter 16, just like Scaliger in *Cyclometrica* I:5.

Van Ceulen computes the circumference of a circumscribed regular 24-gon in great detail, including all root extractions, so that the computation can be followed by a beginning student of arithmetic. Thus Van Ceulen tailored his discussion to what he considered to be Scaliger's level [Van Ceulen, 1596, 61b]. Thus, the circumference of the 24-gon turns out to be less that $\sqrt{10}$ times the diameter, and therefore the rectification of the circle in *Cyclometrica* I:6 [Scaliger, 1594b, 31–33] cannot be correct. Using both inscribed and circumscribed 128-gons and 120-gons, Van Ceulen then shows that Archimedes' estimate $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is correct. This means that Scaliger's statement $\pi = \sqrt{10} > 3\frac{1}{7}$ and his criticisms of Archimedes in the Scholium to *Cyclometrica* I:6 [Scaliger, 1594b, 37] are unfounded. Van Ceulen then shows that if the circle is equal to $1\frac{1}{5}$ times the regular hexagon, its circumference cannot be $\sqrt{10}$ times the diameter. He concludes that Scaliger's circle rectification in *Cyclometrica* I:6 and his quadrature of the circle in *Cyclometrica* II:2 are contradictory [Van Ceulen, 1596, 62a]. Van Ceulen then shows that the area of an inscribed regular 32-gon is greater than $1\frac{1}{5}$ times the area of an inscribed hexagon; as a consequence, Scaliger's *segment of the hexagon* is greater than Scaliger's *triangle* (one-thirtieth of a hexagon). Using the estimate $\pi = 3\frac{141592653}{1000000000}$, which had been proved earlier in *Vanden Circkel*, Van Ceulen argues that even the area of 35 *segments of the hexagon* is already greater than the area of the circle. Thus Scaliger's circle quadrature in *Cyclometrica* II:2 is incorrect.

Then follow the 16 propositions with their answers, which have been discussed in Section 3 of this paper.

After his refutation of Proposition 16, Van Ceulen presents a geometrical construction of regular polygons with odd numbers (5, 7, 9, 11, 13, 15, 17, etc.) of sides [Van Ceulen, 1596, 65a]. Van Ceulen does not inform the reader about the origin of the construction, which is found in *Cyclometrica* I: 12–13 [Scaliger, 1594b, 45–62]. Scaliger "proved" that the construction is exact, but Van Ceulen shows that in the case of the heptagon, the construction produces only approximate results. He states that for the other polygons, the results are approximate as well.

[Van Ceulen, 1596, 65b] continues with the construction of a cyclic quadrilateral (that is, a quadrilateral that can be inscribed in a circle) with four numerically given sides; the solution involves the determination of the two diagonals of the quadrilateral and the diameter of the circumscribing circle. The problem is also discussed in *Cyclometrica* I: 16 [Scaliger, 1594b, 62–64], with a "proof" of an incorrect method for finding the diameter of the circumscribing circle. In modern notation, Scaliger's rule boils down to $D = \frac{1}{2}\left(\sqrt{a^2+b^2} + \sqrt{c^2+d^2}\right)$, where a, b, c, d are the sides of the cyclic quadrilateral and D is the diameter of the circumscribing circle.

Van Ceulen refutes this rule in an interesting way. He begins with the example $a = 60, b = 50, c = 40, d = 32$, and states without proof that the correct diameter in this case is[4] $\sqrt{4386\frac{3613546}{3824439}}$. Van Ceulen says: "If I would use here the means which I have found written [i.e, Scaliger's rule in the *Cyclometrica*], the diameter would come out as a Binomium, which is impossible in this case."[5] Scaliger's rule produces the diameter $\sqrt{1525} + \sqrt{656}$, which is indeed a binomium according to the classification in Euclid's *Elements* X. Thus far,

[4] I have corrected the misprint 3610546 for 3613546.

[5] "Soo verre ick hier wilde ghebruycken de middelen die ick beschreven ghevonden hebbe / soude comen voor den Diameter een *Binomium*, welck in desen onmoghelijck is" [Van Ceulen, 1596, 66a].

Van Ceulen's argument shows neither the correctness of his own value nor the incorrectness of Scaliger's result.

Then Van Ceulen takes the example $a = 18, b = 9, c = 8, d = 6$, and says that the method that he had "found written" (i.e., Scaliger's rule) produces a diameter of the circumscribing circle less than the longest segment of the cyclic quadrilateral, which is of course absurd. Scaliger's method indeed produces $D = \frac{9}{2}\sqrt{5} + 5 = 15.06\ldots < 18$. Van Ceulen says that in this case the correct diameter is $\sqrt{359\frac{239}{3335}}$. Now one can permute the order of the sides in such a way that the diameter remains the same (make $a = 6, b = 18, c = 9, d = 8$). Then Scaliger's method may produce a different value $3\sqrt{10} + \frac{1}{2}\sqrt{145} = 15.51\ldots$ for the diameter, so another absurdity emerges. Here ends Van Ceulen's criticism. Thus, Van Ceulen refutes Scaliger's rule by showing that the rule produces incorrect results in the suitably chosen numerical example $a = 18, b = 9, c = 8, d = 6$.

The same numerical example is also mentioned, together with the (complicated) general rule for finding the diameter, in Van Ceulen's *Arithmetische en Geometrische Fondamenten*, which was published posthumously [Van Ceulen, 1615a, 203–211]. Van Ceulen first determines the diagonal of the cyclic quadrilateral.[6] Then the diameter of the circumscribing circle can be found by the standard method for finding the circumscribed circle of a triangle whose three sides are numerically known.

Van Ceulen concludes Chapter 21 of *Vanden Circkel* by a construction of two mean proportionals between two given lines, which he found in a book on arithmetic by Hendricus Grammatheus (Heinrich Schreyber, ca. 1495–ca. 1525). The two mean proportionals are determined in an incorrect way in Scaliger's *Mesolabium* [Scaliger, 1594c]. Here ends Chapter 21 of *Vanden Circkel*.

Although Van Ceulen did not mention Scaliger's name in *Vanden Circkel*, the purpose of Chapter 21 was of course no secret to his friends. On July 1, 1597, Van Roomen wrote to Clavius that Van Ceulen's book (*Vanden Circkel*) had appeared, and that "he refutes the main points of Scaliger" [Bockstaele, 1976, 124]. In the Latin translation [Van Ceulen, 1615b, 188] of the passage on the cyclic quadrilateral in [Van Ceulen, 1615a, 203–211], Willebrord Snellius mentions the false rule for the cyclic quadrilateral in Scaliger's *Cyclometrica*. Snellius adds that he was encouraged by Ludolph van Ceulen to work on the subject [De Wreede, 2007, 279]. Snellius had been a pupil of both Van Ceulen and Scaliger at the same time [De Wreede, 2007, 320], and he was a teenage boy when the *Cyclometrica* and *Vanden Circkel* were published.

5. The interaction between Van Ceulen and Scaliger

We will now analyze the interaction between Van Ceulen and Scaliger more systematically.

Scaliger is only one of four mathematicians whose work Van Ceulen discusses in *Vanden Circkel* [Van Ceulen, 1596] without mentioning their names. The three others are [a] Simon

[6] In modern terms, if the sides of the cyclic quadrilateral are a, b, c, d, such that a is opposite c and b is opposite d, the two diagonals are $\sqrt{(ab + cd)(ac + bd)/(ad + bc)}$ and $\sqrt{(ad + bc)(ac + bd)/(ab + cd)}$.

van der Eycke, [b] an unidentified Dutch surveyor, and [c] Andreas Helmreich. We will briefly discuss the ways in which these three mathematicians are mentioned.

a. In the dedication to Prince Maurits, Van Ceulen says that "[i]n the years 1584 and 1586 two circle quadratures, inconsistent with one another, have been published by a lover (lief-hebber) [of mathematics], against which I have printed (not without great effort) two booklets in which these are refuted by the truth."[7] The two booklets are Van Ceulen's refutations [Van Ceulen, 1585] and [Van Ceulen, 1586] of the two circle quadratures by the arithmetic teacher Simon Van der Eycke, on whom see [Bierens de Haan, 1878, 131–140]. Thus Van der Eycke must have been the anonymous "lover of mathematics."
b. In the preface of *Vanden Circkel*, Van Ceulen writes: "Twenty-two years ago, a book was given to me, which was written by a surveyor who is famous in these countries, in his own hand. In this book I have found many errors, including the following. … " Van Ceulen then cites a wrong rule for finding the area of a triangle from the numerical values of the three sides. According to Van Ceulen, the anonymous author stated in the manuscript that he had learned the (incorrect) rule from Ian de Haze, a deceased surveyor from Aelst in Flanders, about whom I have not found further information. I have not been able to identify the Dutch surveyor whose manuscript Van Ceulen must have received in 1574.
c. In the tenth problem in Chapter 19, Van Ceulen determines the surface area of a segment of a circle with arc 66 rods, base 60 rods, and height 10 rods. Van Ceulen says that the problem was mentioned to him by a certain Master Eduwaert Leon, and he states that the correct area, which is $408\frac{757237}{1000000}$ square rods, is a little over 46 square rods less than the amount which "the honourable man [i.e., Leon] showed me as it was written in a printed book, by an experienced Mathematician in Halle, in the country of Sachsen, and printed in Leipzig Anno 1591, in which there are many true pieces [i.e., propositions]."[8]

The anonymous mathematician can be identified as Andreas Helmreich from Halle, whose book *Von Feldmessen nach der Geometrei* was printed in Leipzig in 1591. Not much is known about the life of Helmreich, whose first publication dates back to 1557 but who was still alive on Feb. 15, 1591, as appears from the preface of [Helmreich, 1591]. The fourteenth problem in the sixth part of the book [Helmreich, 1591, T ia] is a computation of the area of a segment of a circle whose arc is 66 rods, base 60 rods, and height 10 rods. The outcome is 455 square rods, and most of the difference of 46 rods from Van Ceulen's outcome is explained by the inaccurate approximation of $\sqrt{10}$ which Helmreich used in his computation, as Van Ceulen explains in *Vanden Circkel* [Van Ceulen, 1596, 56b]. Van Ceulen explains that the base and the height already determine the length of the arc. (In modern terms, if the base is b and the height is h, the arc is $a = 2r \arctan \frac{b}{2r-2h}$ rods

[7] "In den Jare 84, ende 86, zijn van een lief-hebber twee Quadratueren des Circkels teghen malcander strijdende in 't licht ghebracht / daer teghen ick (niet sonder groote oorsake) twee Boucxkens hebbe laten drucken / daar mede de selve met waerheydt weder-leydt werden" [Van Ceulen, 1596, dedication].

[8] "Als den Lof-waerdighen Man mijn toonde in een ghedruckt Bouck, beschreven / van eenen hervaren *Mathematicus* tot *Halle*, inden Lande van *Sassen*, ende tot *Leipsig, Anno 1591* ghedruckt / daer inne veel waerachtige stucken beschreven zijn" [Van Ceulen, 1596, 56a].

with $r = \frac{1}{2}\left(h + \frac{b^2}{4h}\right)$, so the problem is overdetermined. For $b = 60$ and $h = 10$ we have $a = 64.35\ldots$ rather than $a = 66$.)

Van Ceulen states that the following Problem 11 in Chapter 19 of *Vanden Circkel* was the 15th example of the sixth part of the same book, in agreement with [Helmreich, 1591, T iii b]. The problem concerns the determination of an area in the shape of a heart, but Van Ceulen only states the area without giving any details about the figure. The figure is drawn with all details and numerical data by Helmreich, but Van Ceulen's solution is inconsistent with these data. Problem 12 in Chapter 19 begins with a quotation in high German. Van Ceulen says that the quotation is from the same book which had been mentioned before, and the quotation agrees almost literally with [Helmrich, 1591, V iv a].[9] Thus there is no doubt that Van Ceulen had Helmreich's book in front of him when he wrote *Vanden Circkel*, and that it was his conscious decision not to reveal the name of Andreas Helmreich to his readers.

