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Regularity for harmonic maps into certain pseudo-Riemannian manifolds

Miaomiao Zhu

Mathematics Institute, University of Warwick, CV4 7AL, Coventry, UK

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Abstract

In this article, we investigate the regularity for certain elliptic systems without an L^2-antisymmetric structure. As applications, we prove some regularity results for weakly harmonic maps from the unit ball $B = B(m) \subset \mathbb{R}^m$ $(m \geqslant 2)$ into certain pseudo-Riemannian manifolds.

Résumé

Dans cet article on étudie la régularité des solutions de quelques systèmes elliptiques non munis de structure antisymétrique L^2. On applique cette étude à la démonstration de résultats de régularitté d'applications harmoniques de la boule unité $B = B(m) \subset \mathbb{R}^m$ $(m \geqslant 2)$, dans des variétés pseudo-riemanniennes.

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1. Introduction

In the recent papers by Rivière [36] and Rivière and Struwe [39], the following regularity results for elliptic systems with an L^2-antisymmetric structure are established:

Theorem 1.1. *(See Rivière [36] for $m = 2$, Rivière and Struwe [39] for $m \geqslant 3$.) Let $B = B(m) \subset \mathbb{R}^m$ $(m \geqslant 2)$ be the unit ball. There exists $\epsilon_m > 0$ such that for every $\Omega \in L^2(B, so(n) \otimes \wedge^1 \mathbb{R}^m)$ and for every weak solution $u \in W^{1,2}(B, \mathbb{R}^n)$ of the following elliptic system*:

$$-\operatorname{div} \nabla u = \Omega \cdot \nabla u, \tag{1.1}$$

satisfying

E-mail address: Miaomiao.Zhu@warwick.ac.uk.

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$$\sup_{B_R(x)\subset B}\left(R^{2-m}\int\limits_{B_R(x)}|\nabla u|^2+|\Omega|^2\right)^{\frac{1}{2}}<\epsilon_m, \tag{1.2}$$

we have that u is Hölder continuous in B.

One of the main applications of the above results is the regularity theory for harmonic map systems into closed Riemannian manifolds, where the L^2-antisymmetric property of the potential Ω in (1.1) relies on the fact that the target manifolds are compact and Riemannian. For classical regularity results of weakly harmonic maps, see e.g. the books by Hélein [22] and Lin and Wang [30] and references therein.

In this paper, we shall study the regularity for weakly harmonic maps from the unit ball $B = B(m) \subset \mathbb{R}^m$ $(m \geqslant 2)$ into certain pseudo-Riemannian manifolds from different points of view. Analytically, it is interesting to know how the structure of the harmonic map system is affected when the target manifolds become pseudo-Riemannian. As we will see later, in general, the L^2-antisymmetric structure for harmonic map systems into closed Riemannian manifolds may not be preserved any more when the target manifolds become non-compact or non-Riemannian. Therefore, we would like to explore the extent to which the results developed by Rivière [36] and Rivière and Struwe [39] can be generalized to elliptic systems without an L^2-antisymmetric structure. Geometrically, considering the link between harmonic maps into $\mathbb{S}^4_1 \subset \mathbb{R}^5_1$ and the conformal Gauss maps of Willmore surfaces in $\mathbb{S}^3$ (see Bryant [6]; see also [21,35,3,4]), and the regularity results for weak Willmore immersions established by Rivière [37], we are strongly encouraged to find a method to study the regularity for weakly harmonic maps into $\mathbb{S}^4_1$ and then extend it to the cases of more general targets. Physically, it is known that harmonic maps play an important role in string theory (see e.g. [11,27]). One of the most significant results in string theory is the AdS/CFT correspondence (anti-de-Sitter space/conformal field theory correspondence) proposed in 1997 by Maldacena [31]. In view of the recent work on minimal surfaces in anti-de-Sitter space and its applications in theoretical physics (see e.g. Alday and Maldacena [1]), we are interested in extending the regularity theory for harmonic maps into closed Riemannian manifolds to the cases that the targets are some model spacetimes (which are non-compact and Lorentzian) considered in general relativity (see e.g. [28,34]), for instance, standard stationary Lorentzian manifolds, de-Sitter space $\mathbb{S}^n_1$ (also denoted by dS_n) and anti-de-Sitter space $\mathbb{H}^n_1$ (also denoted by AdS_n).

In the present work, we solve these problems by using the theory of integrability by compensation developed in [47,33,10,14,15] and some conservation laws, due to the symmetries of the target manifolds considered. We point out that our results partially realize the perspectives (proposed by Rivière [37, pp. 3–4]) of the regularity theory for elliptic systems. For some other generalizations of the methods of Rivière [36] and Rivière and Struwe [39], see Lamm and Rivière [29], Struwe [44], Duzaar and Mingione [12] and Rivière [38]. For some other analytic aspects of harmonic maps into pseudo-Riemannian manifolds, see e.g. Hélein [23].

First, we observe that, by slightly adapting the techniques used by Rivière and Struwe [39], similar regularity results as in Theorem 1.1 extend to certain elliptic systems with a potential a priori in L^2 but not necessary antisymmetric. To see this, recall that for $1 \leqslant s < \infty$, the Morrey norm $\|\cdot\|_{M^s_s(B)}$ of a function $f \in L^s_{\mathrm{loc}}(B)$ is

$$\|f\|_{M^s_s(B)}=\sup_{B_R(x)\subset B}\left(R^{s-m}\int\limits_{B_R(x)}|f|^s\right)^{\frac{1}{s}},$$

then we have the following:

Theorem 1.2. *For $m \geqslant 2$ and for any $\Lambda > 0$, there exists $\epsilon_{m,\Lambda} > 0$ such that for every $\Theta \in L^2(B, so(n) \otimes \wedge^1\mathbb{R}^m)$, $\zeta \in W^{1,2}(B, \mathrm{M}(n) \otimes \wedge^2\mathbb{R}^m)$, $F \in W^{1,2} \cap L^\infty(B, \mathrm{M}(n))$, $G \in W^{1,2} \cap L^\infty(B, \mathrm{M}(n))$ and $Q \in W^{1,2} \cap L^\infty(B, \mathrm{GL}(n))$ and for every weak solution $u \in W^{1,2}(B, \mathbb{R}^n)$ of the following elliptic system*:

$$-\operatorname{div}(Q\nabla u)=\Theta\cdot Q\nabla u+F\operatorname{curl}\zeta\cdot G\nabla u \tag{1.3}$$

satisfying

$$\|\nabla u\|_{M^2_2(B)}+\|\Theta\|_{M^2_2(B)}+\|\operatorname{curl}\zeta\|_{M^2_2(B)}+\|\nabla Q\|_{M^2_2(B)}+\|\nabla F\|_{M^2_2(B)}+\|\nabla G\|_{M^2_2(B)}<\epsilon_{m,\Lambda}, \tag{1.4}$$

and

$$|Q| + |Q^{-1}| + |F| + |G| \leqslant \Lambda, \quad a.e.\ in\ B, \tag{1.5}$$

we have that u is Hölder continuous in B.

The result in Theorem 1.2 was partially obtained by Hajlasz, Strzelecki and Zhong [18, Theorem 1.2] for the case $m = 2$, $\Theta \equiv 0$, $Q \equiv I_n$ and by Schikorra [41, Remark 3.4] for the case $m \geqslant 2$, $\zeta \equiv 0$.

Note that the elliptic system (1.3) can be written as

$$-\operatorname{div}(Q\nabla u) = \{\Theta + F \operatorname{curl}\zeta(GQ^{-1})\} \cdot (Q\nabla u) \tag{1.6}$$

or equivalently as

$$-\operatorname{div}\nabla u = \{Q^{-1}\nabla Q + Q^{-1}(\Theta + F \operatorname{curl}\zeta(GQ^{-1}))Q\} \cdot \nabla u. \tag{1.7}$$

Considering Q as a kind of gauge transformation, we interpret the elliptic system (1.7) as follows: its potential

$$Q^{-1}\nabla Q + Q^{-1}(\Theta + F \operatorname{curl}\zeta(GQ^{-1}))Q$$

is gauge equivalent to a new one

$$\Theta + F \operatorname{curl}\zeta(GQ^{-1})$$

which can be decomposed into an antisymmetric part Θ and an almost divergence free part $F \operatorname{curl}\zeta(GQ^{-1})$.

As an application of Theorem 1.2, we shall study the regularity for weakly harmonic maps into standard stationary Lorentzian manifolds. A standard stationary Lorentzian manifold (see e.g. [28,34]) is a product manifold $\mathbb{R} \times M$ equipped with a metric

$$g = -(\beta \circ \pi_M)(\pi_{\mathbb{R}}^* dt + \pi_M^*\omega) \otimes (\pi_{\mathbb{R}}^* dt + \pi_M^*\omega) + \pi_M^* g_M, \tag{1.8}$$

where $(\mathbb{R}, dt^2)$ is the 1-dimensional Euclidean space, (M, g_M) is a closed Riemannian manifold of class C^3, β is a positive C^2 function on M, ω is a C^2 1-form on M, $\pi_{\mathbb{R}}$ and π_M are the natural projections on $\mathbb{R}$ and M, respectively. For simplicity of notations, we shall write the metric (1.8) as

$$g = -\beta(dt + \omega)^2 + g_M. \tag{1.9}$$

By Nash's embedding theorem, we embed (M, g_M) isometrically into some Euclidean space $\mathbb{R}^n$. Then, there exist a tubular neighborhood $V_\delta M$ of radius $\delta > 0$ of M in $\mathbb{R}^n$ and a C^2 projection map Π from $V_\delta M$ to M (see Hélein's book [22, Chapter 1]). Moreover, we pull back β and ω via the projection Π and obtain $\Pi^*\beta \in C^2(V_\delta M, (0, \infty))$ and $\Pi^*\omega \in C^2(\Omega^1(V_\delta M))$, respectively. For simplicity, we shall still denote $\Pi^*\beta$ and $\Pi^*\omega$ by β and ω, respectively. Write $\omega = \sum_{i=1}^n \omega_i(y)\,dy^i$, $y = (y^1, \ldots, y^n) \in V_\delta M \subset \mathbb{R}^n$, where $\omega_i \in C^2(V_\delta M)$.

