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Fuzzy LINMAP method for multiattribute decision making under fuzzy environments

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Abstract

The Linear Programming Technique for Multidimensional Analysis of Preference (LINMAP) developed by Srinivasan and Shocker [V. Srinivasan, A.D. Shocker, Linear programming techniques for multidimensional analysis of preference, Psychometrika 38 (1973) 337–342] is one of the existing well-known methods for multiattribute decision making (MADM) problems. However, the LINMAP only can deal with MADM problems in crisp environments. Fuzziness is inherent in decision data and decision making processes, and linguistic variables are well suited to assessing an alternative on qualitative attributes using fuzzy ratings. The aim of this paper is further extending the LINMAP method to develop a new methodology for solving MADM problems under fuzzy environments. In this methodology, linguistic variables are used to capture fuzziness in decision information and decision making processes by means of a fuzzy decision matrix. A new vertex method is proposed to calculate the distance between trapezium fuzzy number scores. Consistency and inconsistency indices are defined on the basis of preferences between alternatives given by the decision maker. Each alternative is assessed on the basis of its distance to a fuzzy positive ideal solution (FPIS) which is unknown. The FPIS and the weights of attributes are then estimated using a new linear programming model based upon the consistency and inconsistency indices defined. Finally, the distance of each alternative to the FPIS can be calculated to determine the ranking order of all alternatives. A numerical example is examined to demonstrate the implementation process of this methodology. Also it has

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been proved that the methodology proposed in this paper can deal with MADM problems under not only fuzzy environments but also crisp environments.

Keywords: Linear programming technique for multidimensional analysis of preference (LINMAP); Fuzzy multiattribute decision making; Linguistic variable; Fuzzy number; Linear programming; Preference information

1. Introduction

Multiattribute decision making (MADM) problems are an important type of multicriteria decision making (MCDM) problems [3,14–17,25,26] and are wide spread in real life decision situations [1,2,5,6,8–13,18–22]. Recently, lots of literatures [1,2,4,5,7,9,11–13,16–23,25–28] investigate on MADM problems using fuzzy sets and achieved a great progress.

A MADM problem is to find a best compromise solution from all feasible alternatives assessed on multiple attributes, both quantitative and qualitative [8,16–18]. Suppose the decision maker has to choose one of or rank n alternatives x_j $(j = 1, 2, \dots, n)$ based on m attributes f_i $(i = 1, 2, \dots, m)$. Denote an alternative set by $X = \{x_1, x_2, \dots, x_n\}$ and an attribute set by $F = \{f_1, f_2, \dots, f_m\}$. In general, attributes can be classified into two types: benefit attributes and cost attributes. In other words, the attribute set F can be divided into two subsets: F^1 and F^2, where F^k $(k = 1, 2)$ is the subset of benefit attributes and cost attributes, respectively. Furthermore, $F = F^1 \cup F^2$ and $F^1 \cap F^2 = \emptyset$, where $\emptyset$ is empty set. Then the MADM model can be built as follows

$$\begin{aligned}
&\max\{f_i(x_j) \mid i \in F^1\},\\
&\min\{f_i(x_j) \mid i \in F^2\},\\
&\text{s.t.} \quad x_j \in X.
\end{aligned}$$

The alternative set X and the attribute set F are finite, so it is very convenient to denote the score of alternative x_j $(j = 1, 2, \dots, n)$ on attribute f_i $(i = 1, 2, \dots, m)$ by f_{ij}, i.e., $f_{ij} = f_i(x_j)$. Then a MADM problem can be concisely expressed as the following decision matrix:

$$\boldsymbol{Y} = (f_{ij})_{m\times n} = \begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{pmatrix} \end{array}.$$

In decision making process, different attributes have different importance. Suppose ω_i $(i = 1, 2, \dots, m)$ is the relative weight of attribute f_i, where $\omega_i \geqslant 0$ $(i = 1, 2, \dots, m)$ and $\sum_{i=1}^{m} \omega_i = 1$. Denote a weight vector by $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_m)^T$.

The above MADM problem can be dealt with using several existing methods such as the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) developed by Huang and Yoon [14], the Linear Programming Technique for Multidimensional Analysis of Preference (LINMAP) developed by Srinivasan and Shocker [24] and the nonmetric Multidimensional Scaling (MDS). The TOPSIS and LINMAP methods are two well-known MADM methods, though they require different types of information [4]. In the TOPSIS method, the decision matrix $\boldsymbol{Y}$ and the weight vector $\boldsymbol{\omega}$ are given as crisp values a priori; a positive ideal solution (PIS) and a negative ideal solution (NIS) are generated from $\boldsymbol{Y}$ directly; the best compromise alternative is then defined as the one that has the shortest distance to the PIS and the farthest from the NIS. However, in the LINMAP method, the weight vector $\boldsymbol{\omega}$ and the PIS are unknown a priori. The LINMAP method is based on pairwise comparisons of alternatives given by the decision maker and generates the best compromise alternative as the solution that has the shortest distance to the PIS.

In the LINMAP method, all the decision data are known precisely or given as crisp values. However, under many conditions, crisp data are inadequate or insufficient to model real-life decision problems [3,4,7,16,19,23,27,28]. Indeed, human judgments including preference information are vague or fuzzy in nature and as such it may not be appropriate to represent them by accurate numerical values. A more realistic approach could be to use linguistic variables to model human judgments [3,7,16,23,27,28]. In this paper, we further extend the LINMAP method to develop a new methodology for solving multiattribute decision making problems in a fuzzy environment [1,2,4]. In this methodology, linguistic variables are used to capture fuzziness in decision information and decision making processes by means of a fuzzy decision matrix. A new vertex method is proposed to calculate the distance between trapezium fuzzy scores. Consistency and inconsistency indices are defined on the basis of preferences between alternatives given by the decision maker. Each alternative is assessed on the basis of its distance to a fuzzy positive ideal solution (FPIS) which is unknown. The FPIS and the weights of attributes are then estimated using a new linear programming model based upon the consistency and inconsistency indices defined. Finally, the distance of each alternative to the FPIS can be calculated to determine the ranking order of all alternatives. The lower value of the distance for an alternative indicates that the alternative is closer to the FPIS.

The paper is organized as follows. In next Section, the basic definitions and notations of trapezium fuzzy numbers and linguistic variables are defined as well as the fuzzy distance formula and the normalization method. Section 3 defines consistency and inconsistency indices between preferences of alternatives given by the decision maker and the results of the decision making model, and presents a new linear programming model to solve such multiattribute decision making problems. The developed methodology is also illustrated with a real life example in Section 4. A short concluding remark is given in Section 5.

2. Basic concepts and definitions

2.1. Concepts and notations of trapezium fuzzy numbers

A fuzzy number $\tilde{m}$ is a special fuzzy subset on the set $\mathbf{R}$ of real numbers which satisfy the following conditions [4,9,16]:

(1) There exists a $x_0 \in \mathbf{R}$ so that the degree of its membership $\mu_{\tilde{m}}(x_0) = 1$;
(2) Membership function $\mu_{\tilde{m}}(x)$ is left and right continuous.