Neither Van der Eycke nor Helmreich are mentioned anywhere in *Vanden Circkel*, but the work contains many explicit references to fellow mathematicians with whom Van Ceulen had good relationships, such as Simon Stevin (1548–1620), Rudolf Snellius (1546–1613), and Nicolaus Petri of Deventer (died 1602). Van Ceulen also included critical references to deceased mathematicians such as Carolus Bovelli (died 1553) and Jacob Köbel (1470–1533) [Katscher, 1979, 107]. It seems that in *Vanden Circkel*, Van Ceulen refrains from mentioning the names of the contemporary mathematicians whom he criticizes. The only partial exception is "Niclaes Reymers of Henstede in Ditmarsen", or Nicolaus Reimers Ursus (1551–1600), who is widely known to modern historians in connection with Tycho Brahe. Reimers is criticized in Problem 12 in Chapter 19, which is the problem that Van Ceulen first quotes in High German from [Helmrich, 1591, V iv a], where reference is also made to Reimers. Reimers' problem is the determination of the area of a lune consisting of two circular arcs with lengths 9152 and 8415 rods, central thickness 609 rods, and distance between the end points 7560 rods. Reimers had designed this problem as a test problem for an examination for surveyors. Van Ceulen first shows that Reimers' problem is overdetermined, because three of the data (namely the distance 7560 between the two endpoints, the inner arc 8145 and the central thickness 609) determine the fourth, that is the outer arc, which must be $9154\frac{68816}{100000}$ rods rather than 9152 rods, if Van Ceulen's own accurate value for π is used. Van Ceulen then solves the problem with this modification and finds an area of $3,506,933\frac{11}{25}$ square rods. Helmreich found the solution of 3,505,306 square rods, but Van Ceulen does not mention this value.

Van Ceulen's discussion of Reimers' problem seems to be partly based on Reimers' *Geodaesia Ranzoviana* of 1583 [Katscher, 1979, 105], which I have not been able to consult. Van Ceulen may have wanted to cite Reimers explicitly because the test problem could not have

[9] Van Ceulen: "Op 't laetste in den voornoemden Bouck / heeft den Auteur ghestelt een Vraghe / gheproponeert vanden welhervarenen Geometer *Niclaes Reymers* van *Henstede* in *Ditmarsen:* Aldus. *Es ist ein Feldt, gelegen in form eines Neuvven Monden, desselben ausser Ecke ist lanck 9152, De inner Ecke 8415. In seyn breytiste mittel breyt 609. Zwissen seyne beyden Hornneren de vveyte 7560 Ruten. VVieviel heldt desselbighe Feldt*" [Van Ceulen, 1591, 56b].
Helmreich: "Folget das Exempel vom Felde / in form eines newen Monden / So der Erbar und Wolgefarte Herr Nicolaus Reimer / *Mathematicus* und berühmter *Geometer* zu Hartstadt in Ditmahr / etc. in seinem deutschen Büchlein von Feldmessen / Anno 83. in Druck außgangen gesetzt / und zu Resolvieren unnd Demonstrieren auffgeben / Als. Es ist ein Feld gelegen / in form eines newen Monden / desselben ausser ecke ist lang 9152. die inner ecke 8415. in seinem breittesten mittel 609. breit / zwischen seinen Hörnern die weit 7560. Ruten. Ist die Frage / Wie viel Ruten dasselbige Feld helt /" [Helmreich, 1591, V iv a].

been identified otherwise, or because, as Van Ceulen says, "the same Niclaes [Reimers] truthfully admonishes the surveyors who use easy and false rules, so that some of them measure too much and others too little."[10] The refutation of errors of incompetent surveyors is a recurring theme in *Vanden Circkel*.

Thus, the fact that Van Ceulen did not mention Scaliger by name is consistent with his normal policy in *Vanden Circkel* toward living mathematicians whose work he criticizes. However, his treatment of Scaliger is different in the following ways:

1. Van Ceulen's praise of Scaliger is in contrast with the more neutral or critical treatment of the other three anonymous mathematicians in *Vanden Circkel*. In the subtitle in the middle of Chapter 21, Van Ceulen refers to Scaliger as a "highly learned man" with an "illustrious mind." Between Propositions 15 and 16, van Ceulen says: "These and many other mathematical questions have been given to me to answer, and some of them will be treated in more detail in my great work. For I have to admit that I have learned a lot by them."
2. Van Ceulen minimizes the importance of Scaliger's errors. Scaliger's Propositions 14 and 15 are incorrect, but Van Ceulen makes a small modification in order to correct them, and then he proves the corrected propositions by numbers. In his refutation of Proposition 16, Van Ceulen says: "It [i.e., Proposition 16] appears to be [correct], and the difference [in area between the circle and 36 segments] is not easy to notice." See also the passage on the heptagon which will be quoted presently.
3. Van Ceulen sprinkles Chapter 21 with six references to the circle rectification by Carolus Bovelli (Charles Bouelles, 1470–1553), on whom see [Busard, 1973]. The references are misleading, as I will now show. Thus a casual reader of Chapter 21 of *Vanden Circkel* could get the impression that Bovelli rather than Scaliger was the target of Van Ceulen's criticism.

In 1503, Bovelli published the *Geometricae introductionis libri sex* (Introduction to Geometry in Six Books), which also appeared in French (1542, 1551) and Dutch (1547) translations. Book 4 of the work [Bouelles, 1551, 32b–37b] is about circle rectification. Bovelli says that once when he was on a bridge in Paris, he looked at the wheels of a chariot, and he convinced himself (by a geometrical argument) that during a quarter of a rotation, the covered distance is equal to the radius of the wheel times in modern terms $\sqrt{1^2 + \left(\frac{5}{4}\right)^2 - \left(\frac{1}{4}\right)^2} = \frac{1}{2}\sqrt{10}$. He went back to his home in Paris and checked what he had found by ruler and compass on a copper plate. It follows that $\pi = \sqrt{10}$, in agreement with Scaliger's circle rectification in *Cyclometrica* I:6.

In Chapter 9 of *Vanden Circkel* [Van Ceulen, 1596, 8b–9a], Van Ceulen had already refuted Bovelli's circle rectification, with a detailed discussion of his figure and a reference to him. Nevertheless, Van Ceulen begins Chapter 21 by the statement: "Although it has been shown above that Carolus Bovelli missed (the mark), I will show here in another way (a method) by which the truth can be seen."[11] Then Van Ceulen starts to work in a circle with diameter 16, just like Scaliger in *Cyclometrica* I:5, whereas Bovelli does not assign a numerical value to his diameter.

[10] "den selven Niclaes met waerheyt berispet de ongefundeerde Land-meters / welcke door lichte ende valsche regulen / den eenen te veel / en den anderen te weynigh meten" [Van Ceulen, 1596, 57a].
[11] "Hoe wel boven ghenoegh bewesen is / dat *Carolus Bovelli* ghemist heeft / Ick sal nochtans hier op andere maniere thoonen / daer uyt de waerheyt te mercken is" [Van Ceulen, 1596, 61a–b].

At the end of the refutation of Scaliger's circle quadrature in Proposition 16, which quadrature is equivalent to $\pi = \frac{9}{5}\sqrt{3}$, Van Ceulen says: "my feeling is that this important matter cannot be found by observing the rotation of a wheel, and by means of a compass on a copper plate, as Bovelli did in Paris." This reference to Bovelli is misleading because his and Scaliger's circle rectification, which is equivalent to $\pi = \sqrt{10}$, is inconsistent with Scaliger's circle quadrature, as Van Ceulen had pointed out before.

Van Ceulen continues: "He (i.e., Bovelli) has also found the side of a heptagon inscribed in the circle, which side would be equal to half of the side of the equilateral triangle inscribed in the circle. This seems correct if one measures it by means of the compass. But using numbers, things are shown to be different. These and other sides, as the sides of the 5. 7. 9. 11. 13. 15. 17-gon can be found more accurately in the following way. But not perfectly." Van Ceulen then presents Scaliger's method rather than Bovelli's approximate construction [Bouelles, 1551, 32a–b].

Van Ceulen's exceptionally courteous treatment of Scaliger is best explained by the difference in social status, and by the plausible assumption that Van Ceulen observed the proper forms of his day. Scaliger was a professor at the University of Leiden with a high international reputation and good relations with the city authorities, whereas the arithmetic teacher and fencing master Van Ceulen had a much lower social status.

Van Ceulen's praise of Scaliger could easily have carried a hidden ironical message to expert readers such as Adriaan van Roomen. I now list some possible further examples of irony. Bovelli, whom Van Ceulen seemed to criticize in Chapter 21, was French, just like Scaliger, who was the real target of Van Ceulen's criticism, and the work [Bouelles, 1551] was not very deep. In the preface to *Vanden Circkel*, Van Ceulen probably thought of Scaliger's scholarly and linguistic abilities when he wrote: "But those who want (to criticize) and are not competent in this (subject), may scorn my awkward and simple way of writing; but the rest will remain valid for them." [12] In Chapter 9 of *Vanden Circkel*, Van Ceulen introduces Bovelli's circle rectification, and he remarks: "If the diameter is 1, the circumference would be $\sqrt{10}$, as the Indian [mathematicians] considered to be correct, that is, the diameter would be to the circumference in the proportion of 1 to $\sqrt{10}$. Also, the square of the circumference [of the circle] would be ten times as much as the square of the diameter, as has recently been demonstrated geometrically, but imperfectly. For I, who am simple in this, will prove in the same way that the square of the circumference is eleven times the square of the diameter, etc."[13] The recent incorrect geometrical demonstration must have been the one by Scaliger, who is thus related to mathematicians from India rather than Greece. Van Ceulen adds that Scaliger's method could equally well be used to prove $\pi = \sqrt{11}$.

Apart from Adriaan van Roomen, not many readers could have understood the hidden irony because *Vanden Circkel* was written in Dutch so its circulation was necessarily lim-

[12] "Maar den ghenen die willen / ende gheen verstandt hebben van desen / die moghen mijn onordentlijck ende simpel maniere van schrijven verachten / de reste sal voor haer-luijden bestaende blijven" [Van Ceulen, 1596, preface].

[13] "Als den Diameter doet 1. soude syn omloop zijn $\sqrt{10}$. welcke de Indianen hebben voor goet ghehouden / ofte den Diameter soude een Proportie hebben teghen den omloop / als 1 teghen $\sqrt{10}$. Item het Quadraet des omloops soude thien-mael soo groot zijn / als het Quadraet des Diameters / welck onlangs Geometrice ghedemunstreert is: Maer onvolcomen. Dan ick (als slecht in desen) wil op de selve maniere bewijsen / dat het Quadraet des omloops elf-mael soo groot is / als het Quadraet des Diameters / etc" [Van Ceulen, 1596, 9a].

ited. Chapter 21 was included in the second edition [Van Ceulen, 1615c] but not in the Latin translation [Van Ceulen, 1619], which ended with Chapter 14.

