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Point estimation of simultaneous methods for solving polynomial equations: A survey (II)

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Dedicated to Professor Stephen Smale on the occasion of his 75th birthday

Abstract

The construction of computationally verifiable initial conditions which provide both the guaranteed and fast convergence of the numerical root-finding algorithm is one of the most important problems in solving nonlinear equations. Smale's "point estimation theory" from 1981 was a great advance in this topic; it treats convergence conditions and the domain of convergence in solving an equation $f(z)=0$ using only the information of f at the initial point z_0. The study of a general problem of the construction of initial conditions of practical interest providing guaranteed convergence is very difficult, even in the case of algebraic polynomials. In the light of Smale's point estimation theory, an efficient approach based on some results concerning localization of polynomial zeros and convergent sequences is applied in this paper to iterative methods for the simultaneous determination of simple zeros of polynomials. We state new, improved initial conditions which provide the guaranteed convergence of frequently used simultaneous methods for solving algebraic equations: Ehrlich–Aberth's method, Ehrlich–Aberth's method with Newton's correction, Börsch-Supan's method with Weierstrass' correction and Halley-like (or Wang–Zheng) method. The introduced concept offers not only a clear insight into the convergence analysis of sequences generated by the considered methods, but also explicitly gives their order of convergence. The stated initial conditions are of significant practical importance since they are computationally verifiable; they depend only on the coefficients of a given polynomial, its degree n and initial approximations to polynomial zeros.

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Keywords: Point estimation theory; Localization of zeros; Polynomial zeros; Simultaneous methods; Guaranteed convergence; Initial conditions

1. Point estimation theory based on sequence approach

One of the most important problems in solving nonlinear equations is stating such initial conditions which provide the guaranteed convergence of the applied numerical algorithm. Evidently, only those conditions which depend on attainable data are useful from a practical point of view. First results which deal with computationally verifiable initial conditions providing the guaranteed convergence were stated and developed in [12,31–35]. The research on

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this topic was later continued in [3,7,8,11,13,17–30,37,38,36], and other papers. This approach, often referred to as "point estimation theory", considers convergence conditions and the domain of convergence in solving an equation $f(z)=0$ using only the information of f at the initial point z_0. In this way, it overcomes difficulties that appear in the traditional treating the convergence conditions based on the asymptotical convergence analysis. This analysis is only of theoretical importance since it involves (in the estimation procedure) some unknown parameters as constants, even the (unknown) roots of equation, or uses the terminology as "sufficiently good (close enough) approximations" without quantitative (and computationally verifiable) characterization of the closeness of these approximations to the roots. A review of Smale's point estimation theory and related results can be found in [22,23, Chapters 1–3] and we omit details in this paper.

The study of a general problem of the construction of initial conditions and the choice of initial approximations furnishing guaranteed convergence of a root-finding method is very difficult, even in the case of algebraic polynomials. In this particular case, these conditions should depend only on the coefficients of a given polynomial $P(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ of degree n and the vector of initial approximations $\mathbf{z}^{(0)}=(z_1^{(0)},\ldots,z_n^{(0)})$. More details about the point estimation theory and its applications to algebraic polynomials can be found in the aforementioned papers and the references cited therein.

The paper [22] gives a survey of results concerning the guaranteed convergence of some frequently used iterative methods for the simultaneous determination of polynomial zeros as Durand–Kerner's method, Börsch-Supan's method, the square-root one parameter family. That study uses the concept of convergent iterative corrections proposed in [18]. In this paper, which can be regarded as the continuation of research presented in [22], we state another approach to the convergence analysis in the light of Smale's point estimation theory and based on convergent sequences and some results concerned with the localization of polynomial zeros. Using this approach we improve computationally verifiable initial conditions for several iterative methods which belong to the class of the most efficient and often used simultaneous methods for finding polynomial zeros. The introduced concept presents not only a clear insight into the convergence analysis of sequences produced by the considered methods, but also explicitly gives their order of convergence, which is the advantage in reference to the approach exposed in [22] and some other papers. It is worth noting that the aim of this paper is not only the demonstration of the point estimation theory applied to algebraic polynomials, but also the significant improvement of initial conditions for the four frequently used simultaneous methods for finding polynomial zeros.

The essential question in stating initial convergence conditions is how to express these conditions. The requested form should be computationally verifiable and, in addition, it must take into account some important properties as distribution of zeros, their separation and closeness to initial approximations. Let $I_n:=\{1,\ldots,n\}$ be the index set. For $i\in I_n$ and $m=0,1,\ldots$ let us introduce the quantity

$$W_i^{(m)}=\frac{P(z_i^{(m)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i^{(m)}-z_j^{(m)})}\quad (i\in I_n,\ m=0,1,\ldots)$$

which is often called Weierstrass' correction since it appeared in Weierstrass' paper [40]. As shown in [18], the above requirements can be fulfilled in a satisfactory way by expressing initial conditions in the form

$$w^{(0)}\leqslant c_n d^{(0)}, \tag{1.1}$$

where

$$w^{(0)}=\max_{1\leqslant i\leqslant n}|W_i^{(0)}|,\quad d^{(0)}=\min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}}|z_i^{(0)}-z_j^{(0)}|$$

and c_n is a real quantity depending only on the polynomial degree n. The use of form (1.1) is justified since it involves the requested properties; indeed, if the initial approximations are close enough, then the minimal distance $d^{(0)}$ can be regarded as a measure of separation of the zeros, while $w^{(0)}$ is related to the closeness of approximations to the zeros.

In [36] Wang and Zhao improved Smale's result for Newton's method and applied it to the Durand–Kerner method for the simultaneous determination of polynomial zeros. Their approach also led in a natural way to form (1.1). In both cases the quantity c_n is expressed as $c_n=1/(an+b)$, where a and b are suitably chosen positive constants.

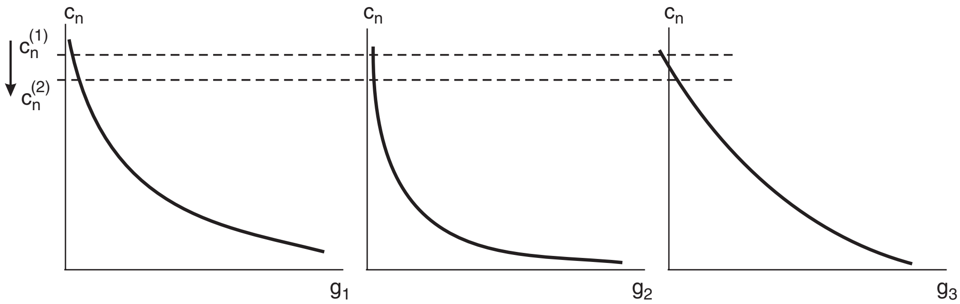

Fig. 1. The choice of i-factor c_n iteratively.

It turned out that initial conditions of this form are also convenient for other simultaneous methods for solving polynomial equations, as shown in the subsequent papers [2,3,17–30,37,36,42], etc. For these reasons, in the convergence analysis of simultaneous methods considered in this paper, we will also use initial conditions of form (1.1). The quantity c_n will be called the *inequality factor*, or i-factor for brevity. We emphasize that in the last years a special attention has been paid to the increase of i-factor c_n for the following obvious reason. From (1.1) we notice that a greater value of c_n allows a greater value of $|W_i^{(0)}|$. This means that initial approximations can be chosen more roughly, which is of evident interest in practical realization.

In this paper we will discuss as good as possible values of the i-factor c_n appearing in the initial condition (1.1) for some efficient and frequently used iterative methods for the simultaneous determination of polynomial zeros. We study the choice of "almost optimal" factor c_n. The notion "almost optimal" i-factor arises from (i) the presence of a system of (say) k inequalities and (ii) the use of computer arithmetic of finite precision:

(i) In the convergence analysis it is necessary to provide the validity of k substantial successive inequalities $g_1(c_n) \geqslant 0, \ldots, g_k(c_n) \geqslant 0$ (in this order), where all $g_i(c_n)$ are monotonically decreasing functions of c_n (see Fig. 1). The optimal value c_n would be determined as unique solution of the corresponding equations $g_i(c_n) = 0$. Since all equations cannot be satisfied simultaneously, we are constrained to find such c_n which makes the inequalities $g_i(c_n) \geqslant 0$ as sharp as possible. Since $g_i(c_n) \geqslant 0$ succeeds $g_j(c_n) \geqslant 0$ for $j < i$, we first find c_n so that the inequality $g_1(c_n) \geqslant 0$ is as sharp as possible and check the validity of all remaining inequalities $g_2(c_n) \geqslant 0, \ldots, g_k(c_n) \geqslant 0$. If some of them is not valid, we decrease c_n and repeat the process until all inequalities are satisfied. For demonstration, we give a particular example on Fig. 1. The third inequality $g_3(c_n) \geqslant 0$ is not satisfied for $c_n^{(1)}$ so that c_n takes a smaller value $c_n^{(2)}$ satisfying all three inequalities. In practice, the choice of c_n is performed iteratively, using some programming package, in our paper *Mathematica* 5.0.

