Locally strongly convex hypersurfaces with constant affine mean curvature

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Received 12 August 2003; received in revised form 2 January 2004
Available online 18 January 2005
Communicated by O. Kowalski

Abstract

Let \( x : M \rightarrow \mathbb{A}^{n+1} \) be a locally strongly convex hypersurface, given by a strictly convex function \( x_{n+1} = f(x_1, \ldots, x_n) \) defined in a convex domain \( \Omega \subset \mathbb{A}^n \). We consider the Riemannian metric \( G^\# \) on \( M \), defined by \( G^\# = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} \, dx_i \, dx_j \). In this paper we prove that if \( M \) is a locally strongly convex surface with constant affine mean curvature and if \( M \) is complete with respect to the metric \( G^\# \), then \( M \) must be an elliptic paraboloid.

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MSC: 53A15

Keywords: Bernstein property; Affine mean curvature

Introduction

In affine differential geometry there are two notions of completeness:

(1) affine completeness, that is, the completeness of the Blaschke metric \( G \);
(2) Euclidean completeness, that is, the completeness of the Riemannian metric on \( M \) induced from a Euclidean metric on \( \mathbb{A}^{n+1} \).

Let \( x : M \to \mathbb{A}^{n+1} \) be a locally strongly convex hypersurface, given by a strictly convex function

\[
x_{n+1} = f(x_1, \ldots, x_n)
\]
defined in a convex domain \( \Omega \subset \mathbb{A}^n \). Following E. Calabi and A.V. Pogorelov (see [1] and [6]), we consider the Riemannian metric \( G^# \) on \( M \), defined by

\[
G^# = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} \, dx_i \, dx_j.
\]

This is the relative metric with respect to the relative normalization \( e_{n+1} = (0, \ldots, 0, 1) \). Thus there is another notion of completeness:

(3) \( G^# \)-completeness, that is, the completeness of the metric \( G^# \).

Example 1.

\[
f(x_1, x_2) = \left(1 + x_1^2 + x_2^2\right)^{1/2},
\]

\[
M = \left\{ (x_1, x_2, f(x_1, x_2)) \mid (x_1, x_2) \in \mathbb{R}^2 \right\}.
\]

Then \( M \) is Euclidean complete and affine complete, but is not \( G^# \)-complete.

Example 2.

\[
f(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2),
\]

\[
M = \left\{ (x_1, x_2, f(x_1, x_2)) \mid (x_1, x_2) \in \mathbb{R}^2 \right\}.
\]

Then \( M \) is Euclidean complete, affine complete and \( G^# \)-complete.

Example 3.

\[
f(x_1, x_2) = 1 - (1 - x_1^2 - x_2^2)^{1/2},
\]

\[
M = \left\{ (x_1, x_2, f(x_1, x_2)) \mid x_1^2 + x_2^2 < 1 \right\}.
\]

Then \( M \) is not \( G^# \)-complete.

Put \( \Phi = \frac{1}{n} \sum G^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \). It is easy to see that \( \frac{(n+2)^2}{4n} \Phi \) is the norm of the Tchebychev vector field with respect to the relative normalization \( e_{n+1} = (0, \ldots, 0, 1) \). The following example shows that there is a class of hypersurfaces with \( \Phi \) bounded above.

Example 4. Let \( x : \to M \) be locally strongly convex hypersurface given by a strictly convex function \( f(x_1, \ldots, x_n) \) defined for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n \). Suppose that \( f \) satisfies

\[
d^2 f(x_1, \ldots, x_n) \to \sum_{i,j} \delta_{ij} \, dx_i \, dx_j,
\]
as \( (x_1, \ldots, x_n) \to \infty \). It is easy to see that \( \Phi \) is bounded.
Our main results can be stated as follows:

**Theorem 1.** Let $x_{n+1} = f(x_1, \ldots, x_n)$ be a strictly convex function defined in a convex domain $\Omega \subset \mathbb{A}^n$. If $M = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) | (x_1, \ldots, x_n) \in \Omega\}$ is an Euclidean complete hypersurface with constant affine mean curvature, and if $\Phi$ is bounded then $M$ is complete with respect to the metric $G^\#$.

**Theorem 2.** Let $x_3 = f(x_1, x_2)$ be a strictly convex function defined in a domain $\Omega \subset \mathbb{A}^2$. If $M = \{(x_1, x_2, f(x_1, x_2)) | (x_1, x_2) \in \Omega\}$ is $G^\#$-complete surface with constant affine mean curvature, then $M$ must be an elliptic paraboloid.

**Remark.** In [3] we have obtained a classification of locally strongly convex, Euclidean complete surfaces with constant affine mean curvature.

**Theorem 3.** Let $x_4 = f(x_1, x_2, x_3)$ be a strictly convex function defined in a convex domain $\Omega \subset \mathbb{A}^3$. If $M = \{(x_1, x_2, x_3, f(x_1, x_2, x_3)) | (x_1, x_2, x_3) \in \Omega\}$ is an Euclidean complete, affine maximal hypersurface and if $\Phi$ is bounded then $M$ must be an elliptic paraboloid.