The exchanges between Scaliger and Van Ceulen can partly be understood in terms of their different interpretations of the word *Mathematicus*. Scaliger's approach to mathematics was determined by his humanist scholarship and ideals, which included the emulation, reconstruction, and further development of ancient Greek science. The *Cyclometrica*, whose full title is *Cyclometrica Elementa*, is modeled after the *Elements* of Euclid. Following the example of Euclid's definitions, Scaliger defined many new mathematical terms in Latin and ancient Greek, and his proofs imitate the style of Euclid. In Book X of the *Elements*, Euclid classifies irrational quantities in a qualitative rather than a quantitative way, and the *Elements* contain no numerical approximations of irrational square roots. Scaliger seems to have concluded that numerical approximations and geometry belonged to completely separate disciplines. Thus he proudly "proves" in *Cyclometrica* I:5 that the numerical value of the perimeter of a regular 12-gon inscribed in a circle is greater than the numerical value of the length of the circumference of that circle [Scaliger, 1594b, 28–31]. At the end of the "proof" he remarks: "This paradox in Geometry is noble, and as we have already mentioned, it was not noticed by Archimedes himself. Otherwise it is not doubtful that the circumference is greater than (a straight line) subtended by it. But using numbers, it is observed differently."[14] Scaliger's ideal *Mathematicus* must have been an ivory-tower expert in Greek mathematics, in the style of the third century B.C., before the advent of Archimedes with his numerical estimates of irrational quantities. Scaliger shows no awareness of practical applications of geometry, such as surveying, and he does not seem to have taught his mathematics to students.

Van Ceulen was a mathematical practitioner without academic training. He earned his living as an arithmetic teacher of phenomenal calculating abilities, and he was competent in surveying, although there is no evidence that he was officially employed as a surveyor. Van Ceulen's native language was high German, and thus he was familiar with the theorems and constructions in the first six Books of Euclid's *Elements* in the high German paraphrase [Xylander, 1562]. The first low German (i.e., Dutch) version of Euclid's *Elements* appeared only in 1604. In Book III of Van Ceulen's posthumously published *Arithmetische en Geometrische Fondamenten* [Van Ceulen, 1615a], the first three Books of the *Elements* are summarized in a way that shows little understanding of the formal structure of Euclid's *Elements* on the part of Van Ceulen. Van Ceulen's likely source [Xylander, 1562] was also imperfect in this respect. In Book V of the *Arithmetische en Geometrische Fondamenten* [Van Ceulen, 1615a, 205–207], Van Ceulen messed up Viète's geometric construction of a cyclic quadrilateral from four given sides [Viète, 1595; Viète, 1646, 275–283], although Van Ceulen was perfectly able to derive the diagonals of a cyclic quadrilateral by correct geometric reasoning from the numerical values of the sides. Van Ceulen believed that many geometrical theorems can be proved "using numbers, " that is to say, by means of numerical calculations, often involving (possibly nested) irrational square roots. Thus Van Ceulen and Scaliger were mathematically opposite in many respects.

In the end of Section 1 we saw that Scaliger did not consider Van Ceulen a *Mathematicus*. Thus he cannot have cared very much about Van Ceulen's proofs or refutations "by numbers," which in Scaliger's opinion belonged to a discipline different from geometry. Yet the fact that Scaliger sent his 16 questions to Van Ceulen and requested his answer can be

[14] "Nobile est hoc paradoxon in Geometria, et ipsi, ut iam tetigimus, Archimedei non animadversum. Alioquin non dubium est, quin peripheria sit maior subtendente sua. Sed per numeros aliter deprehendetur" [Scaliger, 1594b, 30].

interpreted as an implicit form of respect. It is likely that Scaliger was not completely sure of himself, and he must have been hurt by criticisms by academically trained mathematicians, who were *Mathematici* according to his own standards. An example is François Viète, who in [Viète, 1594] criticizes Scaliger's mathematics, shows that much of his new terminology is faulty, and imitates his style. Of course Scaliger's use of the word *Mathematicus* was his private reinterpretation of a term that had been used in the late 16th century for mathematical practitioners and academically trained mathematicians alike.

For Van Ceulen, a key concept in dealing with mathematicians (including Scaliger) was that of honour. In the passage by Van Roomen which we have quoted at the end of Section 1, Van Ceulen urged Scaliger to take better care of his honour. Perhaps Van Ceulen did not like to mention the names of the mathematicians whom he criticized in order not to treat them with dishonor. In *Vanden Circkel*, Van Ceulen says, with respect to different solutions to the problem of Nicolaus Reimers: "The difference is very small — but in these matters no effort should be spared to give not an approximate but a complete and true answer to every lover [of mathematics] to what has been asked or what he has accepted to solve. The surveyors should give due attention to their honour and their oath in measurements and computations."[15] Thus, for Van Ceulen truth and honour were related, and more important than being a *Mathematicus* in Scaliger's sense. In Proposition 16 of Chapter 21 of *Vanden Circkel*, Scaliger said at the end of his "proof": "There is no one who is to some extent a *Mathematicus* and who does not understand this ...". Van Ceulen replied at the end of his refutation: "All who repeat my computation will find the truth. ..."

Scaliger never withdrew his quadrature, and he continued to angrily criticize his adversaries. Already in 1595, the son of Scaliger's publisher in Leiden wrote that Scaliger had "become more a figure of fun than of hatred" [Grafton, 1993, 378], and Van Ceulen probably agreed with this judgement. Van Ceulen and Scaliger both lived in the city of Leiden and they must have found a way to live with one another. In 1598 they were both appointed, together with Simon Stevin and others, by the States of Holland to a committee that was asked to judge the methods of Plancius for determining longitude at sea [Dijksterhuis, 1943, 13]. In 1600, a new engineering school in the Dutch language was attached to Leiden University, and Van Ceulen was appointed as one of its two professors. Thus the two adversaries ended up as colleagues.

Acknowledgments

I am grateful to Jantien Dopper (Utrecht), Steven Wepster (Utrecht), and Liesbeth de Wreede (Leiden) for remarks on an earlier version of this paper, and to the two anonymous referees for their thoughtful comments. I thank Dirk van Miert (Amsterdam) for showing his unpublished edition of Scaliger's correspondence to me. Figs. 6 and 7 were drawn by Steven Wepster.

Appendix

The following passage is taken from Chapter 21 of [Van Ceulen, 1596, 63a–65a], Dutch transcription and English translation by Jan P. Hogendijk. The reprint edition [Van Ceulen,

[15] "Het verschil is gantsch cleyn / men behoort in sulcke saecken gheen arbeydt te sparen / om oock niet naer by: maar volcomen ende ware Antwoordt / elcke Lief-hebber te gheven / op't ghene ghevraght wert / ofte dat hy aenneemt te solveeren / De Land-meters moeten haer Eer ende Eedt in't meten ende rekenen betrachten / etc."[Van Ceulen, 1596, 57b].

1615c, 87b–90b] has also been consulted but the differences are unimportant and have not been noted. The Latin translation [Van Ceulen, 1619] by Willibrord Snellius does not contain Chapter 21. A notation such as [1596:64a] indicates the beginning of f. 64a in the 1596 edition. Text in frakturs font in the Dutch original is printed in Roman font in the transcription and translation. Words in Roman font in the original are printed in italics. The text contains some Latin words which have been left unchanged in the translation. Abbreviations such as eñ have been written out (as "ende") in the transcription. Figs. 4 and 5 are photos of the figures in [Van Ceulen, 1596, 63a, 64a] which appear in exactly the same form in [Van Ceulen, 1615c, 87b, 89a]. Figs. 6 and 7 are modern redrawings.

The mathematical symbolism in nested square roots in the transcription is nearly exactly as in the original. In the translation a more modern symbolism is used. Thus $\sqrt{.2} - \sqrt{3}$ in the Dutch transcription is rendered as $\sqrt{2-\sqrt{3}}$ in the translation. See Section 2 of this paper for an explanation of some of the curious mathematical terminology which is used in this passage. My own additions to the transcription and translation are in square brackets. Parentheses occur in the original.

Dutch transcription

[1615c:85a] Het XXI. CAPITTEL. Daer inne bewesen werdt / dat het Quadraet des omloops eenes Circkels cleynder is dan thien-mael der middel-linie Quadraet. Item / dat 36 Triangels / welcker 30 doen den ghelijckhouckighen 6 houck / te cleyn / ende 36 *Segmenten* des 6 houcx te groot zijn voor't vermogen des Circkels / ende dat 35 der *Segmenten* grooter zijn.

. . .

[1615c:87b] Hier volghen nu eenighe konstighe stucken den Circkel aengaende / Gheproponeert / ende gevonden door een hoogh-gheleerdt Man: Daerinne syn / door-luchtigh verstandt ghemerckt werdt / welcke stucken aen mijn ghesonden zijn / begheerdt mijn meninghe daer van te weten: Daerom ick door't onder-soucken de selve beantwoordt hebbe / ende meest door ghetal goedt ghevonden / Als volgt: [Fig. 4]

In desen Circkel / wiens middel-linie doet 2, syn *AGB*, *ABC*, *ACD*, ende *ADS* Triangels des 6 houcx / $AHI\,\frac{1}{5}$ van *ABC*.

Den ghelijcksydighen Trianghel *KLM* is ghemaeckt van den dubbelt *OF* (ofte een syde des selven is twee mael soo lanck als *FO*) een syde des Quadraets *KN* is ghelijck *OF*. Het Quadraet *PX* is soo groot als den ghelijcksydighen Triangel *KLM*, ende dat Quadraet *KN* t'samen. Noch twee *Segmenten* des 12 houcx / ende den Triangel *GFB* doet soo veel als een afgesneden stuck des 6 houcx (als ooghen-schijnlijck is) Om nu volghende vraghen te beantwoorden / is van noode te weten hoe groot de voornoemde stucken zijn / welcke licht te vinden zijn door't gheene hier vooren geleert is. Den Triangel *ABC* is $\frac{1}{6}$ des 6 houcx / ende doet $\sqrt{\frac{3}{16}}$. Den Triangel *AHI* is $\frac{1}{5}$ van *ABC*, ende doet $\sqrt{\frac{3}{400}}$. De ghelijckvoetighe Triangel *GFB* doet $\frac{1}{2} - \sqrt{\frac{3}{16}}$. Den Triangel *KML* doet $\sqrt{9\frac{3}{16}} - 3$. Dat Quadraet *KN* doet $1\frac{3}{4} - \sqrt{3}$, ende het Quadraet *PX* is groot $\sqrt{1\frac{11}{16}} - 1\frac{1}{4}$. Item $\frac{2}{5}$ uyt desen is $\sqrt{\frac{27}{100}} - \frac{1}{2}$, de helfte van desen / ofte $\frac{1}{5}$ uyt *PX* doet $\sqrt{\frac{27}{400}} - \frac{1}{4}$. Dit wert ghenoemt een Irrationale / welcker vijve in't Quadraet *PX* begrepen zijn. Noch *OF* doet $1 - \sqrt{\frac{3}{4}}$. *GF* [1615c:88a] een syde des 12 houcx doet $\sqrt{.2} - \sqrt{3}$, ofte $\sqrt{1\frac{1}{2}} - \sqrt{\frac{1}{2}}$. Soo verre men van dese syde (als *GF*) eenen gelijcksydighen Triangel maect / sal de Perpendiculaer (welcke uyt eenen winckel op't midden van

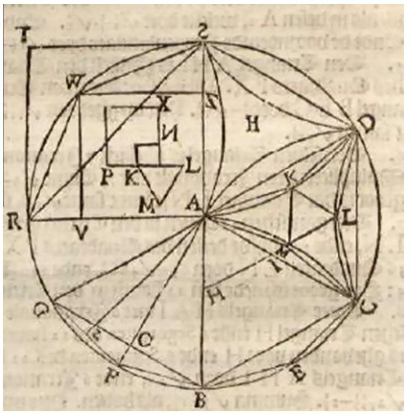

Fig. 4. Figure for Propositions 1–15 in [Van Ceulen, 1596, 63a].

een syde ghetrocken is) doen $\sqrt{.1\frac{1}{2}-\sqrt{1\frac{11}{16}}}$, ofte $\sqrt{1\frac{1}{8}}-\sqrt{\frac{3}{8}}$, ende den Triangel sal groot zijn $\sqrt{\frac{3}{4}}-\frac{3}{4}$, ende zijn dubbelt $\sqrt{3}-1\frac{1}{2}$. De Perpendiculaer welcke uyt den *Centro A* op't midden van de syde des 12 houcx ghetrocken is / doet $\sqrt{\frac{3}{8}}+\sqrt{\frac{1}{8}}$. Alsoo zijn nu de ghetallen ghevonden / daer door volghende Vraghen (aen my ghedaen) op te lossen zijn.

