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Two regularization methods for a Cauchy problem for the Laplace equation

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Abstract

A Cauchy problem for the Laplace equation in a rectangle is considered. Cauchy data are given for $y = 0$, and boundary data are for $x = 0$ and $x = \pi$. The solution for $0 < y \leqslant 1$ is sought. We propose two different regularization methods on the ill-posed problem based on separation of variables. Both methods are applied to formulate regularized solutions which are stably convergent to the exact one with explicit error estimates.

Keywords: Ill-posed problem; Cauchy problem for Laplace equation; Regularization

1. Introduction

Consider the Cauchy problem for the Laplace equation in a rectangle: determine the solution $u(x, y)$ for $0 < y \leqslant 1$ from the input data $\varphi(\cdot) := u(\cdot, 0)$, when $u(x, y)$ satisfies

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi,\ 0 < y < 1, \tag{1}$$

$$u(x, 0) = \varphi(x), \quad 0 \leqslant x \leqslant \pi, \tag{2}$$

$$u_y(x, 0) = 0, \quad 0 \leqslant x \leqslant \pi, \tag{3}$$

$$u(0, y) = u(\pi, y) = 0, \quad 0 \leqslant y \leqslant 1. \tag{4}$$

Physically, φ can only be measured, there will be measurement errors, and we would actually have as data some function $\varphi^\delta \in L^2(0, \pi)$, for which

$$\left\|\varphi^\delta - \varphi\right\| = \left\|\varphi^\delta(\cdot) - u(\cdot, 0)\right\| \leqslant \delta \tag{5}$$

where the constant $\delta > 0$ represents a bound on the measurement error, $\|\cdot\|$ denotes the L^2-norm.

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Problem (1)–(4) is well known to be severely ill-posed: a small perturbation in the data φ may cause dramatically large errors in the solution $u(x, y)$ for $0 < y \leqslant 1$. We give, however, here an explicit example to emphasize this fact [12].

It is easy to verify that the function

$$u_m(x, y) = \frac{a}{m^j} \sin(mx) \cosh(my) \tag{6}$$

is the exact solution of problem (1)–(4) with

$$u_m(x, 0) = \varphi_m(x) = \frac{a}{m^j} \sin(mx), \tag{7}$$

where m, j are positive integers, and $a \in \mathbb{R}$, $a \neq 0$. Although $\sup_{x\in(0,\pi)} |\varphi_m(x)|$ tends to zero as $m \to \infty$, we have $\sup_{x\in(0,\pi)} |u_m(x, y)| \to \infty$ $(m \to \infty)$ for fixed $y > 0$. Thus system (1)–(4) is an ill-posed problem that is impossible to solve using classical numerical methods and requires special techniques, i.e., regularizations [9,19], to be employed.

The stability and convergence analysis of regularization methods for above Cauchy problem can rarely be found in the literature though a series of papers contains numerical examples (see, e.g., [2–5,10–14,18,20,21,24]). Among these, the works of Falk [10], Falk and Monk [11] and Han [13] contain error estimates and convergence results of discrete form. About some stability and convergence estimates of continuous form, one can also refer to the references [15–17].

We shall use two different regularization methods to construct stable solution of the problem (1)–(4) and then obtain error estimates for them. Both methods of proving stability estimates are constructive: we construct stable solutions to the problem that can be numerically implemented. However, we do not pursue this aspect in this paper, as our aim here is to obtain stability estimates only. The numerical computation will be considered in our future research.

The first is the perturbation method. That is to say, we do a modification of the equation, where a fourth-order mixed derivative term is added,

$$u_{xx} + u_{yy} - \mu^2 u_{xxyy} = 0 \tag{8}$$

which we learned from Eldén [6]. In [6], Eldén considered a standard inverse heat conduction problem and the idea initially came from Weber [27]. In (8) the choice of μ is based on some *a priori* knowledge about the magnitude of the errors in the data. We will study the properties of the system (8), (2)–(4), considered as a Cauchy problem in the y variable and as an approximation to the system (1)–(4) in Section 2. The modified equation (8) is popular and interesting. Weber [27] and Beck et al. [1] (P_{252}) applied the perturbation method to compute the inverse heat conduction problems (IHCP). Eldén [6,7] also applied this kind of method to analyze the IHCP and obtained some stability and convergence estimates. The perturbation method can be also applied to problem with variable coefficients [27] which is actually its main advantage over methods based on an integral equation formulation. In nature, the perturbation method transfers an ill-posed problem to an approximate well-posed problem which can be discretized using standard techniques, e.g., finite differences. For the numerical implementation of the perturbation method, one can refer to the references [1,22]. Our aim here is to discuss the stability and convergence analysis of regularization methods.