To study the regularity for weakly harmonic maps into $(\mathbb{R} \times M, g)$, we consider the space

$$W^{1,2}(B, \mathbb{R} \times M) := \{(t, u) \in W^{1,2}(B, \mathbb{R}) \times W^{1,2}(B, \mathbb{R}^n) \mid u(x) \in M \text{ a.e. } x \in B\}. \tag{1.10}$$

For a map $(t, u) \in W^{1,2}(B, \mathbb{R} \times M)$, we define the following Lagrangian:

$$E(t, u) = -\frac{1}{2}\int_B \beta(u)|\nabla t + \omega_i(u)\nabla u^i|^2 + \frac{1}{2}\int_B |\nabla u|^2. \tag{1.11}$$

Definition 1.1. A map $(t, u) \in W^{1,2}(B, \mathbb{R} \times M)$ is called a weakly harmonic map from B into $(\mathbb{R} \times M, g)$, if it is a critical point of the Lagrangian functional (1.11).

The Euler–Lagrange equation (see Section 3) for a weakly harmonic map $(t, u) \in W^{1,2}(B, \mathbb{R} \times M)$ from B into $(\mathbb{R} \times M, g)$ is an elliptic system of the form (1.3), which can be geometrically interpreted as follows: the antisymmetric term Θ corresponds to the Riemannian structure of the closed spacelike hypersurfaces $\{t\} \times M$ and the divergence free term $\operatorname{curl}\zeta$ corresponds to the following conservation law

$$\operatorname{div}\{\beta(u)(\nabla t + \omega_i(u)\nabla u^i)\} = 0, \quad \text{in } \mathcal{D}'(B), \tag{1.12}$$

due to the symmetry of the target generated by the timelike Killing vector field ∂_t. Applying Theorem 1.2, we have the following ϵ-regularity result:

Theorem 1.3. *For $m \geqslant 2$, there exists $\epsilon_m > 0$ depending on $(\mathbb{R} \times M, g)$ such that any weakly harmonic map $(t, u) \in W^{1,2}(B, \mathbb{R} \times M)$ from B into $(\mathbb{R} \times M, g)$ satisfying*

$$\|\nabla t\|_{M_2^2(B)} + \|\nabla u\|_{M_2^2(B)} < \epsilon_m, \tag{1.13}$$

is Hölder continuous (and as smooth as the regularity of the target permits) in B.

In dimension $m = 2$, we notice that the Morrey norm $\|\cdot\|_{M_2^2}$ reduces to the norm $\|\cdot\|_{L^2}$. Therefore, by conformal invariance and rescaling in the domain, we obtain the following regularity result:

Theorem 1.4. *For $m = 2$, any weakly harmonic map $(t, u) \in W^{1,2}(B, \mathbb{R} \times M)$ from B into $(\mathbb{R} \times M, g)$ is Hölder continuous (and as smooth as the regularity of the target permits) in B.*

In Theorem 1.4, if the target $(\mathbb{R} \times M, g)$ is a standard static Lorentzian manifold (see e.g. [28,34]), namely, the 1-form ω in the metric g (see (1.9)) vanishes identically, then the corresponding regularity result was proved by Isobe [24] (using Hélein's method of moving frame [22]).

Next, we shall consider, in a certain sense, elliptic systems of the form (1.1) with the potential Ω a priori only in L^p for some $1 < p < 2$. Note that, if Ω is not in L^2, then the right hand side of (1.1) is not in L^1 and thus the equation makes no sense any more (not even in the distribution sense!). However, we observe that, if in addition, Ω is divergence free, namely,

$$\operatorname{div} \Omega = 0, \quad \text{in } \mathcal{D}'(B), \tag{1.14}$$

then Eq. (1.1) can be written in the following form:

$$-\operatorname{div}(\nabla u + \Omega u) = 0, \quad \text{in } \mathcal{D}'(B). \tag{1.15}$$

This new form (1.15) has the advantage that it is still meaningful in the distribution sense if Ω is a priori only in L^p for some $1 < p < 2$. Moreover, under the further assumption that the Morrey norms $\|\nabla u\|_{M_p^p(B)}$ and $\|\Omega\|_{M_p^p(B)}$ are sufficiently small, the Hölder continuity of the weak solution u holds.

Theorem 1.5. *For $m \geqslant 2$ and for any $1 < p < \frac{m}{m-1}$, there exists $\epsilon_{m,p} > 0$ such that for any $\Omega \in L^p(B, \mathrm{M}(n) \otimes \wedge^1 \mathbb{R}^m)$ satisfying (1.14) and for any weak solution $u \in W^{1,2}(B, \mathbb{R}^n)$ of the elliptic system (1.15) satisfying*

$$\|\nabla u\|_{M_p^p(B)} + \|\Omega\|_{M_p^p(B)} < \epsilon_{m,p}, \tag{1.16}$$

we have that u is Hölder continuous in B.

As applications of Theorem 1.5, we shall study the regularity for weakly harmonic maps into pseudospheres and pseudohyperbolic spaces. For this purpose, we recall some facts about these target spaces and refer to O'Neill's book [34] for more details.

Let $n \in \mathbb{N}$ and let $\nu \in \mathbb{N}$ satisfy $0 \leqslant \nu \leqslant n$. Denote

$$\mathcal{E} = (\varepsilon_{ij}) := \begin{pmatrix} -I_\nu & 0 \\ 0 & I_{n+1-\nu} \end{pmatrix}. \tag{1.17}$$

The pseudo-Euclidean space $\mathbb{R}^{n+1}_\nu$ of signature $(\nu, n+1-\nu)$ is the space $\mathbb{R}^{n+1}$ equipped with a metric

$$\langle v, w\rangle_{\mathbb{R}^{n+1}_\nu} := v^T \mathcal{E} w = -\left(v^1 w^1 + \cdots + v^\nu w^\nu\right) + \left(v^{\nu+1} w^{\nu+1} + \cdots + v^{n+1} w^{n+1}\right)$$

for all $v = (v^1, \dots, v^{n+1})^T \in \mathbb{R}^{n+1}$ and $w = (w^1, \dots, w^{n+1})^T \in \mathbb{R}^{n+1}$. The pseudoshpere $\mathbb{S}^n_\nu$ in $\mathbb{R}^{n+1}_\nu$ is defined as

$$\mathbb{S}^n_\nu := \left\{ y \in \mathbb{R}^{n+1}_\nu \;\middle|\; \langle y, y\rangle_{\mathbb{R}^{n+1}_\nu} = y^T \mathcal{E} y = 1 \right\} \tag{1.18}$$

with the induced metric. In particular, $\mathbb{S}^n_0 \subset \mathbb{R}^{n+1}_0$ is the standard sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and $\mathbb{S}^n_1 \subset \mathbb{R}^{n+1}_1$ is the de-Sitter space dS_n in general relativity. The linear isometries of $\mathbb{R}^{n+1}_\nu$ form the group

$$\mathrm{O}(\nu, n+1-\nu) = \left\{ P \in \mathrm{GL}(n+1) \;\middle|\; P^T = \mathcal{E} P^{-1} \mathcal{E} \right\}. \tag{1.19}$$

Denote by $\mathrm{SO}^+(\nu, n+1-\nu)$ the identity component of $\mathrm{O}(\nu, n+1-\nu)$. The lie algebra of $\mathrm{SO}^+(\nu, n+1-\nu)$ is

$$so(\nu, n+1-\nu) = \left\{ A \in \mathrm{GL}(n+1) \;\middle|\; A^T = -\mathcal{E} A \mathcal{E} \right\}. \tag{1.20}$$

Using the isometric embedding $\mathbb{S}^n_\nu \subset \mathbb{R}^{n+1}_\nu$, we set

$$W^{1,2}\big(B, \mathbb{S}^n_\nu\big) := \left\{ u = \big(u^1, u^2, \ldots, u^{n+1}\big)^T \in W^{1,2}\big(B, \mathbb{R}^{n+1}_\nu\big) \;\middle|\; u^T \mathcal{E} u = 1 \text{ a.e. in } B \right\}. \tag{1.21}$$

For a map $u \in W^{1,2}(B, \mathbb{S}^n_\nu)$, we define the following Lagrangian:

$$E(u) := \frac{1}{2} \int_B (\nabla u)^T \mathcal{E} \nabla u = -\frac{1}{2} \int_B \big(|\nabla u^1|^2 + \cdots + |\nabla u^\nu|^2 \big) + \frac{1}{2} \int_B \big(|\nabla u^{\nu+1}|^2 + \cdots + |\nabla u^{n+1}|^2 \big). \tag{1.22}$$

Definition 1.2. A map $u \in W^{1,2}(B, \mathbb{S}^n_\nu)$ is called a weakly harmonic map from B into $\mathbb{S}^n_\nu$, if it is a critical point of the Lagrangian functional (1.22).

Denote

$$\mathcal{F} = (\varsigma_{ij}) := \begin{pmatrix} -I_{\nu+1} & 0 \\ 0 & I_{n-\nu} \end{pmatrix}. \tag{1.23}$$

The pseudohyperbolic space $\mathbb{H}^n_\nu$ in $\mathbb{R}^{n+1}_{\nu+1}$ is defined as

$$\mathbb{H}^n_\nu := \left\{ y \in \mathbb{R}^{n+1}_{\nu+1} \;\middle|\; \langle y, y \rangle_{\mathbb{R}^{n+1}_{\nu+1}} = y^T \mathcal{F} y = -1 \right\} \tag{1.24}$$

with the induced metric. In particular, $\mathbb{H}^n_0 \subset \mathbb{R}^{n+1}_1$ is a hyperboloid containing two copies of the hyperbolic space $\mathbb{H}^n$ and $\mathbb{H}^n_1 \subset \mathbb{R}^{n+1}_2$ is the anti-de-Sitter space AdS_n in general relativity.