1. *De eerste Vraghe*: Oft niet den Triangel met de dry cromme Linien besloten (ende inder Figuer *H* gheteeckent met *ASD*) met 6 *Segmenten* des 12 houcx soo groot zy (ofte vermagh soo veel) als tweemael den ghelijcksydighen Triangel / gemaeckt van een syde des 12 houcx / in den Circkel gheschreven? *Antvvoordt, Iae.* Voor den gelijcksydighen Triangel is boven ghevonden $\sqrt{\frac{3}{4}}-\frac{3}{4}$, syn dubbelt doet $\sqrt{3}-1\frac{1}{2}$. Ende de Figuer *H* met 6 Segmenten des 12 houcx / doen t'samen den ghelijcksydighen Triangel [1596:63b] *SAD*, weynigher dry ghelijckbeenighe *GFB*, daer boven voor gevonden is $\frac{1}{2}-\sqrt{\frac{3}{16}}$. Dese met 3 gemultipliceert / comt $1\frac{1}{2}-\sqrt{1\frac{11}{16}}$ genomen van den Triangel *SAD* (welcke doet $\sqrt{\frac{3}{16}}$) sal resten voor de Figuer *H* ende 6 *Segmenten* des 12 houcx $\sqrt{3}-1\frac{1}{2}$. Soo veel is mede boven gevonden voor den dubbelen Triangel / is mede *Geometrice* te bewijsen.
2. Dat Quadraet der syden eenes 12 houcx / als hier *GF*, is viermael soo groot als den gelijckbeenigen Triangel *GFB*. *Dit is alsoo:* Dan het Quadraet *GF* doet $2-\sqrt{3}$, ende den Triangel *GFB* doet $\frac{1}{2}-\sqrt{\frac{3}{16}}$. Dese gemultipliceert met 4. Comt $2-\sqrt{3}$, als boven.
3. Soo van den halven Diameter / als in desen *AF*, ende van de Perpendiculaer *AO* werdt een recht Linie ghemaeckt / ende dan *AO* de Perpendiculaer ghedeelt werdt / in reden *AF* ende *AO*, het minste deel zal dry-mael soo groot zijn als de Linie *OF*, (Mijn dunckt dat de eerste woorden in desen ghebruyckt niet van noode en zijn) *Antvvoordt*. Deelt $\sqrt{\frac{3}{4}}$ (soo lanck is *AO*) naer de begheerde Proportie / ghy sult vinden voor den grootsten deel $\sqrt{12}-3$, ende voor den cleynsten $3-\sqrt{6\frac{3}{4}}$. Nu doet *OF* $1-\sqrt{\frac{3}{4}}$. Dese ghemultipliceert met 3, Comt $3-\sqrt{6\frac{3}{4}}$. *Recht.*
4. Soo een ghelijcksydighe Triangel ghemaeckt werdt van *GF*, syn syde met de Perpendiculaer (die uyt een winckel op de syde ghetrocken is) doen t'samen soo veel als de

Perpendiculaer die uyt den *Centro* des Circkels op een syde des 12 houcx getrocken is / als in desen *A*2, welcke doet $\sqrt{\frac{3}{8}} + \sqrt{\frac{1}{8}}$. Ende een syde des 12 houcx doet $\sqrt{1\frac{1}{2}} - \sqrt{\frac{1}{2}}$. Ende de voornoemde Perpendiculaer doet $\sqrt{1\frac{1}{8}} - \sqrt{\frac{3}{8}}$. Summa $\sqrt{\frac{3}{8}} + \sqrt{\frac{1}{8}}$ als boven.

5. Den Triangel *AHI* is ghelijck den Triangel *BEC*, ende 2 Irrationale ofte $\frac{2}{5}$ des Quadraets *PX*. *Dit is alsoo:* Dan den Trianghel *AHI* doet $\sqrt{\frac{3}{400}}$, ende den Triangel *BEC* doet $\frac{1}{2} - \sqrt{\frac{3}{16}}$. Hier by ghedaen $\sqrt{\frac{27}{100}} - \frac{1}{2}$ (soo veel doen 2 Irrationale) Comt t'samen $\sqrt{\frac{3}{400}}$. [1615c:88b]
6. Den selven Triangel is ghelijck 7 Irrationale / ende den Quadraet *LN* t'samen: Multipliceert een Irrationale met 7, Comt $\sqrt{3\frac{123}{400}} - 1\frac{3}{4}$. Hier by ghedaen $1\frac{3}{4} - \sqrt{3}$, (soo groot is dat Quadraet *LN*) comt t'samen $\sqrt{\frac{3}{400}}$, soo groot is mede den Triangel *AHI*.
7. Den gantschen 12 houck in den Circkel geschreven / is soo groot als 36 Quadraten *LN*, ende 240 vijfde deelen des Quadraets *PX*. (ofte Irrationale) *Is goedt:* Oorsake 36 Quadraten *LN* doen $63 - \sqrt{3888}$, ende 240 Irrationale doen $\sqrt{3888} - 60$. Summa 3: Soo groot is mede den 12 houck in den Circkel gheschreven.
8. Twee Triangels *HAI* met 6 Irrationale / doen effen soo veel als den cromlinischen Triangel *H* ende 6 *Segmenten* des 12 houcx. *Recht.*[16] boven in der eerster Vraghe is ghevonden voor *H* ende 6 *Segmenten* des 12 houcx / t'samen $\sqrt{3} - 1\frac{1}{2}$. Item twee Triangels *AHI* doen $\sqrt{\frac{3}{100}}$, ende 6 Irrationale (ofte $\frac{6}{5}$ des Quadraets *PX*) doen $\sqrt{2\frac{43}{100}} - 1\frac{1}{2}$. Summa $\sqrt{3} - 1\frac{1}{2}$, als boven. Hier uyt is openbaer / dat twee ghelijcksydige Triangels / ghemaeckt van een syde des 12 houcx / soo groot zijn als twee Trianghels *AHI*, ende 6 Irrationale.
9. Den Triangel *AHI*, met 8 Irrationale / doen soo veel als de rest / welck blijft als 1 *Segment* des 6 houcx ghenomen werdt van den cromlinighen Triangel *H*, welck ghenoemt werdt de reste des *Segments*, ende noch 8 *Segmenten* des 12 houcx. *Antvvoort.* Den 6 houck in den Circkel doet $\sqrt{6\frac{3}{4}}$: Daerom doen 6 *Segmenten* soo veel als den Circkel / weynigher $\sqrt{6\frac{3}{4}}$, de dry doen half so veel. Dese ghenomen van $\sqrt{\frac{3}{16}}$, als den gantschen Triangel *ASD*, Rest voor den Triangel *H* $\sqrt{3}$, weynigher den halven Circkel. Hier van genomen 1 *Segment* (als $\frac{1}{6}$ des Circkels[17] weynigher $\sqrt{\frac{3}{16}}$), Rest voor de reste des *Segments* $\sqrt{4\frac{11}{16}}$, weynigher $\frac{2}{3}$ des Circkels. Item de 12 *Segmenten* des 12 houcx doen den Circkel / weynigher 3, dan moeten de 8 *Segmenten* doen $\frac{2}{3}$ van den Circkel / weynigher 2. Summa 8 *Segmenten* des 12 houcx / ende de reste des *Segments* $\sqrt{4\frac{11}{16}} - 2$. Item een Irrationale doet $\sqrt{\frac{27}{400}} - \frac{1}{4}$. Multipliceert met 8, Comt $\sqrt{4\frac{8}{25}} - 2$. Hier toe ghedaen $\sqrt{\frac{3}{400}}$ (als den Triangel *AHI*) comt t'samen $\sqrt{4\frac{11}{16}} - 2$, als boven.

[16] The word *Recht* is incorrectly printed in frakturs in both editions.
[17] I have restored a misplaced parenthesis in both editions to its correct position. The two editions read: (als $\frac{1}{6}$ des Circkels) weynigher $\sqrt{\frac{3}{16}}$, Rest ...

10. De 4 Triangels *AHI*, met twee Irrationale / zijn ghelijck den / Triangel *H*, ende twee *Segmenten* des 12 houcx / met 2 *Segmenten* des 6 houcx t'samen. *Antvvoordt.* 4 Triangels *AHI* doen $\sqrt{\frac{3}{25}}$. Hier by ghedaen 2 Irrationale / comt t'samen $\sqrt{\frac{3}{4}} - \frac{1}{2}$, Item substraheert den Triangel *AKD* van den gantschen *ACD*, Rest $\sqrt{\frac{3}{4}} - \frac{1}{2}$ voor den Triangel *H*, ende 2 *Segmenten* des 12 houcx mede 2 *Segmenten* des 6 houcx t'samen / [1596:64a] De oorsaecke is lichter te mercken dan in't voorgaende.
11. Dat Quadraet des Diameters / ende 40 Triangels *AHI*, ende den ingheschreven 12 houck. Item den ingheschreven 6 houck / zijn 4 grootheyden / welcke staen in een Continue Proportie / als in desen den Diameter des Circkels doet 2, syn Quadraet 4, ende 40 Triangels doen $\sqrt{12}$. Den 12 houck 3, den 6 houck $\sqrt{6\frac{3}{4}}$. Als 4 tegen $\sqrt{12}$, Alsoo $\sqrt{12}$ teghen 3, ende 3 teghen $\sqrt{6\frac{3}{4}}$. [1615c:89a]
12. Den 12 houck is soo groot als 6 Quadraten van de *Apotome WY*, ende 12 Triangels *WSR*, *Goedt. WY* doet $1 - \sqrt{\frac{1}{2}}$, syn Quadraet $1\frac{1}{2} - \sqrt{2}$. Dese met 6 ghemultipliceert / comt $9 - \sqrt{72}$. Den Triangel *RWS* doet $\sqrt{\frac{1}{2}} - \frac{1}{2}$. Multipliceert met 12, comt $\sqrt{72} - 6$. Summa 3, Soo groot is den 12 houck.
13. Een syde[18] des 8 houcx bestaet mede uyt 12 Triangels ende 4 Quadraten / *Is mede goet.* De 12 Triangels doen $\sqrt{72} - 6$, ende 4 Quadraten doen $6 - \sqrt{32}$. Summa $\sqrt{8}$, Soo lanck is een syde[19] des 8 houcx *RW*.
14. Een viercant om den Circkel gheschreven (welcke viermael soo groot is als het Quadraet *TRAS*) doet 48 Quadraten *OF*, ende 256 *Segmenten* des 12 houcx / ende 64 Irrationale. *Dit vvaer alsoo:* Soo verre tusschen een *Segment* des 12 houcx / ende een Irrationale gheen differentie ware: dan de 48 Quadraten *LN* (welcke syde is *OF*) doen $84 - \sqrt{6912}$. De 256 Irrationale (ende niet *Segmenten*) doen $\sqrt{4423\frac{17}{25}} - 64$. Item de 64 Irrationale doen $\sqrt{276\frac{12}{25}} - 16$. Summa voor 320 Irrationale $\sqrt{6912} - 80$. Hier toe ghedaen soo veel ghecomen is van 48 Quadraten *LN*, Comt t'samen 4, Soo groot is mede den omgheschreven 4 houck.
15. 36 Triangels / welcker 30 doen soo veel als den 6 houck / doen soo veel als 36 Quadraten *LN*, ende 204 Irrationale (ende niet *Segmenten* des 12 houcx) Ende noch 48 Irrationale / *Goet:* Dan 204 Irrationale doen $\sqrt{2809\frac{2}{25}} - 51$, ende 48 der selver doen $\sqrt{155\frac{13}{25}} - 12$. Item de 36 Quadraten *LN* doen $63 - \sqrt{3888}$. Summa $\sqrt{9\frac{18}{25}}$, soo groot zijn 36 Triangels / den Circkel is grooter. Dese / ende noch veel constighe Vragen zijn mijn te beantwoorden ghegheven / daer van in mijn groote werck meerder ghehandelt sal werden: Dan ick moet bekennen / niet weynigh daer uyt gheleerdt te hebben. Ende soo verre een *Segment* des 12 houcx soo veel vermochte als $\frac{1}{5}$ deel des Quadraets *PX*, (ofte een Irrationale) ende niet meer / soo waer't onwijsselick ghedaen van mijn / dat ick gheen gheloof moste gheven / t'ghene my lest te beantwoorden ghegheven is / met sulcken voorstel / ende bedenckinghe / als volght:
Den Circkel is groot 36 *Segmenten* des 6 houcx / die in hem beschreven werdt. Om sulcx te bewijsen: [1615c:89b]

[18] The word "syde" in both editions is mathematically incorrect. The area of the octagon must have been intended.
[19] Again the word "syde" is mathematically incorrect.