The second is the truncation method. Separation of variables leads to the solution of problem (1)–(4)

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin(nx) \cosh(ny) \tag{9}$$

where

$$c_n = \frac{2}{\pi} \int_0^{\pi} \varphi(t) \sin(nt)\, dt. \tag{10}$$

For guaranteeing the convergence of the series, the coefficient c_n must decay rapidly since $\cosh(ny)$ tends to ∞ as $n \to \infty$ for fixed $y > 0$. Small errors in the components of large n can blow up and completely destroy the solution for $y > 0$. A natural way to stabilize the problem is to eliminate all the components of large n from the solution and instead consider (9) only for $n \leqslant N$. Then we get a regularized solution

$$w(x,y)=\sum_{n=1}^{N}c_n\sin(nx)\cosh(ny). \tag{11}$$

The positive integer N plays the role of regularization parameter. We will study the properties of (11) as an approximation to (9) in Section 3. Here we want to point out that the idea of truncation was applied to analyze and compute a one-dimensional (1D) IHCP by Eldén et al. [8] in which they called it Fourier method. Trong et al. also applied the idea of truncation to the 1D and 2D source identification problems [25,26]. Regińska et al. applied the idea of truncation to a Cauchy problem for the Helmholtz equation [23]. It is interesting that, for a problem whose solution has the explicit expression, three kinds of methods: truncated singular value decomposition [9], truncated method in the present paper and Hào's mollification method [15] have the similar idea. The difference is that: truncated singular value decomposition studies the general theory of ill-posed operator equation in abstract spaces from the viewpoint of operator theory. Truncated method in the present paper is usually used to consider a concrete problem using the needed tools, e.g., [8,23] using the Fourier transform techniques, the present paper using the separation of variables. Hào's mollification method is also used to consider the concrete problem but through mollifying the input data. The idea of the truncated method in the present paper seems to be more direct and simple when considering a concrete problem. Moreover, using the truncated method one can easily obtain the error estimate which has a good convergence rate. This fact has been confirmed in [8,23] (see also Section 3).

2. The perturbation method

In this section we consider the system

$$v^\delta_{xx}+v^\delta_{yy}-\mu^2v^\delta_{xxyy}=0,\quad 0<x<\pi,\ 0<y<1, \tag{12}$$

$$v^\delta(x,0)=\varphi^\delta(x),\quad 0\leqslant x\leqslant\pi, \tag{13}$$

$$v^\delta_y(x,0)=0,\quad 0\leqslant x\leqslant\pi, \tag{14}$$

$$v^\delta(0,y)=v^\delta(\pi,y)=0,\quad 0\leqslant y\leqslant 1. \tag{15}$$

From the following discussions we could find that this system is well posed, i.e., its solution $v^\delta(x,y)$ for any fixed $y>0$ is dependent continuously on the data φ^δ. Moreover, it is an approximation of the exact solution $u(x,y)$. The approximation error depends continuously on the measurement error for fixed $0<y\leqslant 1$.

Separation of variables leads to the solution

$$v^\delta(x,y)=\sum_{n=1}^{\infty}c_n^\delta\sin(nx)\cosh\left(\frac{ny}{\sqrt{1+\mu^2n^2}}\right) \tag{16}$$

where

$$c_n^\delta=\frac{2}{\pi}\int_0^\pi\varphi^\delta(t)\sin(nt)\,dt. \tag{17}$$

We will derive a bound on the difference between the solutions (9) and (16). However, before doing that, we need to assume that $\|u(\cdot,1)\|$ is bounded, i.e.,

$$\big\|u(\cdot,1)\big\|\leqslant E \tag{18}$$

where $E>0$ is a constant. This is essentially necessary in order to obtain any meaning error estimates for approximating the exact solution. If we make no assumptions about the solution of (1)–(4), then we can only bound the error between the regularized solution and the approximation. The relation between any two regularized solutions (16) is given by the following lemma.