Generally, a fuzzy number $\tilde{m}$ can be written as

$$\mu_{\tilde{m}}(x) = \begin{cases} L(x) & (l \leqslant x \leqslant m), \\ R(x) & (m \leqslant x \leqslant r), \end{cases}$$

where $L(x)$ is an increasing function of $x \in [l, m]$ and right continuous, $0 \leqslant L(x) \leqslant 1$; $R(x)$ is a decreasing function of $x \in [m, r]$ and left continuous, $0 \leqslant R(x) \leqslant 1$. m is called a mode of $\tilde{m}$, and l and r are called the low and upper limits of $\tilde{m}$, respectively, depicted as in Fig. 1. This kind of fuzzy numbers is often called L–R fuzzy numbers [16].

Let $\tilde{m} = (l, m_1, m_2, r)$ be a trapezium fuzzy number, where the membership function $\mu_{\tilde{m}}$ of $\tilde{m}$ is

$$\mu_{\tilde{m}}(x) = \begin{cases} \frac{x-l}{m_1-l} & (l \leqslant x < m_1), \\ 1 & (m_1 \leqslant x \leqslant m_2), \\ \frac{r-x}{r-m_2} & (m_2 < x \leqslant r). \end{cases}$$

The closed interval $[m_1, m_2]$ is the mode of $\tilde{m}$. l and r are the low and upper limits of $\tilde{m}$, respectively, depicted as in Fig. 2.

It is easy to see that a trapezium fuzzy number $\tilde{m} = (l, m_1, m_2, r)$ is reduced to a real number m if $l = m_1 = m_2 = r$. Conversely, a real number m can be written as a trapezium fuzzy number $\tilde{m} = (m, m, m, m)$.

If $m_1 = m_2$ then $\tilde{m} = (l, m, r)$ is called a triangular fuzzy number, where $m = m_1 = m_2$. In other words, a triangular fuzzy number has the following membership function

$$\mu_{\tilde{m}}(x) = \begin{cases} \frac{x-l}{m-l} & (l \leqslant x < m), \\ \frac{r-x}{r-m} & (m \leqslant x \leqslant r), \end{cases}$$

depicted as in Fig. 3. So a triangular fuzzy number is a special case of a trapezium fuzzy number.

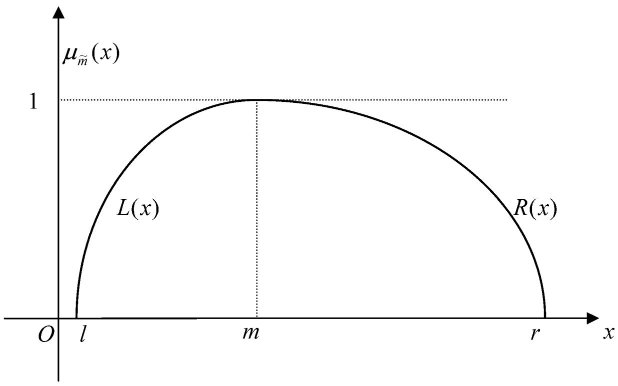

Fig. 1. A fuzzy number.

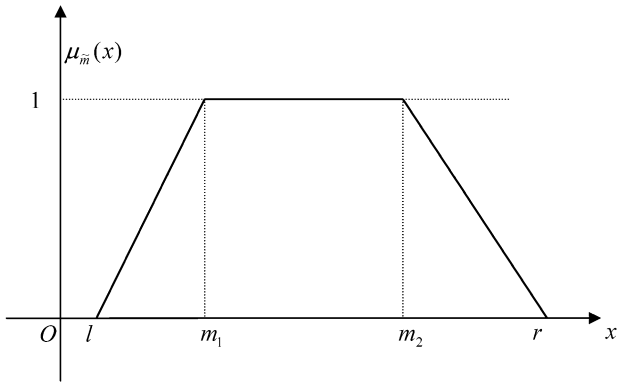

Fig. 2. A trapezium fuzzy number.

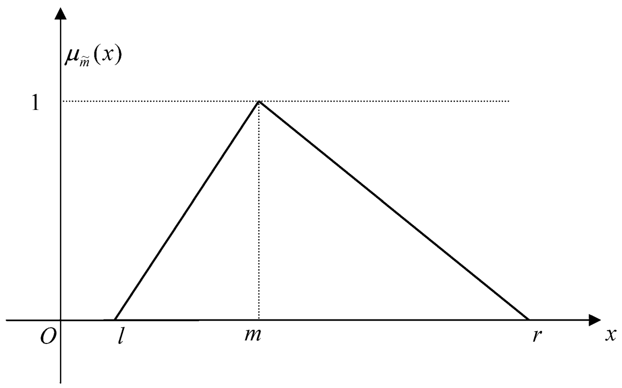

Fig. 3. A triangular fuzzy number.

Similarly, it is easy to see that a triangular fuzzy number $\tilde{m} = (l, m, r)$ is reduced to a real number m if $l = m = r$. Conversely, a real number m can be written as a triangular fuzzy number $\tilde{m} = (m, m, m)$.

$\tilde{m} = (l, m_1, m_2, r)$ is called a positive trapezium fuzzy number if $l \geqslant 0$ and one of l, m_1, m_2 and r is nonzero. Furthermore, $\tilde{m} = (l, m_1, m_2, r)$ is called a normalized positive trapezium fuzzy number if it is a positive trapezium fuzzy number and $l \geqslant 0$, $r \leqslant 1$.

For the sake of simplicity and without loss of generality, assume that all fuzzy numbers are trapezium fuzzy numbers throughout the paper unless otherwise stated.

2.2. Linguistic variable

A linguistic variable is a variable whose values are linguistic terms.

The concept of linguistic variable is very useful in dealing with situations which are too complex or too ill-defined to be reasonably described in conventional quantitative expressions. For example, the ratings of alternatives on qualitative attribute “reliability” could be expressed using linguistic variables such as “very low,” “low,” “medium,” “high,” "very high," etc. Such linguistic values can also be represented using positive trapezium fuzzy numbers. For example, "very low," "low," "medium," "high" and "very high" can be represented by positive trapezium fuzzy numbers $(0, 0.1, 0.2, 0.3)$, $(0.1, 0.2, 0.3, 0.4)$, $(0.3, 0.4, 0.5, 0.6)$, $(0.5, 0.6, 0.7, 0.8)$ and $(0.7, 0.8, 0.9, 1)$, respectively.

2.3. Distance between two trapezium fuzzy numbers

Let $\tilde{m} = (m_1, m_2, m_3, m_4)$ and $\tilde{n} = (n_1, n_2, n_3, n_4)$ be two trapezium fuzzy numbers. Then the vertex method is defined to calculate the distance between them as follows:

$$d(\tilde{m}, \tilde{n}) = \sqrt{\frac{1}{6}\big[(m_1 - n_1)^2 + 2(m_2 - n_2)^2 + 2(m_3 - n_3)^2 + (m_4 - n_4)^2\big]}, \tag{1}$$

which is easily proved to be metric (omitted).