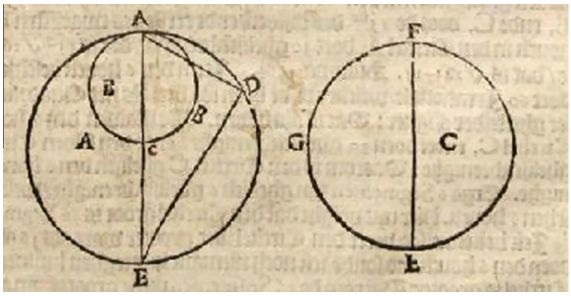

Fig. 5. Figure for Proposition 16 in [Van Ceulen, 1596, 64a].

16. [Fig. 5] Laet den Circkel *ADE* grooter zijn dan den Circkel *ABC* 36 ghelijckbeenighe / ende de rechte Linie *AD* zy ghelijck den Diameter *AC* / door de 47*ste* des eersten *Euclijdes.* De Quadraten *DA*, ende *DE* zijn ghelijck t'samen / den[20] Quadraet *AE*, ende door de 2*de* des twaelfsten *Euclijdes*, de Circkels beschreven / om de Diameters *AD*, ende *DE* zijn t'samen soo groot als den Circkel *ADE*. Laet nu *EF* des Circkels *C* middel-linie ghelijck zijn *DE*. Den Circkel *C* sal dan groot zijn 36 ghelijckbeenighe: Wederomme de selve Circkel *EGF*, ofte *C*, ende den Circkel *ABC*, ofte *B*, zijn tsamen ghelijck den Circkel *A*, ende door de 15*ste* des vijfden *Euclides.* De 6 houck gheschreven in den Circkel *C*, ende den 6 houck gheschreven in den Circkel *B* t'samen zijn ghelijck den 6 houck gheschreven inden Circkel *A*: Maer den 6 houck in den Circkel *A* vermach / ofte is groot / 30 ghelijckbeenighe / ende 60 Irrationale / Ende den 6 houck gheschreven in den Circkel *B* is ghelijck 60 Irrationale. [1596:64b] Ergo den 6 houck / geschreven in den Circkel *C*, sal groot sijn 30 gelijckbeenige: Maer de Circkel *C* is groot 36 ghelijckbeenighe: Daerom is hy ghelijck den 6 houck / in hem beschreven / ende 6 ghelijckbeenighe: Dat is / hy vermagh / ende is ghelijck den 6 houck / ende den 6 *Segmenten*. Ergo 6 *Segmenten* zijn ghelijck de 6 ghelijkbeenighe / welcker 30 doen in den Circkel *C*, den gantschen 6 houck: Daerom den Circkel vermagh / en is ghelijck 36 *Segmenten*. Daer en is niemandt die dit niet en begrijpt / die tamelijk *Mathematicus* is / etc.

Hier op hebbe ick gheantwoordt / als volght: (ende het voorgaende bewesen met ghetallen) Laet den Diameter des Circkels *A* doen 2, (soo veel mede gheset voor den Diameter in der voorgaender Figuer) dan moet den Diameter *FE* des Circkels *C* (soo verre syn 6 houck 30 gelijckbeenighe groot is) doen $\sqrt{\cdot}\,\sqrt{533\frac{1}{3}} - 20$, Ende den Diameter des cleynen Circkels doet $\sqrt{.24} - \sqrt{533\frac{1}{3}}$, soo verre 36 Trianghels (welcker 30 inden Circkel *A* doen den 6 houck) soo groot zijn als den Circkel.[21] Is dat waer / dan moeten 36 *Segmenten* des 6 houcx / den vermoghen des Circkels ghenough doen / welck ick bewijsen sal op voorgaende manieren / ende mede met ghetallen. Den Diameter des Circkels *A* is 2, (als voorseydt) daer van is ghesneden den Circkel *B*: Alsoo dat syn rest gelijck zy 36 gelijckbeenige / die in der Figuer voor desen geteeckent is met *GFB*, ofte *BEC*, dat is $18 - \sqrt{243}$, Dan sal den cleynen Circkel groot zijn $\sqrt{349\frac{23}{25}} - 18$. *Oorsaecke:* Ghenomen van den Circkel *A* 36 ghelijckbeenige / sullen resten 6 ghelijcksydighe (die in der voorga-

[20] I have corrected the printer's error "dien," which appeared in both editions.
[21] The text is confusing; see footnote to the translation.

ender Figuer geteeckent zijn met *KML*) met 6 gelijckbeenige / ende 12 *Segmenten* des 12 houcx. Nu doet een gelijcksydighe $\sqrt{9\frac{3}{16}}-3$, ende de 6 doen $\sqrt{330\frac{3}{4}}-18$. Item een ghelijckbeenighe doet $\frac{1}{2}-\sqrt{\frac{3}{16}}$, Comt voor de ses $3-\sqrt{6\frac{3}{4}}$. Als mede tusschen 12 *Segmenten* des 12 houcx / ende 12 Irrationale gheen differentie waer / soo soude comen voor een *Segment* des 12 $\sqrt{\frac{27}{400}}-\frac{1}{4}$, ende voor de 12 $\sqrt{9\frac{18}{25}}-3$. Summa voor den Circkel *B*, als boven. *Anders:* Substraheert dat vermoghen des Circkels *C* van den vermogen des Circkels *A*, [1615c:90a] Rest mede voor den Circkel *B* als boven. Noch in den Circkel *A* is de Corda *ED* gelijck den Diameter *FE*, des Circkels *C*. Ende de Corda *AD* is gelijck den Diameter *AC*, des Circkels *B*, Dan moet (door de 47[ste] des eersten *Euclijdes*) dat Quadraet *AD*, met den Quadraet *DE* t'samen gelijck zijn den Quadraet des Diameters des Circkels *A*, Ende naer dien den winckel *ADE* gerecht is (door de 31[ste] des derden *Euclijdes*) Daerom den Circkel *C*, ende den Circkel *B* t'samen / moeten soo groot zijn als den Circkel *A*. (door de 2[de] des twaelfden *Euclijdes*) Ende den 6 houck gheschreven in den Circkel *A*, is soo groot als de twee 6 houcken / in de Circkels *B*, ende *C*, door de 15[ste] des vijfden / ende eerste des twaelfsten *Euclijdes*: Maer den 6 houck in den Circkel *A*, doet 30 ghelijckbeenighe / dat is $15-\sqrt{168\frac{3}{4}}$, ende 60 Irrationale / dat is $\sqrt{243}-15$. Summa $\sqrt{6\frac{3}{4}}$. Item den 6 houck beschreven in den Circkel *B*, doet 60 Irrationale / welcker vijve doen soo veel als het Quadraet *PX* in der voor deser ghestelder Figuer: Dat is als boven. Derhalven den 6 houck beschreven in den Circkel *C*, moet doen 30 ghelijck beenighe / ende den selven Circkel is soo groot als 36 ghelijckbeenighe: Daerom is den Circkel *C* ghelijck den 6 houck / ende 6 ghelijckbeenighe. Ergo 6 *Segmenten* zijn ghelijck 6 ghelijckbeenighe / welcker 30 doen den gantschen 6 houck. Hier uyt volght dat den Circkel groot is 36 *Segmenten* des 6 houcx / etc.

Ick bekenne / soo verre den Circkel niet grooter ware als 36 Triangels / welcker 30 doen den 6 houck / soo soude ick noch niemant hier teghen kunnen segghen. Maer den Circkel is grooter: Daerom de 36 *Segmenten* mede grooter zijn als den Circkel / als boven bewesen. Het schijnt / ende het verschil is niet licht te mercken: Naedemael twee Triangels *AHI* in der voor deser Figueren gheteyckent / welcker 30 den 6 houck maken / met 6 Irrationale t'samen / dat $1\frac{1}{5}$ des Quadraets *PX*, welcke Quadraet mede soo groot is / als een ghelijksydighe / ende het Quadraet van de *Apotome OF*, soo groot zijn / ende voorseker soo veel doen als de Figuer *H*, met 6 *Segmenten* des 12 houcx. Dat is de Figuer gheteeckent met *AKDLCM*, die soo groot is als 3 ghelijckbeenighe / ende een ghelijcksydighe. Dese is ghemaeckt van de Figuer *H*, ende 6 *Segmenten* des 12 houcx. Item twee Triangels *AHI*, ende 6 Irrationale / werden mede gemaeckt van de Figuer *H*, ende 6 *Segmenten* des 12 houcx. Hier door is niet te besluyten / dat [1596:65a] de twee Triangels *AHI* ghelijck souden zijn de spatie / ofte den inghehoghen Triangel *H*, ende de 6 Irrationale soo groot souden zijn als 6 *Segmenten* des 12 houcx / *gantsch niet.* De twee Triangels doen t'samen $\sqrt{\frac{3}{100}}$, (Verstaet *AHI*) dat is meerder dan de Figuer *H*, ende de 6 Irrationalen doen $\sqrt{2\frac{43}{100}}-1\frac{1}{2}$, dit is minder dan 6 *Segmenten* des 12 houcx. Summa 2 Trianghels *AHI*, ende 6 Irrationalen doen $\sqrt{3}-1\frac{1}{2}$. Soo veel is mede gevonden voor dat spatie *H*, ende 6 *Segmenten* des 12 houcx. Dese ghelijckheydt comt uyt gheenderhande oorsaecke: dan soo veel twee Triangels *AHI* grooter zijn als het spatie *H*, effen soo veel zijn 6 *Segm-*

enten des 12 houcx grooter dan de 6 Irrationale. [1615c:90b] Ten laetsten / dat een *Segment* grooter is dan een Irrationale / is op veelderhande manieren te bewijsen: Ick sal tot desen gebruycken de grootheydt des 48 houcx / in den Circkel beschreven / daar boven voor ghevonden is $\sqrt{.288 - \sqrt{31104} + \sqrt{10368}}$. dat is meerder als $3\frac{132629}{1000000}$. Hier van ghenomen het vermoghen des 12 houcx / Rest $\frac{132629}{1000000}$. Dit ghedivideert door 12, Comt $\frac{110524}{10000000}$, Soo groot is een Figuer welcke in dat *Segment* des 12 houcx beschreven is / welck *Segment* noch 4 *Segmenten* des 48 houcx grooter is. Nu doet $\frac{1}{5}$ des Quadraets *PX*, ofte een Irrationale $\sqrt{\frac{27}{400}} - \frac{1}{4}$, dat is minder als $\frac{98077}{10000000}$, dit is noch $\frac{12447}{10000000}$ minder als het gheene welck bewijselijck te cleyn is / als het Segment des 12 houcx. Alle die mijn naer rekenen / sullen de waerheydt bevinden: Mijn ghevoelen is dat dese ghewichtighe saecke / niet door op merkinghe eenes Radts omloops / ende met een Passer op een coperen Tafel te vinden is / als *Bovelli* tot *Parijs* vondt. Desen heeft mede ghevonden de syde eenes 7 houcx / inden Circkel beschreven / welcke soude zijn de helfte eener syde des ghelijcksydighen Triangels / in den Circkel beschreven. Dit schijnt / alsmen't met de Passer meet / goedt: Maer door ghetallen wert anders bewesen.