Lemma 2.1. *Suppose that we have two regularized solutions v^1 and v^2 defined by* (16) *with data φ^1 and φ^2, satisfying $\|\varphi^1-\varphi^2\|\leqslant\delta$. If we select $\mu=1/\ln(2E/\delta)$, then for fixed $0<y<1$ we get the error bound*

$$\big\|v^1(\cdot,y)-v^2(\cdot,y)\big\|\leqslant(2E)^y\delta^{1-y}. \tag{19}$$

Proof. From (16) we have

$$\varphi^1(x) = v^1(x,0) = \sum_{n=1}^{\infty} c_n^1 \sin(nx), \qquad \varphi^2(x) = \sum_{n=1}^{\infty} c_n^2 \sin(nx) \tag{20}$$

where

$$c_n^1 = \frac{2}{\pi}\int_0^{\pi} \varphi^1(t)\sin(nt)\,dt, \qquad c_n^2 = \frac{2}{\pi}\int_0^{\pi} \varphi^2(t)\sin(nt)\,dt. \tag{21}$$

Thus the condition $\|\varphi^1 - \varphi^2\| \leqslant \delta$ is equivalent to

$$\frac{\pi}{2}\sum_{n=1}^{\infty}\left(c_n^1 - c_n^2\right)^2 \leqslant \delta^2. \tag{22}$$

Consequently

$$\left\|v^1(\cdot,y) - v^2(\cdot,y)\right\|^2 = \frac{\pi}{2}\sum_{n=1}^{\infty}\left(c_n^1 - c_n^2\right)^2 \cosh^2\left(\frac{ny}{\sqrt{1+\mu^2 n^2}}\right) \leqslant \cosh^2(y/\mu)\delta^2 \leqslant e^{2y/\mu}\delta^2.$$

The choice of parameter $\mu = 1/\ln(2E/\delta)$ leads to

$$\left\|v^1 - v^2\right\| \leqslant (2E)^y \delta^{1-y}. \qquad \square$$

From Lemma 2.1 we see that the solution defined by (16) depends continuously on the input data φ^δ. Next we will investigate the difference between the solutions (9) and (16) with the same exact data φ.

Lemma 2.2. *Let u and v be the solutions* (9) *and* (16) *with the same exact data φ, and let $\mu = 1/\ln(2E/\delta)$. Suppose that $\|u(\cdot,1)\| \leqslant E$. Then for fixed $0 < y < 1$, we have*

$$\left\|u(\cdot,y) - v(\cdot,y)\right\| \leqslant C\frac{E}{\ln^2(2E/\delta)}, \tag{23}$$

where $C = \frac{1}{2}(\frac{3}{(1-y)e})^3$.

Proof. From (9) the assumption $\|u(\cdot,1)\| \leqslant E$ is equivalent to

$$\left\|u(\cdot,1)\right\|^2 = \frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2\cosh^2(n) \leqslant E^2. \tag{24}$$

Consequently,

$$\left\|u(\cdot,y) - v(\cdot,y)\right\| = \sqrt{\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2\left(\cosh(ny) - \cosh\left(\frac{ny}{\sqrt{1+\mu^2n^2}}\right)\right)^2} \leqslant \sup_{n\geqslant 1} A(n)E \tag{25}$$

where

$$A(n) = \left|\frac{\cosh(ny) - \cosh(\tau y)}{\cosh(n)}\right|, \quad \tau = n/\sqrt{1+\mu^2n^2}. \tag{26}$$