1. Preliminaries

Let $\mathbb{A}^{n+1}$ be the unimodular affine space of dimension $n + 1$, $M$ be a connected and oriented $C^\infty$ manifold of dimension $n$, and $x : M \to \mathbb{A}^{n+1}$ a locally strongly convex hypersurface. We choose a local unimodular affine frame field $x, e_1, e_2, \ldots, e_n, e_{n+1}$ on $M$ such that

- $e_1, \ldots, e_n \in T_x M$,
- $\det(e_1, \ldots, e_n, e_{n+1}) = 1$,
- $G_{ij} = \delta_{ij}$,
- $e_{n+1} = Y$,

where we denote by $G_{ij}$ and $Y$ the Blaschke metric and the affine normal vector field, respectively. Denote by $U$, $A_{ijk}$ and $B_{ij}$ the affine conormal vector field, the Fubini–Pick tensor and the affine Weingarten tensor with respect to the frame field $x, e_1, \ldots, e_n$, and by $R_{ij}$ denotes the Ricci curvature. We have the following local formulas (see [5]):

\begin{align*}
  x_{,ij} &= \sum A_{ijk} e_k + \delta_{ij} Y, \\
  U_{,ij} &= - \sum A_{ijk} U_{,k} - B_{ij} U, \\
  \Delta x &= nY, \\
  \Delta U &= - nL_1 U, \\
  \sum A_{ijk} &= 0, \\
  R_{ij} &= \sum A_{mij} A_{mlj} + \frac{n-2}{2} B_{ij} + \frac{n}{2} L_1 \delta_{ij}, \\
  Y_{,i} &= - \sum B_{ij} e_j,
\end{align*}

(1.1) (1.2) (1.3) (1.4) (1.5) (1.6) (1.7)
where $L_1$ denotes the affine mean curvature, and “,” denotes the covariant differentiation with respect to the Blaschke metric.

Let $x: M \to A^{n+1}$ be given by a locally strongly convex function

$$x_{n+1} = f(x_1, \ldots, x_n)$$

defined in a convex domain $\Omega \subset A^n$. We choose the following unimodular affine frame field:

$$e_1 = \left(1, 0, \ldots, 0, \frac{\partial f}{\partial x_1}\right),$$

$$e_2 = \left(0, 1, 0, \ldots, 0, \frac{\partial f}{\partial x_2}\right),$$

$$\ldots$$

$$e_n = \left(0, 0, \ldots, 1, \frac{\partial f}{\partial x_n}\right),$$

$$e_{n+1} = (0, 0, \ldots, 0, 1).$$

Then, the Blaschke metric is given by (see [5])

$$G = \left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} \sum \frac{\partial^2 f}{\partial x_i \partial x_j} \ dx_i \ dx_j.$$

The affine conormal vector field $U$ can be identified with

$$U = \left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} \left(-\frac{\partial f}{\partial x_1}, \ldots, -\frac{\partial f}{\partial x_n}, 1\right).$$

The formula $\Delta U = -nL_1 U$ implies that $M$ is a locally strongly convex hypersurface with constant affine mean curvature $L_1 \equiv L$ if and only if $f$ satisfies the following fourth order PDE:

$$\Delta \left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} = -nL \left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)},$$

where $\Delta$ denotes the Laplacian with respect to the Blaschke metric, which is defined by

$$\Delta = \frac{1}{(\det(G_{kl}))^{1/2}} \sum \frac{\partial}{\partial x_i} \left(G^{ij} (\det(G_{kl}))^{1/2} \frac{\partial f}{\partial x_j}\right).$$

Denote $\rho = [\det(\frac{\partial^2 f}{\partial x_i \partial x_j})]^{-1/(n+2)}$. Then

$$\Phi = \|\text{grad } \rho\|^2_{\mathcal{G}} \rho$$

and

$$\Delta \rho = -nL_1 \rho.$$  \hfill (1.10)

We calculate $\Delta f$ and $\Delta \Phi$. Introduce the Legendre transformation:

$$\xi_i = \frac{\partial f}{\partial x_i}.$$
\[ u = \sum x_i \frac{\partial f}{\partial x_i} - f. \]

By a direct calculation we get (see [5, p. 91 (2.1.2.15)]; also [2]):

\[ Y = \frac{1}{n}(\Delta x_1, \ldots, \Delta x_n, \Delta f) = \left( \rho^{-2} \frac{\partial \rho}{\partial x_1}, \ldots, \rho^{-2} \frac{\partial \rho}{\partial x_n}, \rho^{-1} + \rho^{-2} \sum \xi_i \frac{\partial \rho}{\partial \xi_i} \right). \]

It follows that

\[ \Delta f = \frac{n}{\rho} + n \frac{\rho}{\rho^2} \sum \xi_i \frac{\partial \rho}{\partial \xi_i} = \frac{n}{\rho} + \frac{n}{\rho^2} \sum \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial f}{\partial \xi_i} \frac{\partial \rho}{\partial \xi_j}. \]

In terms of the coordinates \((\xi_1, \ldots, \xi_n)\) the Blaschke metric is

\[ G_{ij} = \rho \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \]

and \(\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}\) is the inverse matrix of \(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}\) (see [5, p. 91 (2.1.2.15)]). It follows from (1.11) that

\[ \Delta f = \frac{n}{\rho} + n \frac{\rho}{\rho^2} \langle \text{grad} \rho, \text{grad} f \rangle, \]

where \(\langle \cdot, \cdot \rangle\) denotes the inner product with respect to the Blaschke metric. Now we calculate \(\Delta \Phi\). Let \(p \in M\). We choose an orthonormal frame field around \(p\).

\[
\Phi = \frac{\sum \rho_{ij}^2}{\rho}, \quad \Phi_{ij} = \frac{2 \sum \rho_{ij} \rho_{ji}}{\rho} - \rho_i \frac{\sum \rho_{ij}^2}{\rho^2},
\]

\[
\Delta \Phi = \frac{2 \rho_{ij}^2 + 2 \sum \rho_{ij} \rho_{jii}}{\rho^2} - 4 \frac{\rho_{ij} \rho_{ji} \rho_{jii}}{\rho^2} + n L_1 \frac{\sum \rho_{ij}^2}{\rho^2} \sim + \frac{\| \text{grad} \rho \|^4}{\rho^3}.
\]

In the case \(\Phi(p) = 0\), it is easy to get at \(p\)

\[ \Delta \Phi \geq \frac{2 \rho_{ij}^2}{\rho}. \]