Equation (1) is an effective and simple method to calculate the distance between two trapezium fuzzy numbers.

Note that if both $\tilde{m}$ and $\tilde{n}$ are real numbers then the distance measurement $d(\tilde{m}, \tilde{n})$ is identical to the Euclidean distance. In fact, suppose that both $\tilde{m} = (m_1, m_2, m_3, m_4)$ and $\tilde{n} = (n_1, n_2, n_3, n_4)$ are two real numbers and let $m_1 = m_2 = m_3 = m_4 = m$ and $n_1 = n_2 = n_3 = n_4 = n$. The distance measurement $d(\tilde{m}, \tilde{n})$ can be calculated as

$$\begin{aligned} d(\tilde{m}, \tilde{n}) &= \sqrt{\frac{1}{6}\big[(m_1 - n_1)^2 + 2(m_2 - n_2)^2 + 2(m_3 - n_3)^2 + (m_4 - n_4)^2\big]} \\ &= \sqrt{\frac{1}{6}\big[(m - n)^2 + 2(m - n)^2 + 2(m - n)^2 + (m - n)^2\big]} \\ &= \sqrt{(m - n)^2} = |m - n|. \end{aligned}$$

Furthermore, it is easily seen that two trapezium fuzzy numbers $\tilde{m}$ and $\tilde{n}$ are identical if and only if the distance measurement $d(\tilde{m}, \tilde{n}) = 0$.

If $\tilde{m} = (m_1, m_2, m_3)$ and $\tilde{n} = (n_1, n_2, n_3)$ be two triangular fuzzy numbers then Eq. (1) can be rewritten as follows:

$$d(\tilde{m}, \tilde{n}) = \sqrt{\frac{1}{6}\big[(m_1 - n_1)^2 + 4(m_2 - n_2)^2 + (m_3 - n_3)^2\big]}. \tag{2}$$

2.4. The normalization method

In this paper, we discuss the following fuzzy multiattribute decision making (FMADM) problem.

Suppose there exist n possible alternatives $x_1, x_2, \ldots, x_n$ from which the decision maker has to choose on the basis of m attributes $f_1, f_2, \ldots, f_m$, both quantitative and qualitative [8,16,23]. Suppose that the rating of alternative x_j $(j = 1, 2, \ldots, n)$ on attribute f_i $(i = 1, 2, \ldots, m)$ given by the decision maker is a trapezium fuzzy number

$\tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr})$. Hence, a fuzzy multiattribute decision making problem can be concisely expressed in matrix format as follows:

$$\tilde{\boldsymbol{F}} = (\tilde{f}_{ij})_{m\times n} = \begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{pmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \cdots & \tilde{f}_{1n} \\ \tilde{f}_{21} & \tilde{f}_{22} & \cdots & \tilde{f}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{f}_{m1} & \tilde{f}_{m2} & \cdots & \tilde{f}_{mn} \end{pmatrix} \end{array}, \tag{3}$$

which is referred to as a fuzzy decision matrix usually used to represent the fuzzy multiattribute decision making problem.

Since the physical dimensions and measurements of the m attributes are different, so the fuzzy decision matrix $\tilde{\boldsymbol{F}}$ needs to be normalized. In this paper, we choose the following normalization formula

$$\tilde{r}_{ij} = \left(\frac{a_{ijl}}{a_{ir}^{\max}}, \frac{a_{ijm_1}}{a_{im_2}^{\max}}, \frac{a_{ijm_2}}{a_{im_1}^{\max}} \wedge 1, \frac{a_{ijr}}{a_{il}^{\max}} \wedge 1\right) \quad \left(i \in F^1\right) \tag{4}$$

and

$$\tilde{r}_{ij} = \left(\frac{a_{il}^{\min}}{a_{ijr}}, \frac{a_{im_1}^{\min}}{a_{ijm_2}}, \frac{a_{im_2}^{\min}}{a_{ijm_1}} \wedge 1, \frac{a_{ir}^{\min}}{a_{ijl}} \wedge 1\right) \quad \left(i \in F^2\right), \tag{5}$$

where F^1 and F^2 are the set of benefit attributes and cost attributes, respectively, and

$$\begin{aligned}
a_{il}^{\max} &= \max\{a_{ijl} \mid \tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}),\ j = 1, 2, \ldots, n\}, \\
a_{il}^{\min} &= \min\{a_{ijl} \mid \tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}),\ j = 1, 2, \ldots, n\}, \\
a_{im_1}^{\max} &= \max\{a_{ijm_1} \mid \tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}),\ j = 1, 2, \ldots, n\}, \\
a_{im_1}^{\min} &= \min\{a_{ijm_1} \mid \tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}),\ j = 1, 2, \ldots, n\}, \\
a_{im_2}^{\max} &= \max\{a_{ijm_2} \mid \tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}),\ j = 1, 2, \ldots, n\}, \\
a_{im_2}^{\min} &= \min\{a_{ijm_2} \mid \tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}),\ j = 1, 2, \ldots, n\}, \\
a_{ir}^{\max} &= \max\{a_{ijr} \mid \tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}),\ j = 1, 2, \ldots, n\}
\end{aligned}$$

and

$$a_{ir}^{\min} = \min\{a_{ijr} \mid \tilde{f}_{ij} = (a_{ijl}, a_{ijm_1}, a_{ijm_2}, a_{ijr}),\ j = 1, 2, \ldots, n\}.$$

Denote $\tilde{r}_{ij}$ by $\tilde{r}_{ij} = (r_{ijl}, r_{ijm_1}, r_{ijm_2}, r_{ijr})$ for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. It is easily seen that all $\tilde{r}_{ij}$ are trapezium fuzzy numbers. Furthermore, all $\tilde{r}_{ij} \in [0, 1]$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, i.e., each $\tilde{r}_{ij}$ is a normalized positive trapezium fuzzy number.

Using Eqs. (4) and (5), Eq. (3) can be transformed into the following normalized positive trapezium fuzzy number decision matrix

$$\tilde{\boldsymbol{R}} = (\tilde{r}_{ij})_{m\times n} = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_m \end{array} & \begin{pmatrix} \tilde{r}_{11} & \tilde{r}_{12} & \cdots & \tilde{r}_{1n} \\ \tilde{r}_{21} & \tilde{r}_{22} & \cdots & \tilde{r}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{r}_{m1} & \tilde{r}_{m2} & \cdots & \tilde{r}_{mn} \end{pmatrix} \end{array}. \tag{6}$$

3. Fuzzy LINMAP model and method

3.1. Consistency and inconsistency measurements

Let $\tilde{\boldsymbol{R}}_j = (\tilde{r}_{1j}, \tilde{r}_{2j}, \ldots, \tilde{r}_{mj})^T$ express a normalized positive trapezium fuzzy number vector for alternatives x_j $(j = 1, 2, \ldots, n)$, where $\tilde{r}_{ij} = (r_{ijl}, r_{ijm_1}, r_{ijm_2}, r_{ijr})$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ is a normalized positive trapezium fuzzy number. Sometimes $\tilde{\boldsymbol{R}}_j$ is called an alternative. In other words, $\tilde{\boldsymbol{R}}_j$ and x_j have the same meaning.