We now estimate $A(n)$. Note that $n \geqslant \tau = n/\sqrt{1+\mu^2n^2}$,

$$\begin{aligned} A(n) &= \left|\frac{\cosh(ny) - \cosh(\tau y)}{\cosh(n)}\right| = \frac{(e^{ny}+e^{-ny})/2 - (e^{\tau y}+e^{-\tau y})/2}{(e^n+e^{-n})/2} = \frac{(e^{ny}-e^{\tau y}) - (e^{ny}-e^{\tau y})/e^{(n+\tau)y}}{e^n+e^{-n}} \\ &\leqslant \frac{e^{ny}-e^{\tau y}}{e^n} = e^{-n(1-y)}\left(1 - e^{-(n-\tau)y}\right). \end{aligned} \tag{27}$$

Using the inequality $1 - e^{-r} \leqslant r$ $(r \geqslant 0)$, note that $0 < y < 1$, we have

$$A(n) \leqslant e^{-n(1-y)}(n-\tau)y \leqslant e^{-n(1-y)}(n-\tau). \tag{28}$$

Now, since $\sqrt{1+\mu^2n^2} \leqslant 1 + \frac{1}{2}\mu^2n^2$, we get

$$n - \tau = n - n/\sqrt{1+\mu^2n^2} = n \cdot \frac{\sqrt{1+\mu^2n^2}-1}{\sqrt{1+\mu^2n^2}} \leqslant \frac{1}{2}\mu^2n^3, \tag{29}$$

so (28) becomes

$$A(n) \leqslant \frac{1}{2}\mu^2n^3e^{-n(1-y)}.$$

The function $h(s) := s^3e^{-s(1-y)}$ attains its maximum

$$h_{\max} = h\left(\frac{3}{1-y}\right) = \left(\frac{3}{(1-y)e}\right)^3$$

at $s = \frac{3}{1-y}$, and then

$$A(n) \leqslant \frac{1}{2}\mu^2h_{\max} = \frac{1}{2}\left(\frac{3}{(1-y)e}\right)^3\frac{1}{\ln^2(2E/\delta)}. \tag{30}$$

The lemma now follows by combining (25) and (30). □

Now we are ready to formulate the main result of this section:

Theorem 2.3. *Suppose that u is given by* (9) *with exact data φ and that v^δ is given by* (16) *with measured data φ^δ. If we have a bound $\|u(\cdot,1)\| \leqslant E$, and the measured function φ^δ satisfies $\|\varphi - \varphi^\delta\| \leqslant \delta$ and if we choose $\mu = 1/\ln(2E/\delta)$, then for fixed* $0 < y < 1$, *we get the error bound*

$$\left\|u(\cdot,y) - v^\delta(\cdot,y)\right\| \leqslant (2E)^y\delta^{1-y} + C\frac{E}{\ln^2(2E/\delta)}. \tag{31}$$

Proof. Let v be the solution defined by (16) with exact data φ. Then the theorem is straightforward by using the triangle inequality $\|u - v^\delta\| \leqslant \|u - v\| + \|v - v^\delta\|$ and the two previous lemmas. □

From Theorem 2.3 we find that (16) is an approximation of the exact solution $u(x,y)$. The approximation error depends continuously on the measurement error for fixed $0 < y < 1$. However, as $y \to 1$, the accuracy of the regularized solution becomes progressively lower. This is common in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution.

To retain the continuous dependence of the solution at $y = 1$, instead of (18), we introduce a stronger *a priori* assumption,

$$\left\|\left.\frac{\partial^p u(\cdot,y)}{\partial y^p}\right|_{y=1}\right\| \leqslant E \tag{32}$$

where $p > 0$ is an integer.

Theorem 2.4. *Suppose that u is given by* (9) *with exact data φ and that v^δ is given by* (16) *with measured data φ^δ. If we have an* a priori *bound* (32), *and the measured function φ^δ satisfies $\|\varphi - \varphi^\delta\| \leqslant \delta$. The parameter $\mu \in (0,1)$ is chosen as*

$$\mu = \frac{1}{\ln(\frac{2E}{\delta}(\ln\frac{2E}{\delta})^{-p})}. \tag{33}$$

Then for $p > 0$, we get the error bound

$$\left\|u(\cdot,1) - v^\delta(\cdot,1)\right\| \leqslant \frac{2E}{(\ln\frac{2E}{\delta})^p} + \varepsilon \tag{34}$$

where $\varepsilon := \max\{\mu^{2p/3}, \frac{1}{2}\mu^2\}E$.

