Center conditions and bifurcation of limit cycles at degenerate singular points in a quintic polynomial differential system

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Abstract

The center problem and bifurcation of limit cycles for degenerate singular points are far to be solved in general. In this paper, we study center conditions and bifurcation of limit cycles at the degenerate singular point in a class of quintic polynomial vector field with a small parameter and eight normal parameters. We deduce a recursion formula for singular point quantities at the degenerate singular points in this system and reach with relative ease an expression of the first five quantities at the degenerate singular point. The center conditions for the degenerate singular point of this system are derived. Consequently, we construct a quintic system, which can bifurcates 5 limit cycles in the neighborhood of the degenerate singular point. The positions of these limit cycles can be pointed out exactly without constructing Poincaré cycle fields. The technique employed in this work is essentially different from more usual ones. The recursion formula we present in this paper for the calculation of singular point quantities at degenerate singular point is linear and then avoids complex integrating operations.

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1. Introduction

Consider polynomial differential systems

\[
\begin{align*}
\frac{dx}{dt} &= P(x, y), \\
\frac{dy}{dt} &= Q(x, y),
\end{align*}
\]

where \( P(x, y) \) and \( Q(x, y) \) are real coprime polynomials. If the maximum of the degrees of \( P \) and \( Q \) is \( m \), then we say that system (1.1) is of degree \( m \). The polynomial systems of degree 5 are called quintic systems.

One of the classical problems in the qualitative theory of planar analytic differential systems is the study of the local phase portrait at the singularities to characterize when a singular point is of focus-center type. Recall that a singular point is said to be of focus-center if it is either a focus or a center. In what follows, this problem will be called the focus-center problem or the monodromy problem. Of course, if the linear part of the singular point is non-degenerate (i.e., its determinant does not vanish) the characterization is well known. The Lyapunov–Poincaré theory was developed to solve this problem in the case where the singular point is non-degenerate [6,10,13,18,27,28]. The problem has also been solved when the linear part is degenerate but not identically zero [1,3]. Hence the main difficulties in solving the focus-center problem appear when the singular point has an identically zero linear part. On the other hand, once we know that a singular point is of focus-center type, one comes across another classical problem, usually called the center problem or the stability problem, which is of distinguishing a center from a focus. Until fairly recently, necessary and sufficient conditions for a center were known in relatively few instances. If the singular point has a nilpotent linear part, there are some results on the center problem [19]. But if the singular point has a zero linear part then there are very few results on the center problem and the problem to decide whether a degenerate singular point of focus-center type is either a center or a focus is very complicated in comparison to the case of non-degenerate singular points [11,12].

Another main problem in the qualitative theory of real planar differential equations is the determination of limit cycles. Limit cycles of planar vector fields were defined by Poincaré. A limit cycle is of system (1.1) is a real isolated periodic solution in the set of all periodic solutions. At the end of the 1920s Van Der Pol [25], Lienard [16] and Andronov [2] proved that a closed trajectory of a self-sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After these works, the non-existence, existence, uniqueness and other properties of limit cycles were studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, economists, etc. [7,26]. This activity reflects the breadth of interest in Hilbert 16th problem: compute \( H(n) \) such that the number of limit cycles of any polynomial differential system of degree \( n \) is less than or equal to \( H(n) \), and the fact that such systems are often used in mathematical models. Hilbert’s Problem remains unsolved even for \( n = 2 \). The study of appearance of limit cycles by varying the coefficients of a planar polynomial vector fields goes back to Poincaré. It was mentioned by Hilbert as one method to solve his famous 16th problem. Several bifurcations in the finite domain have been studied, in order to create vector fields with the maximum possible number of limit cycles. The simplest, Hopf bifurcation, describes the simultaneous appearance of several limit cycles.
from a non-degenerate singular point. Although this method is completely algorithmic it requires enormous calculations and few complete answers are known: the maximum number $M = 3$ for $n = 2$ [4]. $M = 5$ for $n = 3$ in the case of symmetric cubic vector fields [5,22,23], and conjectured to be $M = 8$ in the general case of cubic systems [15]. Other bifurcation methods in the finite domain have been studied: homoclinic loop bifurcation, etc., for which there are only partial results even in the quadratic case [9,14,15,20,21]. The bifurcation of limit cycles from non-degenerate singular point in the finite domain is an issue that has stimulated a great deal of effort and consequently the literature on them [24], but if the singular point is degenerate with a zero linear part then the results on bifurcation of limit cycles are very slight [8,13] and in the general case, the question of the existence of a uniform bound for the number of limit cycles to be produced in these polynomial vector fields of a given degree is still open. In this paper, we are concerned with the determination of center-focus and the appearance of limit cycles from the degenerate singular point in a quintic system

\[
\begin{align*}
\frac{dx}{dt} &= (-y + \delta x + xQ - yP)x^2 + y^2 - yR, \\
\frac{dy}{dt} &= (x + \delta y + xP + yQ)x^2 + y^2 + xR,
\end{align*}
\]