(ii) Since computer arithmetic of finite precision is employed, the optimal value (the exact solution of $g_i(c_n) = 0$, if it exists for some i) cannot be represented exactly so that c_n should be decreased for few bits to satisfy the inequalities $g_i(c_n) > 0$. The required conditions (in the form of inequalities $g_i(c_n) \geqslant 0$) are still satisfied with great accuracy. We stress that this slight decrease of the i-factor c_n in reference to the optimal value is negligible from a practical point of view. For this reason, the constants a and b appearing in $c_n = 1/(an + b)$ are rounded to one decimal place for all four methods considered in this paper,

The entries of c_n, obtained in this way and presented in this paper, are increased (and, thus, improved) compared with those given in the literature, which means that newly established initial conditions for the guaranteed convergence of the considered methods are weakened (see Fig. 2 in Section 7).

In what follows we will present in short the basic idea and concept of the convergence analysis involving initial conditions of form (1.1) which guarantee the convergence of the considered methods.

Let $z_1^{(m)}, \ldots, z_n^{(m)}$ be approximations to the simple zeros $\zeta_1, \ldots, \zeta_n$ of a polynomial P, generated by some iterative method for the simultaneous determination of zeros at the mth iterative step and let $u_i^{(m)} = z_i^{(m)} - \zeta_i$ $(i \in I_n)$. Our main goal is to study the convergence of the sequences $\{u_1^{(m)}\}, \ldots, \{u_n^{(m)}\}$ under the initial condition (1.1). We will use the initial condition $w^{(0)} < c_n d^{(0)}$, where $w^{(m)}$ is the maximal Weierstrass correction and $d^{(m)}$ is the minimal distance between approximations at the mth iteration. In our convergence analysis the main attention will be devoted to the choice of c_n which guarantees the convergence of the considered simultaneous methods.

Throughout this paper a closed disk with center c and radius r will be denoted by the parametric notation $\{c; r\}$. For simplicity, we will often omit the iteration index m and denote entries in the latter $(m+1)$st iteration by the symbol $\widehat{\ }$. In our analysis we will use the following result proved in [23].

Theorem 1.1. *If the i-factor c_n appearing in* (1.1) *is not greater than* $1/(2n)$, *then the disks defined by*

$$Z_i^{(0)} = \left\{ z_i^{(0)}; \frac{1}{1-nc_n} |W_i^{(0)}| \right\} \quad (i \in I_n)$$

are mutually disjoint and each of them contains one and only one zero of P.

The point estimation approach presented in this paper consists of the following main steps:

1. If $c_n \leqslant 1/(2n)$ and (1.1) holds, from Theorem 1.1 it follows that the inequalities

$$|u_i^{(0)}| = |z_i^{(0)} - \zeta_i| < \frac{|W_i^{(0)}|}{1-nc_n} \tag{1.2}$$

are valid for each $i \in I_n$. These inequalities have an important role in the estimation procedure involved in the convergence analysis of the sequences $\{z_i^{(m)}\}$ which are produced by the considered simultaneous method.

2. In the next step we derive the inequalities

$$d < \tau_n \hat{d} \quad \text{and} \quad |\widehat{W}_i| < \beta_n |W_i|,$$

which involve the minimal distances and the maximal Weierstrass corrections at two successive iterative steps. The i-factor c_n appearing in (1.1) has to be chosen to provide such values of τ_n and β_n which furnish the following implication:

$$w < c_n d \Rightarrow \widehat{w} < c_n \hat{d}.$$

This has the essential role in the proof of convergence theorems by induction. Let us note that the above implication will hold if $\tau_n \beta_n < 1$.

3. In the final step we derive the inequalities of the form

$$|u_i^{(m)}| \leqslant \gamma(n, d^{(m)}) |u_i^{(m)}|^p \left(\sum_{j \neq i} |u_j^{(m)}|^q \right)^r \tag{1.3}$$

for $i = 1, \ldots, n$ and $m = 0, 1, \ldots,$ and prove that all sequences $\{|u_1^{(m)}|\}, \ldots, \{|u_n^{(m)}|\}$ tend to 0 under condition (1.1) (with suitably chosen c_n), which means that $z_i^{(m)} \to \zeta_i$ $(i \in I_n)$. The order of convergence of these sequences is obtained from (1.3) and it is equal to $p + qr$.

2. Some auxiliary results

In order to study iterative methods which do not involve Weierstrass' corrections W_i, appearing in the initial conditions of form (1.1), it is necessary to establish a suitable relation between $P(z_i)/P'(z_i)$ and W_i. Applying the logarithmic derivative to $P(t)$, represented by the Lagrangian interpolation formula

$$P(t) = \left(\sum_{j=1}^{n} \frac{W_j}{t - z_j} + 1 \right) \prod_{j=1}^{n} (t - z_j) \tag{2.1}$$

for distinct complex numbers $z_1, \ldots, z_n$, one obtains

$$\frac{P'(t)}{P(t)} = \sum_{j \neq i} \frac{1}{t - z_j} + \frac{\sum_{j \neq i} (W_j/(t-z_j)) + 1 - (t - z_i) \sum_{j \neq i} (W_j/(t-z_j)^2)}{W_i + (t - z_i)[\sum_{j \neq i} (W_j/(t-z_j)) + 1]}.$$

Putting $t = z_i$ in this formula we get (see [5])

$$\frac{P'(z_i)}{P(z_i)} = \sum_{j \neq i} \frac{1}{z_i - z_j} + \frac{\sum_{j \neq i} (W_j/(z_i - z_j)) + 1}{W_i}. \tag{2.2}$$

In the next sections we will apply the three-stage procedure described in Section 1 to some frequently used simultaneous methods. This procedure needs certain estimates of the same type and, to avoid the repetition, we give them in the following lemma.

Lemma 2.1. *For distinct complex numbers* $z_1, \ldots, z_n, \hat{z}_1, \ldots, \hat{z}_n$ *let*

$$d = \min_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} |z_i - z_j|, \qquad \hat{d} = \min_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} |\hat{z}_i - \hat{z}_j| \quad (i \in I_n),$$

and let the inequality

$$|\hat{z}_i - z_i| \leqslant \lambda_n d \quad (i \in I_n) \tag{2.3}$$

hold. Then

$$|\hat{z}_i - z_j| \geqslant (1 - \lambda_n) d \quad (i, j \in I_n), \tag{2.4}$$

$$|\hat{z}_i - \hat{z}_j| \geqslant (1 - 2\lambda_n) d \quad (i, j \in I_n), \tag{2.5}$$

and

$$\left| \prod_{j \neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j} \right| \leqslant \left(1 + \frac{\lambda_n}{1 - 2\lambda_n} \right)^{n-1}. \tag{2.6}$$

The proofs of the above assertions are based on triangular inequalities and the definition of the minimal distance, and we omit them.

3. The Ehrlich–Aberth method

In this section we will use the Newton and Weierstrass correction given, respectively, by

$$N_i^{(m)} = \frac{P(z_i^{(m)})}{P'(z_i^{(m)})} \quad \text{and} \quad W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod_{j \neq i} (z_i^{(m)} - z_j^{(m)})} \qquad (i \in I_n;\ m = 0, 1, \ldots).$$

We are concerned here with one of the most efficient numerical methods for the simultaneous approximation of all zeros of a polynomial, given by the iterative formula:

$$z_i^{(m+1)} = z_i^{(m)} - \frac{1}{1/N_i^{(m)} - \sum_{j \neq i} (1/(z_i^{(m)} - z_j^{(m)}))} \quad (i \in I_n;\ m = 0, 1, \ldots). \tag{3.1}$$

Although this method was first suggested by Maehly [14] in 1954 for refinement of the Newton method and used by Börsch-Supan [3] in finding a posteriori error bounds for the zeros of polynomials, it is more often referred to as Ehrlich–Aberth's method. Ehrlich [9] proved the cubic convergence of this method and Aberth [1] gave an important contribution to its practical realization.

Our aim is to state practically applicable initial conditions of form (1.1) which enable a guaranteed convergence of the Ehrlich–Aberth method (3.1), shorter the E–A method in the sequel. As mentioned above, in our analysis we will sometimes omit iteration index m and new entries in the later $(m+1)$st iteration will be additionally stressed by the symbol $\widehat{\ }$ ("*hat*"). For example, instead of $z_i^{(m)}, z_i^{(m+1)}, W_i^{(m)}, W_i^{(m+1)}, d^{(m)}, d^{(m+1)}, N_i^{(m)}, N_i^{(m+1)}$, etc., we will write $z_i, \hat{z}_i, W_i, \widehat{W}_i, d, \hat{d}, N_i, \widehat{N}_i$. According to this we denote

$$w = \max_{1 \leqslant i \leqslant n} |W_i|, \quad \widehat{w} = \max_{1 \leqslant i \leqslant n} |\widehat{W}_i|.$$

This denotation will be also used in the next sections.