Proof. From (9) and (32), we have

$$\left\|\frac{\partial^p u(\cdot,y)}{\partial y^p}\bigg|_{y=1}\right\|^2=\begin{cases}\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2n^{2p}\cosh^2(n), & p \text{ is even},\\ \frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2n^{2p}\sinh^2(n), & p \text{ is odd}\end{cases}\leqslant E^2.$$

In the following, we only discuss the case that p is even, i.e.,

$$\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2n^{2p}\cosh^2(n)\leqslant E^2, \tag{35}$$

since the procedure of the proof is completely similar when p is odd.

Note that u is defined by (9) and v^δ is defined by (16), we have

$$\begin{aligned}\left\|u(\cdot,1)-v^\delta(\cdot,1)\right\| &\leqslant \left\|u(\cdot,1)-v(\cdot,1)\right\|+\left\|v(\cdot,1)-v^\delta(\cdot,1)\right\|\\ &=\sqrt{\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2\left(\cosh(n)-\cosh\left(\frac{n}{\sqrt{1+\mu^2n^2}}\right)\right)^2}\\ &\quad+\sqrt{\frac{\pi}{2}\sum_{n=1}^{\infty}\left(c_n-c_n^\delta\right)^2\cosh^2\left(\frac{n}{\sqrt{1+\mu^2n^2}}\right)}.\end{aligned}$$

Now the conditions (35) and $\|\varphi^\delta-\varphi\|\leqslant\delta$ (see also (22)) lead to

$$\left\|u(\cdot,1)-v^\delta(\cdot,1)\right\|\leqslant\sup_{n\geqslant1}\widetilde{A}(n)E+\sup_{n\geqslant1}\widetilde{B}(n)\delta \tag{36}$$

where

$$\widetilde{A}(n)=\left|\frac{\cosh(n)-\cosh(\tau)}{n^p\cosh(n)}\right|,\qquad \widetilde{B}(n)=\cosh(\tau),\quad \tau=\frac{n}{\sqrt{1+\mu^2n^2}}.$$

We start by estimating the second term on the right-hand side of (36). Since $\cosh(\cdot)$ is a monotone increasing function in the interval $[0,\infty)$ and μ is chosen in (33), we have

$$\widetilde{B}(n)\delta=\cosh\left(n/\sqrt{1+\mu^2n^2}\right)\delta\leqslant\cosh(1/\mu)\delta\leqslant e^{1/\mu}\delta=2E\left(\ln\frac{2E}{\delta}\right)^{-p}. \tag{37}$$

We now consider the first term on the right-hand side of (36). Taking the similar procedure of (27), then

$$\left|\frac{\cosh(n)-\cosh(\tau)}{\cosh(n)}\right|\leqslant 1-e^{-(n-\tau)}.$$

So

$$\widetilde{A}(n)\leqslant\left(1-e^{-(n-\tau)}\right)/n^p. \tag{38}$$

For estimating (38), we now distinguish between two cases.

Case I: for large values of n, i.e., for $n\geqslant\frac{1}{\mu^{2/3}}$, note that $n\geqslant\tau=n/\sqrt{1+\mu^2n^2}$, we have

$$\widetilde{A}(n)\leqslant\frac{1}{n^p}\leqslant\mu^{\frac{2p}{3}}. \tag{39}$$

Case II: for $n<\frac{1}{\mu^{2/3}}$, using the inequalities $1-e^{-r}\leqslant r$ $(r\geqslant0)$ and (29), we have from (38) that

$$\widetilde{A}(n)\leqslant\frac{1}{2}\mu^2n^{3-p}. \tag{40}$$

If $0<p<3$, from (40), we have

$$\widetilde{A}(n)\leqslant\frac{1}{2}\mu^2\left(\frac{1}{\mu^{2/3}}\right)^{3-p}=\frac{1}{2}\mu^{\frac{2p}{3}}. \tag{41}$$