Now we assume that \(\Phi(p) \neq 0\) and choose an orthonormal frame field such that, at \(p \in M\), \(\rho_1 = \| \text{grad} \rho \|_G > 0, \rho_i = 0, \forall i > 1\). Then we have

\[
\Delta \Phi = \sum \rho_{ij}^2 + 2 \sum \rho_{ij} \rho_{jii} - 4 \rho_{ij} \rho_{ji} \rho_{jii} + n L_1 \frac{\rho_{ij}^2}{\rho^2} + 2 \frac{\rho_{ij}^2}{\rho^3}.
\]

Applying the Schwarz inequality and using (1.5) we get

\[
\sum A_{m/l}^2 \geq A_{111}^2 + \sum_{m>1} A_{mm1}^2 \geq A_{111}^2 + \frac{1}{n-1} \left( \sum_{m>1} A_{mm1} \right)^2 = A_{111}^2 + \frac{1}{n-1} A_{111}^2 = \frac{n}{n-1} A_{111}^2.
\]

Taking the \((n + 1)\)th component of \(U_{ij} = - \sum A_{ijk} U_{k} - B_{ij} U\) we have

\[ \rho_{11} = - A_{111} \rho_1 - B_{11} \rho. \]
Using the formula (1.6) and (1.16) we get
\[
2 \sum \rho_{,j} \rho_{,jii} = 2 \sum \rho_{,j} (\Delta \rho)_{,j} + 2R_{11}\rho_{,11}^2 + 2 \sum A_{m1l}^2 \rho_{,11}^2 + (n - 2)B_{11}\rho_{,11}^2 + nL_{1}\rho_{,11}^2
\]
\[
= 2 \sum A_{m1l}^2 \rho_{,11}^2 - (n - 2)\frac{\rho_{11}^2}{\rho} - (n - 2)A_{111} \frac{\rho_{,11}^3}{\rho} - nL_{1}\rho_{,11}^2
\]
\[
\geq -(n - 2)\frac{\rho_{11}^2}{\rho} - \frac{(n - 2)^2(n - 1)\rho_{,11}^3}{8n} - nL_{1}\rho_{,11}^2.
\] (1.17)

Substituting (1.17) into (1.14) we obtain
\[
\Delta \Phi \geq \frac{2}{\rho} \sum \rho_{,ij}^2 - (n + 2)\frac{\rho_{11}^2 \rho_{,11}}{\rho} + \left(2 - \frac{(n - 2)^2(n - 1)}{8n}\right) \frac{\rho_{,11}^4}{\rho^3}
\]
\[
\geq \frac{2\delta}{\rho} \sum \rho_{,ij}^2 + \left(2 - \frac{(n - 2)^2(n - 1)}{8n} - \frac{(n + 2)^2}{8(1 - \delta)}\right) \frac{\Phi^2}{\rho},
\] (1.18)

where \(\delta > 0\) is a small number. For \(n = 2\), we have, by (1.17),
\[
2 \sum \rho_{,j} \rho_{,jii} \geq -2L_{1}\rho_{,11}^2.
\] (1.19)

Substituting (1.19) into the formula
\[
\Delta \Phi = \frac{2}{\rho} \sum \rho_{,ij}^2 + 2 \sum \rho_{,j} \rho_{,jii} - 4 \frac{\rho_{11}^2 \rho_{,11}}{\rho^2} + 2L_{1}\rho_{,11}^3 + 2\rho_{,11}^4,
\]
we get
\[
\Delta \Phi \geq \frac{2\rho_{,11}^2 + 2\rho_{,12}^2}{\rho} - 4 \frac{\rho_{11}^2 \rho_{,11}}{\rho^2} + 2\rho_{,11}^4.
\] (1.20)

Note that
\[
\sum \frac{\Phi_{,i}^2}{\Phi} = 4 \sum \frac{\rho_{1i}^2}{\rho} + \frac{\rho_{,11}^4}{\rho^3} - 4 \frac{\rho_{1i}^2 \rho_{,11}}{\rho^2},
\] (1.21)

and
\[
\sum \frac{\Phi_{,i} \rho_{,i}}{\rho} = 2 \frac{\rho_{,1i}^3 \rho_{,11}}{\rho^2} - \frac{\rho_{,11}^4}{\rho^3}.
\] (1.22)

Then (1.20), (1.21) and (1.22) together give us
\[
\frac{\Delta \Phi}{\Phi} \geq \frac{\Phi_{,i}^2}{\Phi} - \frac{\Phi_{,i} \rho_{,i}}{\rho} + \frac{1}{2} \frac{\Phi_{,i} \rho_{,i}}{\rho^2} + \frac{1}{2} \frac{\Phi_{,i}^2}{\Phi^2} + \frac{1}{4} \frac{\Phi}{\rho}.
\] (1.23)

For \(L_{1} = 0\), we have
\[
\sum \rho_{,ij}^2 \geq \rho_{,11}^2 + \sum_{i>1} \rho_{,ii}^2 + 2 \sum_{i>1} \rho_{,ii}^2
\]
\[
\geq \rho_{,11}^2 + \frac{1}{n - 1} \left( \sum_{i>1} \rho_{,ii} \right)^2 + 2 \sum_{i>1} \rho_{,ii}^2 = \frac{n}{n - 1} \rho_{,11}^2 + 2 \sum_{i>1} \rho_{,ii}^2.
\] (1.24)
It follows from (1.14), (1.17) and (1.24) that
\[
\Delta \Phi \geq \frac{2n}{n-1} \frac{\rho_{,11}^2 - (n+2)\frac{\rho_{,11}^2}{\rho}}{\rho} + \left(2 - \frac{(n-2)^2(n-1)}{8n}\right) \frac{\rho_1^4}{\rho^3}.
\] (1.25)