(1.2)

where

\[
\begin{align*}
P &= (A_{30} + A_{21} + A_{12})x^2 - 2(B_{30} - B_{12})xy - (A_{30} - A_{21} + A_{12})y^2, \\
Q &= (B_{30} - B_{21} + B_{12})y^2 - 2(A_{30} - A_{12})xy - (B_{30} + B_{21} + B_{12})x^2, \\
R &= 2A_{03}(x^4 - 6x^2y^2 + y^4) + 8B_{03}xy(x^2 - y^2).
\end{align*}
\]

(1.3)

The technique employed in this paper is essentially different from more usual ones [13,27] and benefits greatly from the availability of computer algebra – a fact that will be demonstrated in this paper.

This paper is organized as follows. In Section 2 we definite the generalized Lyapunov constants for degenerate singular points. Some basic results on generalized Lyapunov constants for degenerate singular points are given, which is necessary for investigating center conditions and bifurcations of limit cycles. In Section 3, the algebraic equivalent relations between generalized Lyapunov constants and singular point quantities for degenerate singular point are discussed and a recursive formula for the computation of singular point quantity is presented. The algorithm for the singular quantities is recursive and then avoids complex integrating operations and solving equations. The algorithm can be readily done with using computer algebra system such as Mathematica or Maple. By the computation of singular point quantities, the conditions for the degenerate singular point $O(0,0)$ to be a center and the stability criterions are derived. In Section 4, we study the appearance of limit cycles in the neighborhood of the degenerate singular point. We construct a quintic system with a small parameter and eight normal parameters, which can bifurcates 5 limit cycles from the degenerate singular point as the parameters being suitable values.
2. Generalized Lyapunov constant for degenerate singular points

Consider the following real planar analytic autonomous differential system

$$\frac{dx}{dt} = \sum_{k=2n+1}^{\infty} X_k(x, y) = X(x, y), \quad \frac{dy}{dt} = \sum_{k=2n+1}^{\infty} Y_k(x, y) = Y(x, y), \quad (2.1)$$

where $n$ is a natural number, $X_k(x, y), Y_k(x, y)$ are homogeneous polynomials of degree $k$. $X(x, y)$ and $Y(x, y)$ are analytic at the origin. Suppose that there exists $\sigma > 0$, such that

$$xY_{2n+1}(x, y) - yX_{2n+1}(x, y) \geq \sigma(x^2 + y^2)^{n+1}. \quad (2.2)$$

From (2.2), the origin is a center or a focus of system (2.1), and it is a degenerate singular point (or a higher-degree singular point) if $n > 0$. By means of transformation $x = r \cos \theta, y = r \sin \theta$, system (2.1) can be transformed into

$$\frac{dr}{d\theta} = r^{2n+1} \sum_{k=0}^{\infty} \psi_{2n+2+k}(\theta) r^k, \quad \frac{d\theta}{dt} = r^{2n} \sum_{k=0}^{\infty} \psi_{2n+2+k}(\theta) r^k, \quad (2.3)$$

where

$$\left\{ \begin{array}{l}
\varphi_{k+1}(\theta) = \cos \theta X_k(\cos \theta, \sin \theta) + \sin \theta Y_k(\cos \theta, \sin \theta), \\
\psi_{k+1}(\theta) = \cos \theta Y_k(\cos \theta, \sin \theta) - \sin \theta X_k(\cos \theta, \sin \theta),
\end{array} \right. \quad (2.4)$$

$k = 0, 1, 2, \ldots$ Particularly, from (2.2), we have $\psi_{2n+2}(\theta) \geq \sigma > 0$.