If $p \geqslant 3$, note that $n \geqslant 1$, from (40), we have

$$\widetilde{A}(n) \leqslant \frac{1}{2}\mu^2. \tag{42}$$

Summarizing (39), (41) and (42), we complete the estimate of the first term on the right-hand side of (36), i.e.,

$$\widetilde{A}(n)E \leqslant \max\left\{\mu^{2p/3}, \frac{1}{2}\mu^2\right\}E =: \varepsilon, \quad p > 0. \tag{43}$$

The theorem now follows from (36), (37) and (43). □

Remark 2.5. Since the regularization parameter $\mu \to 0$ as the measured error $\delta \to 0$, we can easily find that, for $p > 0$, $\varepsilon \to 0$ $(\delta \to 0)$. Thus

$$\lim_{\delta \to 0} \left\| u(\cdot, 1) - v^\delta(\cdot, 1) \right\| = 0, \quad p > 0.$$

Remark 2.6. We separately consider the case $0 < y < 1$ (Theorem 2.3) and the case $y = 1$ (Theorem 2.4), in order to emphasize the following facts. For the case $0 < y < 1$, the *a priori* bound for $\|u(\cdot, 1)\|$ is sufficient. However, for the case $y = 1$, the stronger *a priori* bound for $\|\frac{\partial^p u(\cdot, y)}{\partial y^p}|_{y=1}\|$ where $p > 0$ must be imposed. In the next section we unite both cases using the truncation method.

3. The truncation method

In this section we study the properties of the regularized solution

$$w^\delta(x, y) = \sum_{n=1}^{N} c_n^\delta \sin(nx)\cosh(ny) \tag{44}$$

where

$$c_n^\delta = \frac{2}{\pi}\int_0^\pi \varphi^\delta(t)\sin(nt)\,dt. \tag{45}$$

From the following discussions we could find that $w^\delta(x, y)$ is also an approximation of the exact solution $u(x, y)$. The approximation error depends continuously on the measurement error for fixed $0 < y \leqslant 1$.

We also assume that the *a priori* bound on the solution

$$\left\| \frac{\partial^p u(\cdot, y)}{\partial y^p}\bigg|_{y=1} \right\| \leqslant E \tag{46}$$

holds. As in the proof of Theorem 2.4, we only discuss the case that p is even. Consequently, (46) is equivalent to

$$\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2 n^{2p}\cosh^2(n) \leqslant E^2. \tag{47}$$

The following theorem shows that the regularized solution $w^\delta(x, y)$ is a good approximation to the exact solution $u(x, y)$.

Theorem 3.1. *Suppose that u is given by* (9) *with exact data φ and that w^δ is given by* (44) *with measured data φ^δ. If the* a priori *bound* (47) *holds, and the measured data φ^δ satisfies $\|\varphi^\delta - \varphi\| \leqslant \delta$, and if we choose $N = [a]$ where $[a]$ with square bracket denotes the largest integer less than or equal to a,*

$$a = \ln\left(\frac{2E}{\delta}\left(\ln\frac{2E}{\delta}\right)^{-p}\right), \tag{48}$$

then for $0 < y \leqslant 1$, $p > 0$, we get the error bound

$$\left\|u(\cdot, y) - w^\delta(\cdot, y)\right\| \leqslant \left(1 + \left(\frac{\ln\frac{2E}{\delta}}{\ln\frac{2E}{\delta} + \ln(\ln\frac{2E}{\delta})^{-p}}\right)^p\right)(2E)^y\delta^{1-y}\left(\ln\frac{2E}{\delta}\right)^{-py}. \tag{49}$$