Then (1.20), (1.21) and (1.25) together give us
\[
\Delta \Phi \geq \frac{n}{2(n-1)} \sum \frac{\Phi_i^2}{\Phi} - \frac{n^2-n-2}{2(n-1)} \sum \rho_i \frac{\rho_{,i}}{\rho} + \left(2 - \frac{(n-2)^2(n-1)}{8n} - \frac{n^2-2}{2(n-1)}\right) \frac{\Phi^2}{\rho}.
\] (1.26)

Denote by $\Delta^g$ and $\text{Ric}^g$ the Laplacian and the Ricci curvature with respect to the metric $G^g$, respectively. By definition of Laplacian and a direct calculation we get
\[
\frac{\|\nabla \rho\|_{G^g}^2}{\rho} = \|\nabla \rho\|_{G}^2;
\] (1.27)
\[
\Delta^g r = \rho \Delta r - \frac{n-2}{2} \langle \nabla \rho, \nabla r \rangle_{G^g}.
\] (1.28)

where $r$ is the geodesic distance function with respect to the metric $G^g$ on $M$. Denote $f_{ijk} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}$.

We have the following formula (see [6, p. 38]):
\[
\text{Ric}^g_{ik} = \frac{1}{4} \sum f_{jl}^m (f_{hil} f_{mj} - f_{hil} f_{mjl}),
\] (1.29)
where $(f^{ij})$ denotes the inverse matrix of $(f_{ij})$.

2. Proof of Theorem 1

Suppose that $x : M \to A^{n+1}$ is a locally strongly convex hypersurface with constant affine mean curvature, given as a graph of a strictly convex function $f(x_1, \ldots, x_n)$ defined in a convex domain $\Omega \subset A^n$. Let $p \in M$ be any fixed point. By adding a linear function we may assume that $\rho$ has coordinates $(0, \ldots, 0, 0)$ and
\[
f(0) = 0, \quad \frac{\partial f}{\partial x_i}(0) = 0, \quad 1 \leq i \leq n.
\]

The key point of the proof of Theorem 1 is to estimate $\rho \frac{\|\nabla f\|_{G}^2}{(1+f)^2}$. We shall show that $\rho \frac{\|\nabla f\|_{G}^2}{(1+f)^2}$ is bounded if $\Phi$ is bounded. Since $M$ is Euclidean complete, for any number $C > 0$, the set
\[
M_C = \\{ (x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \in M \mid f(x_1, \ldots, x_n) \leq C \}
\]
is compact. To estimate $\rho \frac{\|\nabla f\|_{G}^2}{(1+f)^2}$, we consider the following function
\[
F = \exp\left\{ -m \frac{\rho}{C-f} + \Psi \right\} \rho \frac{\|\nabla f\|_{G}^2}{(1+f)^2}
\] (2.1)
defined on $M_C$, where
\[
\Psi = \exp\{\Phi\},
\]
and $m$ is a positive constant to be determined later. Clearly, $F$ attains its supremum at some interior point $p^*$ of $M_C$. We can assume that $\|\text{grad } f\|_G > 0$ at $p^*$. Choose a local orthonormal frame field of the Blaschke metric $e_1, \ldots, e_n$ on $M$ such that, at $p^*$, $f_1 = \|\text{grad } f\|_G > 0$, $f_i = 0$ ($i \geq 2$). Then, at $p^*$,

$$F_j = 0,$$

$$\sum F_{,ii} \leq 0. \quad (2.2)$$

We now calculate both expressions $(2.2)$ and $(2.3)$ explicitly. To simplify expressions we denote

$$g = \frac{m}{(C - f)^2}, \quad g' = \frac{m}{(C - f)^3}.$$

By $(2.2)$ and $(2.3)$, we have

$$2 \sum f_{,j} f_{,ji} + \left(-gf_{,i} - 2\frac{f_{,i}}{1 + f} + \frac{\rho_{,i}}{\rho} + \Psi_{,i}\right) \sum f_{,j}^2 = 0, \quad (2.4)$$

$$2 \sum f_{,ij}^2 + 2 \sum f_{,j} f_{,jii} + \sum \left(-gf_{,i} - 2\frac{f_{,i}}{1 + f} + \frac{\rho_{,i}}{\rho} + \Psi_{,i}\right) 2f_{,j} f_{,ji}$$

$$+ \left[-2g' \sum f_{,j}^2 - g \Delta f + 2 \sum f_{,j}^2 - \Delta f \frac{\Delta f}{1 + f} - \sum \frac{\rho_{,j}^2}{\rho^2} + \frac{\Delta \rho}{\rho} + \Delta \Psi\right] \sum f_{,j}^2 \leq 0. \quad (2.5)$$

Let us simplify $(2.5)$. From $(2.4)$ we have

$$2f_{,i1} = \left(gf_{,i} + 2\frac{f_{,i}}{1 + f} - \frac{\rho_{,i}}{\rho} - \Psi_{,i}\right) f_{,i}. \quad (2.6)$$

Applying Schwarz inequality we get

$$2 \sum f_{,ij}^2 \geq 2 \sum f_{,ii}^2 + 4 \sum f_{,i1}^2 \geq 2 \left(\frac{n}{n - 1} - \delta\right) f_{,11}^2 + 4 \sum f_{,i1}^2 = 2 - \frac{2\delta(n - 1)}{\delta(n - 1)^2}(\Delta f)^2, \quad (2.7)$$

for any $0 < \delta < 1$. Substituting $(2.6)$ and $(2.7)$ into $(2.4)$ we obtain

$$\left[2 \left(\frac{n}{n - 1} - \delta\right) - 4\right] f_{,11}^2 + 2 \sum f_{,j} f_{,jii} - 2 - \frac{2\delta(n - 1)}{\delta(n - 1)^2}(\Delta f)^2$$

$$+ \left[-2g' f_{,1i}^2 - g \frac{\Delta f}{1 + f} + 2 \frac{f_{,1i}^2}{(1 + f)^2} - \frac{\Delta f}{1 + f} - \frac{\Phi}{\rho} - nL_1 + \Delta \Psi\right] f_{,1i}^2 \leq 0. \quad (2.8)$$