In order to define the generalized Lyapunov constants, we write system (2.3) in the form

$$\frac{dr}{d\theta} = r \sum_{k=0}^{\infty} R_k(\theta) r^k, \quad (2.5)$$

where the function on the right hand side of Eq. (2.5) is convergent in the range $\theta \in [-4\pi, 4\pi], \, |r| < r_0$, and

$$R_k(\theta + \pi) = (-1)^k R_k(\theta), \quad k = 0, 1, 2, \ldots. \quad (2.6)$$

For sufficient small $h$, let

$$\Delta(h) = r(2\pi, h) - h, \quad r = r(\theta, h) = \sum_{m=1}^{\infty} v_m(\theta) h^m \quad (2.7)$$

be the Poincaré successor function and the solution of (2.5), satisfying the initial-value condition $r|_{\theta=0} = h$. From (2.5) and (2.7), we have a series of differential equations with $v_m(\theta)$ as follows

$$\left\{ \begin{array}{l}
v_1(\theta) = \exp \int_0^\theta R_0(\varphi) d\varphi, \\
v_m(\theta) = R_0(\theta) v_m(\theta) + \sum_{k=2}^{m} R_{k-1}(\theta) \Omega_{m,k}(\theta),
\end{array} \right. \quad (2.8)$$

where, for any $m \geq 2$, $v_m(0) = 0$, and

$$\Omega_{m,k}(\theta) = \sum_{i_1+i_2+\cdots+i_k=m} \frac{m!}{i_1! i_2! \cdots i_k!} v_{i_1}(\theta) v_{i_2}(\theta) \cdots v_{i_k}(\theta) \quad (2.9)$$
is the coefficient of $h^m$ in the series $r^k(\theta, h)$. Particularly, we have
\[ \Omega_{m,1}(\theta) = v_m(\theta), \quad \Omega_{m,m}(\theta) = v_1^m(\theta) \] (2.10)
and we can derive successively $v_k(\theta)$ from (2.8).

**Lemma 2.1** (Theorem 1.1 in [17]). For any positive integer $m$, among $v_{2m}(\pi)$ and $v_k(2\pi)$’s, $v_1(\pi)$’s, there exists expression of the relation
\[ [1 + v_1(\pi)]v_{2m}(2\pi) = \xi_m(0)[v_1(2\pi) - 1] + \sum_{k=1}^{m-1} \xi_m^{(k)}v_{2k+1}(2\pi), \] (2.11)
where $\xi_m^{(k)}$ are all polynomials of $v_1(\pi), v_2(\pi), \ldots, v_{2m}(\pi)$ and $v_1(2\pi), v_2(2\pi), \ldots, v_{2m}(2\pi)$ with rational coefficients.

**Definition 2.1.** For system (2.1), $n > 0$, if $v_1(2\pi) \neq 1$, the origin is called the degenerate rough focus. If $v_1(2\pi) = 1$ and there exists a positive integer $m$, such that $v_{2m+1}(2\pi) \neq 0$ (2.12) then the origin is called the degenerate weak focus of order $m$, and $v_{2m+1}(2\pi)$ is called the $m$th generalized Lyapunov Constant (focal value of the origin). If $v_1(2\pi) = 1$, and for any positive integer $m$, $v_{2m+1}(2\pi) = 0$, then the origin is called the center.

**Theorem 2.1.** For system (2.1), $n > 0$ and sufficiently small $h > 0$, if $v_1(2\pi) < 1$, then the origin is a degenerate stable focus. If $v_1(2\pi) > 1$, then the origin is a degenerate unstable focus. If $v_1(2\pi) = 1$ and $v_{2m+1} = 0$ for any positive $m$, then the trajectories of system (2.1) in the neighborhoods of the origin are all closed.

Now we consider system (2.1) with
\[ X_{2n+1}(x, y) = -y(x^2 + y^2)^n, \quad Y_{2n+1}(x, y) = x(x^2 + y^2)^n. \] (2.13)
System (2.1) becomes the following form
\[
\begin{align*}
\frac{dx}{dt} &= -y(x^2 + y^2)^n + \sum_{k=2n+2}^{\infty} X_k(x, y) = X(x, y), \\
\frac{dy}{dt} &= x(x^2 + y^2)^n + \sum_{k=2n+2}^{\infty} Y_k(x, y) = Y(x, y),
\end{align*}
\] (2.14)
relevantly, in system (2.11)
\[ \varphi_{2n+2}(\theta) \equiv 0, \quad \psi_{2n+2}(\theta) \equiv 1, \quad v_1(\theta) \equiv 1. \] (2.15)