Suppose that the fuzzy positive ideal solution be $\tilde{\boldsymbol{a}}^* = (\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_m^*)^T$ which is unknown a priori and needs to determine, where $\tilde{a}_i^* = (a_{il}^*, a_{im_1}^*, a_{im_2}^*, a_{ir}^*)$ $(i = 1, 2, \ldots, m)$ is a positive trapezium fuzzy number on attribute f_i.

Using Eq. (1), the square of the weighted Euclidean distance between the alternative $\tilde{\boldsymbol{R}}_j = (\tilde{r}_{1j}, \tilde{r}_{2j}, \ldots, \tilde{r}_{mj})^T$ and the FPIS $\tilde{\boldsymbol{a}}^* = (\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_m^*)^T$ can be calculated as

$$S_j = \sum_{i=1}^{m} \omega_i \big[d(\tilde{r}_{ij}, \tilde{a}_i^*)\big]^2. \tag{7}$$

It is easily seen that S_j can be written explicitly as

$$S_j = \sum_{i=1}^{m} \frac{\omega_i}{6}\big[(a_{ijl} - a_{il}^*)^2 + 2(a_{ijm_1} - a_{im_1}^*)^2 + 2(a_{ijm_2} - a_{im_2}^*)^2 + (a_{ijr} - a_{ir}^*)^2\big], \tag{8}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is a weight vector which is unknown a priori and needs to determine.

Assume that the decision maker gives the preference relations between alternatives by $\Omega = \{(k, j) \mid x_k \succcurlyeq x_j,\ (k, j = 1, 2, \ldots, n)\}$ from his/her knowledge and experience, where the symbol "$\succcurlyeq$" is a preference relation given by the decision maker. $x_k \succcurlyeq x_j$ means that either the decision maker prefers the alternative x_k to x_j or the decision maker is indifferent between x_k and x_j. If the weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)^T$ and the fuzzy positive ideal solution $\tilde{\boldsymbol{a}}^* = (\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_m^*)^T$ are chosen by the decision maker already, then using Eq. (7) the decision maker can calculate the square of the weighted Euclidean distance between each pair of alternative $(k, j) \in \Omega$ and the fuzzy positive ideal solution $\tilde{\boldsymbol{a}}^* = (\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_m^*)^T$ as follows:

$$S_k = \sum_{i=1}^{m} \omega_i \big[d(\tilde{r}_{ik}, \tilde{a}_i^*)\big]^2$$

and

$$S_j = \sum_{i=1}^{m} \omega_i \big[d(\tilde{r}_{ij}, \tilde{a}_i^*)\big]^2.$$

For each pair of alternatives $(k, j) \in \Omega$, the alternative x_k is closer to the FPIS than the alternative x_j if $S_j \geqslant S_k$. So the ranking order of alternatives x_k and x_j determined by S_j and S_k based on $(\boldsymbol{\omega}, \tilde{\boldsymbol{a}}^*)$ is consistent with the preference given by the decision maker. Conversely, if $S_j < S_k$, then $(\boldsymbol{\omega}, \tilde{\boldsymbol{a}}^*)$ is not chosen properly since it results in that ranking order of alternatives x_k and x_j determined by S_j and S_k based on $(\boldsymbol{\omega}, \tilde{\boldsymbol{a}}^*)$ is inconsistent with the preferences given by the decision maker. Therefore, $(\boldsymbol{\omega}, \tilde{\boldsymbol{a}}^*)$ should be chosen so that the ranking order of alternatives x_k and x_j determined by S_j and S_k is consistent with the preference provided by the decision maker.

We define an index $(S_j - S_k)^-$ to measure inconsistency between the ranking order of alternatives x_k and x_j determined by S_j and S_k and the preferences given by the decision maker preferring x_k to x_j as follows

$$(S_j - S_k)^- = \begin{cases} S_k - S_j & (S_j < S_k), \\ 0 & (S_j \geqslant S_k). \end{cases} \tag{9}$$

Obviously, the ranking order of alternatives x_k and x_j determined by S_j and S_k based on $(\boldsymbol{\omega}, \tilde{\boldsymbol{a}}^*)$ is consistent with the preferences given by the decision maker if $S_j \geqslant S_k$. Hence, $(S_j - S_k)^-$ is defined to be 0. On the other hand, the ranking order of alternatives x_k and x_j determined by S_j and S_k based on $(\boldsymbol{\omega}, \tilde{\boldsymbol{a}}^*)$ is inconsistent with the preferences given by the decision maker if $S_j < S_k$. Hence, $(S_j - S_k)^-$ is defined to be $S_k - S_j$. Then, the inconsistency index can be rewritten as

$$(S_j - S_k)^- = \max\{0, S_k - S_j\}.$$

Then, a total inconsistency index of the decision maker is defined as

$$B = \sum_{(k,j)\in\Omega} (S_j - S_k)^- = \sum_{(k,j)\in\Omega} \max\{0, S_k - S_j\}. \tag{10}$$

In a similar way, an index $(S_j - S_k)^+$ to measure consistency between the ranking order of alternatives x_k and x_j determined by S_j and S_k and the preferences given by the decision maker preferring x_k to x_j can be defined as follows:

$$(S_j - S_k)^+ = \begin{cases} S_j - S_k & (S_j \geqslant S_k), \\ 0 & (S_j < S_k). \end{cases} \tag{11}$$

This equation mention above can be rewritten as

$$(S_j - S_k)^+ = \max\{0, S_j - S_k\}.$$

Hence, a total consistency index of the decision maker is defined as

$$G = \sum_{(k,j)\in\Omega} (S_j - S_k)^+ = \sum_{(k,j)\in\Omega} \max\{0, S_j - S_k\}. \tag{12}$$

3.2. Fuzzy LINMAP model and method

To determine $(\boldsymbol{\omega}, \tilde{\boldsymbol{a}}^*)$, we construct the following mathematical programming as follows:

$$\begin{aligned}
&\max\{G\}\\
&\quad G - B \geqslant h,\\
&\quad \sum_{i=1}^{m} \omega_i = 1,\\
&\quad \omega_i \geqslant \varepsilon \quad (i = 1, 2, \ldots, m),
\end{aligned} \tag{13}$$

where $h > 0$ is given by the decision maker a priori and $\varepsilon > 0$ is sufficiently small which ensures that the weights generated are not zero as it may be the case in the LINMAP method [24].

The aim of Eq. (13) is to maximize the total consistency index G of the decision maker under the condition in which the total consistency index G is greater than or equals to the total inconsistency index B by a given value $h > 0$.