Dese / ende andere syden / als de syden des 5, 7, 9, 11, 13, 15, 17 houcx / zijn op volghende maniere naerder te vinden: Maer niet volcomen. Besiet desen volghenden Circkel / . . .

English translation

[1615c:85a] "The XXI. Chapter. In which is proved that the square of the circumference of a circle is less than ten times the square of the diameter. Also that 36 triangles, 30 of which amount to the equiangular hexagon, are less, and 36 *Segments* of the hexagon are greater than the area of the circle, and that 35 of the segments are greater (than the area)." [1596:61b]

. . .

[1596:63a; 1615c:87b] Here follow some pieces in the art (of mathematics), concerning the circle, proposed and found by a highly learned man, in which (pieces) his illustrious mind is evident. These pieces have been sent to me and he desires to have my opinion of them. Therefore I have answered them by investigating them, and I have found most of them to be correct by means of numbers, as follows:

[Fig. 6] In this circle, whose diameter amounts to 2, *AGB*, *ABC*, *ACD*, and *ADS* are triangles of the hexagon, and / *AHI* is $\frac{1}{5}$ of *ABC*. The equilateral triangle *KLM* is constructed from the double of *OF* (that is, a side of it is twice as long as *FO*).[22] A side of the square *KN* is equal to *OF*. The square *PX* is equal in size to the equilateral triangle *KLM* together with the square *KN*. Further, two *Segments* of the dodecagon and triangle *GFB* amount to the same as one cut-off piece of the hexagon[23] (as is clear to the eye). In order to answer the following questions, it is necessary to know the sizes of the above-mentioned pieces. They are easily found by means of what has been explained before:

[22] *F* is the midpoint of arc *BG*, and *AF* intersects *BG* at *O*.
[23] The "piece cut off of the heptagon" is actually the area between arc *GB* and line *GB* in Scaliger's terminology; see Section 2.

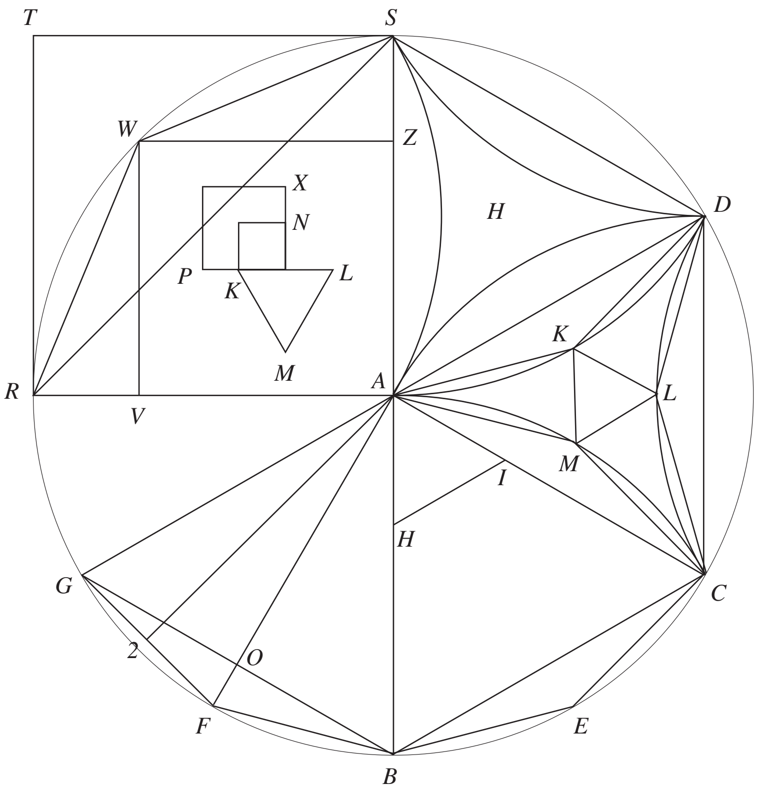

Fig. 6. Figure for Propositions 1–15.

The triangle *ABC* is $\frac{1}{6}$ of the hexagon / and it amounts to $\sqrt{\frac{3}{16}}$. The triangle *AHI* is $\frac{1}{5}$ of *ABC*, and amounts to $\sqrt{\frac{3}{400}}$. The isosceles triangle *GFB* amounts to $\frac{1}{2} - \sqrt{\frac{3}{16}}$. The triangle *KML* amounts to $\sqrt{9\frac{3}{16}} - 3$. The square *KN* amounts to $1\frac{3}{4} - \sqrt{3}$, and the size of the square *PX* is $\sqrt{1\frac{11}{16}} - 1\frac{1}{4}$. Also, $\frac{2}{5}$ of this is $\sqrt{\frac{27}{100}} - \frac{1}{2}$, and half of this, that is $\frac{1}{5}$ of *PX*, amounts to $\sqrt{\frac{27}{400}} - \frac{1}{4}$. This is called an Irrational, five of which are contained in the square *PX*. Further, *OF* amounts to $1 - \sqrt{\frac{3}{4}}$. *GF*, [1615c:88a] a side of the dodecagon, amounts to $\sqrt{2 - \sqrt{3}}$, that is $\sqrt{1\frac{1}{2} - \sqrt{\frac{1}{2}}}$. If one constructs on this side (as *GF*) an equilateral triangle, the perpendicular (which is drawn from an angular point to the midpoint of a side) will amount to $\sqrt{1\frac{1}{2} - \sqrt{1\frac{11}{16}}}$, that is $\sqrt{1\frac{1}{8} - \sqrt{\frac{3}{8}}}$, and the size of the triangle will be $\sqrt{\frac{3}{4}} - \frac{3}{4}$, and its double will be $\sqrt{3} - 1\frac{1}{2}$. The perpendicular which is drawn from the *Centro A* to the midpoint of the side of the dodecagon amounts to $\sqrt{\frac{3}{8}} + \sqrt{\frac{1}{8}}$. Thus the numbers have now been found by means of which the following questions (posed to me) can be solved.

1. *The first question*: Whether the triangle contained by the three curved lines (drawn in the figure *H* as *ASD*)[24] together with 6 *Segments* of the dodecagon, is equal in size to (that is, has the same area as) twice the equilateral triangle constructed on a side of the dodecagon inscribed in the circle? *Answer, Yes.* Above, for the equilateral triangle has been found (the area) $\sqrt{\frac{3}{4}} - \frac{3}{4}$, and its double amounts to $\sqrt{3} - 1\frac{1}{2}$. And the figure *H* together with 6 Segments of the dodecagon amount to the equilateral triangle [1596:63b] *SAD* less three isosceles triangles *GFB*, for which [triangle the area] has been found above to be $\frac{1}{2} - \sqrt{\frac{3}{16}}$. This multiplied by 3 comes out as $1\frac{1}{2} - \sqrt{1\frac{11}{16}}$, [and if this is] taken away from the triangle *SAD* (which amounts to $\sqrt{\frac{3}{16}}$), there will remain for the figure *H* and 6 *Segments* of the dodecagon $\sqrt{3} - 1\frac{1}{2}$. This much has also been found above for the double of the triangle. [This] can also be proved *Geometrice.*
2. The square of [one of the] the sides of the dodecagon, such as *GF* here, is four times as much as the isosceles triangle *GFB*. *This is true:* For the square *GF* amounts to $2 - \sqrt{3}$, and the triangle *GFB* amounts to $\frac{1}{2} - \sqrt{\frac{3}{16}}$. This, multiplied by 4, comes out as $2 - \sqrt{3}$ as above.
3. If a straight line is constructed from the semidiameter, in this [Figure] *AF*, and the perpendicular *AO*, and the perpendicular *AO* is then divided according to the ratio [between] *AF* and *AO*, the lesser part will be three times as much as the line *OF*. (I think that the first words used in this[25] are unnecessary.) *Answer*. Divide $\sqrt{\frac{3}{4}}$ (this long is *AO*) according to the desired proportion, and you will find for the greater part $\sqrt{12} - 3$, and for the lesser $3 - \sqrt{6\frac{3}{4}}$. Now *OF* amounts to $1 - \sqrt{\frac{3}{4}}$. This multiplied by 3 comes out as $3 - \sqrt{6\frac{3}{4}}$. *Right.*
4. If an equilateral triangle is constructed on *GF*, its side together with the perpendicular (drawn from an angular point onto the side) amount to the perpendicular drawn from the *Centro* of the circle onto the side of the dodecagon, as in this [figure] *A*2, which amounts to $\sqrt{\frac{3}{8}} + \sqrt{\frac{1}{8}}$. And a side of the dodecagon amounts to $\sqrt{1\frac{1}{2} - \sqrt{\frac{1}{2}}}$. And the above-mentioned perpendicular amounts to $\sqrt{1\frac{1}{8} - \sqrt{\frac{3}{8}}}$. Sum $\sqrt{\frac{3}{8}} + \sqrt{\frac{1}{8}}$ as above.
5. The triangle *AHI* is equal to the triangle *BEC* and 2 Irrationals, that is $\frac{2}{5}$ of the square *PX*. *This is true:* For the triangle *AHI* amounts to $\sqrt{\frac{3}{400}}$, and the triangle *BEC* amounts to $\frac{1}{2} - \sqrt{\frac{3}{16}}$. To this is added $\sqrt{\frac{27}{100}} - \frac{1}{2}$ (2 Irrationals amount to this much); together it comes out as $\sqrt{\frac{3}{400}}$. [1615c:88b]
6. The same triangle is equal to 7 Irrationals together with the square *LN*: Multiply an Irrational by 7, it comes out as $\sqrt{3\frac{123}{400}} - 1\frac{3}{4}$. To this is added $1\frac{3}{4} - \sqrt{3}$ (this much is the square *LN*); together it comes out as $\sqrt{\frac{3}{400}}$, this much is also the triangle *AHI.*

24 It would have been correct to say "drawn in the figure *ASD* as *H*."

25 Van Ceulen means the words "a straight line is constructed from the semidiameter, in this Figure *AF*, and the perpendicular *AO*".