**Definition 2.2.** For system (2.11) and (2.14), all $a_{ab}$’s, $b_{ab}$’s are called parameters. For any positive integer $m \geq 3$, if there exist $\xi_2, \xi_3, \ldots, \xi_{m-1}$, such that
\[ v_m(2\pi) + \sum_{k=2}^{m-1} \xi_k v_k(2\pi) = V, \] (2.16)
where $\xi_k$'s are all polynomials of $a_{\alpha \beta}$'s, $b_{\alpha \beta}$'s with complex coefficient, then $V$ and $v_m(2\pi)$ are called algebraic equivalence, we denote $V \sim v_m(2\pi)$. For any constant $\lambda \neq 0$ we denote by $\lambda^{-1} V \sim v_m(2\pi)$ the algebraic equivalence of $V$ and $\lambda v_m(2\pi)$.

From (2.8), it is readily verify the following result by mathematical induction.

**Lemma 2.2.** For system (2.14), we have $v_1(\theta) \equiv 1$ and for any positive integer $m$

$$v_m(\theta) = \sum_{k=1}^{s(m)} \xi_m(k) \tilde{v}_{m,k}(\theta),$$

(2.17)

where $s(m)$ is a positive integer, and $\xi_m(k)$ is a polynomial of $a_{\alpha \beta}$, $b_{\alpha \beta}$ with complex coefficients, and $\tilde{v}_{m,k}(\theta)$ is independent of $a_{\alpha \beta}$, $b_{\alpha \beta}$.

By Theorem 2.1 and Lemma 2.2, we have

**Theorem 2.2.** For system (2.14), we have $v_2(2\pi) = 0$, and for any positive integer $m$, we have $v_{2m}(2\pi) \sim 0$. Moreover, if $V \sim v_{2m+1}(2\pi)$, then there exist $\xi_1, \xi_2, \ldots, \xi_{m-1}$, such that

$$v_{2m+1}(2\pi) + \sum_{k=1}^{m-1} \xi_k v_{2k+1}(2\pi) = V,$$

(2.18)

where $\xi_k$'s are all polynomials of $a_{\alpha \beta}$'s, $b_{\alpha \beta}$'s with complex coefficient.

It is readily verify that

**Theorem 2.3.** For system (1.2)$_{$\delta=0}$, the following assertions hold

(i) $v_2(2\pi) = 0,$

(ii) $v_{2m}(2\pi) = \sum_{k=1}^{m-1} \xi_m(k) v_{2k+1}(2\pi),$

(2.19)

(2.20)

where $\xi_m(k) (k = 1, 2, \ldots, m - 1)$ is a polynomial of coefficients of system (1.2)$_{$\delta=0}$.

3. Singular point quantity and algorithm

Form Theorem 2.2, in order to be able to apply Theorem 2.1 to obtain the center condition for degenerate singular points in system (1.2) we need an easy way of computing $v_{2m+1}(2\pi)$ ($m = 1, 2, \ldots$).

Following [17], let us introduce here a complex affine change of variables and a rescaling of the time

$$z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}.$$

(3.1)
System (1.2) can be transformed into

\[
\begin{align*}
\frac{dz}{dT} &= z(1-i\delta) + a_{4,-1}z^4w^{-1} + a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3, \\
\frac{dw}{dT} &= -w(1+i\delta w) - b_{4,-1}w^4z^{-1} - b_{30}w^3 - b_{21}w^2z - b_{12}zw^2 - b_{03}z^3,
\end{align*}
\]  
(3.2)

where \(a_{4,-1} = b_{03}, b_{4,-1} = a_{03}\), and for \((k, j) \in \{(3, 0), (2, 1), (1, 2), (0, 3)\}\),

\[a_{kl} = A_{kl} + iB_{kl}, \quad b_{kl} = A_{kl} - iB_{kl}.
\]  
(3.3)

For any positive integer \(k\), we denote

\[f_k(z, w) = \sum_{\alpha + \beta = k} c_{\alpha, \beta}z^\alpha w^\beta
\]

a homogeneous polynomial of degree \(k\) with \(c_{0,0} = 1, c_{k,k} = 0, k = 1, 2, \ldots\). We prove the following result.