Now we calculate $\Delta \Psi$. We have, by $(1.13)$ and $(1.18)$,

$$\Delta \Phi \geq \frac{2\delta}{\rho} \sum \rho_{,ij}^2 + \left[2 - \frac{(n - 2)^2(n - 1)}{8n} - \frac{(n + 2)^2}{8(1 - \delta)}\right] \frac{\Phi^2}{\rho}. \quad (2.9)$$

Then we get

$$\Psi_{,i} = \Psi \Phi_{,i}, \quad (2.10)$$

and

$$\Delta \Psi \geq \Psi \sum \Phi_{,i}^2 + \Psi \frac{2\delta}{\rho} \sum \rho_{,ij}^2 + \Psi \left[2 - \frac{(n - 2)^2(n - 1)}{8n} - \frac{(n + 2)^2}{8(1 - \delta)}\right] \frac{\Phi^2}{\rho}. \quad (2.11)$$
Let us now compute the terms $\sum f_{j} f_{j,iii}$. An application of the Ricci identity shows that

$$\sum f_{j} f_{j,iii} = 2 \sum f_{j} (\Delta f)_{,j} + 2 \sum R_{ij} f_{i} f_{,j}.$$ 

We use (1.6) and (1.7) to obtain

$$\sum f_{j} f_{j,iii} = -2nB_{11} f_{i,1}^{2} + 2 \sum A_{m1}^{2} f_{i,1}^{2} + (n - 2)B_{11} f_{i,1}^{2} + nL_{1} f_{i,1}^{2}$$

$$= (2 - 2\delta) \sum A_{m1}^{2} f_{i,1}^{2} + 2\delta \sum A_{m1}^{2} f_{i,1}^{2} + (n + 2) \frac{\rho_{11}}{\rho} f_{i,1}^{2}$$

$$+ (n + 2) \sum A_{11k} \frac{\rho_{k}}{\rho} f_{i,1}^{2} + nL_{1} f_{i,1}^{2}$$

$$\geq (2 - 2\delta) \sum A_{m1}^{2} f_{i,1}^{2} + (n + 2) \frac{\rho_{11}}{\rho} f_{i,1}^{2} - \frac{(n + 2)^{2}}{8\delta} \frac{\Phi}{\rho} f_{i,1}^{2} + nL_{1} f_{i,1}^{2}. \quad (2.12)$$

Taking the $(n + 1)$th component of

$$x_{ij} = \sum A_{ijk} x_{k} + \frac{\Delta x}{n} \delta_{ij}$$

we have

$$f_{ij} = A_{ij1} f_{1} + \frac{\Delta f}{n} \delta_{ij},$$

$$\sum f_{ij}^{2} = \sum \left( A_{ij1} f_{1} + \frac{\Delta f}{n} \delta_{ij} \right)^{2} = \sum A_{ij1}^{2} f_{1}^{2} + \frac{1}{n} (\Delta f)^{2}. \quad (2.13)$$

Combination of (2.12) and (2.13) gives

$$2 \sum f_{j} f_{j,iii} \geq (2 - 2\delta) \sum f_{ij}^{2} - \frac{2 - 2\delta}{n} (\Delta f)^{2} + (n + 2) \frac{\rho_{11}}{\rho} f_{1}^{2} - \frac{(n + 2)^{2}}{8\delta} \frac{\Phi}{\rho} f_{1}^{2} + nL_{1} f_{1}^{2}$$

$$\geq (2 - 2\delta) \left( \frac{n}{n - 1} - \delta \right) f_{11}^{2} + (n + 2) \frac{\rho_{11}}{\rho} f_{1}^{2} - \frac{(n + 2)^{2}}{8\delta} \frac{\Phi}{\rho} f_{1}^{2} + nL_{1} f_{1}^{2}$$

$$- (1 - \delta) \left[ \frac{2 - 2\delta(n - 1)}{\delta(n - 1)^{2}} + \frac{2}{n} \right] (\Delta f)^{2}$$

$$\geq (2 - 2\delta) \left( \frac{n}{n - 1} - \delta \right) f_{11}^{2} - \frac{(n + 2)^{2}}{8\delta} \frac{\Phi}{\rho} f_{1}^{2} + nL_{1} f_{1}^{2}$$

$$- (1 - \delta) \left[ \frac{2 - 2\delta(n - 1)}{\delta(n - 1)^{2}} + \frac{2}{n} \right] (\Delta f)^{2} - \Psi \frac{2\delta}{\rho} \sum \rho_{ij} f_{1}^{2} - \frac{(n + 2)^{2}}{8\delta \Psi} f_{1}^{2}. \quad (2.14)$$
Inserting (2.11), (2.14) into (2.8) and using (2.6) we get
\[
\left( \frac{1}{n-1} - \delta \frac{3n-2}{2(n-1)} \right) \left( gf,1 + 2 \frac{f,1}{1+f} - \frac{\rho,1}{\rho} - \Psi,1 \right)^2 f,1^2
- \left[ (2 - \delta)(n+2) \Phi + 2 \frac{f,1}{1+f} \right] f,1^2 + \Psi \sum \Phi,1 f,1^2
+ \Psi \left[ 2 - \frac{(n-2)^2(n-1)}{8n} - \frac{(n+2)^2}{8(1-\delta)} \right] \frac{\phi^2}{\rho} f,1^2 \leq 0. \tag{2.15}
\]