Using the isometric embedding $\mathbb{H}^n_\nu \subset \mathbb{R}^{n+1}_{\nu+1}$, we set

$$W^{1,2}\big(B, \mathbb{H}^n_\nu\big) := \left\{ u = \big(u^1, u^2, \ldots, u^{n+1}\big)^T \in W^{1,2}\big(B, \mathbb{R}^{n+1}_{\nu+1}\big) \;\middle|\; u^T \mathcal{F} u = -1 \text{ a.e. in } B \right\}. \tag{1.25}$$

For a map $u \in W^{1,2}(B, \mathbb{H}^n_\nu)$, we define the following Lagrangian:

$$E(u) := \frac{1}{2} \int_B (\nabla u)^T \mathcal{F} \nabla u = -\frac{1}{2} \int_B \big(|\nabla u^1|^2 + \cdots + |\nabla u^{\nu+1}|^2 \big) + \frac{1}{2} \int_B \big(|\nabla u^{\nu+2}|^2 + \cdots + |\nabla u^{n+1}|^2 \big). \tag{1.26}$$

Definition 1.3. A map $u \in W^{1,2}(B, \mathbb{H}^n_\nu)$ is called a weakly harmonic map from B into $\mathbb{H}^n_\nu$, if it is a critical point of the Lagrangian functional (1.26).

Notice that the following anti-isometry (see O'Neill's book [34])

$$\begin{aligned} \sigma: \qquad & \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_{n-\nu+1} \\ & (y_1, \ldots, y_{n+1}) \mapsto (y_{\nu+1}, \ldots, y_{n+1}, y_1, \ldots, y_\nu) \end{aligned}$$

induces an anti-isometry from $\mathbb{S}^n_\nu$ to $\mathbb{H}^n_{n-\nu}$. In the sequel, we shall only consider the cases of $\mathbb{S}^n_\nu$ $(0 \leqslant \nu \leqslant n)$.

To proceed, we recall that a weakly harmonic map $u \in W^{1,2}(B, \mathbb{S}^n)$ satisfies the following conservation laws (due to Shatah [42] and Chen [9]. See also Rubinstein, Sternberg and Keller [40] and Hélein's book [22]):

$$\operatorname{div}\big(u^i \nabla u^j - u^j \nabla u^i\big) = 0, \quad \text{in } \mathcal{D}'(B),\ \forall i, j = 1, 2, \ldots, n+1, \tag{1.27}$$

which can be interpreted by Noether theorem, using the symmetries of $\mathbb{S}^n$. Note that the pseudospheres $\mathbb{S}^n_\nu$ $(1 \leqslant \nu \leqslant n)$ have isometry groups $\mathrm{O}(\nu, n+1-\nu)$ and hence they are all maximally symmetric. With the help of the symmetric properties, we are able to extend the conservation laws (1.27) to weakly harmonic maps into these more general targets.

Proposition 1.1. *Let $m \geqslant 2$. Let $u \in W^{1,2}(B, \mathbb{S}^n_\nu)$ $(1 \leqslant \nu \leqslant n)$ be a weakly harmonic map. Then the conservation laws* (1.27) *hold.*

For a weakly harmonic map $u \in W^{1,2}(B, \mathbb{S}^n_\nu)$ $(0 \leqslant \nu \leqslant n)$, we define the following matrix valued vector field

$$\Theta = (\Theta^{ij}) := (u^i \nabla u^j - u^j \nabla u^i), \quad i, j = 1, 2, \ldots, n+1. \tag{1.28}$$

In the case of a compact target $\mathbb{S}^n$, Θ is in $L^2(B, \mathrm{M}(n) \otimes \wedge^1 \mathbb{R}^m)$ and u weakly solves the following elliptic system (see Hélein [19,22])

$$-\operatorname{div} \nabla u = \Theta \cdot \nabla u.$$

Since Θ is divergence free (due to the conservation laws (1.27)), the continuity of u in dimension $m = 2$ follows immediately from Wente's lemma [47].

However, in the case of a non-compact target $\mathbb{S}^n_\nu$ $(1 \leqslant \nu \leqslant n)$, Θ is only in $L^p(B, \mathrm{M}(n) \otimes \wedge^1 \mathbb{R}^m)$ for any $1 < p < 2$. Proposition 1.1 indicates that Θ is still divergence free. In what follows, we show that u is a weak solution of an elliptic system of the form (1.15) with its potential satisfying (1.14). Moreover, by making use of the conservation laws (1.27), we are able to estimate $\|\Theta\|_{M^p_p(B_{1/2})}$ by $\|\nabla u\|_{M^p_p(B)}$, where $B_{1/2} = B_{1/2}(m) \subset \mathbb{R}^m$ $(m \geqslant 2)$ is the ball centered at 0 and of radius $1/2$.

Proposition 1.2. *Let $m \geqslant 2$. Let $u \in W^{1,2}(B, \mathbb{S}^n_\nu)$ $(1 \leqslant \nu \leqslant n)$. Then*

$$\nabla u + \Theta \mathcal{E} u = 0, \quad \textit{a.e. in } B, \tag{1.29}$$

where Θ is defined as in (1.28). *Consequently, we have*

$$-\operatorname{div}(\nabla u + \Theta \mathcal{E} u) = 0 \quad \textit{in } \mathcal{D}'(B). \tag{1.30}$$

Furthermore, suppose that u is weakly harmonic and for any fixed $1 < p < \frac{m}{m-1}$ there holds $\|\nabla u\|_{M^p_p(B)} < \infty$, then we have the following estimate:

$$\|\Theta\|_{M^p_p(B_{1/2})} \leqslant C \|\nabla u\|^2_{M^p_p(B)}. \tag{1.31}$$

Since $\mathcal{E}$ is a constant matrix, applying Theorem 1.5 with $\Omega = \Theta \mathcal{E}$ and using a rescaling of the domain gives the following ϵ-regularity result:

Theorem 1.6. *For $m \geqslant 2$ and for any $1 < p < \frac{m}{m-1}$, there exists $\epsilon_{m,p} > 0$ such that any weakly harmonic map $u \in W^{1,2}(B, \mathbb{S}^n_\nu)$ $(1 \leqslant \nu \leqslant n)$ satisfying*

$$\|\nabla u\|^2_{M^p_p(B)} < \epsilon_{m,p} \tag{1.32}$$

is Hölder continuous (*and hence smooth*) *in B.*

In dimension $m = 2$, a straightforward calculation gives that $\|\nabla u\|_{M^p_p(B)} \leqslant \|\nabla u\|_{L^2(B)}$ for any $1 < p < 2$. Therefore, by conformal invariance, we have

Theorem 1.7. *For $m = 2$, any weakly harmonic map $u \in W^{1,2}(B, \mathbb{S}^n_\nu)$ $(1 \leqslant \nu \leqslant n)$ is Hölder continuous* (*and hence smooth*) *in B.*

In particular, we prove that any weakly harmonic map from a disc into the de-Sitter space $\mathbb{S}^n_1$ or the anti-de-Sitter space $\mathbb{H}^n_1 \cong \mathbb{S}^n_{n-1}$ is smooth. Also, we give an alternative proof of the Hölder continuity of weakly harmonic maps from a disc into the hyperbolic space $\mathbb{H}^n$ (one component of $\mathbb{H}^n_0 \cong \mathbb{S}^n_n$) without using the fact that the target has non-positive sectional curvature (for a proof using the curvature property, we refer to Jost's book [26]). We expect that the results in Theorems 1.6 and 1.7 can be extended in the same spirit of Hélein's setting in [20] to certain homogeneous pseudo-Riemannian manifolds.

Furthermore, we observe that the methods used in the proofs of Proposition 1.2 and Theorem 1.5 can be applied to study the ϵ-regularity of maps in the spaces of distributions of lower regularity. This motivates us to extend the notion of generalized (weakly) harmonic maps from a disc into the standard sphere $\mathbb{S}^n$ (introduced by Almeida [2]) to the cases that the targets are pseudospheres $\mathbb{S}^n_\nu$ $(1 \leqslant \nu \leqslant n)$ (see Section 5). To see this, we recall the notion of generalized (weakly) harmonic maps into $\mathbb{S}^n$.

Definition 1.4. (See Almeida [2].) For $m = 2$, a map $u \in W^{1,1}(B, \mathbb{S}^n)$ is called a generalized (weakly) harmonic map if (1.27) hold.

Generalized (weakly) harmonic maps into $\mathbb{S}^n$ might be not continuous. However, there are some ϵ-regularity results for such maps. Almeida [2] showed that any generalized harmonic map $u \in W^{1,1}(B, \mathbb{S}^n)$ with $\|\nabla u\|_{L^{(2,\infty)}}$ small is smooth (an alternative proof was given by Ge [16]). Moser [32] proved that any generalized harmonic map $u \in W^{1,p}_{\mathrm{loc}}(B, \mathbb{S}^n)$ with $p \in (1, 2)$ is smooth if p is sufficiently close to 2, and $\|u\|_{\mathrm{BMO}}$ is small. Strzelecki [46] showed that any generalized harmonic map $u \in W^{1,p}_{\mathrm{loc}}(B, \mathbb{S}^n)$ with $p \in (1, 2)$ is smooth provided that $\|u\|_{\mathrm{BMO}}$ is small.

To extend the notion of generalized (weakly) harmonic maps into the pseudospheres $\mathbb{S}^n_\nu$ $(1 \leqslant \nu \leqslant n)$, we observe that a $W^{1,1}$ map from a disc into any of these non-compact targets is not a priori in L^∞ and hence the conservation laws (1.27) make no sense for such a map. Therefore, we need to require that the map u belongs to the Sobolev space $W^{1,\frac{4}{3}}$ so that

$$u^i \nabla u^j - u^j \nabla u^i \in L^1_{\mathrm{loc}}(B), \quad \forall i, j = 1, 2, \ldots, n+1,$$

and hence the conservation laws (1.27) become meaningful.