First we present a lemma concerned with the localization of polynomial zeros.

Lemma 3.1. *Assume that the following condition*

$$w < c_n d, \quad c_n = \begin{cases} \dfrac{1}{2n+1.4}, & 3 \leqslant n \leqslant 7, \\ \dfrac{1}{2n}, & n \geqslant 8, \end{cases} \tag{3.2}$$

is satisfied. Then each disk $\{z_i; 1/(1-nc_n)|W_i|\}$ $(i \in I_n)$ *contains one and only one zero of* P.

The above assertion follows from Theorem 1.1 under condition (3.2).

Lemma 3.2. *Let* $z_1, \ldots, z_n$ *be disjoint approximations to the zeros* $\zeta_1, \ldots, \zeta_n$ *of a polynomial* P *of degree* n, *and let* $\hat{z}_1, \ldots, \hat{z}_n$ *be new respective approximations obtained by the E–A method* (3.1). *Then the following formula is valid*:

$$\widehat{W}_i = -(\hat{z}_i - z_i)^2 \sum_{j \neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} \prod_{j \neq i} \left(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\right). \tag{3.3}$$

Proof. From the iterative formula (3.1) one obtains

$$\frac{1}{\hat{z}_i - z_i} = \sum_{j \neq i} \frac{1}{z_i - z_j} - \frac{P'(z_i)}{P(z_i)},$$

so that, using (2.2),

$$\begin{aligned} \frac{W_i}{\hat{z}_i - z_i} &= W_i \left(\sum_{j \neq i} \frac{1}{z_i - z_j} - \frac{P'(z_i)}{P(z_i)} \right) = -W_i \left[\frac{1}{W_i} \left(\sum_{j \neq i} \frac{W_j}{z_i - z_j} + 1 \right) \right] \\ &= -\sum_{j \neq i} \frac{W_j}{z_i - z_j} - 1. \end{aligned}$$

According to this we have

$$\begin{aligned} \sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1 &= \frac{W_i}{\hat{z}_i - z_i} + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j} + 1 \\ &= -\sum_{j \neq i} \frac{W_j}{z_i - z_j} - 1 + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j} + 1 \\ &= -(\hat{z}_i - z_i) \sum_{j \neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)}. \end{aligned}$$

Taking into account the last expression, returning to (2.1) we find for $t = \hat{z}_i$

$$\begin{aligned} P(\hat{z}_i) &= \left(\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1 \right) \prod_{j=1}^{n} (\hat{z}_i - z_j) \\ &= -(\hat{z}_i - z_i)^2 \sum_{j \neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} \prod_{j \neq i} (\hat{z}_i - z_j). \end{aligned}$$

After dividing by $\prod_{j \neq i} (\hat{z}_i - \hat{z}_j)$ and some rearrangement, we obtain formula (3.3). □

Let us introduce the abbreviations:

$$\rho_n = \frac{1}{1-nc_n}, \quad \gamma_n = \frac{1}{1-\rho_n c_n - (n-1)(\rho_n c_n)^2},$$

$$\lambda_n = \rho_n c_n (1-\rho_n c_n)\gamma_n, \quad \beta_n = \frac{(n-1)\lambda_n^2}{1-\lambda_n}\left(1+\frac{\lambda_n}{1-2\lambda_n}\right)^{n-1}.$$

Lemma 3.3. *Let $z_1, \ldots, z_n$ be approximations produced by the E–A method* (3.1) *and let $u_i = z_i - \zeta_i$, $\hat{u}_i = \hat{z}_i - \zeta_i$. If $n \geqslant 3$ and inequality* (3.2) *holds, then*

(i) $d < \frac{1}{1-2\lambda_n}\hat{d}$;
(ii) $\widehat{w} < \beta_n w$;
(iii) $\widehat{w} < c_n \hat{d}$;
(iv) $|\hat{u}_i| \leqslant \frac{\gamma_n}{d^2}|u_i|^2 \sum_{j\neq i}|u_j|$.

Proof. According to the initial condition (3.2) and Lemma 3.1 we have

$$|u_i| = |z_i - \zeta_i| \leqslant \rho_n |W_i| \leqslant \rho_n w < \rho_n c_n d. \tag{3.4}$$

Having in mind (3.4) and the definition of the minimal distance d we find

$$|z_j - \zeta_i| \geqslant |z_j - z_i| - |z_i - \zeta_i| > d - \rho_n c_n d = (1-\rho_n c_n)d. \tag{3.5}$$

Using the identity

$$\frac{P'(z_i)}{P(z_i)} = \sum_{j=1}^{n} \frac{1}{z_i - \zeta_j} = \frac{1}{u_i} + \sum_{j\neq i} \frac{1}{z_i - \zeta_j}, \tag{3.6}$$

from (3.1) we get

$$\begin{aligned} \hat{u}_i = \hat{z}_i - \zeta_i &= z_i - \zeta_i - \frac{1}{1/u_i + \sum_{j\neq i}(1/(z_i-\zeta_j)) - \sum_{j\neq i}(1/(z_i - z_j))} \\ &= u_i - \frac{u_i}{1-u_i S_i} = -\frac{u_i^2 S_i}{1-u_i S_i}, \end{aligned} \tag{3.7}$$

where

$$S_i = \sum_{j\neq i} \frac{u_j}{(z_i-\zeta_j)(z_i-z_j)}.$$

Using the definition of d and the bounds (3.4) and (3.5), we estimate

$$|u_i S_i| \leqslant |u_i| \sum_{j\neq i} \frac{|u_j|}{|z_i - \zeta_j||z_i - z_j|} < \rho_n c_n d \cdot \frac{(n-1)\rho_n c_n d}{(1-\rho_n c_n)d \cdot d} = \frac{(\rho_n c_n)^2(n-1)}{1-\rho_n c_n}. \tag{3.8}$$

Now, by (3.4) and (3.8), we find from (3.1):

$$\begin{aligned} |\hat{z}_i - z_i| &= \left|\frac{u_i}{1-u_i S_i}\right| \leqslant \frac{|u_i|}{1-|u_i S_i|} < \frac{|u_i|}{1-((\rho_n c_n)^2(n-1)/(1-\rho_n c_n))} \\ &< \frac{\rho_n c_n (1-\rho_n c_n)}{1-\rho_n c_n - (\rho_n c_n)^2(n-1)} d = \rho_n c_n (1-\rho_n c_n)\gamma_n d = \lambda_n d, \end{aligned} \tag{3.9}$$

and also

$$|\hat{z}_i - z_i| < (1-\rho_n c_n)\gamma_n |u_i| < (1-\rho_n c_n)\gamma_n \rho_n |W_i|. \tag{3.10}$$

Having in mind (3.9), according to Lemma 2.1 we conclude that the estimates $|\hat{z}_i - z_j| > (1-\lambda_n)d$ and $|\hat{z}_i - \hat{z}_j| > (1-2\lambda_n)d$ $(i \in I_n)$ hold. From the last inequality we find

$$\frac{d}{\hat{d}} < \frac{1}{1-2\lambda_n} \quad \text{for every } n \geqslant 3, \tag{3.11}$$

which proves assertion (i) of Lemma 3.3.

Using the starting inequality $w/d < c_n$ and the bounds (3.9), (3.10), (2.4)–(2.6), we estimate the quantities involved in (3.3):

$$|\widehat{W}_i| \leqslant |\hat{z}_i - z_i|^2 \sum_{j\neq i} \frac{|W_j|}{|\hat{z}_i - z_j||z_i - z_j|} \prod_{j\neq i}\left(1 + \frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\right)$$
$$< \frac{(n-1)\lambda_n^2}{(1-\lambda_n)}\left(1 + \frac{\lambda_n}{1-2\lambda_n}\right)^{n-1}|W_i| = \beta_n|W_i|,$$

where β_n is the term in front of $|W_i|$ depending on n.

Therefore, we have

$$\widehat{w} < \beta_n w \tag{3.12}$$

so that, by (3.2), (3.11) and (3.12), we estimate

$$\widehat{w} < \beta_n w < \beta_n c_n d < \frac{\beta_n}{1-2\lambda_n} c_n \hat{d}.$$

Since

$$\frac{\beta_n}{1-2\lambda_n} < 0.95 < 1 \quad \text{for all } 3 \leqslant n \leqslant 7,$$

and

$$\frac{\beta_n}{1-2\lambda_n} < 0.78 < 1 \quad \text{for all } n \geqslant 8,$$

we have

$$\widehat{w} < c_n \hat{d}, \quad n \geqslant 3.$$

In this way we have proved assertions (ii) and (iii) of Lemma 3.3.