**Theorem 3.1.** For system (3.2)\(|\delta = 0\), we can derive successively the terms of the following formal series

\[F(z, w) = zw \sum_{k=0}^{\infty} f_k(z, w)(zw)^{5k},
\]  
(3.4)

such that

\[
\frac{dF}{dT} = \sum_{m=1}^{\infty} \mu_m (zw)^{m+2}.
\]  
(3.5)

with

\[
\mu_m = \sum_{k+j=4} \left[ (3-k+m)a_{k,j-1} - (3-j+m)b_{j,k-1} \right]
\times c_{5m-3k-2j+10,5m-2k-3j+10}
\]  
(3.6)

and

\[
c_{\alpha, \beta} = \frac{1}{5(\beta - \alpha)} \sum_{k+j=4} \left[ (3\alpha - 2\beta - 5k + 15)a_{k,j-1} - (3\beta - 2\alpha - 5j + 15)b_{j,k-1} \right]
\times c_{\alpha-3k-2j+10,\beta-2k-3j+10}.
\]  
(3.7)

**Proof.** For system (3.2)\(|\delta = 0\), differentiating both sides of (3.4) with respect to \(T\) along the trajectories of the differential system, we have

\[
\frac{dF}{dT} = \sum_{m=1}^{\infty} (zw)^{5(1-m)} \left[ \frac{\partial f_m}{\partial z} z - \frac{\partial f_m}{\partial w} w + H_m \right],
\]  
(3.8)

where

\[
\frac{\partial f_m}{\partial z} z - \frac{\partial f_m}{\partial w} w = \sum_{\alpha+\beta=5m} (\alpha - \beta)c_{\alpha, \beta}z^\alpha w^\beta
\]  
(3.9)
and for any positive integer $m$

$$H_{5m} = \sum_{k=1}^{m-1} (zw)^{2(m-k-1)} \left( \frac{\partial f_{5k}}{\partial z} z + (1 - 2k) f_{5k} \right) w Z_{3+m-k}$$

$$- \left( \frac{\partial f_{5k}}{\partial w} w + (1 - 2k) f_{5k} \right) z W_{3+m-k} \right)$$

(3.10)

with

$$Z_{3+m-k} = \sum_{p+q=3+m-k} a_{p,q-1} z^p w^{q-1},$$

$$W_{3+m-k} = \sum_{p+q=3+m-k} b_{q,p-1} w^q z^{p-1}. \quad (3.11)$$

Computing carefully and matching the coefficients of like powers of $zw$, we obtain the recursion relations (3.6) and (3.7). \(\square\)

**Definition 3.1.** For system $(3.2)_{|\delta=0}$, $\mu_m$ is called the $m$th singular point quantity at the degenerate singular point.

By Theorem 2.2, we have the following theorem.

**Theorem 3.2.** For system $(1.2)_{|\delta=0}$, the following assertion holds

$$v_{2m+1}(2\pi) = i\pi \left( \mu_m + \sum_{k=1}^{m-1} \eta_m^{(k)} \mu_k \right). \quad (3.12)$$

where $\eta_m^{(k)}$ is a polynomial with rational coefficients in the variables given by the coefficients of the system.

Theorem 3.2 indicates that

$$\mu_m \sim \frac{1}{i\pi} v_{2m+1}(2\pi), \quad m = 1, 2, \ldots. \quad (3.13)$$

By Theorem 3.2, we can be able to reduce the problem of determining whether the degenerate singular point of $(1.2)_{|\delta=0}$ is a center or a weak focus to the calculating of the singular point quantities of the origin of system $(3.2)_{|\delta=0}$. From Theorem 3.1, the algorithm for singular point quantity $\mu_m$ is recursive and then avoids complex integrating operations and solving equations. The algorithm can be readily done with using computer algebra system such as Mathematica or Maple. Because all $c_{a, b}$ and $\mu_m$ are polynomials with rational coefficient, we can use computer to calculate the singular point quantities without errors. In order to calculate the singular point quantities of the origin of system $(3.2)_{|\delta=0}$ and simplify them quickly, as well as make use of the extended symmetric principle to obtain the conditions of integrability of the system, we need to find out all the Lie-invariants (see [17]) of system $(3.2)_{|\delta=0}$.