Definition 1.5. For $m = 2$, a map $u \in W^{1,\frac{4}{3}}(B, \mathbb{S}^n_\nu)$ $(1 \leqslant \nu \leqslant n)$ is called a generalized (weakly) harmonic map if (1.27) hold.

Analogously to Theorem 1.6, we have the following ϵ-regularity result.

Theorem 1.8. *For $m = 2$ and for any $\frac{4}{3} < p < 2$, there exists $\epsilon_p > 0$ such that any generalized* (*weakly*) *harmonic map $u \in W^{1,\frac{4}{3}}(B, \mathbb{S}^n_\nu)$ $(1 \leqslant \nu \leqslant n)$ satisfying*

$$\|\nabla u\|^2_{M^p_p(B)} < \epsilon_p \tag{1.33}$$

is Hölder continuous (*and hence smooth*) *in B.*

Finally, we study the regularity for an elliptic system of the form (1.1) with $\Omega \in L^2(B, so(1, 1) \otimes \wedge^1\mathbb{R}^2)$ in dimension $m = 2$ and show by constructing an example that weak solutions in $W^{1,2}$ to such an elliptic system might be not in L^∞.

The paper is organized as follows. In Section 2, we prove Theorems 1.2 and 1.5. In Section 3, we apply Theorem 1.2 to prove the ϵ-regularity (Theorem 1.3) of weakly harmonic maps into standard stationary Lorentzian manifolds. In Section 4, we first show Propositions 1.1 and 1.2. Then we prove the regularity results (Theorems 1.6 and 1.7) for weakly harmonic maps into pseudospheres. In Section 5, the ϵ-regularity result (Theorem 1.8) for generalized (weakly) harmonic maps from a disc into pseudospheres is proved. In Section 6, we study an elliptic system with an L^2-$so(1, 1)$ structure in dimension $m = 2$.

Notation. For a 2-vector field $\xi = \xi_{ij}\partial_{x_i} \wedge \partial_{x_j}$, $\operatorname{curl}\xi$ denotes the vector field $(\sum_i \partial_{x_i}\xi_{ij})\partial_{x_j}$ and $d\xi$ denotes the 3-vector field $(\partial_{x_k}\xi_{ij})\partial_{x_k} \wedge \partial_{x_i} \wedge \partial_{x_j}$. A constant C may depend on m, n and p.

2. Proofs of Theorems 1.2 and 1.5

In this section, we will prove Theorems 1.2 and 1.5.

First, combining the div-curl inequality by Coifman, Lions, Meyer and Semmes [10] (see Müller [33] for an earlier contribution), the Hardy-BMO duality by Fefferman [14] (see also Fefferman and Stein [15] and Stein [43]) and the observation that the Morrey spaces $M^s_s(\mathbb{R}^m)$ ($1 \leqslant s < \infty$) are contained in the space $\mathrm{BMO}(\mathbb{R}^m)$ (due to Evans [13]), we give the following lemma (see Proposition III.2 in Bethuel [5], Lemma 3.1 in Schikorra [41] and Strzelecki [45, pp. 234–235]; see also Chanillo [7] and Chanillo and Li [8]).

Lemma 2.1. *Let $m \geqslant 2$, $1 \leqslant s < \infty$ and $1 < p < \infty$. Let $1 < q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For any ball $B_R(x) \subset \mathbb{R}^m$, $f \in W^{1,p}(B_R(x))$, $g \in W^{1,q}(B_R(x), \wedge^2\mathbb{R}^m)$ satisfying*

$$f|_{\partial B_R(x)} = 0 \quad \text{or} \quad g|_{\partial B_R(x)} = 0, \tag{2.1}$$

and $h \in W^{1,s}(B_{2R}(x))$ satisfying

$$\|\nabla h\|_{M^s_s(B_{2R}(x))} < \infty, \tag{2.2}$$

there holds:

$$\int\limits_{B_R(x)} (\nabla f \cdot \operatorname{curl} g)h \leqslant C\|\nabla f\|_{L^p(B_R(x))}\|\operatorname{curl} g\|_{L^q(B_R(x))}\|\nabla h\|_{M^s_s(B_{2R}(x))}, \tag{2.3}$$

where $C = C_{m,s,p} > 0$ is a uniform constant independent of $R > 0$.

Next, with the help of the above lemma, we follow the approach used by Rivière and Struwe [39] to prove Theorems 1.2 and 1.5.

Proof of Theorem 1.2. Fix $m \geqslant 2$ and $\Lambda > 0$. Choose $\epsilon_{m,\Lambda} > 0$ sufficiently small, then by assumption (1.4) and the existence of Coulomb gauge (due to Rivière [36] for $m = 2$ and Rivière and Struwe [39] for $m \geqslant 3$), we conclude that there are $P \in W^{1,2}(B, \mathrm{SO}(n))$ and $\xi \in W^{1,2}_0(B, so(n) \otimes \wedge^2\mathbb{R}^m)$ with $d\xi = 0$ such that

$$P^{-1}\nabla P + P^{-1}\Theta P = \operatorname{curl}\xi \quad \text{in } B, \tag{2.4}$$

and the following estimate holds

$$\|\nabla P\|_{M^2_2(B)} + \|\nabla\xi\|_{M^2_2(B)} \leqslant C\|\Theta\|_{M^2_2(B)} \leqslant C\epsilon_{m,\Lambda}. \tag{2.5}$$

Using (2.4), we rewrite the system (1.3) as

$$\begin{aligned} -\operatorname{div}\big(P^{-1}Q\nabla u\big) &= \big(P^{-1}\nabla P + P^{-1}\Theta P\big)\cdot P^{-1}Q\nabla u + P^{-1}F\operatorname{curl}\zeta G\cdot\nabla u \\ &= \operatorname{curl}\xi\cdot P^{-1}Q\nabla u + P^{-1}F\operatorname{curl}\zeta G\cdot\nabla u. \end{aligned} \tag{2.6}$$

Write $P^{-1} = (P_{ij})$, $\Theta = (\Theta^{ij})$, $\zeta = (\zeta^{ij})$, $F = (F^{ij})$, $G = (G^{ij})$ and $Q = (Q^{ij})$. Then the above equation can be written as

$$\begin{aligned} -\operatorname{div}\Big(\sum_{j,k} P_{ij}Q^{jk}\nabla u^k\Big) &= \sum_{j,k,l}\operatorname{curl}\xi^{ij}\cdot P_{jk}Q^{kl}\nabla u^l + \sum_{j,k,l,r} P_{ij}F^{jk}\operatorname{curl}\zeta^{kl}\cdot G^{lr}\nabla u^r \\ &= \sum_{j,k,l} P_{jk}Q^{kl}\operatorname{curl}\xi^{ij}\cdot\nabla u^l + \sum_{j,k,l,r} P_{ij}F^{jk}G^{lr}\operatorname{curl}\zeta^{kl}\cdot\nabla u^r. \end{aligned} \tag{2.7}$$

Since $P \in W^{1,2}(B, \mathrm{SO}(n))$, $F \in W^{1,2} \cap L^\infty(B, \mathrm{M}(n))$, $G \in W^{1,2} \cap L^\infty(B, \mathrm{M}(n))$ and $Q \in W^{1,2} \cap L^\infty(B, \mathrm{GL}(n))$, we have $P^{-1}Q \in W^{1,2} \cap L^\infty(B, \mathrm{GL}(n))$, $P_{ij}F^{jk}G^{lr} \in W^{1,2} \cap L^\infty(B)$. Using the assumption (1.5), one can verify that

$$\begin{aligned} &\big\|\nabla\big(P^{-1}Q\big)\big\|_{M^2_2(B)} + \sum_{i,k,l,r}\big\|\nabla\big(P_{ij}F^{jk}G^{lr}\big)\big\|_{M^2_2(B)} \\ &\quad\leqslant C(\Lambda)\big(\|\nabla P\|_{M^2_2(B)} + \|\nabla Q\|_{M^2_2(B)} + \|\nabla F\|_{M^2_2(B)} + \|\nabla G\|_{M^2_2(B)}\big). \end{aligned} \tag{2.8}$$

Here and in the sequel, $C(\Lambda) > 0$ is a constant also depending on Λ.

Combining (2.5), (2.8) and assumption (1.4), we get

$$\|\nabla u\|_{M_2^2(B)} + \|\nabla(P^{-1}Q)\|_{M_2^2(B)} + \sum_{i,j,k,l,r} \|\nabla(P_{ij}F^{jk}G^{lr})\|_{M_2^2(B)} + \|\nabla\xi\|_{M_2^2(B)} + \|\operatorname{curl}\zeta\|_{M_2^2(B)} \leqslant C(\Lambda)\epsilon_{m,\Lambda}. \tag{2.9}$$

On the other hand, since P^{-1} takes values in $\mathrm{SO}(n)$, it follows from assumption (1.5) that

$$C(\Lambda)^{-1}|\nabla u| \leqslant |P^{-1}Q\nabla u| = |Q\nabla u| \leqslant C(\Lambda)|\nabla u|, \quad \text{a.e. in } B. \tag{2.10}$$

Similarly to the approach by Rivière and Struwe [39, Section 3, Proof of Theorem 1.1, pp. 459–460] (see also Schikorra [41, pp. 510–511]), we apply Hodge decomposition (see [25]) to $P^{-1}Q\nabla u$, use (2.7), (2.9), (2.10), Lemma 2.1, and take $\epsilon_{m,\Lambda} > 0$ sufficiently small to get the Morrey type estimates for ∇u. Finally, we apply an iteration argument as in [17] to obtain the Hölder continuity of u in B. □