We choose the following values for \(\delta\) and \(m\):
\[
0 < \delta < \frac{1}{3n-2},
\]
\[
m = \frac{40(1 + (\frac{1}{n-1} - \delta \frac{3n-2}{2(n-1)})e^K)}{\frac{1}{n-1} - \delta \frac{3n-2}{2(n-1)}} C,
\]
where \(K = \sup_{x \in \Omega} \Phi(x)\). To simplify expression we denote
\[
a_1 = \frac{1}{n-1} - \delta \frac{3n-2}{2(n-1)}, \quad a_2 = (2 - \delta) \frac{2 - 2\delta(n-1)}{\delta(n-1)^2} + (1 - \delta) \frac{2}{n},
\]
\[
a_3 = \frac{8\delta + (n+2)^2}{8\delta}, \quad a_4 = \frac{(n+2)^2}{8\delta}, \quad a_5 = \frac{(n-2)^2(n-1)}{8n} + \frac{(n+2)^2}{8(1-\delta)} - 2.
\]
Recall that \(\Psi,1 = \Psi \Phi,1\). Then we have
\[
a_1 \left( gf,1 + 2 \frac{f,1}{1+f} - \frac{\rho,1}{\rho} - \Psi,1 \right)^2 f,1^2 + \Psi \sum \Phi,1 f,1^2
\geq a_1 \left( gf,1 + 2 \frac{f,1}{1+f} - \frac{\rho,1}{\rho} \right)^2 f,1^2
\geq \frac{9}{10} a_1 \left( gf,1 + 2 \frac{f,1}{1+f} \right)^2 f,1^2 - \frac{a_1}{1+a_1} \Phi f,1^2.
\]
Inserting (2.16) into (2.15) we get
\[
\frac{9}{10} a_1 \left( gf,1 + 2 \frac{f,1}{1+f} \right)^2 f,1^2 - a_2(\Delta f)^2 - \left( a_3 + 9 \frac{a_1}{1+a_1} \right) \Phi f,1^2
- \frac{a_4}{\rho} f,1^2 + \left( -2g' f,1^2 - g \Delta f + 2 \frac{f,1^2}{(1+f)^2} - 2 \frac{\Delta f}{1+f} \right) f,1^2 - e^K a_5 \Phi f,1^2 \leq 0. \tag{2.17}
\]
Multiply to both sides of (2.17) by \( \frac{\rho^2}{(1+f)^2} \). Then, we obtain

\[
\begin{align*}
9 \frac{a_1}{10 + a_1 e^k} \left( g f_1 + 2 \frac{f_1}{1 + f} \right)^2 \rho^2 \frac{f_1^2}{(1 + f)^2} - a_2 \rho^2 \left( \frac{\Delta f}{1 + f} \right)^2 - \left( a_3 + 9 \frac{a_1}{1 + a_1} \right) \phi \rho \frac{f_1^2}{(1 + f)^2} \\
- a_4 \rho \frac{f_1^2}{(1 + f)^2} - 2g \rho^2 \frac{f_1^2}{(1 + f)^2} - g \phi (\Delta f) \rho \frac{f_1^2}{(1 + f)^2} + 2 \rho^2 \frac{f_1^4}{(1 + f)^4} \\
- 2 \rho \left( \frac{\Delta f}{1 + f} \right) \rho \frac{f_1^2}{(1 + f)^2} - e^\kappa \rho \frac{f_1^2}{(1 + f)^2} \leq 0.
\end{align*}
\]

To further estimate this expression we use (1.12) and the definition of \( g' \) and \( g \); we have the following inequalities:

\[
\begin{align*}
2g' \rho^2 \frac{f_1^4}{(1 + f)^2} &\leq \frac{a_1}{10 + a_1 e^k} \left( g f_1 + 2 \frac{f_1}{1 + f} \right)^2 \rho^2 \frac{f_1^2}{(1 + f)^2}, \\
g \phi (\Delta f) \rho \frac{f_1^2}{(1 + f)^2} &\leq \frac{a_1}{10 + a_1 e^k} \left( g f_1 + 2 \frac{f_1}{1 + f} \right)^2 \rho^2 \frac{f_1^2}{(1 + f)^2} + \frac{5(1 + a_1 e^k)n^2}{2a_1} \\
&+ \frac{2(1 + a_1 e^k)n^2}{2a_1} \phi \frac{f_1^2}{(1 + f)^2}, \\
\rho^2 \left( \frac{\Delta f}{1 + f} \right)^2 &\leq 2n^2 + 2n^2 \phi \rho \frac{f_1^2}{(1 + f)^2}, \\
2 \rho \left( \frac{\Delta f}{1 + f} \right) \rho \frac{f_1^2}{(1 + f)^2} &\leq 2n \phi \frac{f_1^2}{(1 + f)^2} + \rho^2 \frac{f_1^4}{(1 + f)^4} + n^2 \phi \rho \frac{f_1^2}{(1 + f)^2}.
\end{align*}
\]

Substituting these inequalities into (2.18) we get

\[
\begin{align*}
\frac{3}{5} \frac{a_1}{1 + a_1 e^k} \left( g f_1 + 2 \frac{f_1}{1 + f} \right)^2 \rho^2 \frac{f_1^4}{(1 + f)^2} + \rho^2 \frac{f_1^4}{(1 + f)^4} \\
- \left[ \left( 2a_2 n^2 + a_3 + \frac{5(1 + a_1 e^k)n^2}{2a_1} + n^2 \right) \phi + 2a_4 + e^\kappa a_5 \Phi \right] \rho \frac{f_1^2}{(1 + f)^2} \\
- \left( 2a_2 n^2 + \frac{5(1 + a_1 e^k)n^2}{2a_1} \right) \leq 0.
\end{align*}
\]