7. The whole dodecagon inscribed in the circle is as much as 36 squares[26] *LN*, and 240 fifth parts of the square *PX* (or Irrationals). *Is correct:* because 36 squares *LN* amount to $63 - \sqrt{3888}$, and 240 Irrationals amount to $\sqrt{3888} - 60$. Sum 3: This much is also the dodecagon inscribed in the circle.
8. Two triangles *HAI* together with 6 Irrationals amount to exactly the same as the curvilinear triangle *H* and 6 *Segments* of the dodecagon. *Right.* Above in the first question there has been found for *H* together with 6 *Segments* of the dodecagon $\sqrt{3} - 1\frac{1}{2}$. Also, two triangles *AHI* amount to $\sqrt{\frac{3}{100}}$, and 6 Irrationals (that is, $\frac{6}{5}$ of the square *PX*) amount to $\sqrt{2\frac{43}{100}} - 1\frac{1}{2}$. Sum $\sqrt{3} - 1\frac{1}{2}$, as above. From this it is evident that two equilateral triangles constructed on a side of the dodecagon are equal in size to two triangles *AHI* and 6 Irrationals.
9. The triangle *AHI* together with 8 Irrationals amounts to the same as [a.] the remainder if 1 *Segment* of the hexagon is taken away from the curvilinear triangle *H*, which [remainder] is called the remainder of the *Segment*, together with [b.] 8 *Segments* of the dodecagon. *Answer.* The hexagon in the circle amounts to $\sqrt{6\frac{3}{4}}$: therefore 6 *Segments* amount to the circle less $\sqrt{6\frac{3}{4}}$, and the three [segments] amount to half as much. If these are taken from $\sqrt{\frac{3}{16}}$, as the whole triangle *ASD*, there remains for triangle *H* $\sqrt{3}$ less half the circle. From this is taken 1 *Segment* (as $\frac{1}{6}$ of the circle less $\sqrt{\frac{3}{16}}$). There remains, as the remainder of the *Segment*, $\sqrt{4\frac{11}{16}}$ less $\frac{2}{3}$ of the circle. Also, the 12 *Segments* of the dodecagon amount to the circle less 3; so the 8 *Segments* must amount to $\frac{2}{3}$ of the circle less 2. Sum: 8 *Segments* of the dodecagon and the remainder of the *Segment* [amount to] $\sqrt{4\frac{11}{16}} - 2$. Also, an Irrational amounts to $\sqrt{\frac{27}{400}} - \frac{1}{4}$. Multiply by 8, it comes out as $\sqrt{4\frac{8}{25}} - 2$. To this is added $\sqrt{\frac{3}{400}}$ (as the triangle *AHI*), and together it comes out as $\sqrt{4\frac{11}{16}} - 2$, as above.
10. The 4 triangles *AHI* together with two Irrationals are equal to the triangle *H* and two *Segments* of the dodecagon together with 2 *Segments* of the hexagon. *Answer.* 4 triangles *AHI* amount to $\sqrt{\frac{3}{25}}$. [If] 2 Irrationals are added to this, together it comes out as $\sqrt{\frac{3}{4}} - \frac{1}{2}$. Also, subtract the triangle *AKD* from the whole *ACD*, the remainder is $\sqrt{\frac{3}{4}} - \frac{1}{2}$, for the triangle *H* together with 2 *Segments* of the dodecagon and 2 *Segments* of the hexagon. [1596:64a] The cause is more easily noticed than above.
11. The square of the diameter, 40 triangles *AHI*, the inscribed dodecagon, and also the inscribed hexagon are four magnitudes in a continued proportion, as in this [figure]: the diameter of the circle amounts to 2, its square to 4, and 40 triangles amount to $\sqrt{12}$, the dodecagon to 3, the hexagon to $\sqrt{6\frac{3}{4}}$. As 4 is to $\sqrt{12}$, so is $\sqrt{12}$ to 3, and 3 to $\sqrt{6\frac{3}{4}}$. [1615c:89a]

[26] Points *L* and *N* occur in Fig. 4 but the figure does not display the square *LN*, which is congruent to the square *KN*.

12. The dodecagon is equal in size to 6 squares of the *Apotome*[27] WY, and 12 triangles WSR. *Good.* WY amounts to $1 - \sqrt{\frac{1}{2}}$, its square to $1\frac{1}{2} - \sqrt{2}$. This multiplied by 6 comes out as $9 - \sqrt{72}$. The triangle RWS amounts to $\sqrt{\frac{1}{2}} - \frac{1}{2}$. Multiply by 12, it comes out as $\sqrt{72} - 6$. Sum 3, this much is the dodecagon.
13. A side[28] of the octagon also consists of 12 triangles and 4 squares. *Is also good.* The 12 triangles amount to $\sqrt{72} - 6$, and 4 squares amount to $6 - \sqrt{32}$. Summa $\sqrt{8}$, this long is a side [i.e., the area] of the octagon RW.
14. A square circumscribed about the circle (which is four times as much as the square $TRAS$) amounts to 48 squares OF and 256 *Segments* of the dodecagon and 64 Irrationals. *This is true* if there were no difference [in size] between a *Segment* of the dodecagon and an Irrational: for the 48 squares LN (whose side is OF) amount to $84 - \sqrt{6912}$. The 256 Irrationals (and not *Segments*) amount to $\sqrt{4423\frac{17}{25}} - 64$. Also, the 64 Irrationals amount to $\sqrt{276\frac{12}{25}} - 16$. Sum for 320 Irrationals: $\sqrt{6912} - 80$. [If] to this is added the outcome of 48 squares LN, the [total] outcome is 4. This much is also the circumscribed quadrilateral [i.e., square].
15. 36 triangles such that 30 amount to the hexagon, amount to 36 squares LN and 204 Irrationals (and not *Segments* of the dodecagon) and another 48 Irrationals. / *Good:* For 204 Irrationals amount to $\sqrt{2809\frac{2}{25}} - 51$, and 48 of them amount to $\sqrt{155\frac{13}{25}} - 12$. Also the 36 squares LN amount to $63 - \sqrt{3888}$. Sum, $\sqrt{9\frac{18}{25}}$, this much are 36 triangles; the circle is greater. These and many other mathematical questions have been given to me to answer, and some of them will be treated in more detail in my great work. For I have to admit that I have learned a lot by them. If a *Segment* of the dodecagon would amount to the same as $\frac{1}{5}$ part of the square PX (that is, an Irrational) and not more, then it would be unwise of me to reject the last [proposition] which has been given me to answer, with a theorem and counter-argument as follows: The circle is equal to 36 *Segments* of the hexagon inscribed in it. To prove this: [1615c:89b]
16. [Fig. 7] Let the circle ADE be greater than the circle ABC by 36 isosceles [triangles], and let the straight line AD be equal to the diameter AC. By the 47^{th} of the first of *Euclid*, the squares of DA and DE are together equal to the square of AE, and by the 2^{nd} of the twelfth of *Euclid*, the circles described on the diameters AD and DE are together equal to the circle ADE. Now let EF, the diameter of the circle C, be equal to DE. Then the circle C wil be equal to 36 isosceles [triangles]. Again, the same circle EGF, that is C, and the circle ABC, that is B, are together equal to the circle A, and by the 15^{th} of the fifth of *Euclid*, the hexagon inscribed in the circle C and the hexagon inscribed in the circle B are together equal to the hexagon inscribed in the circle A. But the hexagon in the circle A has the same area as, that is, is equal in size to, 30 isosceles [triangles] and 60 Irrationals, and the hexagon inscribed in the circle B is equal to

[27] Y is the midpoint of RS, and the word *Apotome* is a technical term from the Euclidean classification of irrational magnitudes.
[28] The length of the "side" RW of the octagon is $\sqrt{2 - \sqrt{2}}$ but the area of the octagon is $4 \cdot \frac{1}{2} RS \cdot AW = \sqrt{8}$. Thus "area" must have been intended instead of "side," which occurs in both the 1596 and 1615 editions.

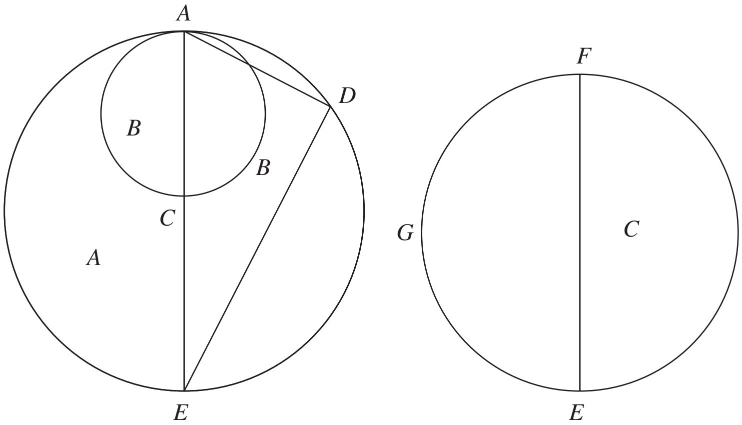

Fig. 7. Figure for Proposition 16.

> 60 Irrationals. [1596:64b] Ergo the hexagon inscribed in the circle *C* will be equal to 30 isosceles [triangles]. But the circle *C* is equal to 36 isosceles [triangles]: Therefore it is equal to the hexagon inscribed in it and 6 isosceles [triangles]: That is, it has the same area as, and is equal to, the hexagon and the 6 *Segments*. Ergo 6 *Segments* are equal to the 6 isosceles [triangles], 30 of which amount to the whole hexagon in the circle *C*. Therefore the circle has the same area as, and is equal to 36 *Segments*. There is no one who is to some extent *Mathematicus* and does not understand this, etc.

To this I have answered as follows (and I have demonstrated the preceding by means of numbers). Let the diameter of the circle *A* amount to 2 (this much has also been put for the diameter in the previous figure). Then the diameter *FE* of the circle *C* (in so far as its hexagon is equal to 30 isosceles [triangles]) must amount to $\sqrt{\sqrt{533\frac{1}{3}} - 20}$. And the diameter of the small circle amounts to $\sqrt{24 - \sqrt{533\frac{1}{3}}}$, in so far as 36 triangles (30 of which amount to the hexagon in circle *A*) are equal in size to the circle.[29] If this is true, then 36 *Segments* of the hexagon must exhaust the area of the circle, which I will demonstrate in the same way as above, and using numbers as well. The diameter of the circle *A* is 2, (as above). From this the circle *B* is cut off, in such a way that its remainder is equal to 36 isosceles [triangles], which have been drawn in the preceding figure as *GFB* or *BEC*, that is $18 - \sqrt{243}$. Then the small circle will amount to $\sqrt{349\frac{23}{25}} - 18$. *The cause:* if from the circle *A* 36 isosceles [triangles] are taken away, there will remain 6 equilateral [triangles] (which have been drawn in the preceding figure as *KML*) and 6 isosceles [triangles] and 12 *Segments* of the dodecagon. Now an equilateral [triangle] amounts to $\sqrt{9\frac{3}{16}} - 3$, and the 6 [equilateral triangles] amount to $\sqrt{330\frac{3}{4}} - 18$. Also, an isosceles [triangle] amounts to $\frac{1}{2} - \sqrt{\frac{3}{16}}$, so the

[29] The text is confusing. All of Van Ceulen's computations are based on Assumption 2 in my Section 3 above, to the effect that the inscribed hexagon in the small circle is equal in area to 60 "irrationals." As a consequence, the inscribed hexagon in the greater circle *C* is equal to 30 "isosceles" triangles. Perhaps Van Ceulen stated Assumption 1 (of my Section 3) here because he was trying to be nice to Scaliger.

six come out as $3 - \sqrt{6\frac{3}{4}}$. If there were no difference between 12 *Segments* of the dodecagon / and 12 Irrationals either,[30] then one *Segment* of the dodecagon would come out as $\sqrt{\frac{27}{400}} - \frac{1}{4}$, the 12 as $\sqrt{9\frac{18}{25}} - 3$. The sum for the circle *B* [will be] as above.