Proof of Theorem 1.5. Fix any $1 < p < \frac{m}{m-1}$. Since $\operatorname{div}\Omega = 0$, by Hodge decomposition, there exists $\xi \in W^{1,p}(B, \mathrm{M}(n) \otimes \wedge^2\mathbb{R}^m)$ such that

$$\Omega = \operatorname{curl}\xi. \tag{2.11}$$

Let $B_{2R}(x_0) \subset B$ and let $w \in W^{1,2}(B_R(x_0), \mathbb{R}^n)$ be solving

$$\begin{cases} -\operatorname{div}\nabla w = 0, & \text{in } B_R(x_0), \\ w = u, & \text{on } \partial B_R(x_0). \end{cases} \tag{2.12}$$

Then $v := u - w \in W^{1,2}(B_R(x_0), \mathbb{R}^n)$ solves

$$\begin{cases} -\operatorname{div}(\nabla v + \Omega u) = 0, & \text{in } B_R(x_0), \\ v = 0, & \text{on } \partial B_R(x_0). \end{cases} \tag{2.13}$$

Let $q = \frac{p}{p-1} > m$ be the conjugate exponent of p. For any $\varphi \in W_0^{1,q}(B_R(x_0))$ with $\|\varphi\|_{W^{1,q}(B_R(x_0))} \leqslant 1$, using assumption (1.16), Lemma 2.1, (2.11) and (2.13), we estimate for each i,

$$\begin{aligned} \int_{B_R(x_0)} \nabla v^i \cdot \nabla\varphi &= -\int_{B_R(x_0)} (\Omega^{ij}u^j) \cdot \nabla\varphi \\ &= -\int_{B_R(x_0)} (\operatorname{curl}\xi^{ij} \cdot \nabla\varphi)u^j \\ &\leqslant C\|\operatorname{curl}\xi^{ij}\|_{L^p(B_R(x_0))}\|\nabla\varphi\|_{L^q(B_R(x_0))}\|\nabla u\|_{M_p^p(B_{2R}(x_0))} \\ &\leqslant C\|\Omega^{ij}\|_{L^p(B_R(x_0))}\|\nabla u\|_{M_p^p(B_{2R}(x_0))} \\ &\leqslant CR^{\frac{m}{p}-1}\|\Omega^{ij}\|_{M_p^p(B)}\|\nabla u\|_{M_p^p(B_{2R}(x_0))} \\ &\leqslant CR^{\frac{m}{p}-1}\epsilon_{m,p}\|\nabla u\|_{M_p^p(B_{2R}(x_0))}. \end{aligned} \tag{2.14}$$

Since $v|_{\partial B_R(x_0)} = 0$, by duality (similarly to Rivière and Struwe [39]) there holds:

$$\|\nabla v\|_{L^p(B_R(x_0))} \leqslant C \sup_{\varphi \in W_0^{1,q}(B_R(x_0)),\, \|\varphi\|_{W^{1,q}} \leqslant 1} \int_{B_R(x_0)} \nabla v \cdot \nabla\varphi. \tag{2.15}$$

Combining (2.14) and (2.15) gives

$$\|\nabla v\|_{L^p(B_R(x_0))} \leqslant CR^{\frac{m}{p}-1}\epsilon_{m,p}\|\nabla u\|_{M_p^p(B_{2R}(x_0))}. \tag{2.16}$$

Next, we see from (2.12) that w is harmonic in $B_R(x_0)$ and hence ∇w is also harmonic in $B_R(x_0)$. By Campanato estimates for harmonic functions (see [17]), we have that for any $r < R$ the following holds:

$$\int_{B_r(x_0)} |\nabla w|^p \leqslant C\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |\nabla w|^p. \tag{2.17}$$

Using that fact that $u = v + w$ and combining (2.16), (2.17), we estimate

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla u|^p &\leqslant C \int_{B_r(x_0)} |\nabla w|^p + C \int_{B_r(x_0)} |\nabla v|^p \\
&\leqslant C\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |\nabla w|^p + C \int_{B_R(x_0)} |\nabla v|^p \\
&\leqslant C\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |\nabla u|^p + C \int_{B_R(x_0)} |\nabla v|^p \\
&\leqslant C\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |\nabla u|^p + CR^{m-p}(\epsilon_{m,p})^p \|\nabla u\|^p_{M^p_p(B_{2R}(x_0))}.
\end{aligned} \tag{2.18}$$

For the rest of the proof, we can apply the same arguments as in Schikorra [41, pp. 510–511], similarly to Rivière and Struwe [39, Proof of Theorem 1.1, pp. 459–460], to obtain the Morrey type estimates for ∇u. The Hölder continuity of u in B follows immediately from an iteration argument as in [17]. □

Remark 2.1. By slightly modifying the proof, we will see that the regularity result in Theorem 1.5 still holds if the elliptic system (1.15) is replaced by the following:

$$-\operatorname{div}(Q\nabla u + \Omega u) = 0, \quad \text{in } \mathcal{D}'(B) \tag{2.19}$$

with $Q \in W^{1,2} \cap L^\infty(B, \mathrm{GL}(n))$ satisfying $|Q| + |Q^{-1}| \leqslant \Lambda$, a.e. in B, for some uniform constant $\Lambda > 0$. The proof relies on applying Hodge decomposition to $Q\nabla u$ to get the Morrey type estimates for ∇u as is done by Rivière and Struwe in [39]. (See also Schikorra [41].)

3. Harmonic maps into standard stationary Lorentzian manifolds

In this section, we shall first show that the Euler–Lagrangian equations for weakly harmonic maps into standard stationary Lorentzian manifolds are elliptic systems of the form (1.3) and then apply Theorem 1.2 to prove the ϵ-regularity (Theorem 1.3) for such maps.

Proof of Theorem 1.3. Let $(t, u) \in W^{1,2}(B, \mathbb{R} \times M)$ be a weakly harmonic map from B into $(\mathbb{R} \times M, g)$, where the metric g is defined as in (1.9). For any $s \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R})$ and for any $v \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}^n)$, we have that

$$t_\epsilon = t + \epsilon s \quad \text{and} \quad u_\epsilon = \Pi(u + \epsilon v) \tag{3.1}$$

are well defined for sufficiently small $\epsilon > 0$. Hence $(t_\epsilon, u_\epsilon) \in W^{1,2}(B, \mathbb{R} \times M)$ gives an admissible variation for (t, u). By Definition 1.1, there holds

$$\frac{d}{d\epsilon} E(t_\epsilon, u_\epsilon)\bigg|_{\epsilon=0} = 0, \quad \forall s \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}),\ \forall v \in W^{1,2}_0 \cap L^\infty\big(B, \mathbb{R}^n\big). \tag{3.2}$$

A straightforward calculation gives

$$\int_B \left\{-\frac{1}{2}(\nabla\beta \cdot w)\big|\nabla t + \omega_i \nabla u^i\big|^2 - \beta\big(\nabla t + \omega_i \nabla u^i\big) \cdot \big(\nabla s + \omega_j \nabla w^j + (\nabla\omega_k \cdot w)\nabla u^k\big) + \nabla u \cdot \nabla w\right\} = 0, \tag{3.3}$$

where $w = d\Pi(u)v$, $v \in W^{1,2}_0 \cap L^\infty(B, \mathbb{R}^n)$.

To deduce the Euler–Lagrangian equations, we shall choose appropriate admissible variations in (3.3).

First, taking $s \in W_0^{1,2} \cap L^\infty(B)$ and $v \equiv 0$ in (3.3), we obtain

$$0 = \int_B -\beta(u)\big(\nabla t + \omega_i(u)\nabla v^i\big) \cdot \nabla s.$$

Since $s \in W_0^{1,2} \cap L^\infty(B)$ is arbitrarily chosen, we get the following conservation law

$$-\operatorname{div}\big\{\beta(u)\big(\nabla t + \omega_i(u)\nabla u^i\big)\big\} = 0. \tag{3.4}$$

Next, taking $w = d\Pi(u)v$, $v \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^n)$ and $s \equiv 0$ in (3.3) gives

$$\begin{aligned}
0 &= \int_B \left\{ -\frac{1}{2}(\nabla\beta \cdot w)\big|\nabla t + \omega_i \nabla u^i\big|^2 - \beta\big(\nabla t + \omega_i \nabla u^i\big) \cdot \big(\omega_j \nabla w^j + (\nabla\omega_k \cdot w)\nabla u^k\big) + \nabla u \cdot \nabla w \right\} \\
&= \int_B \left\{ -\frac{1}{2}\left(\frac{\partial\beta}{\partial y^j} \cdot w^j\right)\big|\nabla t + \omega_i \nabla u^i\big|^2 - \beta\big(\nabla t + \omega_i \nabla u^i\big) \cdot \left(\omega_j \nabla w^j + \nabla u^k \frac{\partial\omega_k}{\partial y^j} \cdot w^j\right) + \nabla u \cdot \nabla w \right\} \\
&= \int_B \left\{ -\operatorname{div}\big(\nabla u^j\big) \cdot w^j + \beta\big(\nabla t + \omega_i \nabla u^i\big) \cdot \nabla u^k \left(\frac{\partial\omega_j}{\partial y^k} - \frac{\partial\omega_k}{\partial y^j}\right) \cdot w^j - \frac{1}{2}\big|\nabla t + \omega_i \nabla u^i\big|^2 \frac{\partial\beta}{\partial y^j} \cdot w^j \right\}.
\end{aligned} \tag{3.5}$$

where in the last step we have used (3.4) and integration by parts. Denote $H := (H^1, \ldots, H^n)$ with

$$H^j := \beta\big(\nabla t + \omega_i \nabla u^i\big) \cdot \nabla u^k \left(\frac{\partial\omega_j}{\partial y^k} - \frac{\partial\omega_k}{\partial y^j}\right) - \frac{1}{2}\frac{\partial\beta}{\partial y^j}\big|\nabla t + \omega_i \nabla u^i\big|^2. \tag{3.6}$$

Then (3.5) becomes

$$0 = \int_B (-\operatorname{div}\nabla u + H) \cdot d\Pi(u)v, \quad \forall v \in W_0^{1,2} \cap L^\infty\big(B, \mathbb{R}^n\big).$$