Using the already derived bounds we find

$$|\hat{u}_i| \leqslant \frac{|u_i|^2|S_i|}{1-|u_i S_i|} < \frac{|u_i|^2}{1-((\rho c_n)^2(n-1)/(1-\rho_n c_n))} \sum_{j\neq i} \frac{|u_j|}{|z_i - \zeta_j||z_i - z_j|}$$
$$< \frac{1-\rho_n c_n}{1-\rho_n c_n - (\rho_n c_n)^2(n-1)} |u_i|^2 \sum_{j\neq i} \frac{|u_j|}{(1-\rho_n c_n)d \cdot d}$$
$$< \frac{1}{(1-\rho_n c_n - (\rho_n c_n)^2(n-1))d^2} |u_i|^2 \sum_{j\neq i} |u_j|,$$

wherefrom

$$|\hat{u}_i| < \frac{\gamma_n}{d^2}|u_i|^2 \sum_{j\neq i}|u_j|. \tag{3.13}$$

This strict inequality is derived assuming that $u_i \neq 0$ (see Remark 2 in this section). If we include the case $u_i = 0$ then it follows

$$|\hat{u}_i| \leqslant \frac{\gamma_n}{d^2}|u_i|^2 \sum_{j\neq i}|u_j|$$

and assertion (iv) of Lemma 3.3 is proved. □

Remark 1. The assertions of form (i)–(iv) of Lemma 3.3 will be presented in Sections 4–6 for the three another methods, but for different i-factor c_n and specific entries of $\lambda_n, \beta_n, \gamma_n$.

Now we give the convergence theorem for the E–A method (3.1) which involves only initial approximations to the zeros, the polynomial coefficients and the polynomial degree n.

Theorem 3.1. *Under the initial condition*

$$w^{(0)} < c_n d^{(0)}, \tag{3.14}$$

where c_n is given by (3.2), *the E–A method* (3.1) *is convergent with the third order of convergence*.

Proof. The convergence analysis is based on the estimate procedure of the error $u_i^{(m)} = z_i^{(m)} - \zeta_i$. The proof is by induction with the argumentation used for inequalities (i)–(iv) of Lemma 3.3. Since the initial condition (3.14) coincides with (3.2), all estimates given in Lemma 3.3 are valid for the index $m = 1$. Actually, this is the part of the proof with respect to $m = 1$. Furthermore, inequality (iii) again reduces to the condition of form (3.2) and, therefore, assertions (i)–(iv) of Lemma 3.3 hold for the next index, and so on. All estimates and bounds for the index m are derived essentially in the same way as for $m = 0$. In fact, the implication

$$w^{(m)} < c_n d^{(m)} \;\Rightarrow\; w^{(m+1)} < c_n d^{(m+1)}$$

plays a key role in the convergence analysis of the E–A method (3.1) because it involves the initial condition (3.14) which causes the validity of all inequalities given in Lemma 3.3 for all $m = 0, 1, \dots$. Especially, regarding (3.11) and (3.13), we have

$$\frac{d^{(m)}}{d^{(m+1)}} < \frac{1}{1 - 2\lambda_n} \tag{3.15}$$

and

$$|u_i^{(m+1)}| \leqslant \frac{\gamma_n}{(d^{(m)})^2} |u_i^{(m)}|^2 \sum_{j \neq i} |u_j^{(m)}| \quad (i \in I_n) \tag{3.16}$$

for each iteration index $m = 0, 1, \dots$ if (3.14) holds.

Substituting

$$t_i^{(m)} = \left[\frac{(n-1)\gamma_n}{(1 - 2\lambda_n)(d^{(m)})^2}\right]^{1/2} |u_i^{(m)}|,$$

inequalities (3.16) become

$$t_i^{(m+1)} \leqslant \frac{(1 - 2\lambda_n) d^{(m)}}{(n-1) d^{(m+1)}} [t_i^{(m)}]^2 \sum_{j \neq i} t_j^{(m)},$$

wherefrom, by (3.15),

$$t_i^{(m+1)} < \frac{[t_i^{(m)}]^2}{n-1} \sum_{j \neq i} t_j^{(m)} \quad (i \in I_n). \tag{3.17}$$

By virtue of (3.4) we find

$$t_i^{(0)} = \sqrt{\frac{(n-1)\gamma_n}{(1 - 2\lambda_n)(d^{(0)})^2}} |u_i^{(0)}| < \rho_n c_n d^{(0)} \sqrt{\frac{(n-1)\gamma_n}{(1 - 2\lambda_n)(d^{(0)})^2}}$$

$$= \rho_n c_n \sqrt{\frac{(n-1)\gamma_n}{1 - 2\lambda_n}}$$

for each $i = 1, \ldots, n$. Taking

$$t = \max_{1 \leqslant i \leqslant n} t_i^{(0)} < \rho_n c_n \sqrt{\frac{(n-1)\gamma_n}{1-2\lambda_n}},$$

we come to the inequalities

$$t_i^{(0)} \leqslant t < 0.570 < 1 \quad (3 \leqslant n \leqslant 7)$$

and

$$t_i^{(0)} \leqslant t < 0.432 < 1 \quad (n \geqslant 8)$$

for all $i = 1, \ldots, n$. According to this we conclude from (3.17) that the sequences $\{t_i^{(m)}\}$ (and, consequently, $\{|u_i^{(m)}|\}$) tend to 0 for all $i = 1, \ldots, n$. Therefore, the E–A method (3.1) is convergent.

Taking into account that the quantity $d^{(m)}$ which appears in (3.16) is bounded and tends to $\min_{i \neq j} |\zeta_i - \zeta_j|$, and setting $u^{(m)} = \max_{1 \leqslant i \leqslant n} |u_i^{(m)}|$, from (3.16) we obtain

$$|u_i^{(m+1)}| \leqslant u^{(m+1)} < \frac{(n-1)\gamma_n}{(d^{(m)})^2} |u^{(m)}|^3,$$

which proves the cubical convergence. □

Remark 2. As usual in the convergence analysis of iterative methods (see, e.g. [10]), we could assume that the errors $u_i^{(m)} = z_i^{(m)} - \zeta_i$ $(i \in I_n)$ do not reach 0 for a finite m. However, if $u_i^{(m_0)} = 0$ for some indices $i_1, \ldots, i_k$ and $m_0 \geqslant 0$, we just take $z_{i_1}^{(m_0)}, \ldots, z_{i_k}^{(m_0)}$ as approximations to the zeros $\zeta_{i_1}, \ldots, \zeta_{i_k}$ and do not iterate further for the indices $i_1, \ldots, i_k$. If the sequences $\{u_i^{(m)}\}$ $(i \in I_n \backslash \{i_1, \ldots, i_k\})$ have the order of convergence q, then obviously the sequences $\{u_{i_1}^{(m)}\}, \ldots, \{u_{i_k}^{(m)}\}$ converge with the convergence rate at least q. This remark refers not only to the iterative method (3.1) but also to all methods considered in this paper. For this reason, we do not discuss further this point.

4. Ehrlich–Aberth's method with Newton's corrections

The convergence of the Ehrlich–Aberth method (3.1) can be accelerated using Newton's corrections $N_i^{(m)} = P(z_i^{(m)})/P'(z_i^{(m)})(i \in I_n; m = 0, 1, \ldots)$. In this way the following method for the simultaneous approximation of all simple zeros of a given polynomial P can be established

$$z_i^{(m+1)} = z_i^{(m)} - \frac{1}{1/N_i^{(m)} - \sum_{j \neq i} (1/(z_i^{(m)} - z_j^{(m)} + N_j^{(m)}))} \quad (i \in I_n), \tag{4.1}$$

where $m = 0, 1, \ldots$. This method, proposed by Nourein [16], is an obvious improvement of method (3.1) and it is one of the most efficient iterative methods for the simultaneous determination of polynomial zeros. The order of convergence of the Ehrlich–Aberth method with Newton's corrections (4.1) briefly the EAN method in the sequel, is *four* (see [16]). The great computational efficiency of method (4.1) is the consequence of the increased convergence order (from 3 to 4) with the negligible number of additional numerical operations since already calculated quantities $N_j^{(m)}$ are used in the sum of (4.1).

From Theorem 1.1 the following lemma treating inclusion disks can be stated:

Lemma 4.1. *Let* $z_1, \ldots, z_n$ *be distinct numbers satisfying the inequality*

$$w < c_n d, \quad c_n = \begin{cases} \dfrac{1}{2.2n + 1.9}, & 3 \leqslant n \leqslant 21, \\ \dfrac{1}{2.2n}, & n \geqslant 22. \end{cases} \tag{4.2}$$

Then the disks $\{z_1; (1/(1-nc_n))\,|W_1|\}, \ldots, \{z_n; (1/(1-nc_n))\,|W_n|\}$ *are mutually disjoint and each of them contains exactly one zero of a polynomial* P.

First we give the expression for the improved Weierstrass correction.