Another way: Subtract the area of the circle *C* from the area of the circle *A*, [1615c:90a] then the remainder is also the circle *B* as above. Further the chord *ED* in the circle *A* is equal to the diameter *FE* of the circle *C*, and the chord *AD* is equal to the diameter *AC* of the circle *B*. Then (by the 47th of the first of *Euclid*) the square of *AD* together with the square *DE* must be equal to the square of the diameter of the circle *A*, also because the angle *ADE* is a right angle (by the 31st of the third of *Euclid*). Therefore the circle *C* together with the circle *B* must be equal to the circle *A* (by the 2nd of the twelfth of *Euclid*). And the hexagon inscribed in the circle *A* is equal in size to the two hexagons in the circles *B* and *C*, by the 15th of the fifth, and the first of the twelfth of *Euclid*. But the hexagon in the circle *A* amounts to 30 isosceles [triangles], that is $15 - \sqrt{168\frac{3}{4}}$, and 60 Irrationals, that is $\sqrt{243} - 15$. Sum: $\sqrt{6\frac{3}{4}}$. Also, the hexagon inscribed in the circle *B* amounts to 60 Irrationals, five of which amount to the square *PX* in the previous figure: that is as [has been shown] above. Therefore the hexagon inscribed in the circle *C* must amount to 30 isosceles [triangles], and the same circle is equal to 36 isosceles [triangles].[31] Therefore the circle *C* is equal to the hexagon and 6 isosceles [triangles]. Ergo 6 *Segments* are equal to 6 isosceles triangles, of which 30 amount to the whole hexagon. From this is follows that the circle is equal to 36 *Segments* of the hexagon, etc.

I admit: if the circle were not greater than 36 triangles of which 30 amount to the hexagon, then neither I nor anyone else would be able to refute this. But the circle is greater. Therefore the 36 *Segments* are also greater than the circle, as has been demonstrated above.[32] It appears to be [correct], and the difference [between circle and 36 segments] is not easy to notice. For the two triangles *AHI* in the figure [Fig. 4] drawn before these figures [Fig. 5], 30 of which constitute the hexagon, together with 6 Irrationals, which are $1\frac{1}{5}$ of the square *PX*, which is also equal to an equilateral triangle and the square of the *Apotome OF*, are equal in size to, and certainly amount to the same as, the figure *H* together with 6 *Segments* of the dodecagon. That is the figure drawn as *AKDLCM*, which is equal to 3 isosceles [triangles] and one equilateral [triangle]. This is composed of the figure *H* and 6 *Segments* of the dodecagon. Also, two triangles *AHI* and 6 Irrationals were also composed of the figure *H* and 6 *Segments* of the dodecagon. From this it cannot be concluded that [1596:65a] the two triangles *AHI* would be equal to the space, or the concave triangle *H*, and [that] the 6 Irrationals would be equal to the 6 *Segments* of the dodecagon, *not at all.* The two triangles together amount to $\sqrt{\frac{3}{100}}$ (to wit, *AHI*), which is more than the figure *H*, and the 6 Irrationals amount to $\sqrt{2\frac{43}{100}} - 1\frac{1}{2}$, which is less than 6 *Segments* of the dodecagon. Sum: 2 triangles *AHI* and 6 Irrationals amount to $\sqrt{3} - 1\frac{1}{2}$. This much has also been found for [the sum of] the space *H* and 6 *Segments* of the dodecagon. This equality [between the two triangles

30 This assumption is equivalent to Scaliger's circle quadrature.
31 This is assumption 1 in Section 3 of this paper.
32 Van Ceulen refers to a previous part of Chapter 21 which I have not translated; see Section 4.

AHI and the space *H*] does not result from any cause: for, by as much the two triangles *AHI* are greater than the space *H*, by just as much are 6 *Segments* of the dodecagon greater than the 6 Irrationals. [1615c:90b] Finally, that a *Segment* is greater than an Irrational can be demonstrated in many ways: I will use for this the magnitude of the 48-gon inscribed in a circle, for which has been found above $\sqrt{288 - \sqrt{31104} + \sqrt{10368}}$. This is more than $3\frac{132629}{1000000}$. From this the area of the dodecagon is taken away. The remainder, $\frac{132629}{1000000}$, divided by 12, comes out as $\frac{110524}{10000000}$. This is the size of a figure which is inscribed in the *Segment* of the dodecagon, which *Segment* is still greater [than that figure] by 4 *Segments* of the 48-gon.[33] Now $\frac{1}{5}$ of the square *PX*, that is an Irrational, amounts to $\sqrt{\frac{27}{400}} - \frac{1}{4}$, which is less than $\frac{98077}{10000000}$, which is $\frac{12447}{10000000}$ less than something which can be proved to be less than the *Segment* of the dodecagon. All who repeat my computation will find the truth. My feeling is that this important matter cannot be found by observing the rotation of a wheel, and by means of a compass on a copper plate, as Bovelli did in Paris. He (i.e., Bovelli) has also found the side of a heptagon inscribed in the circle, which side would be half of the side of the equilateral triangle inscribed in the circle. This seems correct if one measures it by means of the compass. But using numbers, things are shown to be different.

These and other sides, as the sides of the 5. 7. 9. 11. 13. 15. 17-gon can be found more accurately in the following way. But not perfectly. Consider the following circle …

References

Bierens de Haan, D., 1876. Bouwstoffen voor de Geschiedenis der Wis- en Natuurkundige Wetenschappen in de Nederlanden. VIII: Ludolph van Ceulen. Verslagen en mededeelingen der Koninklijke Akademie der Wetenschappen, afdeeling Natuurkunde, tweede reeks, vol. 9, 322–369. [Reprinted in [Bierens de Haan, 1878, 123–156]. References will be to the reprint edition.

Bierens de Haan, D., 1877. Bouwstoffen voor de Geschiedenis der Wis- en Natuurkundige Wetenschappen in de Nederlanden. XIV: Josephus Scaliger J.C. Fil. als cirkel-quadrator. Verslagen en mededeelingen der Koninklijke Akademie der Wetenschappen, afdeeling Natuurkunde, tweede reeks, vol. 10, 62–96. Reprinted in [Bierens de Haan, 1878, 280–303].

Bierens de Haan, D., 1878. Bouwstoffen voor de Geschiedenis der Wis- en Natuurkundige Wetenschappen in de Nederlanden [Materials for the History of the Mathematical and Physical Sciences in the Netherlands; in Dutch], reprint edition. Digital edition: www.dbnl.org.

Bockstaele, P., 1976. The correspondence of Adriaan van Roomen. LIAS 3, 85–129, 249–299. Digital version: http://webdoc.ubn.kun.nl/tijd/l/lias/.

Bouelles, C. de, 1551. Geometrie practique. Regnaud Chaudière, Paris. Digital version: gallica.bnf.fr.

Busard, H.L.L., 1973. Article: Charles Bouvelles. In: Gillispie, C.G. (Ed.), . In: Dictionary of Scientific Biography, vol. 2. Scribner's Sons, New York, pp. 360–361.

Busard, H.L.L., 1975. Article: Adriaan van Roomen. In: Gillispie, C.G. (Ed.), . In: Dictionary of Scientific Biography, vol. 11. Scribner's Sons, New York, pp. 532–534.

De Wreede, L., 2007. Willebrord Snellius (1580–1626): a Humanist Reshaping the Mathematical Sciences. Unpublished Ph.D. dissertation. Utrecht. Digital edition: www.igitur-archive.library.uu.nl.

Dijksterhuis, E.J., 1943. Simon Stevin. Martinus Nijhoff, 's Gravenhage.

Goulding, R., 2005. Polemic in the margin: Henry Savile against Joseph Scaliger's quadrature of the circle. In: Jacquart, D., Burnett, C. (Eds.), Scientia in margine: Études sur les marginalia dans les manuscrits scientifiques du moyen âge à la renaissance. Droz, Genève, pp. 241–259.

[33] Van Ceulen's term is inspired by Scaliger's *segment of the hexagon.*

Grafton, A., 1993. Joseph Justus Scaliger: A Study in the History of Classical Scholarship. In: Volume 2: Historical Chronology. Clarendon Press, Oxford.
Helmreich, Andreas, 1591. Vom Feldmessen nach der Geometrei. Leipzig. Digital version: www.digital-collections.de.
Hofmann, J.E., 1970. Introduction to the reprint edition of [Viète, 1646]. Olms, Hildesheim.
Katscher, F., 1971. Einige Entdeckungen über die Geschichte der Zahl Pi sowie Leben und Werk von Christoffer Dybvad und Ludolph van Ceulen. Wien, Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, Denkschriften, 116. Band, 7. Abhandlung.
Scaliger, J., 1594a. Appendix ad Cyclometrica Sua. Apud Franciscum Raphelengium, Lugduni Batavorum.
Scaliger, J., 1594b. Cyclometrica Elementa. Apud Franciscum Raphelengium, Lugduni Batavorum. Digital version: books.google.com.
Scaliger, J., 1594c. Mesolabium. Apud Franciscum Raphelengium, Lugduni Batavorum.
Van Ceulen, L., 1585. Kort claar bewijs dat die nieuwe ghevonden proportie eens circkels jegens zyn diameter te groot is ende overzulcx de quadratura circuli des zelven vinders onrecht zy. Harmen Jansz Muller, Amsterdam. Second edition in [Van Ceulen, 1615c, 103a–105a].
Van Ceulen, L., 1586. Proefsteen ende claerder wederleggingh dat het claarder bewijs (so dat ghenaempt is) op de gheroemde ervindingh vande quadrature des circkels een onrecht te kennen gheven, ende gheen waerachtich bewijs is: hier by gevoeght Een corte verclaringh aengaende het onverstant ende misbruyck inde reductie op simpel interest. Harmen Jansz Muller, Amsterdam. Second edition in [Van Ceulen, 1615c, 105a–108b].
Van Ceulen, L., 1596. Vanden Circkel. Jan Andriesz, Delf. Digital version: gdz.sub.uni-goettingen.de. Second edition in [Van Ceulen, 1615c].
Van Ceulen, L., 1615a. De Arithmetische en Geometrische Fondamenten. Ioost van Colster ende Iacob Marcus, Leiden. Digital version: books.google.com.
Van Ceulen, L., 1615b. Fundamenta Arithmetica et Geometrica, e vernaculo in Latinem translata a Wil. Sn. Apud Iustum a Colster, Lugduni Batavorum.
Van Ceulen, L., 1615c. Vanden Circkel ... Tweede editie. Van nieus oversien ende van alle de voorgaaende fauten verbetert / ende eyndelick vermeerdert met dry Tractaetgens .... Leyden, ghedruckt by Ioris Abrahamsz. van der Marsse, voor Iacob Marcvs, Boeck-vercooper. Digital version: books.google.com.
Van Ceulen, L., 1619. De circulo et adscriptis liber (Willebrord Snellius, Trans.). Jod. à Colster, Lugduno Batavae.
Van Roomen, A., 1597. Adr. Romani In Archimedis circuli dimensionem expositio et analysis: apologia pro Archimede ad Jos. Scaligerum : exercitationes cyclicae contra Jos. Scaligerum, Orontium Finaeum et Raym. Ursum. Wurceburgi.
Viète, F., 1594. Munimen adversus nova cyclometrica, seu Antipelekus. Paris. Reprint ed. in [Viète 1646, 436–446].
Viète, F., 1595. Pseudo-Mesolabum et alia quaedam adiuncta capitula. Paris. Reprint ed. in [Viète 1646, 258–285].
Viète, F., 1646. Opera mathematica in unum volumen congesta ac recognita, opera atque studio Francisci à Schooten. Ex officini Bonaventurae & Abrahami Elzeviriorum, Lugduni Batavorum. Reprint edition, with introduction and index by J.E. Hofmann. Hildesheim: Georg Olms, 1970. Digital version: gallica.bnf.fr.
Xylander, G., 1562. Die erste sechs Bücher Euclidis: vom anfang oder grund der Geometrj. Jakob Kündig, Basel.