Using Eqs. (9)–(12), it follows

$$\begin{aligned}
G - B &= \sum_{(k,j)\in\Omega} (S_j - S_k)^+ - \sum_{(k,j)\in\Omega} (S_j - S_k)^-\\
&= \sum_{(k,j)\in\Omega} \left[(S_j - S_k)^+ - (S_j - S_k)^-\right] = \sum_{(k,j)\in\Omega} (S_j - S_k).
\end{aligned}$$

Combining Eq. (12), Eq. (13) can be rewritten as follows:

$$\begin{aligned}
&\max\left\{\sum_{(k,j)\in\Omega} \max\{0, S_j - S_k\}\right\}\\
&\quad \sum_{(k,j)\in\Omega} (S_j - S_k) \geqslant h,\\
&\quad \sum_{i=1}^{m} \omega_i = 1,\\
&\quad \omega_i \geqslant \varepsilon \quad (i = 1, 2, \ldots, m).
\end{aligned} \tag{14}$$

For each pair of $(k, j) \in \Omega$, let

$$\lambda_{kj} = \max\{0, S_j - S_k\},$$

then for each $(k, j) \in \Omega$

$$\lambda_{kj} \geqslant 0$$

and

$$\lambda_{kj} \geqslant S_j - S_k.$$

Thus, Eq. (14) can be transformed into the following mathematical programming

$$\max\Big\{\sum_{(k,j)\in\Omega}\lambda_{kj}\Big\}$$
$$\begin{aligned}
&\sum_{(k,j)\in\Omega}(S_j - S_k)\geqslant h,\\
&\omega_i\geqslant\varepsilon\quad(i=1,2,\dots,m),\\
&\sum_{i=1}^{m}\omega_i=1,\\
&S_k - S_j+\lambda_{kj}\geqslant 0\quad\big((k,j)\in\Omega\big),\\
&\lambda_{kj}\geqslant 0\quad\big((k,j)\in\Omega\big).
\end{aligned}\tag{15}$$

Using Eq. (8), we can construct the following linear programming model:

$$\max\Big\{\sum_{(k,j)\in\Omega}\lambda_{kj}\Big\}$$
$$\begin{aligned}
&\sum_{i=1}^{m}\omega_i\sum_{(k,j)\in\Omega}\big[\big(a_{ijl}^2-a_{ikl}^2\big)+2\big(a_{ijm_1}^2-a_{ikm_1}^2\big)+2\big(a_{ijm_2}^2-a_{ikm_2}^2\big)\\
&\quad+\big(a_{ijr}^2-a_{ikr}^2\big)\big]-2\Bigg[\sum_{i=1}^{m}v_{il}\sum_{(k,j)\in\Omega}(a_{ijl}-a_{ikl})\\
&\quad+2\sum_{i=1}^{m}v_{im_1}\sum_{(k,j)\in\Omega}(a_{ijm_1}-a_{ikm_1})+2\sum_{i=1}^{m}v_{im_2}\sum_{(k,j)\in\Omega}(a_{ijm_2}-a_{ikm_2})\\
&\quad+\sum_{i=1}^{m}v_{ir}\sum_{(k,j)\in\Omega}(a_{ijr}-a_{ikr})\Bigg]\geqslant 6h,\\
&\sum_{i=1}^{m}\omega_i\big[\big(a_{ikl}^2-a_{ijl}^2\big)+2\big(a_{ikm_1}^2-a_{ijm_1}^2\big)+2\big(a_{ikm_2}^2-a_{ijm_2}^2\big)+\big(a_{ikr}^2-a_{ijr}^2\big)\big]\\
&\quad-2\Bigg[\sum_{i=1}^{m}v_{il}(a_{ikl}-a_{ijl})+2\sum_{i=1}^{m}v_{im_1}(a_{ikm_1}-a_{ijm_1})\\
&\quad+2\sum_{i=1}^{m}v_{im_2}(a_{ikm_2}-a_{ijm_2})+\sum_{i=1}^{m}v_{ir}(a_{ikr}-a_{ijr})\Bigg]+6\lambda_{kj}\geqslant 0\quad\big((k,j)\in\Omega\big),\\
&\omega_i\geqslant\varepsilon\quad(i=1,2,\dots,m),\\
&\sum_{i=1}^{m}\omega_i=1,\\
&\lambda_{kj}\geqslant 0\quad\big((k,j)\in\Omega\big),\\
&v_{il}\geqslant 0,\qquad v_{im_1}\geqslant 0,\qquad v_{im_2}\geqslant 0,\qquad v_{ir}\geqslant 0\quad(i=1,2,\dots,m),
\end{aligned}\tag{16}$$

where

$$
\begin{aligned}
&v_{il} = \omega_i a^*_{il},\\
&v_{im_1} = \omega_i a^*_{im_1},\\
&v_{im_2} = \omega_i a^*_{im_2},\\
&v_{ir} = \omega_i a^*_{ir},
\end{aligned}
\tag{17}
$$

ω_i, v_{il}, v_{im_1}, v_{im_2} and v_{ir} $(i = 1, 2, \ldots, m)$ can be obtained by solving the above linear programming (i.e., Eq. (16)) using the Simplex method. Then, the best values of a^*_{il}, $a^*_{im_1}$, $a^*_{im_2}$ and a^*_{ir} $(i = 1, 2, \ldots, m)$ are computed using Eq. (17) and are denoted as the trapezium fuzzy number $\tilde{a}^*_i = (a^*_{il}, a^*_{im_1}, a^*_{im_2}, a^*_{ir})$ $(i = 1, 2, \ldots, m)$. Hence the ranking order of the alternative set $X = \{x_1, x_2, \ldots, x_n\}$ is generated based on the increasing order of distances $S_j (j = 1, 2, \ldots, n)$ calculated with Eq. (8).

If the ratings of attributes are expressed by triangular fuzzy numbers $\tilde{f}_{ij} = (a_{ijl}, a_{ijm}, a_{ijr})$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, the linear programming model (i.e., Eq. (16)) can be transformed into the following:

$$
\begin{aligned}
&\max\left\{\sum_{(k,j)\in\Omega} \lambda_{kj}\right\}\\
&\sum_{i=1}^{m} \omega_i \sum_{(k,j)\in\Omega} \left[\left(a^2_{ijl} - a^2_{ikl}\right) + 4\left(a^2_{ijm} - a^2_{ikm}\right) + \left(a^2_{ijr} - a^2_{ikr}\right)\right]\\
&\quad - 2\left[\sum_{i=1}^{m} v_{il} \sum_{(k,j)\in\Omega} (a_{ijl} - a_{ikl}) + 4\sum_{i=1}^{m} v_{im} \sum_{(k,j)\in\Omega} (a_{ijm} - a_{ikm})\right.\\
&\quad \left. + \sum_{i=1}^{m} v_{ir} \sum_{(k,j)\in\Omega} (a_{ijr} - a_{ikr})\right] \geqslant 6h,\\
&\sum_{i=1}^{m} \omega_i \left[\left(a^2_{ikl} - a^2_{ijl}\right) + 4\left(a^2_{ikm} - a^2_{ijm}\right) + \left(a^2_{ikr} - a^2_{ijr}\right)\right] - 2\left[\sum_{i=1}^{m} v_{il}(a_{ikl} - a_{ijl})\right.\\
&\quad \left. + 4\sum_{i=1}^{m} v_{im}(a_{ikm} - a_{ijm}) + \sum_{i=1}^{m} v_{ir}(a_{ikr} - a_{ijr})\right] + 6\lambda_{kj} \geqslant 0 \quad \big((k,j) \in \Omega\big),\\
&\omega_i \geqslant \varepsilon \quad (i = 1, 2, \ldots, m),\\
&\sum_{i=1}^{m} \omega_i = 1,\\
&\lambda_{kj} \geqslant 0 \quad \big((k,j) \in \Omega\big),\\
&v_{il} \geqslant 0, \qquad v_{im} \geqslant 0, \qquad v_{ir} \geqslant 0 \quad (i = 1, 2, \ldots, m),
\end{aligned}
\tag{18}
$$