Since $v \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^n)$ is arbitrarily chosen, we have (similarly to the calculations in [20, Chapter 1])

$$-\operatorname{div}\nabla u - A(u)(\nabla u, \nabla u) + d\Pi(u)H = 0, \tag{3.7}$$

where A is the second fundamental form of M in $\mathbb{R}^n$. Let ν_l, $l = d+1, d+2, \ldots, n$, be an orthonormal frame for the normal bundle $T^\perp M$ (and still denote by ν_l the corresponding normal frame along the map u), then we can rewrite (3.7) as follows:

$$-\operatorname{div}\nabla u = \nu_l \nabla\nu_l \cdot \nabla u - H + \langle H, \nu_l\rangle\nu_l, \tag{3.8}$$

where $\langle\cdot\,,\cdot\rangle$ denotes the Euclidean metric on $\mathbb{R}^n$. We have thus obtained the Euler–Lagrangian equations:

$$-\operatorname{div}\big\{\beta(u)\big(\nabla t + \omega_i(u)\nabla u^i\big)\big\} = 0, \tag{3.9}$$

$$-\operatorname{div}\nabla u = \nu_l \nabla\nu_l \cdot \nabla u - H + \langle H, \nu_l\rangle\nu_l. \tag{3.10}$$

To proceed, we write the system of equations (3.9) and (3.10) in the form of (1.3). By Hodge decomposition, we conclude from the conservation law (3.4) that there exists $\eta \in W^{1,2}(B, \wedge^2\mathbb{R}^m)$ such that

$$\beta(u)\big(\nabla t + \omega_i(u)\nabla u^i\big) = \operatorname{curl}\eta. \tag{3.11}$$

Then, by (3.6), we can rewrite Eq. (3.10) as:

$$-\operatorname{div}\nabla u^j = \Theta^{jk} \cdot \nabla u^k + a_{jk}\operatorname{curl}\eta \cdot \nabla u^k + b_j \operatorname{curl}\eta \cdot \beta(u)\big(\nabla t + \omega_i \nabla u^i\big), \tag{3.12}$$

where

$$\Theta^{jk} := \nu_l^j \nabla \nu_l^k - \nu_l^k \nabla \nu_l^j, \tag{3.13}$$

$$a_{jk} := -\left(\frac{\partial \omega_j}{\partial y^k} - \frac{\partial \omega_k}{\partial y^j}\right) + \left(\frac{\partial \omega_i}{\partial y^k} - \frac{\partial \omega_k}{\partial y^i}\right)\nu_l^i \nu_l^j, \tag{3.14}$$

$$b_j := \frac{1}{2\beta^2(u)}\left(\frac{\partial \beta}{\partial y^j} - \frac{\partial \beta}{\partial y^i}\nu_l^i \nu_l^j\right). \tag{3.15}$$

Now we can write the Euler–Lagrangian equations (3.9) and (3.10) as the following elliptic system:

$$-\operatorname{div}\left\{Q \cdot \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix}\right\} = \Theta \cdot Q \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix} + F \operatorname{curl} \zeta \cdot Q \begin{pmatrix} \nabla t \\ \nabla u \end{pmatrix}, \tag{3.16}$$

where

$$Q = \tilde{Q} \circ u, \quad \tilde{Q} = \begin{pmatrix} \beta & \beta\omega \\ 0 & I_n \end{pmatrix}, \quad \omega = (\omega_1, \omega_2, \ldots, \omega_n), \tag{3.17}$$

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & (\Theta^{jk}) \end{pmatrix}, \tag{3.18}$$

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ b_1 & a_{11} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ b_n & a_{n1} & \cdots & a_{nn} \end{pmatrix}, \tag{3.19}$$

$$\zeta = \operatorname{diag}(\eta, \eta, \ldots, \eta). \tag{3.20}$$

Since M is compact, $\beta \in C^2(M, (0, \infty))$ and $\omega \in C^2(\Omega^1(M))$, there exists $\lambda > 0$ depending only on the target $(\mathbb{R} \times M, g)$ such that for any $y \in M$ there hold

$$0 < \lambda^{-1} \leqslant \beta(y), \qquad \left|\beta(y)\right| + \left|\nabla\beta(y)\right| + \left|\nabla^2\beta(y)\right| \leqslant \lambda < \infty,$$
$$\left|\omega(y)\right| + \left|\nabla\omega(y)\right| + \left|\nabla^2\omega(y)\right| \leqslant \lambda < \infty. \tag{3.21}$$

Using the notations (3.13)–(3.15), (3.17)–(3.20) and the above estimates (3.21), we can easily verify that $\Theta \in L^2(B, so(n+1) \otimes \wedge^1\mathbb{R}^m)$, $F \in W^{1,2} \cap L^\infty(B, \mathrm{M}(n+1))$, $\zeta \in W^{1,2}(B, \mathrm{M}(n+1) \otimes \wedge^2\mathbb{R}^m)$, $Q \in W^{1,2} \cap L^\infty(B, \mathrm{GL}(n+1))$ and the following estimates hold:

$$|Q| + |F| \leqslant C_1(\lambda), \quad \text{a.e. in } B, \tag{3.22}$$

and

$$\|\Theta\|_{M_2^2(B)} + \|\nabla F\|_{M_2^2(B)} + \|\nabla Q\|_{M_2^2(B)} \leqslant C_2(\lambda)\|\nabla u\|_{M_2^2(B)}, \tag{3.23}$$

where $C_1(\lambda) > 0$ and $C_2(\lambda) > 0$ are constants also depending on λ.

To estimate $|Q^{-1}|$, we note that

$$\tilde{Q}^{-1} = \begin{pmatrix} \beta^{-1} & -\omega \\ 0 & I_n \end{pmatrix}.$$

Hence, by (3.21), there exists some constant $C_3(\lambda) > 0$ such that

$$\left|Q^{-1}\right| = \left|\tilde{Q}^{-1} \circ u\right| \leqslant C_3(\lambda), \quad \text{a.e. in } B. \tag{3.24}$$

On the other hand, it follows from (3.11) and (3.21) that

$$|\operatorname{curl} \eta| \leqslant C_4(\lambda)\big(|\nabla t| + |\nabla u|\big), \quad \text{a.e. in } B.$$

By (3.20) and the above inequality, we verify that

$$\|\operatorname{curl} \zeta\|_{M_2^2(B)} \leqslant C_5(\lambda)\big(\|\nabla t\|_{M_2^2(B)} + \|\nabla u\|_{M_2^2(B)}\big). \tag{3.25}$$

Combining (3.22) and (3.24) gives

$$|Q|+\left|Q^{-1}\right|+|F|\leqslant C_1(\lambda)+C_3(\lambda),\quad \text{a.e. in } B. \tag{3.26}$$

Combining (3.23) and (3.25) gives

$$\begin{aligned}&\|\nabla t\|_{M_2^2(B)}+\|\nabla u\|_{M_2^2(B)}+\|\operatorname{curl}\zeta\|_{M_2^2(B)}+\|\Theta\|_{M_2^2(B)}+\|\nabla F\|_{M_2^2(B)}+\|\nabla Q\|_{M_2^2(B)}\\&\quad\leqslant\big(1+C_2(\lambda)+C_5(\lambda)\big)\big(\|\nabla t\|_{M_2^2(B)}+\|\nabla u\|_{M_2^2(B)}\big).\end{aligned} \tag{3.27}$$

Take $\Lambda := C_1(\lambda)+C_3(\lambda)>0$, then Λ depends only on $(\mathbb{R}\times M, g)$. Let $\epsilon_{m,\Lambda}>0$ be the small constant (depending on m and Λ) as in Theorem 1.2. Take

$$\epsilon_m := \frac{\epsilon_{m,\Lambda}}{1+C_2(\lambda)+C_5(\lambda)},$$

then $\epsilon_m>0$ depends only on $(\mathbb{R}\times M, g)$. Applying Theorem 1.2 to the elliptic system (3.16), we conclude from (3.26) and (3.27) that (t,u) is Hölder continuous in B if $\|\nabla t\|_{M_2^2(B)}+\|\nabla u\|_{M_2^2(B)}<\epsilon_m$. By standard elliptic regularity theory, (t,u) is as smooth as the regularity of the target $(\mathbb{R}\times M, g)$ permits. $\square$

4. Harmonic maps into pseudospheres $\mathbb{S}^n_\nu$ $(1\leqslant\nu\leqslant n)$

In this section, we shall first prove Propositions 1.1 and 1.2. Then, with the help of these two propositions, we apply Theorem 1.5 to prove the regularity results (Theorems 1.6 and 1.7) for weakly harmonic maps into pseudospheres $\mathbb{S}^n_\nu$ $(1\leqslant\nu\leqslant n)$.