We use the abbreviations:

\[
a := \left( 2a_2 n^2 + a_3 + \frac{5(1 + a_1 e^k)n^2}{2a_1} + n^2 \right) K + 2a_4 + e^\kappa a_5 K^2,
\]

\[
b := 2a_2 n^2 + \frac{5(1 + a_1 e^k)n^2}{2a_1},
\]

and get the following form of the inequality:

\[
\rho^2 \frac{f_1^4}{(1 + f)^4} - a \rho \frac{f_1^2}{(1 + f)^2} - b \leq 0.
\]
The left-hand side is a quadratic expression in $\frac{f^2}{(1+f)^2}$. If one considers its zeroes it follows that
\[ \rho \frac{f^2}{(1+f)^2} \leq a + \sqrt{b}. \]
From (2.1) we thus get, with our special choice of $\delta$ and $m$:
\[ F \leq \exp(e^K)(a + \sqrt{b}), \]
which holds at $p^*$, where $F$ attains its supremum. Hence, at any interior point of $MC$, we have
\[ \rho \frac{\| \text{grad } f \|^2}{(1+f)^2} \leq \exp(e^K)(a + \sqrt{b}) \exp\left\{ \frac{40(1+a_1e^K)}{a_1(C-f)} \right\}. \]
Let $C \to \infty$ then
\[ \rho \frac{\| \text{grad } f \|^2}{(1+f)^2} \leq \exp\left\{ e^K + \frac{40(1+a_1e^K)}{a_1} \right\} (a + \sqrt{b}) := Q, \]
where $Q$ is a constant.
Recall that $G^\# = \frac{1}{\rho} G$. Then, by a direct calculation, we have
\[ \| \text{grad } f \|^2_{G^\#} \leq \rho \frac{\| \text{grad } f \|^2}{(1+f)^2} \leq Q. \] (2.20)
Using the gradient estimate (2.20) we can prove that $M$ is complete with respect to the metric $G^\#$: for any unit speed geodesic starting from $p$
\[ \sigma : [0, S] \to M \]
we have
\[ \frac{df}{ds} \leq \| \text{grad } f \|_{G^\#} \leq \sqrt{Q}(1+f). \]
It follows that
\[ s \geq \frac{1}{\sqrt{Q}} \int_0^{x_{n+1}(\sigma(S))} \frac{df}{(1+f)}. \] (2.21)
Since
\[ \int_0^\infty \frac{df}{1+f} = \infty \]
and $f : \Omega \to \mathbb{R}$ is proper (i.e., the inverse image of any compact set is compact), (2.21) implies that $M$ is complete with respect to the metric $G^\#$. This complete the proof of Theorem 1. \qed
3. Proofs of Theorems 2 and 3

Proof of Theorem 2. Let \( p_0 \in M \) be any fixed point. By adding a linear function we may assume that \( p_0 \) has coordinates \((0, \ldots, 0)\) and
\[
f(0) = 0, \quad \frac{\partial f}{\partial x_i}(0) = 0, \quad 1 \leq i \leq n.
\]
Denote by \( r(p_0, p) \) the geodesic distance function from \( p_0 \) with respect to the metric \( G^\# \). For any positive number \( a \), let \( B_a(p_0) = \{ p \in M \mid r(p_0, p) \leq a \} \). Consider the function
\[
F = (a^2 - r^2)^2 \Phi
\]
defined on \( B_a(p_0) \). Obviously, \( F \) attains its supremum at some interior point \( p^* \). We may assume that \( r^2 \) is a \( C^2 \)-function in a neighborhood of \( p^* \), and \( \Phi > 0 \) at \( p^* \). Then, at \( p^* \),
\[
F_i = 0, \quad \sum F_{ii} \leq 0,
\]
where \( \cdot, \cdot \) denotes the covariant differentiation with respect to the Blaschke metric. We calculate both expression explicitly:
\[
\frac{\Phi_i}{\Phi} - \frac{2(r^2)_i}{a^2 - r^2} = 0, \quad (3.1)
\]
\[
\frac{\Delta \Phi}{\Phi} - \frac{\sum \Phi_i^2}{\Phi^2} - 2\frac{\|\nabla r^2\|_G^2}{(a^2 - r^2)^2} - \frac{2\Delta r^2}{a^2 - r^2} \leq 0. \quad (3.2)
\]
Inserting (3.1) into (3.2) and noting that
\[
\|\nabla r^2\|_G^2 = 4r^2\|\nabla r\|_G^2,
\]
\[
\Delta r^2 = 2\|\nabla r\|_G^2 + 2r \Delta r,
\]
we get
\[
\frac{\Delta \Phi}{\Phi} \leq \frac{6\|\nabla r^2\|_G^2}{(a^2 - r^2)^2} + \frac{2\Delta r^2}{a^2 - r^2} = \frac{24r^2\|\nabla r\|_G^2}{(a^2 - r^2)^2} + \frac{4\|\nabla r\|_G^2}{a^2 - r^2} + \frac{4r \Delta r}{a^2 - r^2}. \quad (3.3)
\]
Since
\[
\frac{4r \Delta r}{a^2 - r^2} = \frac{4r}{a^2 - r^2} \frac{\Delta^g r}{\rho} + \frac{2(n - 2)r \langle \nabla \rho, \nabla r \rangle_{G^g}}{\rho^2}
\]
and
\[
\frac{\|\nabla \rho\|^2_{G^g}}{\rho} = \|\nabla \rho\|_G^2,
\]
\[
\frac{\|\nabla r\|^2_{G^g}}{\rho} = \|\nabla r\|_G^2,
\]
\[
\|\nabla r\|^2_{G^g} = 1,
\]
we obtain from (3.3)
\[
\Delta \Phi = \frac{24r^2 \|
abla r\|_{G^2_r}^2}{(a^2 - r^2)^2 \rho} + 4 \|\nabla \rho\|_{G^2_r}^2 + \frac{4r \Delta_r}{(a^2 - r^2)^2 \rho} + \frac{2(n-2)r \langle \nabla \rho, \nabla r \rangle_{G^2_r}}{a^2 - r^2 \rho^2}
\]
\[
= \frac{24r^2}{(a^2 - r^2)^2 \rho} + \frac{4}{(a^2 - r^2)^2 \rho} + \frac{4r \Delta_r}{(a^2 - r^2)^2 \rho} + \frac{2(n-2)r \langle \nabla \rho, \nabla r \rangle_{G^2_r}}{a^2 - r^2 \rho^2}.
\]
(3.4)