where

$$v_{il} = \omega_i a^*_{il},$$

$$v_{im} = \omega_i a_{im}^*,$$
$$v_{ir} = \omega_i a_{ir}^*, \tag{19}$$

ω_i, v_{il}, v_{im} and v_{ir} $(i = 1, 2, \dots, m)$ can be obtained by solving the above linear programming (i.e., Eq. (18)) using the Simplex method. Then, the best values of a_{il}^*, a_{im}^* and a_{ir}^* $(i = 1, 2, \dots, m)$ are computed using Eq. (19) and are denoted as the triangular fuzzy number $\tilde{a}_i^* = (a_{il}^*, a_{im}^*, a_{ir}^*)$ $(i = 1, 2, \dots, m)$. Hence the ranking order of the alternative set $X = \{x_1, x_2, \dots, x_n\}$ is generated based on the increasing order of distances S_j $(j = 1, 2, \dots, n)$ calculated with Eq. (8).

3.3. Decision process of fuzzy LINMAP method

In the above, the fuzzy LINMAP method is proposed, especially the fuzzy linear programming model is constructed to solve the weight vector and the FPIS. Hence the ranking order of all alternatives is generated once the distances of alternatives from the FPIS.

In sum, an algorithm and decision process of the fuzzy multiattribute decision making with fuzzy set approach is given in the following.

Step 1: The decision maker identifies the evaluation attributes.

Step 2: The decision maker gives the preference relations between alternatives by $\Omega = \{(k, j) \mid x_k \succcurlyeq x_j,\ (k, j = 1, 2, \dots, n)\}$.

Step 3: Choose the appropriate linguistic variables for the linguistic ratings of alternatives on attributes.

Step 4: Pool the decision maker's opinion to get the linguistic rating $\tilde{f}_{ij}$ of alternative x_j under attribute f_i.

Step 5: Construct the fuzzy decision matrix $\tilde{\boldsymbol{F}}$ and the normalization positive trapezium fuzzy number decision matrix $\tilde{\boldsymbol{R}}$.

Step 6: Construct the linear programming Eq. (16).

Step 7: Solve Eq. (16) using the Simplex method of the linear programming.

Step 8: Obtain ω_i and $\tilde{a}_i^* = (a_{il}^*, a_{im_1}^*, a_{im_2}^*, a_{ir}^*)$ $(i = 1, 2, \dots, m)$ using Eq. (17), hence obtain the weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_m)^T$ and the fuzzy positive ideal solution $\tilde{\boldsymbol{a}}^* = (\tilde{a}_1^*, \tilde{a}_2^*, \dots, \tilde{a}_m^*)^T$.

Step 9: Calculate the distance S_j $(j = 1, 2, \dots, n)$ of alternative x_j from the FPIS $\tilde{\boldsymbol{a}}^*$ using Eq. (8).

Step 10: According to the increasing order of the distances S_j $(j = 1, 2, \dots, n)$, the best alternative from the alternative set X is determined and the ranking order of all alternatives is generated.

Compared with the LINMAP method [24], Eqs. (16) and (17) can be used in fuzzy decision-making environments with linguistic ratings. Furthermore, to avoid the situation of $\omega_i = 0$ as it may be the case in the LINMAP method, the constraints $\omega_i \geqslant \varepsilon$ and $\sum_{i=1}^m \omega_i = 1$ are added to Eq. (16). It is easy to show that Eqs. (16) and (17) are reduced to the linear programming model of the LINMAP method in a crisp environment if the fuzzy ratings $\tilde{f}_{ik}$ and $\tilde{f}_{ij}$ (or $\tilde{r}_{ik}$ and $\tilde{r}_{ij}$) are reduced to the crisp ratings f_{ik} and f_{ij} (or r_{ik} and r_{ij}), respectively.

4. A numerical example

An extended air-fighter selection problem [14] is investigated in this section.

Suppose one country D plans to buy air-fighters from another country H. The Defense Department of the country H would provide the country D with characteristic data for four candidate air-fighters x_1, x_2, x_3 and x_4. The decision maker takes into consideration the following six attributes in evaluating the air-fighters, including maximum speed (f_1), cruise radius (f_2), maximum loading (f_3), price (f_4), reliability (f_5) and maintenance (f_6). f_5 and f_6 are qualitative attributes and their ratings are expressed using linguistic variables.

The data and ratings of all air-fighters on every attribute are given by the decision maker as in Table 1.

The corresponding relations between linguistic variables and positive trapezium fuzzy numbers are given in Table 2.

The linguistic variables are depicted as in Fig. 4.

Assume that the decision maker provide his/her preferences between air-fighters as follows:

$$\Omega = \{(1,2), (3,2), (1,3), (3,4)\}. \tag{20}$$

We can obtain the following fuzzy decision matrix according to Tables 1 and 2

$$\tilde{\boldsymbol{F}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 2.0 & 2.5 & 1.8 & 2.2 \\ 1.5 & 2.7 & 2.0 & 1.8 \\ 2.0 & 1.8 & 2.1 & 2.0 \\ 5.5 & 6.5 & 4.5 & 5.0 \\ (0.3,0.4,0.5,0.6) & (0.1,0.2,0.3,0.4) & (0.5,0.6,0.7,0.8) & (0.3,0.4,0.5,0.6) \\ (0.7,0.8,0.9,1) & (0.3,0.4,0.5,0.6) & (0.5,0.6,0.7,0.8) & (0.3,0.4,0.5,0.6) \end{pmatrix} \end{array}. \tag{21}$$

Notice that real numbers can be written as trapezium fuzzy numbers such as $2.0 = (2.0, 2.0, 2.0, 2.0)$ and $1.8 = (1.8, 1.8, 1.8, 1.8)$.