Proof of Proposition 1.1. Fix $i\neq j\in\{1,2,\dots,n+1\}$. Let $E_{ij}\in so(n+1)$ be the matrix whose (i,j)-component is 1, (j,i)-component is -1 and all the other components are 0. Let $\mathcal{E}$ be the matrix defined as in (1.17). Then one verifies that $E_{ij}\mathcal{E}\in so(\nu,n+1-\nu)$ and $e^{E_{ij}\mathcal{E}}\in \mathrm{O}(\nu,n+1-\nu)$ (see e.g. [34]). For any $\varphi\in C_0^\infty(B)$, define

$$R_t := e^{t\varphi E_{ij}\mathcal{E}}\in C_0^\infty\big(B,\mathrm{O}(\nu,n+1-\nu)\big). \tag{4.1}$$

Using the property of an element in the group $\mathrm{O}(\nu,n+1-\nu)$ (see (1.19)), we have

$$\langle R_tu,R_tu\rangle_{\mathbb{R}^{n+1}_\nu}=(R_tu)^T\mathcal{E}R_tu=u^TR_t^T\mathcal{E}R_tu=u^T\mathcal{E}u=1,\quad\text{a.e. in } B. \tag{4.2}$$

It follows that $R_tu\in W^{1,2}(B,\mathbb{S}^n_\nu)$. Since u is weakly harmonic, by Definition 1.2, we calculate

$$\begin{aligned}0=\frac{d}{dt}\bigg|_{t=0}E(R_tu)&=\int_B\big(\nabla(R_0u)\big)^T\mathcal{E}\frac{d}{dt}\bigg|_{t=0}\big(\nabla(R_tu)\big)\\&=\int_B(\nabla u)^T\mathcal{E}\big(\nabla(\varphi E_{ij}\mathcal{E}u)\big)\\&=\int_B(\nabla u)^T\mathcal{E}E_{ij}\mathcal{E}u\nabla\varphi+(\nabla u)^T\mathcal{E}E_{ij}\mathcal{E}\nabla u\varphi\\&=\int_B(\nabla u)^T\mathcal{E}E_{ij}\mathcal{E}u\nabla\varphi\\&=(\varepsilon_{ii}\varepsilon_{jj})\int_B\big(u^i\nabla u^j-u^j\nabla u^i\big)\nabla\varphi,\end{aligned} \tag{4.3}$$

where we have used the fact that $\mathcal{E}E_{ij}\mathcal{E}\in so(n+1)$ and hence

$$(\nabla u)^T\mathcal{E}E_{ij}\mathcal{E}\nabla u=0\quad\text{a.e. in } B.$$

Since $\varphi\in C_0^\infty(B)$ is arbitrary and $\varepsilon_{ii}\varepsilon_{jj}$ is either 1 or -1 (see (1.17)), we conclude from (4.3) that the conservation laws (1.27) hold for $i\neq j$.

The case of $i=j$ is trivial. This completes the proof. $\square$

Proof of Proposition 1.2. First, by definition of the space $W^{1,2}(B,\mathbb{S}^n_\nu)$ (see (1.21)), we have

$$u^j\varepsilon_{jk}u^k = 1 \quad \text{a.e. in } B. \tag{4.4}$$

Taking ∇ on both sides of (4.4) gives

$$\nabla u^j\varepsilon_{jk}u^k = 0 \quad \text{a.e. in } B. \tag{4.5}$$

Recall that (see (1.28)) $\Theta = (\Theta^{ij}) = (u^i\nabla u^j - u^j\nabla u^i)$. Combining (4.4) and (4.5), we calculate

$$\begin{aligned}\nabla u^i + \Theta^{ij}\varepsilon_{jk}u^k &= \nabla u^i + \left(u^i\nabla u^j - u^j\nabla u^i\right)\varepsilon_{jk}u^k\\ &= \nabla u^i\left(1 - u^j\varepsilon_{jk}u^k\right) + u^i\left(\nabla u^j\varepsilon_{jk}u^k\right)\\ &= 0 \quad \text{a.e. in } B.\end{aligned} \tag{4.6}$$

This proves (1.29).

Since $u \in W^{1,2}(B,\mathbb{R}^{n+1})$, one verifies that $\nabla u^i + \Theta^{ij}\varepsilon_{jk}u^k \in L^1(B)$ for each i. Taking $-\operatorname{div}$ on both sides of (4.6) gives

$$-\operatorname{div}(\nabla u + \Theta\mathcal{E}u) = 0, \quad \text{in } \mathcal{D}'(B).$$

Next, we assume that u is weakly harmonic and for any fixed $1 < p < \frac{m}{m-1}$ there holds $\|\nabla u\|_{M^p_p(B)} < \infty$. We shall derive the estimate (1.31).

Let $q = \frac{p}{p-1} > m$ be the conjugate exponent of p. Let $B_R(x_0) \subset B_{1/2}$. For any $\Phi \in L^q(B_R(x_0), \wedge^1\mathbb{R}^m)$ with $\|\Phi\|_{L^q(B_R(x_0))} \leqslant 1$ and for any $0 < \rho < R$, let $\tau = \tau(\rho) \in C_0^\infty(B_R(x_0), [0,1])$ be a cut-off function satisfying

$$\tau \equiv 1, \quad \text{on } B_\rho(x_0),$$

then $\tau\Phi$ is supported in $B_R(x_0)$ and vanishes on $\partial B_R(x_0)$. By Hodge decomposition, there exist $\alpha \in W_0^{1,q}(B_R(x_0))$, $\beta \in W_0^{1,q}(B_R(x_0), \wedge^2\mathbb{R}^m)$ and a harmonic $h \in C^\infty(B_R(x_0), \wedge^1\mathbb{R}^m)$ such that

$$\tau\Phi = \nabla\alpha + \operatorname{curl}\beta + h. \tag{4.7}$$

Moreover, we have

$$\|\nabla\alpha\|_{L^q(B_R(x_0))} + \|\nabla\beta\|_{L^q(B_R(x_0))} \leqslant C\|\tau\Phi\|_{L^q(B_R(x_0))} \leqslant C\|\Phi\|_{L^q(B_R(x_0))} \leqslant C, \tag{4.8}$$

where $C > 0$ is a constant independent of ρ and R. Recall that $\tau \in C_0^\infty(B_R(x_0))$; we get $h|_{\partial B_R(x_0)} = (\tau\Phi)|_{\partial B_R(x_0)} = 0$. Since h is harmonic, it follows that $h \equiv 0$ in $B_R(x_0)$.

Since u is weakly harmonic, by Proposition 1.1, $\Theta = (\Theta^{ij}) = (u^i\nabla u^j - u^j\nabla u^i)$ is divergence free. Then, using (4.7), (4.8) and the fact that $h \equiv 0$ in $B_R(x_0)$, and applying Lemma 2.1, we estimate for fixed $i, j \in \{1, 2, \ldots, n+1\}$,

$$\begin{aligned}\int_{B_R(x_0)} \left(\tau\Theta^{ij}\right)\cdot\Phi &= \int_{B_R(x_0)} \Theta^{ij}\cdot(\tau\Phi)\\ &= \int_{B_R(x_0)} \Theta^{ij}\cdot(\nabla\alpha + \operatorname{curl}\beta)\\ &= \int_{B_R(x_0)} \Theta^{ij}\cdot\operatorname{curl}\beta\\ &= \int_{B_R(x_0)} \left(u^i\nabla u^j - u^j\nabla u^i\right)\cdot\operatorname{curl}\beta\\ &= \int_{B_R(x_0)} \left\{\left(\nabla u^j\cdot\operatorname{curl}\beta\right)u^i - \left(\nabla u^i\cdot\operatorname{curl}\beta\right)u^j\right\}\\ &\leqslant C\|\nabla u\|_{L^p(B_R(x_0))}\|\operatorname{curl}\beta\|_{L^q(B_R(x_0))}\|\nabla u\|_{M^p_p(B_{2R}(x_0))}\\ &\leqslant C\|\nabla u\|_{L^p(B_R(x_0))}\|\nabla\beta\|_{L^q(B_R(x_0))}\|\nabla u\|_{M^p_p(B_{2R}(x_0))}\\ &\leqslant C\|\nabla u\|_{L^p(B_R(x_0))}\|\nabla u\|_{M^p_p(B_{2R}(x_0))}.\end{aligned} \tag{4.9}$$

By duality characterization of L^p functions, we have

$$\left\|\left(\tau\Theta^{ij}\right)\right\|_{L^p(B_R(x_0))} \leqslant C\|\nabla u\|_{L^p(B_R(x_0))}\|\nabla u\|_{M_p^p(B_{2R}(x_0))}. \tag{4.10}$$

It follows that

$$\left\|\Theta^{ij}\right\|_{L^p(B_\rho(x_0))} \leqslant \left\|\left(\tau\Theta^{ij}\right)\right\|_{L^p(B_R(x_0))} \leqslant C\|\nabla u\|_{L^p(B_R(x_0))}\|\nabla u\|_{M_p^p(B_{2R}(x_0))}. \tag{4.11}$$

Since $\rho \in (0, R)$ is arbitrary, let $\rho \nearrow R$, then we get

$$\left\|\Theta^{ij}\right\|_{L^p(B_R(x_0))} \leqslant C\|\nabla u\|_{L^p(B_R(x_0))}\|\nabla u\|_{M_p^p(B_{2R}(x_0))}. \tag{4.12}$$

Furthermore, using the definition of the Morrey norm $\|\nabla u\|_{M_p^p(B)}$ and the fact that $B_{2R}(x_0) \subset B$, we estimate

$$\begin{aligned}\|\Theta\|_{L^p(B_R(x_0))} = \sum_{i,j}\left\|\Theta^{ij}\right\|_{L^p(B_R(x_0))} &\leqslant C\|\nabla u\|_{L^p(B_R(x_0))}\|\nabla u\|_{M_p^p(B_{2R}(x_0))} \\ &\leqslant CR^{\frac{m}{p}-1}\|\nabla u\|_{M_p^p(B)}\|\nabla u\|_{M_p^p(B)} \\ &= CR^{\frac{m}{p}-1}\|\nabla u\|^2_{M_p^p(B)}.\end{aligned}$$

Since the ball $B_R(x_0) \subset B_{1/2}$ is arbitrary, it follows that

$$\|\Theta\|_{M_p^p(B_{1/2})} = \sup_{B_R(x_0)\subset B_{1/2}}\left(R^{p-m}\int_{B_R(x_0)}|\Theta|^p\right)^{\frac{1}{p}} \leqslant C\|\nabla u\|^2_{M_p^p(B)}.$$

Thus, we have completed the proof. □

Proof of Theorem 1.6. Note that $\mathcal{E}$ is a constant matrix. Combining Propositions 1.1, 1.2, Theorem 1.5 and using a rescaling of the domain gives that u is Hölder continuous in B. Moreover, since $\operatorname{div}\Theta = 0$, we can rewrite the equation in (1.30) as

$$-\operatorname{div}\nabla u = \Theta\mathcal{E}\cdot\nabla u.$$

By standard elliptic regularity theory, u is smooth in B. □

Proof of Theorem 1.7. Fix some $1 < p < \frac{m}{m-1} = 2$. By conformal invariance in dimension $m = 2$ and rescaling in the domain, we assume w.l.o.g. that

$$\|\nabla u\|^2_{L^2(B)} < \epsilon_{2,p}, \tag{4.13}$$

where $\epsilon_{2,p}$ is given in Theorem 1.6 with $m = 2$. By a straightforward calculation, it follows that

$$\|\nabla u\|^2_{M_p^p(B)} \leqslant \|\nabla u\|^2_{L^2(B)} < \epsilon_{2,p}. \tag{4.14}$$

Applying Theorem 1.6 with $m = 2$ gives that u is Hölder continuous (and hence smooth) in B. □

5. Generalized (weakly) harmonic maps into $\mathbb{S}^n_\nu$ ($1 \leqslant \nu \leqslant n$)

In this section, we shall prove the ϵ-regularity result (Theorem 1.8) for generalized (weakly) harmonic maps into $\mathbb{S}^n_\nu$ ($1 \leqslant \nu \leqslant n$). Throughout this section, B will denote the unit disc in $\mathbb{R}^2$.