Proof. Subtracting and adding w which is defined by (44) with the exact data φ, and using the triangle inequality, (9) and (44), we get

$$\begin{aligned}\left\|u(\cdot, y) - w^\delta(\cdot, y)\right\| &\leqslant \left\|u(\cdot, y) - w(\cdot, y)\right\| + \left\|w(\cdot, y) - w^\delta(\cdot, y)\right\| \\ &= \sqrt{\frac{\pi}{2}\sum_{n=N+1}^{\infty}(c_n)^2\cosh^2(ny)} + \sqrt{\frac{\pi}{2}\sum_{n=1}^{N}\left(c_n - c_n^\delta\right)^2\cosh^2(ny)} \\ &= \sqrt{\frac{\pi}{2}\sum_{n=N+1}^{\infty}(c_n)^2 n^{2p}\cosh^2(n)\frac{\cosh^2(ny)}{n^{2p}\cosh^2(n)}} + \sqrt{\frac{\pi}{2}\sum_{n=1}^{N}\left(c_n - c_n^\delta\right)^2\cosh^2(ny)}.\end{aligned} \tag{50}$$

Note that

$$\cosh(ny) \leqslant e^{ny}, \qquad \frac{\cosh(ny)}{\cosh(n)} \leqslant 2e^{-n(1-y)}. \tag{51}$$

Combining the conditions $\|\varphi^\delta - \varphi\| \leqslant \delta$ (see also (22)) and (47), we have

$$\left\|u(\cdot, y) - w^\delta(\cdot, y)\right\| \leqslant 2e^{-(N+1)(1-y)}(N+1)^{-p}E + e^{Ny}\delta. \tag{52}$$

Since $N = [a]$, i.e., $N \leqslant a < N + 1$, we can estimate

$$\left\|u(\cdot, y) - w^\delta(\cdot, y)\right\| \leqslant 2e^{-a(1-y)}a^{-p}E + e^{ay}\delta. \tag{53}$$

Now using (48), we arrive at the statement of the theorem. □

Remark 3.2. In Theorem 3.1, if we are only interested in $y = 1$, then, for $p > 0$, we have

$$\left\|u(\cdot, 1) - w^\delta(\cdot, 1)\right\| \leqslant 2E\big(1 + o(1)\big)\left(\ln\frac{2E}{\delta}\right)^{-p} \quad \text{for } \delta \to 0. \tag{54}$$

We now want to compare the regularized solutions (16) and (44) with the exact solution (9) and find an interesting relation for these regularized solutions. From the simple analysis about the exact solution (9) in last paragraph of Section 1, we know that the data error can be arbitrarily amplified by the "kernel" $\cosh(ny)$. That is the reason why the Cauchy problem of Laplace equation is ill-posed. The proposed regularized solutions (16) and (44) can be interpreted as replacing the arbitrarily large kernel $\cosh(ny)$ by the regularized kernels

$$\cosh(ny)\chi_N, \quad \text{where } \chi_N = \begin{cases} 1, & n \leqslant N, \\ 0, & n > N, \end{cases}$$

and

$$\cosh\left(\frac{ny}{1 + \mu^2 n^2}\right).$$

Both above kernels have the following two common properties:

(a) If the parameter N is large or the parameter μ is small, then for small n, both kernels are close to the exact kernel $\cosh(ny)$.
(b) If N or μ is fixed, both kernels are bounded.

Property (a) describes that, for the appropriately chose parameter N or μ, the regularized kernel reserves the information of the exact kernel in the components of small n. These reserved information guarantee the possibility of the regularized solution approximating the exact one. Property (b) describes the degree of continuous dependence, i.e., when the regularized kernel is bounded, the regularized solution will depend continuously on the data. Both properties (a) and (b) guarantee that the regularized solution (16) or (44) is dependent continuously on the data and is the approximation of the exact solution.

Combining the general regularization theory [9,19] and properties (a) and (b), we now give a more general principle of regularization methods for the Cauchy problem of Laplace equation. We suggest that, in order to obtain a regularization method one can construct a new kernel $k(y,n,\alpha)$ and replace $\cosh(ny)$ by $k(y,n,\alpha)$ where the new kernel should satisfy:

(A) If the parameter α is appropriately chosen, then for small n, the kernel $k(y,n,\alpha)$ is close to the exact kernel $\cosh(ny)$;
(B) If α is fixed, $k(y,n,\alpha)$ is bounded.

Following properties (A) and (B), one can construct other kernels. Furthermore, the idea of properties (A) and (B) can be applied to other ill-posed problems when the solution has the similar form of (9), e.g., the inverse heat conduction problem [28]. In this sense, we say that the properties (A) and (B) are useful and interesting.