For the benefit attribute f_5, we have

$$a_{5l}^{\max} = 0.5, \qquad a_{5m_1}^{\max} = 0.6, \qquad a_{5m_2}^{\max} = 0.7, \qquad a_{5r}^{\max} = 0.8.$$

Table 1
Decision information given by the decision maker

Air-fighters	Attributes					
	f_1 (mach)	f_2 (mile × 10^3)	f_3 (lb. × 10^4)	f_4 (\$ × 10^6)	f_5	f_6
x_1	2.0	1.5	2.0	5.5	medium	very high
x_2	2.5	2.7	1.8	6.5	low	medium
x_3	1.8	2.0	2.1	4.5	high	high
x_4	2.2	1.8	2.0	5.0	medium	medium

Table 2
The relations between linguistic variables and positive trapezium fuzzy numbers

Linguistic variables	Trapezium fuzzy numbers
Very high (VH)	(0.7, 0.8, 0.9, 1)
High (H)	(0.5, 0.6, 0.7, 0.8)
Medium (M)	(0.3, 0.4, 0.5, 0.6)
Low (L)	(0.1, 0.2, 0.3, 0.4)
Very low (VL)	(0, 0.1, 0.2, 0.3)

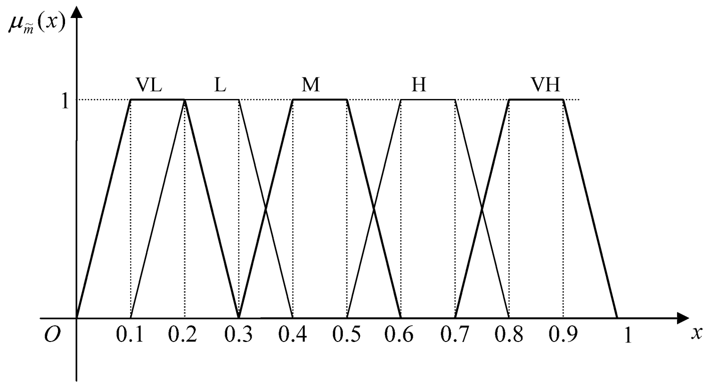

Fig. 4. Linguistic variables.

So the normalization positive trapezium fuzzy numbers of x_j $(j = 1, 2, 3, 4)$ on the attribute f_5 can be calculated as follows using Eq. (4):

$$\tilde{r}_{51} = \left(\frac{a_{51l}}{a_{5r}^{\max}}, \frac{a_{51m_1}}{a_{5m_2}^{\max}}, \frac{a_{51m_2}}{a_{5m_1}^{\max}} \wedge 1, \frac{a_{51r}}{a_{5l}^{\max}} \wedge 1\right) = \left(\frac{0.3}{0.8}, \frac{0.4}{0.7}, \frac{0.5}{0.6} \wedge 1, \frac{0.6}{0.5} \wedge 1\right)$$
$$= (0.375, 0.571, 0.831, 1),$$
$$\tilde{r}_{52} = \left(\frac{a_{52l}}{a_{5r}^{\max}}, \frac{a_{52m_1}}{a_{5m_2}^{\max}}, \frac{a_{52m_2}}{a_{5m_1}^{\max}} \wedge 1, \frac{a_{52r}}{a_{5l}^{\max}} \wedge 1\right) = \left(\frac{0.1}{0.8}, \frac{0.2}{0.7}, \frac{0.3}{0.6} \wedge 1, \frac{0.4}{0.5} \wedge 1\right)$$
$$= (0.125, 0.286, 0.5, 0.8),$$
$$\tilde{r}_{53} = \left(\frac{a_{53l}}{a_{5r}^{\max}}, \frac{a_{53m_1}}{a_{5m_2}^{\max}}, \frac{a_{53m_2}}{a_{5m_1}^{\max}} \wedge 1, \frac{a_{53r}}{a_{5l}^{\max}} \wedge 1\right) = \left(\frac{0.5}{0.8}, \frac{0.6}{0.7}, \frac{0.7}{0.6} \wedge 1, \frac{0.8}{0.5} \wedge 1\right)$$
$$= (0.625, 0.875, 1, 1)$$

and

$$\tilde{r}_{54} = \left(\frac{a_{54l}}{a_{5r}^{\max}}, \frac{a_{54m_1}}{a_{5m_2}^{\max}}, \frac{a_{54m_2}}{a_{5m_1}^{\max}} \wedge 1, \frac{a_{54r}}{a_{5l}^{\max}} \wedge 1\right) = \left(\frac{0.3}{0.8}, \frac{0.4}{0.7}, \frac{0.5}{0.6} \wedge 1, \frac{0.6}{0.5} \wedge 1\right)$$
$$= (0.375, 0.571, 0.831, 1).$$

In a similar way, the fuzzy decision matrix mention above (i.e., Eq. (21)) can be transformed into the following normalization positive trapezium fuzzy number matrix with Eqs. (4) and (5):

$$\tilde{\boldsymbol{R}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$$

$$\tilde{\boldsymbol{R}} = \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} \begin{pmatrix} 0.8 & 1 & 0.72 & 0.88 \\ 0.55 & 1 & 0.74 & 0.67 \\ 0.95 & 0.86 & 1 & 0.95 \\ 0.82 & 0.69 & 1 & 0.9 \\ (0.375, 0.571, 0.831, 1) & (0.125, 0.286, 0.5, 0.8) & (0.625, 0.857, 1, 1) & (0.375, 0.571, 0.831, 1) \\ (0.7, 0.889, 1, 1) & (0.3, 0.444, 0.625, 0.875) & (0.5, 0.667, 0.875, 1) & (0.3, 0.444, 0.625, 0.875) \end{pmatrix}. \tag{22}$$

Notice that real numbers can be written as trapezium fuzzy numbers such as $0.8 = (0.8, 0.8, 0.8, 0.8)$ and $0.67 = (0.67, 0.67, 0.67, 067)$.

Using Eq. (16) and combining with Eqs. (20) and (22), we can construct the linear programming problem:

$$\begin{aligned}
&\max\{\lambda_{12} + \lambda_{32} + \lambda_{13} + \lambda_{34}\} \\
&\quad -0.976\omega_1 - 1.296\omega_2 + 0.423\omega_3 + 0.5826\omega_4 + 0.8992\omega_5 + 1.3021\omega_6 \\
&\qquad + 0.1867v_{1l} + 0.3733v_{1m_1} + 0.3733v_{1m_2} + 0.1867v_{1r} + 0.2767v_{2l} \\
&\qquad + 0.5533v_{2m_1} + 0.5533v_{2m_2} + 0.2767v_{2r} - 0.0767v_{3l} - 0.1533v_{3m_1} \\
&\qquad - 0.1533v_{3m_2} - 0.0767v_{3r} - 0.12v_{4l} - 0.24v_{4m_1} - 0.24v_{4m_2} - 0.12v_{4r} \\
&\qquad - 0.25v_{5l} - 0.5707v_{5m_1} - 0.554v_{5m_2} - 0.1333v_{5r} - 0.3333v_{6l} - 0.742v_{6m_1} \\
&\qquad - 0.6667v_{6m_2} - 0.143v_{6r} \leqslant -1, \\
&\quad 0.36\omega_1 + 0.6975\omega_2 - 0.1629\omega_3 - 0.1963\omega_4 - 0.3091\omega_5 - 0.5118\omega_6 \\
&\qquad - 0.0667v_{1l} - 0.1333v_{1m_1} - 0.1333v_{1m_2} - 0.0667v_{1r} - 0.15v_{2l} - 0.3v_{2m_1} \\
&\qquad - 0.3v_{2m_2} - 0.15v_{2r} + 0.03v_{3l} + 0.06v_{3m_1} + 0.06v_{3m_2} + 0.03v_{3r} \\
&\qquad + 0.0433v_{4l} + 0.0867v_{4m_1} + 0.0867v_{4m_2} + 0.0433v_{4r} + 0.0833v_{5l} \\
&\qquad + 0.19v_{5m_1} + 0.2207v_{5m_2} + 0.0667v_{5r} + 0.1333v_{6l} + 0.2967v_{6m_1} \\
&\qquad + 0.25v_{6m_2} + 0.0477v_{6r} - \lambda_{12} \leqslant 0, \\
&\quad 0.4816\omega_1 + 0.4524\omega_2 - 0.2604\omega_3 - 0.5239\omega_4 - 0.5901\omega_5 - 0.2785\omega_6 \\
&\qquad - 0.0933v_{1l} - 0.1867v_{1m_1} - 0.1867v_{1m_2} - 0.0933v_{1r} - 0.0867v_{2l} \\
&\qquad - 0.1733v_{2m_1} - 0.1733v_{2m_2} - 0.0867v_{2r} + 0.0467v_{3l} + 0.0933v_{3m_1} \\
&\qquad + 0.0933v_{3m_2} + 0.0467v_{3r} + 0.1033v_{4l} + 0.2067v_{4m_1} + 0.2067v_{4m_2} \\
&\qquad + 0.1033v_{4r} + 0.1667v_{5l} + 0.3807v_{5m_1} + 0.3333v_{5m_2} + 0.0667v_{5r} \\
&\qquad + 0.0667v_{6l} + 0.1487v_{6m_1} + 0.1667v_{6m_2} + 0.0477v_{6r} - \lambda_{32} \leqslant 0, \\
&\quad -0.1216\omega_1 + 0.2451\omega_2 + 0.0975\omega_3 + 0.3276\omega_4 + 0.281\omega_5 - 0.2333\omega_6 \\
&\qquad + 0.0267v_{1l} + 0.0533v_{1m_1} + 0.0533v_{1m_2} + 0.0267v_{1r} - 0.0633v_{2l} \\
&\qquad - 0.1267v_{2m_1} - 0.1267v_{2m_2} - 0.0633v_{2r} - 0.0167v_{3l} - 0.0333v_{3m_1} \\
&\qquad - 0.0333v_{3m_2} - 0.0167v_{3r} - 0.06v_{4l} - 0.12v_{4m_1} - 0.12v_{4m_2} - 0.06v_{4r} \\
&\qquad - 0.0833v_{5l} - 0.1907v_{5m_1} - 0.1127v_{5m_2} + 0v_{5r} + 0.0667v_{6l} + 0.0148v_{6m_1} \\
&\qquad + 0.0833v_{6m_2} + 0v_{6r} - \lambda_{13} \leqslant 0,
\end{aligned}$$

$$\begin{aligned}
&0.256\omega_1 - 0.0987\omega_2 - 0.0975\omega_3 - 0.19\omega_4 - 0.281\omega_5 - 0.2785\omega_6 - 0.0267v_{1l}\\
&\quad - 0.0533v_{1m_1} - 0.0533v_{1m_2} - 0.0267v_{1r} + 0.0233v_{2l} + 0.0467v_{2m_1}\\
&\quad + 0.0467v_{2m_2} + 0.0233v_{2r} + 0.0167v_{3l} + 0.0333v_{3m_1} + 0.0333v_{3m_2}\\
&\quad + 0.0167v_{3r} + 0.0333v_{4l} + 0.0667v_{4m_1} + 0.0667v_{4m_2} + 0.0333v_{4r}\\
&\quad + 0.0833v_{5l} + 0.1907v_{5m_1} + 0.1127v_{5m_2} + 0v_{5r} + 0.0667v_{6l}\\
&\quad + 0.1487v_{6m_1} + 0.1667v_{6m_2} + 0.0477v_{6r} - \lambda_{34} \leqslant 0,\\
&\omega_i \geqslant 0.001, \qquad v_{il} \geqslant 0, \qquad v_{im_1} \geqslant 0, \qquad v_{im_2} \geqslant 0, \qquad v_{ir} \geqslant 0\\
&\quad (i = 1, 2, \ldots, 6),\\
&\sum_{i=1}^{6} \omega_i = 1,\\
&\lambda_{12} \geqslant 0, \qquad \lambda_{32} \geqslant 0, \qquad \lambda_{13} \geqslant 0, \qquad \lambda_{34} \geqslant 0.
\end{aligned} \tag{23}$$

Solving Eq. (23) using the existing Simplex method software, we can obtain the optimal solutions as follows:

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_6)^T = (0.194, 0.261, 0.15, 0.164, 0.148, 0.083)^T \tag{24}$$

and

$$\begin{aligned}
\tilde{\boldsymbol{v}} &= (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_6)\\
&= \big(0.487, 0.453, 0.505, 0.502, (0.505, 0.510, 0.510, 0.520),\\
&\quad (0.521, 0.537, 0.537, 0.548)\big).
\end{aligned} \tag{25}$$

Using Eq. (17) and combining with Eqs. (24) and (25), the fuzzy positive ideal solution can be calculated as follows:

$$\begin{aligned}
\tilde{\boldsymbol{a}}^* &= (\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_6^*)^T\\
&= \big(2.510, 1.736, 3.367, 3.061, (0.341, 0.345, 0.345, 0.351),\\
&\quad (0.628, 0.647, 0.647, 0.660)\big)^T.
\end{aligned}$$

The square of the distance of each air-fighter from the FPIS $\tilde{\boldsymbol{a}}^*$ can be calculated using Eq. (8) as follows:

$$S_1 = 6.336, \qquad S_2 = 7.733, \qquad S_3 = 6.102, \qquad S_4 = 6.502.$$

So the ranking order of four air-fighters is generated as follows:

$$x_3 \succ x_1 \succ x_4 \succ x_2.$$

Obviously, the best selection is the air-fighter x_3.

5. Short conclusions

Most multiattribute decision making problems include both quantitative and qualitative attributes which are often assessed using imprecise data and human judgments. Fuzzy set theory is well suited to dealing with such decision problems. In this paper, the LINMAP method [24] is further developed to solve multiattribute decision making problems in fuzzy environments. Linguistic variables as well as crisp numerical values are used to assess qualitative and quantitative attributes. In particular, trapezium fuzzy numbers are used in this paper to assess alternatives with respect to qualitative attributes.

A fuzzy linear programming (FLP) model was constructed to rank alternative decisions using the pairwise comparisons between alternatives, which can be used in both crisp and fuzzy environments. In the FLP model, the normalization constraints on weights are imposed, which ensures that the weights generated are not zero. The technique can be used to generate consistent and reliable ranking order of alternatives in question.

The developed method is illustrated using an air-fighter selection problem [14]. It is expected to be applicable to decision problems in many areas, especially in situations where multiple decision makers are involved and the weights of attributes are not provided a priori.

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