It follows from the technique used in [17].
Theorem 3.3. For system \((3.2)_{\delta=0}\), there exist just 13 elementary Lie-invariants as follows:

\[
\begin{align*}
    a_{21}, & \quad b_{21}, \quad a_{30}b_{30}, \quad a_{12}b_{12}, \quad a_{03}b_{03}, \quad a_{30}a_{12}, \quad b_{30}b_{12}, \\
    a_{30}^2a_{03}, & \quad b_{30}^2b_{03}, \quad a_{30}b_{12}a_{03}, \quad b_{30}a_{12}b_{03}, \quad b_{12}^2a_{03}, \quad a_{12}^2b_{03}.
\end{align*}
\]

4. Center conditions and bifurcation of limit cycles

Using the recursive formulas in Theorem 3.1 to compute the singular point quantities of the origin of system \((3.2)_{\delta=0}\) and simplify them with the constructive theorem of singular point quantities (Theorem 4.2 in [17]), we get:

Theorem 4.1. The first five singular point quantities of the origin of system \((3.2)_{\delta=0}\) are as follows:

\[
\begin{align*}
    \mu_1 &= a_{21} - b_{21}, \\
    \mu_2 &= b_{30}b_{12} - a_{30}a_{12}, \\
    \mu_3 &= \frac{1}{2}[(a_{30} - b_{12})(a_{30} + 3b_{12})a_{03} - (b_{30} - a_{12})(b_{30} + 3a_{12})b_{03}], \\
    \mu_4 &= \frac{1}{3}(a_{21} + b_{21})[(a_{30} - b_{12})a_{30}a_{03} - (b_{30} - a_{12})b_{30}b_{03}], \\
    \mu_5 &= \frac{8}{9}(a_{12}b_{12} - a_{03}b_{03})[(a_{30} - b_{12})a_{30}a_{03} - (b_{30} - a_{12})b_{30}b_{03}].
\end{align*}
\]

In the above expression of \(\mu_k\), we have already let \(\mu_1 = \mu_2 = \cdots = \mu_{k-1}, \quad k = 2, 3, 4, 5.\)

Noting that if \(a_{30}a_{12} = b_{30}b_{12}, \quad |a_{30}| + |b_{30}| \neq 0\), then there exists some \(\lambda\) such that \(a_{12} = \lambda b_{30}, \quad b_{12} = \lambda a_{30}\), from Theorem 4.1 we have:

Theorem 4.2. For system \((3.2)_{\delta=0}\), the first five singular point quantities of the origin are zero if and only if one of the following three conditions holds:

\[
\begin{align*}
    (I) \quad &a_{21} = b_{21}, \quad a_{30}a_{12} = b_{30}b_{12}, \quad a_{30}a_{12}a_{03} = b_{30}b_{12}b_{03}, \quad b_{12}^2a_{03} = a_{12}^2b_{03}, \\
    (II) \quad &a_{21} = b_{21} = 0, \quad a_{30} = -3b_{12}, \quad b_{30} = -3a_{12}, \quad a_{03}b_{03} = a_{12}b_{12}, \\
    (III) \quad &a_{21} = b_{21}, \quad a_{30} = b_{12}, \quad b_{30} = a_{12}.
\end{align*}
\]

Readily verify the following result.

Theorem 4.3.

(1) If (4.1) holds, then system \((3.2)_{\delta=0}\) satisfies the conditions of the extended symmetric principle.
If (4.2) holds, then system (3.2)|_{\delta=0} has an integral factor $g_1^{-1}g_2^{-1}(zw)^{-1/2}$, where
$$g_1 = a_{03}z + w(a_{12}z + a_{03}w)^2, \quad g_2 = b_{03}w + z(b_{12}w + b_{03}z)^2.$$ (4.4)
(3) If (4.3) holds, then system (3.2)|_{\delta=0} has the first integral $zw = \text{Const}$.

From Theorem 4.3, we have

**Theorem 4.4.** For system (3.2)|_{\delta=0}, all the singular point quantities of the origin are zero if and only if the first five singular quantities of the origin are zero, i.e., one of the three conditions of Theorem 4.2 holds. Relevantly, the three conditions of Theorem 4.2 are the integrable conditions in the neighborhood of the origin.

**Corollary 4.5.** The origin of system (1.2) is a center or the trajectories of system (1.2) in the neighborhood of the degenerate singular point are all closed if and only if $\delta = 0$ and one of the three conditions of Theorem 4.2 holds.

**Corollary 4.6.** The highest degree number of fine focus of system (1.2)|_{\delta=0} is 5.

Theorem 4.1–Corollary 4.6 imply the stability criterions for the degenerate singular point of system (1.2).