Denote by \(a^* = r(p_0, p^*)\). In the case \(p^* \neq p_0\) we have \(a^* > 0\). Let
\[
B_{a^*}(p_0) = \{ p \in M \mid r(p_0, p) \leq a^* \}.
\]
For \(n = 2\), we have by (1.23)
\[
\max_{p \in B_{a^*}(p_0)} \Phi(p) = \max_{p \in \partial B_{a^*}(p_0)} \Phi(p).
\]
On the other hand, we have \(a^2 - r^2 = a^2 - a^2 = 0\) on \(\partial B_{a^*}(p_0)\), it follows that
\[
\max_{p \in B_{a^*}(p_0)} \Phi(p) = \Phi(p^*).
\]
For any \(p \in B_{a^*}(p_0)\), by a coordinate transformation we may assume that
\[
R^*_i(p) = 0 \quad \text{for } i \neq j,
\]
and
\[
f_{ij}(x) = \delta_{ij},
\]
where \(p = (x, f(x))\). Then from (1.26), we get
\[
R^*_i(p) = \frac{1}{4} \sum \frac{f^2_{mji}}{f_{mji}^2} + \frac{n+2}{4} \sum f_{mii} \frac{\partial}{\partial x_m} \log \rho
\]
\[
\geq - \frac{(n+2)^2}{16} \sum \frac{\partial}{\partial x_m} \log \rho \cdot \frac{\partial}{\partial x_m} \log \rho = - \frac{(n+2)^2}{16} \Phi(p) \geq - \Phi(p^*).
\]
Therefore, by Laplacian comparison theorem (see [5, Appendix 2]), we have
\[
r \Delta_r r \leq (1 + \sqrt{\Phi(p^*)} \cdot r).
\]
(3.5)

In the case \(p^* = p_0\), we have \(r(p_0, p^*) = 0\). Consequently, from (3.4) and (3.5), it follows that
\[
\frac{\Delta \Phi}{\Phi} \leq \frac{24r^2}{(a^2 - r^2)^2 \rho} + \frac{8}{(a^2 - r^2)^2 \rho} + \frac{4 \sqrt{\Phi(p^*)} \cdot r}{(a^2 - r^2)^2 \rho}.
\]
(3.6)

For \(n = 2\), we have by (1.22)
\[
\frac{\Delta \Phi}{\Phi} \geq \frac{1}{2} \sum \frac{\Phi_{d}}{\Phi_d^2} + \frac{1}{2} \frac{\Phi}{\rho} - \sum \frac{\Phi_d}{\Phi} \frac{\rho}{\rho} \geq - \frac{1}{2} \sum \frac{\Phi_d^2}{\Phi^2} + \frac{1}{2} \frac{\Phi}{\rho} = - \frac{8r^2}{(a^2 - r^2)^2 \rho} + \frac{1}{4} \frac{\Phi}{\rho},
\]
(3.7)
where we used (3.1). Inserting (3.7) into (3.6) we get
\[
\Phi \leq \frac{128r^2}{(a^2 - r^2)^2} + \frac{32}{a^2 - r^2} + \frac{16 \sqrt{\Phi} \cdot r}{a^2 - r^2} \leq \frac{128r^2}{(a^2 - r^2)^2} + \frac{32}{a^2 - r^2} + \frac{1}{2} \Phi + \frac{128r^2}{(a^2 - r^2)^2}.
\]
It follows that
\[ \Phi \leq \frac{512r^2}{(a^2 - r^2)^2} + \frac{64}{a^2 - r^2}. \tag{3.8} \]
Multiply both sides of (3.8) by \((a^2 - r^2)^2\). We obtain, at \(p^*\),
\[ (a^2 - r^2)^2 \Phi \leq 512a^2 + 64a^2 = 576a^2. \tag{3.9} \]
Hence at any interior point of \(B_a(p_0)\) we have
\[ \Phi \leq \frac{576}{a^2(1 - \frac{r^2}{a^2})^2}. \]
Let \(a \to \infty\), then
\[ \Phi \equiv 0. \tag{3.10} \]
It follows that
\[ \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) = \text{constant} > 0. \]
This means that \(M\) is an affine complete parabolic affine sphere. By a result of E. Calabi (see [5]) we conclude that \(M\) must be an elliptic paraboloid.

**Remark.** In [4] we have proved the following theorem.

**Theorem 4.** Let \(x_{n+1} = f(x_1, \ldots, x_n)\) be a strictly convex function defined in a convex domain \(\Omega \subset \mathbb{A}^n\). If \(M = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \mid (x_1, \ldots, x_n) \in \Omega\}\) is an affine maximal hypersurface, and if \(M\) is complete with respect to the metric \(G^\#\), then in the case \(n = 2\) or \(n = 3\), \(M\) must be an elliptic paraboloid.

Using the same method of the proof of Theorem 2 and the differential inequality (1.26) we may give a simple proof of Theorem 4.

**Proof of Theorem 3.** As a consequence of Theorems 1 and 4 above we obtain Theorem 3.

**Acknowledgements**

The authors would like to thank Prof. U. Simon for many valuable discussions.

**References**