4. Generalizations

In problem (1)–(4), although we seek to recover u only for $0<x<\pi$, $0<y<1$, the problem specification includes the Laplace equation for $0<\bar{x}<l$ $(l>0)$, $0<\bar{y}<H$ $(H>0)$. Actually, through a change of variables, $x=\pi\bar{x}/l$, $y=\bar{y}/H$, the region $0<x<\pi$, $0<y<1$ is equivalent to $0<\bar{x}<l$, $0<\bar{y}<H$.

Consider the Cauchy problem with non-homogeneous Cauchy and boundary data,

$$u_{xx}+u_{yy}=f(x,y), \quad 0<x<\pi,\ 0<y<1, \tag{55}$$

$$u(x,0)=\varphi_0(x), \qquad u_y(x,0)=\varphi_1(x), \quad 0\leqslant x\leqslant\pi, \tag{56}$$

$$u(0,y)=g_0(y), \qquad u(\pi,y)=g_1(y), \quad 0\leqslant y\leqslant 1, \tag{57}$$

where f, φ_0, φ_1, g_0, g_1 are known, the solution $u(x,y)$ for $0<y\leqslant 1$ is sought. We cannot immediately use the same techniques for deriving a convergence estimate, since separation of variables will give extra terms that make the estimation more complicated. However, we can define $u=u_1+u_2$, where u_1 satisfies (1)–(4) with modified Cauchy data,

$$u(x,0)=\varphi_0(x)-u_2(x,0), \quad u_y(x,0)=0. \tag{58}$$

The second component u_2 satisfies

$$u_{xx}+u_{yy}=f(x,y), \quad 0<x<\pi,\ 0<y<1, \tag{59}$$

$$u_y(x,0)=\varphi_1(x), \quad 0\leqslant x\leqslant\pi, \tag{60}$$

$$u(x,1)=\frac{(\pi-x)g_0(1)+xg_1(1)}{\pi}, \quad 0\leqslant x\leqslant\pi, \tag{61}$$

$$u(0,y)=g_0(y), \qquad u(\pi,y)=g_1(y), \quad 0\leqslant y\leqslant 1. \tag{62}$$

From the linearity of the Cauchy problem, it follows that u satisfies (55)–(57). The boundary value problem (59)–(62) is well-posed, and small perturbations in the data functions only lead to small changes in the solution u_2. Thus the general Cauchy problem is separated into a well-posed problem for u_2 and an ill-posed problem for u_1. Note that errors in the boundary values (60)–(62) lead to an error in the solution $u_2(x,0)$, and hence indirectly influence the error level in the data for the ill-posed problem.

It is also of interest to consider problems with Neumann data on the boundaries. Assume that we replace the boundary data (4) by

$$u_x(0,y)=u_x(\pi,y)=0, \quad 0\leqslant y\leqslant 1, \tag{63}$$

a careful examination of the proofs of Sections 2 and 3 shows that we obtain exactly the same stability results.

5. Conclusions and discussion

We have proved the convergence results for a Cauchy problem for the Laplace equation in a rectangle using two different regularization methods. For the perturbation method, the equation is modified by adding a fourth order mixed derivative, with a coefficient that serves as a regularization parameter. It is shown that with a certain choice of the parameter, an explicit error estimate of logarithmic type is obtained. With a stronger assumption on the regularity of the solution, the convergence estimate on the data is obtained for the whole domain (i.e., including $y = 1$). For the truncation method, an explicit convergence estimate of Hölder type is obtained. We also discussed the relation of both methods and gave an interesting result which may be helpful when one consider a similar problem.

From Sections 2 and 3, we could find that the results of the truncation method is better and the procedure of the proof is easier than that of the perturbation method. However, the perturbation method also has its advantage. Actually, from Section 2 we know that the problem (12)–(15) is well posed. That is to say, we can discretize it using standard techniques, e.g., finite differences. For this reason, we can also expect to use the perturbation method to solve some nonconstant coefficient problems (e.g., [27]).

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