Lemma 4.2. *Let* $z_1, \ldots, z_n$ *be distinct approximations to the zeros* $\zeta_1, \ldots, \zeta_n$ *of a polynomial* P *of degree* n, *and let* $\hat{z}_1, \ldots, \hat{z}_n$ *be new respective approximations obtained by the EAN method* (4.1). *Then the following formula is valid*:

$$\widehat{W}_i = -(\hat{z}_i - z_i)(W_i \Sigma_{N,i} + (\hat{z}_i - z_i)\Sigma_{W,i}) \prod_{j \neq i} \left(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\right), \tag{4.3}$$

where

$$\Sigma_{N,i} = \sum_{j \neq i} \frac{N_j}{(z_i - z_j + N_j)(z_i - z_j)}, \quad \Sigma_{W,i} = \sum_{j \neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)}.$$

Relation (4.3) is obtained by combining the Lagrangian interpolation formula (2.1) for $t = \hat{z}_i$, the iterative formula (4.1) and identity (2.2), see the proof of Lemma 3.1 and [24].

In the sequel, we will use the abbreviations:

$$\rho_n = \frac{1}{1 - nc_n}, \quad \delta_n = 1 - \rho_n c_n - (n-1)\rho_n c_n,$$

$$\alpha_n = (1 - \rho_n c_n)((1 - \rho_n c_n)^2 - (n-1)\rho_n c_n),$$

$$\gamma_n = \frac{n-1}{\alpha_n - (n-1)^2(\rho_n c_n)^3}, \quad \lambda_n = \frac{\alpha_n \gamma_n \rho_n c_n}{n-1},$$

$$\beta_n = \lambda_n (n-1) \left(\frac{(1 - \rho_n c_n)^2 \rho_n c_n}{\alpha_n} + \frac{\lambda_n}{1 - \lambda_n}\right) \left(1 + \frac{\lambda_n}{1 - 2\lambda_n}\right)^{n-1}.$$

Lemma 4.3. *Let* $z_1, \ldots, z_n$ *be approximations produced by the EAN method* (4.1) *and let* $u_i = z_i - \zeta_i$, $\hat{u}_i = \hat{z}_i - \zeta_i$. *If* $n \geqslant 3$ *and inequality* (4.2) *holds*, *then*

(i) $d < \frac{1}{1-2\lambda_n}\hat{d}$;
(ii) $\widehat{w} < \beta_n w$;
(iii) $\widehat{w} < c_n \hat{d}$;
(iv) $|\hat{u}_i| \leqslant \frac{\gamma_n}{d^3} |u_i|^2 \sum_{j \neq i} |u_j|^2$.

Proof. In regard to (4.2) and Lemma 4.1 it follows that $\zeta_i \in \{z_i; (1/(1-nc_n))|W_i|\}$ $(i \in I_n)$, so that

$$|u_i| = |z_i - \zeta_i| \leqslant \rho_n |W_i| < \rho_n w < \rho_n c_n d. \tag{4.4}$$

According to this and the definition of the minimal distance d we find

$$|z_j - \zeta_i| \geqslant |z_j - z_i| - |z_i - \zeta_i| > d - \rho_n c_n d = (1 - \rho_n c_n) d. \tag{4.5}$$

Using identity (3.6) and estimates (4.4) and (4.5), we obtain

$$\left|\frac{P'(z_i)}{P(z_i)}\right| = \left|\sum_{j=1}^{n} \frac{1}{z_i - \zeta_j}\right| \geqslant \frac{1}{|z_i - \zeta_i|} - \sum_{j \neq i} \frac{1}{|z_i - \zeta_j|} > \frac{1}{\rho_n c_n d} - \frac{n-1}{(1 - \rho_n c_n) d}$$
$$= \frac{1 - \rho_n c_n - (n-1)\rho_n c_n}{(1 - \rho_n c_n)\rho_n c_n d} = \frac{\delta_n}{(1 - \rho_n c_n)\rho_n c_n d}.$$

Hence

$$|N_i| = \left|\frac{P(z_i)}{P'(z_i)}\right| < \frac{(1-\rho_n c_n)\rho_n c_n d}{\delta_n}, \tag{4.6}$$

so that

$$\begin{aligned}|z_i - z_j + N_j| &\geqslant |z_i - z_j| - |N_j| > d - \frac{(1-\rho_n c_n)\rho_n c_n d}{\delta_n}\\ &= \frac{(1-\rho_n c_n)^2 - (n-1)\rho_n c_n}{\delta_n} d = \frac{\alpha_n}{\delta_n(1-\rho_n c_n)} d.\end{aligned} \tag{4.7}$$

Let us introduce

$$S_i = \sum_{j \neq i} \frac{N_j - u_j}{(z_i - \zeta_j)(z_i - z_j + N_j)}, \quad h_j = \sum_{k \neq j} \frac{1}{z_j - \zeta_k}.$$

We start from the iterative formula (4.1) and use identity (3.6) to find

$$\begin{aligned}\hat{u}_i = \hat{z}_i - \zeta_i &= z_i - \zeta_i - \frac{1}{1/u_i + \sum_{j\neq i}(1/(z_i - \zeta_j)) - \sum_{j \neq i}(1/(z_i - z_j + N_j))}\\ &= u_i - \frac{u_i}{1 + u_i \sum_{j \neq i}((N_j - u_j)/(z_i - \zeta_j)(z_i - z_j + N_j))}\\ &= u_i - \frac{u_i}{1 + u_i S_i} = \frac{u_i^2 S_i}{1 + u_i S_i}.\end{aligned} \tag{4.8}$$

Furthermore, we find

$$N_j = \frac{u_j}{1 + u_j h_j}, \quad N_j - u_j = -\frac{u_j^2 h_j}{1 + u_j h_j}, \quad S_i = -\sum_{j \neq i} \frac{u_j^2 h_j/(1 + u_j h_j)}{(z_i - \zeta_j)(z_i - z_j + N_j)}.$$

Using (4.4) and the inequality

$$|h_j| = \left|\sum_{k \neq j} \frac{1}{z_j - \zeta_k}\right| < \frac{n-1}{(1-\rho_n c_n)d},$$

we find

$$\left|\frac{h_j}{1 + u_j h_j}\right| \leqslant \frac{|h_j|}{1 - |u_j||h_j|} < \frac{(n-1)/(1-\rho_n c_n)d}{1 - \rho_n c_n d((n-1)/(1-\rho_n c_n)d)} = \frac{n-1}{\delta_n d}. \tag{4.9}$$

Combining (4.4), (4.5), (4.7) and (4.9), we obtain

$$\begin{aligned}|u_i S_i| &\leqslant |u_i| \sum_{j \neq i} \frac{|u_j|^2 |h_j/(1 + u_j h_j)|}{|z_i - \zeta_j||z_i - z_j + N_j|} < \rho_n c_n d \cdot \frac{(n-1)(\rho_n c_n d)^2((n-1)/\delta_n d)}{(1-\rho_n c_n)d(\alpha_n/\delta_n(1-\rho_n c_n))d}\\ &= \frac{(n-1)^2(\rho_n c_n)^3}{\alpha_n}.\end{aligned} \tag{4.10}$$

Using (4.4) and (4.10), we find from (4.1)

$$\begin{aligned}|\hat{z}_i - z_i| &= \left|\frac{u_i}{1 + u_i S_i}\right| \leqslant \frac{|u_i|}{1 - |u_i S_i|} < \frac{|u_i|}{1 - ((n-1)^2(\rho_n c_n)^3/\alpha_n)} = \frac{\alpha_n}{\alpha_n - (n-1)^2(\rho_n c_n)^3}|u_i|\\ &< \frac{\alpha_n \rho_n c_n \gamma_n}{n-1} d = \lambda_n d\end{aligned}$$

and

$$|\hat{z}_i - z_i| < \frac{\alpha_n}{\alpha_n - (n-1)^2(\rho_n c_n)^3}|u_i| < \frac{\alpha_n \rho_n \gamma_n}{n-1}|W_i| = \frac{\lambda_n}{c_n}|W_i| < \lambda_n d. \tag{4.11}$$

Since (4.11) holds, we apply Lemma 2.1 and obtain from (2.9):

$$d < \frac{1}{1-2\lambda_n}\hat{d}. \tag{4.12}$$

Thus, assertion (i) of Lemma 4.3 is valid.