Now, we consider bifurcation of limit cycles from the degenerate singular point of the perturbed system (1.2).

For polynomial system (1.2), the solution of Eq. (2.5)$r(\theta, h) = r(\theta, h, \delta)$ is analytic at $\delta = 0$ to parameter $\delta$. So we have that
$$v_{2m+1}(2\pi, \delta) = v_{2m+1}(2\pi, 0) + \varphi_m(A_{a, \beta}, B_{a, \beta}, \delta)\delta,$$ (4.5)
where $\varphi_m(A_{a, \beta}, B_{a, \beta}, \delta)$ are analytic at $\delta = 0$, $m = 0, 1, 2, \ldots$

**Theorem 4.7.** If $\delta$ and the coefficients of system (1.2) satisfy
$$\begin{cases}
\delta = \frac{1}{2} \varepsilon^{15+N}, & A_{30} = -3 - \frac{16}{3} \varepsilon^{3}, & B_{30} = \frac{1}{2} \varepsilon^{6+N}, \\
A_{21} = \frac{1}{3} \varepsilon, & B_{21} = \frac{1}{2} \varepsilon^{10+N}, & A_{12} = 1, \\
B_{12} = 0, & A_{03} = 0, & B_{03} = \frac{3}{5} \varepsilon^N
\end{cases}$$ (4.6)
(accordingly, the coefficients of system (3.2) are determined), where $\delta$ is a small parameter and $N$ is a positive integer, then, when $0 < \varepsilon \ll 1$, there are 5 limit cycles in a small enough neighborhood of the origin of system (1.2) which are near the cycles $x^2 + y^2 = \varepsilon^k$, $k = 1, 2, 3, 4, 5$, respectively.

**Proof.** It is easy to get that
$$v_1(2\pi, \delta) - 1 = \pi \varepsilon^{15+N} + o(\varepsilon^{15+N}).$$
According to (3.12) and (4.5), we have
$$v_{2m+1}(2\pi, \delta) = i \pi \mu_m + \varphi_m(A_{a, \beta}, B_{a, \beta}, \delta)\delta, \quad m = 1, 2, \ldots.$$
From Theorem 4.1, after computing carefully, we have

\begin{align*}
v_3(2\pi, \delta) &= -\pi\varepsilon^{10+N} + o(\varepsilon^{10+N}), \\
v_5(2\pi, \delta) &= \pi\varepsilon^{6+N} + o(\varepsilon^{6+N}), \\
v_7(2\pi, \delta) &= -\pi\varepsilon^{3+N} + o(\varepsilon^{3+N}), \\
v_9(2\pi, \delta) &= \pi\varepsilon^{1+N} + o(\varepsilon^{1+N}), \\
v_{11}(2\pi, \delta) &= -\pi\varepsilon^{N} + o(\varepsilon^{N}),
\end{align*}

and

\begin{align*}
a_{21} - b_{21} &= O(\varepsilon^{N}), \\
a_{30}a_{12} - b_{30}b_{12} &= O(\varepsilon^{N}), \\
a_{30}d_{03} - b_{30}b_{03} &= O(\varepsilon^{N}), \\
a_{30}b_{12}d_{03} - b_{30}a_{12}b_{03} &= O(\varepsilon^{N}), \\
b_{12}^2d_{03} - a_{12}^2b_{03} &= O(\varepsilon^{N}).
\end{align*}

Therefore, by the constructive theorem of singular point quantities, we have \( v_{2k+1} = O(\varepsilon^{N}) \) for \( k > 5 \), and the weak bifurcation function is of the form

\[ L(h, \varepsilon) = \pi (\varepsilon^{15} - \varepsilon^{10}h^2 + \varepsilon^8h^4 - \varepsilon^3h^6 + \varepsilon h^8 - h^{10}). \quad (4.7) \]

A reference to Theorem 6.6 in [17] completes the proof of the theorem. \( \square \)

References

[4] N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Amer. Math. Soc. Trans. 100 (154).
[22] Shi Songling, Example of five limit cycles for the system $dx/dt = \lambda x - y + \sum_{i+k=3} a_{ik} x^i y^k$, $dy/dt = x + \lambda y + \sum_{i+k=3} b_{ik} x^i y^k$, Acta Math. Sinica 18 (1975) 300–304.