Proof of Theorem 1.8. Slightly modifying some arguments in the proofs of Proposition 1.2 and Theorem 1.5 will be sufficient to prove this theorem.

Fix any $\frac{4}{3} < p < 2$ and let $u \in W^{1,\frac{4}{3}}(B, \mathbb{S}^n_\nu)$ ($1 \leqslant \nu \leqslant n$) be a generalized (weakly) harmonic map satisfying

$$\|\nabla u\|^2_{M_p^p(B)} < \epsilon_p \tag{5.1}$$

with $\epsilon_p > 0$ being determined later. Then $u \in W^{1,p}(B)$ and hence $\Theta = (\Theta^{ij}) := (u^i \nabla u^j - u^j \nabla u^i) \in L^{p'}(B)$, where $p' = \frac{2p}{4-p} \in (1, p)$. By Definition 1.5, there holds

$$\operatorname{div} \Theta = 0, \quad \text{in } \mathcal{D}'(B). \tag{5.2}$$

Applying similar arguments as in the proof of Proposition 1.2 (with $m = 2$) gives that

$$\nabla u + \Theta \mathcal{E} u = 0 \quad \text{a.e. in } B, \tag{5.3}$$

and

$$\|\Theta\|_{M^{p'}_{p'}(B_{1/2})} \leqslant C_{p'} \|\nabla u\|^2_{M^{p'}_{p'}(B)} \leqslant C_{p'} \|\nabla u\|^2_{M^p_p(B)} \leqslant C_{p'} \epsilon_p. \tag{5.4}$$

Let $B_{2R}(x_0) \subset B_{1/2}$ and let $w \in W^{1,p'}(B_R(x_0), \mathbb{R}^{n+1})$ be solving

$$\begin{cases} -\operatorname{div} \nabla w = 0, & \text{in } B_R(x_0), \\ w = u, & \text{on } \partial B_R(x_0), \end{cases} \tag{5.5}$$

and define $v := u - w \in W^{1,p'}_0(B_R(x_0), \mathbb{R}^{n+1})$.

Let $q' = \frac{p'}{p'-1}$ be the conjugate exponent of p'. Then for any $\varphi \in W^{1,q'}_0(B_R(x_0))$ with $\|\varphi\|_{W^{1,q'}(B_R(x_0))} \leqslant 1$, using (5.3) and (5.5), we get

$$\int_{B_R(x_0)} \nabla v^i \cdot \nabla \varphi = \int_{B_R(x_0)} \nabla u^i \cdot \nabla \varphi - \int_{B_R(x_0)} \nabla w^i \cdot \nabla \varphi = -\varepsilon_{jj} \int_{B_R(x_0)} \Theta^{ij} u^j \cdot \nabla \varphi.$$

Then using (5.1), (5.2), (5.4), Lemma 2.1 and taking $\epsilon_p > 0$ sufficiently small, we can apply the same arguments as in the proof of Theorem 1.5 (with $m = 2$) and use a rescaling of the domain to conclude that u is Hölder continuous and hence smooth (by standard elliptic regularity) in B. □

Furthermore, we observe that the ϵ-regularity result in Theorem 1.8 still holds if the Morrey norm $\|\nabla u\|_{M^p_p(B)}$ is replaced with the Lorentz norm $\|\nabla u\|_{L^{(2,\infty)}(B)}$ (which was used in Almeida [2]). To see this, we recall the following:

Lemma 5.1. *(See Almeida [2, Lemma 9].) Suppose D has finite measure. Let $1 < p < p_1 < \infty$. Then, there is a constant C such that, for all $q, q_1 \in [1, \infty]$ and for any $f \in L^{(p_1,q_1)}(D)$,*

$$\|f\|_{L^{(p,q)}} \leqslant C\big(\mu(D)\big)^{\frac{p_1-p}{pp_1}} \|f\|_{L^{(p_1,q_1)}}. \tag{5.6}$$

Recall that $L^{(p,p)} = L^p$. Consequently, we have

Lemma 5.2. *Let $1 < p < 2$. Then, there is a constant C such that, for any $f \in L^{(2,\infty)}(B)$,*

$$\|f\|_{M^p_p(B)} \leqslant C \|f\|_{L^{(2,\infty)}(B)}. \tag{5.7}$$

Proof. Take $p = q$, $p_1 = 2$, $q_1 = \infty$ in Lemma 5.1 and let D run over all discs $B_R(x_0) \subset B$. □

Combining Theorem 1.8 and Lemma 5.2 gives the following ϵ-regularity result (using the Lorentz norm).

Theorem 5.1. *There exists $\epsilon > 0$ such that any generalized (weakly) harmonic map $u \in W^{1,\frac{4}{3}}(B, \mathbb{S}^n_\nu)$ $(1 \leqslant \nu \leqslant n)$ satisfying*

$$\|\nabla u\|_{L^{(2,\infty)}(B)} < \epsilon \tag{5.8}$$

is smooth in B.

6. Regularity for an elliptic system with a potential in *so*(1, 1)

Throughout this section, B will denote the unit disc in $\mathbb{R}^2$. We consider the elliptic system (1.1) with a potential $\Omega \in L^2(B, so(1,1) \otimes \wedge^1\mathbb{R}^2)$. By Hodge decomposition, there exist $\Omega_1 \in W^{1,2}(B, so(1,1))$ and $\Omega_2 \in W^{1,2}(B, so(1,1) \otimes \wedge^2\mathbb{R}^2)$ such that

$$\Omega = \nabla\Omega_1 + \operatorname{curl}\Omega_2. \tag{6.1}$$

Theorem 6.1. *Let $u \in W^{1,2}(B, \mathbb{R}^2)$ be a weak solution of the elliptic system (1.1) with a potential $\Omega \in L^2(B, so(1,1) \otimes \wedge^1\mathbb{R}^2)$. Decompose Ω as in (6.1). If $\Omega_1 \in L^\infty(B, so(1,1))$, then u is Hölder continuous in B.*

Proof. Since Ω_1 takes values in $so(1,1)$, we can write (see O'Neill's book [34])

$$\Omega_1 = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}, \quad \text{for some } s \in W^{1,2}(B). \tag{6.2}$$

Consequently, we have $\nabla\Omega_1\Omega_1 = \Omega_1\nabla\Omega_1$ and hence $\nabla(e^{\Omega_1}) = e^{\Omega_1}\nabla\Omega_1$. Then we calculate

$$-\operatorname{div}\bigl(e^{\Omega_1}\nabla u\bigr) = -e^{\Omega_1}\nabla\Omega_1 \cdot \nabla u + e^{\Omega_1}\Omega \cdot \nabla u = e^{\Omega_1}\operatorname{curl}\Omega_2 \cdot \nabla u. \tag{6.3}$$

Using (6.2), we get

$$e^{\Omega_1} = \bigl(e^{\Omega_1}\bigr)^T = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}, \qquad e^{-\Omega_1} = \bigl(e^{\Omega_1}\bigr)^{-1} = \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix}.$$

Since $\Omega_1 \in L^\infty(B)$, there exists a constant $\lambda \in (0, \infty)$, such that $|s| \leqslant \lambda$, a.e. in B. Therefore, we have

$$\bigl|e^{\Omega_1}\bigr| + \bigl|\bigl(e^{\Omega_1}\bigr)^{-1}\bigr| \leqslant C(\lambda), \quad \text{a.e. in } B$$

for some constant $C(\lambda) > 0$ depending on λ.

On the other hand, one verifies that $e^{\Omega_1} \in W^{1,2} \cap L^\infty(B, \mathrm{M}(2))$. Recall that $\Omega_2 \in W^{1,2}(B, so(1,1) \otimes \wedge^2\mathbb{R}^2)$. Applying Theorem 1.2 (with $m = 2$ and $\Lambda = C(\lambda)$) to the elliptic system (6.3), using the conformal invariance in dimension $m = 2$ and rescaling in the domain, we get the Hölder continuity of u in B. □

Theorem 6.1 is optimal. To see this, we set

$$s(x) = \log\log\frac{2}{|x|}, \qquad u_1(x) = \log\log\frac{2}{|x|}, \qquad u_2(x) = \log\log\frac{2}{|x|}, \quad x \in B.$$

Then the map $u = (u_1, u_2)^T \in W^{1,2}(B, \mathbb{R}^2)$ is a weak solution to the elliptic system (1.1) with a potential Ω satisfying

$$\Omega = \begin{pmatrix} 0 & \nabla s \\ \nabla s & 0 \end{pmatrix} \in L^2\bigl(B, so(1,1) \otimes \wedge^1\mathbb{R}^2\bigr) \quad \text{and} \quad s \text{ is not in } L^\infty(B).$$

However, u is not in $L^\infty(B)$.

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