Using the starting inequality $w/d < c_n$ and bounds (4.6), (4.7), (4.11), (2.4) and (2.5), for $n \geqslant 3$ we estimate the quantities appearing in (4.3):

$$|W_i||\Sigma_{N,i}| < w \cdot \frac{(n-1)((1-\rho_n c_n)\rho_n c_n d/\delta_n)}{(\alpha_n/\delta_n(1-\rho_n c_n))d \cdot d} < \frac{(n-1)(1-\rho_n c_n)^2 \rho_n c_n^2}{\alpha_n},$$

$$|\hat{z}_i - z_i||\Sigma_{W,i}| < \lambda_n d \cdot \frac{(n-1)c_n d}{(1-\lambda_n)d \cdot d} < \frac{(n-1)\lambda_n c_n}{1-\lambda_n}.$$

According to the last two bounds and (2.6), from (4.3) we estimate

$$\begin{aligned} |\widehat{W}_i| &\leqslant |\hat{z}_i - z_i|(|W_i||\Sigma_{N,i}| + |\hat{z}_i - z_i||\Sigma_{W,i}|)\left|\prod_{j\neq i}\left(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\right)\right| \\ &< \frac{\lambda_n}{c_n}|W_i|\left(\frac{(n-1)(1-\rho_n c_n)^2\rho_n c_n^2}{\alpha_n} + \frac{(n-1)\lambda_n c_n}{1-\lambda_n}\right)\left(1 + \frac{\lambda_n}{1-2\lambda_n}\right)^{n-1} \\ &= \beta_n |W_i|, \end{aligned}$$

that is,

$$\widehat{w} < \beta_n w. \tag{4.13}$$

Therefore, we have proved assertion (ii) of Lemma 4.3.

Since

$$\frac{\beta_n}{1-2\lambda_n} < 0.942 \quad \text{for } 3 \leqslant n \leqslant 21$$

and

$$\frac{\beta_n}{1-2\lambda_n} < 0.943 \quad \text{for } n \geqslant 22,$$

starting from (4.13), by (4.2) and (4.12) we find

$$\hat{w} < \beta_n w < \beta_n c_n d < \frac{\beta_n}{1-2\lambda_n} \cdot c_n \hat{d} < c_n \hat{d},$$

which means that the implication $w < c_n d \;\Rightarrow\; \widehat{w} < c_n d$ holds. This proves (iii) of Lemma 4.3.

Using the above bounds, from (4.8) we obtain

$$\begin{aligned} |\hat{u}_i| &\leqslant \frac{|u_i|^2|S_i|}{1-|u_i S_i|} < \frac{\alpha_n}{\alpha_n - (n-1)^2(\rho_n c_n)^3}|u_i|^2 \sum_{j\neq i} \frac{|u_j|^2|h_j/(1+u_j h_j)|}{|z_i - \zeta_j||z_i - z_j + N_j|} \\ &< \frac{\alpha_n}{\alpha_n - (n-1)^2(\rho_n c_n)^3} \cdot \frac{(n-1)/\delta_n d}{(1-\rho_n c_n)d \cdot (\alpha_n/\delta_n(1-\rho_n c_n))d}|u_i|^2 \sum_{j\neq i}|u_j|^2, \end{aligned}$$

wherefrom (taking into account Remark 2)

$$|\hat{u}_i| \leqslant \frac{\gamma_n}{d^3}|u_i|^2 \sum_{j\neq i} |u_j|^2,$$

which proves (iv) of Lemma 4.3. □

Now we give the convergence theorem for the EAN method (4.1) which involves only initial approximations to the zeros, the polynomial coefficients and the polynomial degree n.

Theorem 4.1. *Let P be a polynomial of degree $n \geqslant 3$ with simple zeros. If the initial condition*

$$w^{(0)} < c_n d^{(0)} \tag{4.14}$$

holds, where c_n is given by (4.2) *then the EAN method* (4.1) *is convergent with the order of convergence four.*

Proof. Similarly to the proof of Theorem 3.1, we apply induction with the argumentation used for inequalities (i)–(iv) of Lemma 4.3. According to (4.14) and (4.2) all estimates given in Lemma 4.3 are valid for the index $m = 1$ which is a part of the proof with respect to $m = 1$. Since inequality (iii) coincides with the condition of form (4.14), assertions (i)–(iv) of Lemma 4.3 are valid for the next index, etc. The implication

$$w^{(m)} < c_n d^{(m)} \;\Rightarrow\; w^{(m+1)} < c_n d^{(m+1)}$$

provides the validity of all inequalities given in Lemma 4.3 for all $m = 0, 1, \ldots$. In particular, we have

$$\frac{d^{(m)}}{d^{(m+1)}} < \frac{1}{1-2\lambda_n} \tag{4.15}$$

and

$$|u_i^{(m+1)}| \leqslant \frac{\gamma_n}{(d^{(m)})^3}|u_i^{(m)}|^2 \sum_{j\neq i} |u_j^{(m)}|^2 \quad (i \in I_n) \tag{4.16}$$

for each iteration index $m = 0, 1, \ldots,$ where

$$\gamma_n = \frac{n-1}{\alpha_n - (n-1)^2(\rho_n c_n)^3}.$$

Let us substitute

$$t_i^{(m)} = \left[\frac{(n-1)\gamma_n}{(1-2\lambda_n)(d^{(m)})^3}\right]^{1/3} |u_i^{(m)}|,$$

then inequalities (4.16) become

$$t_i^{(m+1)} \leqslant \frac{(1-2\lambda_n)d^{(m)}}{(n-1)d^{(m+1)}}[t_i^{(m)}]^2 \sum_{j\neq i} [t_j^{(m)}]^2.$$

Hence, using (4.15), we obtain

$$t_i^{(m+1)} < \frac{1}{n-1}[t_i^{(m)}]^2 \sum_{j\neq i} [t_j^{(m)}]^2 \quad (i \in I_n). \tag{4.17}$$

Using (4.4) we find

$$\begin{aligned} t_i^{(0)} &= \left[\frac{(n-1)\gamma_n}{(1-2\lambda_n)(d^{(0)})^3}\right]^{1/3} |u_i^{(0)}| < \rho_n c_n d^{(0)} \left[\frac{(n-1)\gamma_n}{(1-2\lambda_n)(d^{(0)})^3}\right]^{1/3} \\ &= \rho_n c_n \left[\frac{(n-1)\gamma_n}{1-2\lambda_n}\right]^{1/3}. \end{aligned}$$

Taking $t = \max_{1 \leqslant i \leqslant n} t_i^{(0)}$ we have

$$t_i^{(0)} \leqslant t < 0.626 < 1 \quad (3 \leqslant n \leqslant 21)$$

and

$$t_i^{(0)} \leqslant t < 0.640 < 1 \quad (n \geqslant 22),$$

for each $i = 1, \ldots, n$. In regard to this we conclude from (4.17) that the sequences $\{t_i^{(m)}\}$ and $\{|u_i^{(m)}|\}$ tend to 0 for all $i = 1, \ldots, n$, which means that $z_i^{(m)} \to \zeta_i$. Therefore, the EAN method (4.1) is convergent. Besides, taking into account that the quantity $d^{(m)}$ appearing in (4.16) is bounded and tends to $\min_{\substack{i,j \\ i \neq j}} |\zeta_i - \zeta_j|$ and setting

$$u^{(m)} = \max_{1 \leqslant i \leqslant n} |u_i^{(m)}|,$$

from (4.16) we obtain

$$|u_i^{(m+1)}| \leqslant u^{(m+1)} < (n-1) \frac{\gamma_n}{(d^{(m)})^3} [u^{(m)}]^4,$$

which means that the order of convergence of the EAN method is four. $\square$

5. Börsch-Supan method with Weierstrass' correction

The cubically convergent Börsch-Supan's method

$$z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{j \neq i} (W_j^{(m)}/(z_i^{(m)} - z_j^{(m)}))} \quad (i \in I_n; m = 0, 1, \ldots)$$

presented in [4], can be accelerated in the same way as the Ehrlich–Aberth method (3.1) using Weierstrass' corrections $W_i^{(m)} = P(z_i^{(m)}) / \prod_{j \neq i} (z_i^{(m)} - z_j^{(m)})$. In this way we obtain the following iterative formula (see Nourein [15]):

$$z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{j \neq i} (W_j^{(m)}/(z_i^{(m)} - W_i^{(m)} - z_j^{(m)}))} \quad (i \in I_n; m = 0, 1, \ldots). \tag{5.1}$$

The order of convergence of the Börsch-Supan method with Weierstrass corrections (5.1) is *four* (see, e.g. [6,41]). For brevity, method (5.1) will be referred to as the BSW method. We note that an interesting derivation of the BSW method was given in [6] using the secant method and the Lagrangian interpolation formula (2.1).

Let us introduce the notation:

$$\rho_n = \frac{1}{1 - nc_n}, \quad \gamma_n = \frac{\rho_n (1 + \rho_n c_n)^{2n-2}}{(1 - \rho_n c_n)^2},$$

$$\lambda_n = \rho_n c_n (1 - c_n), \quad \beta_n = \frac{\lambda_n \rho_n c_n^2 (n-1)^2}{(1 - \lambda_n)(1 - c_n)} \left(1 + \frac{\lambda_n}{1 - 2\lambda_n}\right)^{n-1}.$$

Lemma 5.1. *Let $z_1, \ldots, z_n$ be approximations produced by the iterative method* (5.1) *and let $\hat{u}_i = \hat{z}_i - \zeta_i, \hat{d} = \min_{i \neq j} |\hat{z}_i - \hat{z}_j|, \widehat{w} = \max_{1 \leqslant i \leqslant n} |\widehat{W}_i|$. If the inequality*

$$w < c_n d, \quad c_n = \begin{cases} \dfrac{1}{2n+1}, & 3 \leqslant n \leqslant 13, \\[2ex] \dfrac{1}{2n}, & n \geqslant 14 \end{cases} \tag{5.2}$$

holds, then

(i) $d < \frac{1}{1-2\lambda_n} \hat{d}$;

(ii) $|u_i| < \rho_n c_n d$;
(iii) $\widehat{w} < c_n \hat{d}$;
(iv) $|\hat{u}_i| \leqslant \frac{\gamma_n}{d^3} |u_i|^2 (\sum_{j\neq i} |u_j|)^2$.

The proof of this lemma is similar to the proofs of Lemmas 3.3 and 4.3 and will be omitted.

Now we establish initial conditions of practical interest, which guarantee the convergence of the BSW method (5.1).

Theorem 5.1. *If the initial condition given by*

$$w^{(0)} < c_n d^{(0)} \tag{5.3}$$

is satisfied, where c_n is given by (5.2), *then the iterative method* (5.1) *is convergent with the order of convergence four.*

Proof. The proof of this theorem is based on the assertions of Lemma 5.1 with the help of previously presented technique. As in the already stated convergence theorems, the proof goes by induction. By the same argumentation as in the previous proofs, the initial condition (5.3) provides the validity of the inequality $w^{(m)} < c_n d^{(m)}$ for all $m \geqslant 0$, and whence, inequalities (i)–(iv) of Lemma 5.1 also hold for all $m \geqslant 0$. In particular (according to Lemma 5.1(i)), we have

$$\frac{d^{(m)}}{d^{(m+1)}} < \frac{1}{1-2\lambda_n} \tag{5.4}$$

and (according to Lemma 5.1(iv))

$$|u_i^{(m+1)}| \leqslant \frac{\gamma_n}{(d^{(m)})^3} |u_i^{(m)}|^2 \left(\sum_{j\neq i} |u_j^{(m)}| \right)^2 \tag{5.5}$$

for each $i \in I_n$ and all $m = 0, 1, \dots$.

Substituting

$$t_i^{(m)} = \left[\frac{(n-1)^2 \gamma_n}{(1-2\lambda_n)(d^{(m)})^3} \right]^{1/3} |u_i^{(m)}|$$

in (5.5) and using (5.4), we obtain

$$t_i^{(m+1)} < \frac{1}{(n-1)^2} [t_i^{(m)}]^2 \left(\sum_{j\neq i} t_j^{(m)} \right)^2 . \tag{5.6}$$

By assertion (ii) of Lemma 5.1 for the first iteration ($m = 0$) we have

$$t_i^{(0)} = \left[\frac{(n-1)^2 \gamma_n}{(1-2\lambda_n)(d^{(0)})^3} \right]^{1/3} |u_i^{(0)}| < \rho_n c_n \left[\frac{(n-1)^2 \gamma_n}{1-2\lambda_n} \right]^{1/3} . \tag{5.7}$$

Putting $t = \max_i t_i^{(0)}$, we find from (5.7) that $t_i^{(0)} \leqslant t < 0.988 < 1$ for $3 \leqslant n \leqslant 13$, and $t_i^{(0)} \leqslant t < 0.999 < 1$ for $n \geqslant 14$, for each $i = 1, \dots, n$. According to this we infer from (5.6) that the sequences $\{t_i^{(m)}\}$ (and, consequently, $\{|u_i^{(m)}|\}$) tend to 0 for all $i = 1, \dots, n$. Hence the BSW method (5.1) is convergent.

Let $u^{(m)} = \max_i |u_i^{(m)}|$. Since the quantity $d^{(m)}$ involved in (5.5) is bounded and tends to $\min_i |\zeta_i - \zeta_j|$, from (5.5) we get

$$u^{(m+1)} < \frac{\gamma_n}{(d^{(m)})^3} (n-1)^2 [u^{(m)}]^4,$$

which means that the order of convergence of the BSW method is four. $\square$

6. Halley-like method

Using a concept based on Bell's polynomials, Wang and Zheng [39] established a family of iterative methods of the order of convergence $k+2$, where k is the highest order of the derivative of P appearing in the generalized iterative formula. For $k=1$ this family gives the Ehrlich–Aberth method (3.1), and for $k=2$ produces the following iterative method of the fourth order for the simultaneous approximation of all simple zeros of a polynomial P,

$$z_i^{(m+1)} = z_i^{(m)} - \frac{1}{f(z_i^{(m)}) - (P(z_i^{(m)})/2P'(z_i^{(m)}))([S_{1,i}^{(m)}]^2 + S_{2,i}^{(m)})} \quad (i \in I_n;\, m = 0, 1, \dots), \tag{6.1}$$

where

$$f(z) = \frac{P'(z)}{P(z)} - \frac{P''(z)}{2P'(z)}, \qquad S_{r,i}^{(m)} = \sum_{j \neq i} \frac{1}{(z_i^{(m)} - z_j^{(m)})^k} \quad (r = 1, 2).$$

Iterative methods generated for $k \geqslant 3$ are rather complicated and their computational efficiency is not high. For this reason, they are rarely applied in practice and we pay attention to the method (6.1). Let us note that method (6.1) is of Halley's type since the function $f(z)$ appears in the well known Halley iterative method

$$\hat{z}_i = z_i - \frac{1}{P'(z_i)/P(z_i) - P''(z_i)/2P'(z_i)} = z_i - \frac{1}{f(z_i)}.$$

In literature, method (6.1) is also referred to as the Wang–Zheng method.

The convergence analysis of the Halley-like method (6.1) is similar to that given in the previous sections (see, also, [21]) so that it will be presented in short.

Let us introduce the following abbreviations:

$$\rho_n = \frac{1}{1 - nc_n}, \quad \eta_n = \frac{2(1 - n\rho_n c_n)}{1 - \rho_n c_n} - \frac{n(n-1)(\rho_n c_n)^3 (2 - \rho_n c_n)}{(1 - \rho_n c_n)^2},$$

$$\lambda_n = \frac{2\rho_n c_n (1 - \rho_n c_n + (n-1)\rho_n c_n)}{(1 - \rho_n c_n)\eta_n}, \quad \gamma_n = \frac{n(2 - \rho_n c_n)}{\eta_n (1 - \rho_n c_n)^2}.$$

In a similar manner as in Sections 3 and 4, we can prove the following assertions.

Lemma 6.1. *Let $z_1, \dots, z_n$ be approximations produced by the iterative method* (6.1) *and let $\hat{u}_i = \hat{z}_i - \zeta_i$, $\hat{d} = \min_{i \neq j} |\hat{z}_i - \hat{z}_j|$, $\widehat{w} = \max_{1 \leqslant i \leqslant n} |\widehat{W}_i|$. If the inequality*

$$w < c_n d, \quad c_n = \begin{cases} \dfrac{1}{3n + 2.4}, & 3 \leqslant n \leqslant 20, \\ \dfrac{1}{3n}, & n \geqslant 21 \end{cases} \tag{6.2}$$

holds, then

(i) $d < \frac{1}{1 - 2\lambda_n} \hat{d}$;
(ii) $|u_i| < \rho_n c_n d$;
(iii) $\widehat{w} < c_n \hat{d}$;
(iv) $|\hat{u}_i| \leqslant \frac{\gamma_n}{d^3} |u_i|^3 \sum_{j \neq i} |u_j|$.

Now we give the convergence theorem for the iterative method (6.1) under computationally verifiable initial conditions.

Theorem 6.1. *Let P be a polynomial of degree $n \geqslant 3$ with simple zeros. If the initial condition*

$$w^{(0)} < c_n d^{(0)} \tag{6.3}$$

holds, where c_n is given by (6.2), *then the Halley-like method* (6.1) *is convergent with the order of convergence four.*

Proof. The proof of this theorem goes in a similar way as in the previous sections using the assertions of Lemma 6.1. By virtue of the implication (iii) of Lemma 6.1 (that is, $w < c_n d \Rightarrow \hat{w} < c_n \hat{d}$) we conclude by the complete induction that the initial condition (6.3) implies the inequality $w^{(m)} < c_n d^{(m)}$ for each $m = 1, 2, \ldots$. For this reason Lemma 6.1 is valid for all $m \geqslant 0$. In particular (according to (i) and (iv) of Lemma 6.1), we have

$$\frac{d^{(m)}}{d^{(m+1)}} < \frac{1}{1 - 2\lambda_n} \tag{6.4}$$

and

$$|u_i^{(m+1)}| \leqslant \frac{\gamma_n}{(d^{(m)})^3} |u_i^{(m)}|^3 \sum_{j \neq i} |u_j^{(m)}| \quad (i \in I_n) \tag{6.5}$$

for each iteration index $m = 0, 1, \ldots$.

Substituting

$$t_i^{(m)} = \left[\frac{(n-1)\gamma_n}{(1-2\lambda_n)(d^{(m)})^3} \right]^{1/3} |u_i^{(m)}|$$

in (6.5), we find

$$t_i^{(m+1)} \leqslant \frac{(1-2\lambda_n) d^{(m)}}{(n-1) d^{(m+1)}} [t_i^{(m)}]^3 \sum_{j \neq i} [t_j^{(m)}] \quad (i \in I_n).$$

Hence, using (6.4), we obtain

$$t_i^{(m+1)} < \frac{1}{n-1} [t_i^{(m)}]^3 \sum_{j \neq i} t_j^{(m)} \quad (i \in I_n). \tag{6.6}$$

Since $|u_i^{(0)}| < \rho_n c_n d^{(0)}$ (assertion (ii) of Lemma 6.1), we have

$$t_i^{(0)} = \left[\frac{(n-1)\gamma_n}{(1-2\lambda_n)(d^{(0)})^3} \right]^{1/3} |u_i^{(0)}| < \rho_n c_n \left[\frac{(n-1)\gamma_n}{1-2\lambda_n} \right]^{1/3}$$

for each $i = 1, \ldots, n$. Let $t_i^{(0)} \leqslant \max_i t_i^{(0)} = t$. Then

$$t < \rho_n c_n \left[\frac{(n-1)\gamma_n}{1-2\lambda_n} \right]^{1/3} < 0.310 \quad \text{for } 3 \leqslant n \leqslant 20$$

and

$$t < 0.239 \quad \text{for } n \geqslant 21,$$

that is, $t_i^{(0)} \leqslant t < 1$ for all $i = 1, \ldots, n$. Hence we conclude from (6.6) that the sequences $\{t_i^{(m)}\}$ (and, consequently, $\{|u_i^{(m)}|\}$) tend to 0 for all $i = 1, \ldots, n$. Therefore, $z_i^{(m)} \to \zeta_i$ $(i \in I_n)$ and method (6.1) is convergent.

Since the quantity $d^{(m)}$ appearing in (6.5) is bounded and tends to $\min_{i \neq j} |\zeta_i - \zeta_j|$, from (6.5) there follows

$$|u_i^{(m+1)}| \leqslant u^{(m+1)} < (n-1) \frac{\gamma_n}{(d^{(m)})^3} [u^{(m)}]^4,$$

where $u^{(m)} = \max_{1 \leqslant i \leqslant n} |u_i^{(m)}|$. Therefore, the order of convergence of Halley-like method (6.1) is four. □

7. Some computational aspects

In this paper we have improved the convergence conditions of four root-finding methods. For the comparison purpose, let us introduce the *normalized* i-factor $\Omega_n = n \cdot c_n$. The former Ω_n for the considered methods, found in the recent papers cited in Section 1, and the improved (new) Ω_n, proposed in this paper, are given in Table 1.

Table 1

	Former Ω_n	New Ω_n
Ehrlich–Aberth's method (3.1)	$\dfrac{n}{2n+3}$	$\begin{cases} \dfrac{n}{2n+1.4} & (3\leqslant n\leqslant 7), \\ 1/2 & (n\geqslant 8) \end{cases}$
The EAN method (4.1)	$\dfrac{1}{3}$	$\begin{cases} \dfrac{n}{2.2n+1.9} & (3\leqslant n\leqslant 21), \\ 1/2.2 & (n\geqslant 22) \end{cases}$
The BSW method (5.1)	$\dfrac{n}{2n+2}$	$\begin{cases} \dfrac{n}{2n+1} & (3\leqslant n\leqslant 13), \\ 1/2 & (n\geqslant 14) \end{cases}$
Halley-like method (6.1)	$\dfrac{1}{4}$	$\begin{cases} \dfrac{n}{3n+2.4} & (3\leqslant n\leqslant 20), \\ 1/3 & (n\geqslant 21) \end{cases}$

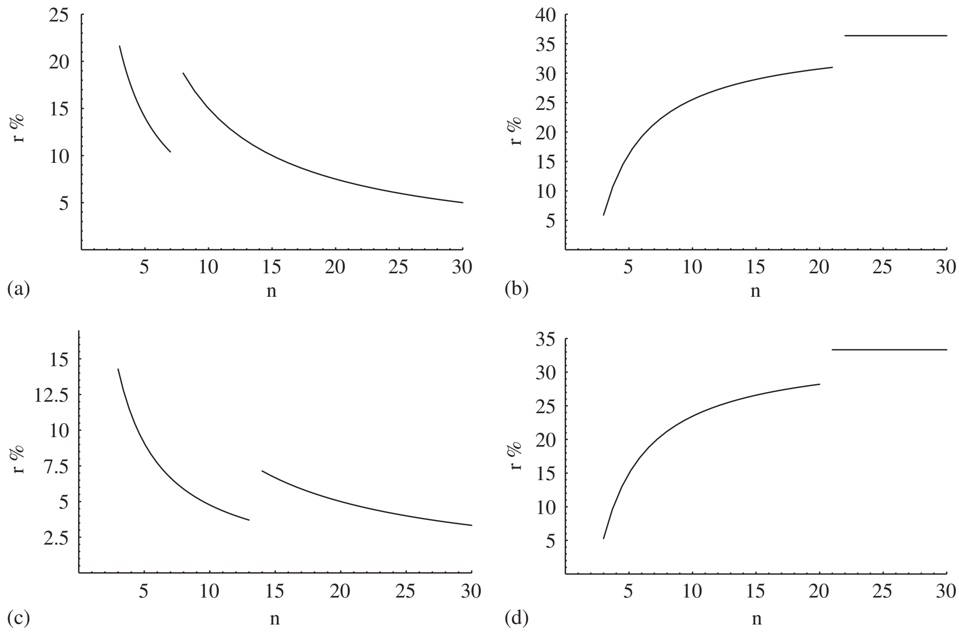

Fig. 2. (a) $r\%$—Ehrlich–Aberth's method (3.1). (b) $r\%$—the EAN method (4.1). (c) $r\%$—the BSW method (5.1). (d) $r\%$—Halley-like method (6.1).

To compare the former $\Omega_n = nc_n$ with the improved values of Ω_n obtained in this paper, we introduce a percentage measure of the improvement

$$r\% = \frac{\Omega_n^{(\mathrm{new})} - \Omega_n^{(\mathrm{former})}}{\Omega_n^{(\mathrm{former})}} \cdot 100.$$

Following Table 1 we calculated $r\%$ for $n \in [3, 30]$ and displayed $r\%$ in Fig. 2 as a function of n for each of the four considered methods. From Fig. 2 we observe that we significantly improved i-factors c_n, especially for the EAN method (4.1) and Halley-like method (6.1).

The values of the i-factor c_n, given in the corresponding convergence theorems for the considered iterative methods, are mainly of theoretical importance. We were constrained to take smaller values of c_n to enable the validity of inequalities appearing in the convergence analysis. However, these theoretical values of c_n can be suitably applied in ranking the considered methods regarding (i) their initial conditions for the guaranteed convergence and (ii) convergence behavior in practice.

Table 2

	c_n	$1.5c_n$	$2c_n$	$3c_n$	$5c_n$	$10c_n$
Ehrlich–Aberth's method (3.1)	3.9	4	4.2	5.4	7.3	13.3
The EAN method (4.1)	3.1	3.2	3.4	5.1	6.1	10.2
The BSW method (5.1)	3	3.1	3.3	4.3	5.8	9.8
Halley-like method (6.1)	3.2	3.4	4.2	5.5	6.7	10.7

As it was mentioned in [22], in practical implementation of simultaneous root-finding methods we can take greater c_n related to that given in the convergence theorems and still preserve both guaranteed and fast convergence. The determination of the range of values of i-factor c_n providing favorable features (guaranteed and fast convergence) is very difficult problem and practical experiments are the only means for obtaining some information on this range. We have tested the considered methods in examples of many algebraic polynomials with degree up to 20 taking initial approximations in such a way that the i-factor has taken the values kc_n for $k = 1$ (theoretical entry applied in the stated initial conditions) and for $k = 1.5$, 2, 3, 5 and 10. The stopping criterion was given by the inequality

$$\max_{1 \leqslant i \leqslant n} |z_i^{(m)} - \zeta_i| < 10^{-15}.$$

In Table 2 we give the average number of iterations (rounded to one decimal place), needed to satisfy this criterion.

From Table 2 we observe that the new i-factor not greater than $2c_n$ mainly preserves the convergence rate related to the theoretical value c_n given in the presented convergence theorems. The entry $3c_n$ is rather acceptable from a practical point of view, while the choice of $5c_n$ doubles the number of iterations. Finally, the value $10c_n$ significantly decreases the convergence rate of all considered methods, although still provides the convergence.

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