Two dimensional numerical simulation of crack kinking from an interface under dynamic loading by time domain boundary element method

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Abstract

The hybrid time-domain boundary element method, together with the multi-region technique, is applied to simulate the dynamic process of propagation and/or kinking of an interface crack in a two-dimensional bi-material. The whole bi-material is divided into two regions along the interface. The traditional displacement boundary integral equations are employed with respect to each region. However, when the crack kinks into the matrix material, the non-hypersingular traction boundary integral equations are used with respect to the part of the crack in the matrix. Crack propagation along the interface is numerically modelled by releasing the nodes in the front of the moving crack-tip controlled by the fracture criterion. Kinking of the interface crack is controlled by a criterion developed from the quasi-static one. Once the crack kinks into the matrix, its propagation is modeled by adding new elements of constant length to the moving crack-tip controlled by a criterion extended from the quasi-static maximum circumferential stress. The numerical results of the crack growth trajectory for different material combinations are computed and compared with the corresponding experimental results. Good agreement between numerical and experimental results implies that the present boundary element numerical method can provide an excellent simulation for the dynamic propagation and deflection of an interface crack.

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1. Introduction

Interfaces between different phases inherently exist in composites and the interfacial failure mode often is the principal one in the process of the manufacture or application. Such composites may be subjected to
dynamic loading or their failure mechanisms may involve dynamic characteristics. For this reason dynamic interfacial failure is in the focus of interest of many experts of materials science and fracture mechanics.

Some fundamental features of dynamic interfacial cracks have been discussed by Willis (1971), Yang et al. (1991) and Wu (1991) among others. The steady state or the singular fields close to the tip of a crack growing along the interface between two anisotropic elastic solids is given in Yang et al. (1991). However, for general transient dynamic interfacial crack problems, analytical solutions are very difficult to obtain due to the mathematical complexity of the problem. Thus numerical methods are essential for investigating the various features of dynamic crack propagation along bi-material interfaces.

Under appropriate loading conditions, an interfacial crack may certainly propagate along the interface. This problem has been studied numerically by Kavaturu and Shukla (1988), Lo et al. (1994, 1995), Xu and Needleman (1996) and experimentally by Lambros and Rosakis (1995,). In each of these studies, criteria were developed to govern the dynamic crack growth along the interface. However, in many cases, an interfacial crack will not grow along the interface but deflect from it either directly at initiation or after some interfacial growth. This phenomenon is usually termed “crack kinking”. He and Hutchinson (1989) developed a fracture criterion (HH-criterion) for determining an interfacial crack kinking under quasi-static loading. Following their work, some researchers such as Geubelle and Knauss (1994) revised this criterion. But until now, no numerical studies on crack kinking from an interface under dynamic loading have been done to the knowledge of the authors. It is still an open question whether the criteria developed for the quasi-static case can be extended to dynamic loading conditions.

In the present paper, interfacial crack kinking under dynamic loading is studied by a time-domain boundary element method (BEM) which combines non-hypersingular time-domain traction boundary integral equations (BIEs) and traditional time-domain displacement BIEs (Lei et al., 2003). The crack growth along the interface is controlled by a dynamic fracture criterion developed by Lo et al. (1994) while crack growth in the homogeneous matrix after kinking is governed by a dynamic fracture criterion which has been used in our previous studies (Lei et al., 2004) for a fast running crack in a bi-material. Crack kinking from the interface under dynamic loading is determined by the fracture criterion (HH-criterion) extended from the quasi-static case (He and Hutchinson, 1989). Numerical results are presented for different material mismatches and compared with the corresponding experimental results by Kimberley and Lambros (2004) to verify the numerical method and the criteria developed in this paper.

2. Problem formulation

Consider the plane strain problem of a pre-existing interfacial crack in a bi-material with arbitrary geometry and loading, see Fig. 1. Both materials are assumed to be isotropic and linearly elastic. In the initial state,
the domains of material 1 and 2 are surrounded by the external boundaries $\partial B_{e1}^1 = \partial B_1^1 + \partial B_u^1 + \partial B_{in}^1$ and $\partial B_{e2}^2 = \partial B_2^2 + \partial B_u^2 + \partial B_{in}^2$, respectively. Here $\partial B_{in}^{1,2}$ denotes the interface; $\partial B_i^j$ is the part of the external boundary with prescribed tractions $t_i(x, t)$; and $\partial B_u^j$ is the remaining external boundary with prescribed displacements $\ddot{u}_i(x, t)$. The unit normal vectors of all boundaries are shown in Fig. 1. Under dynamic loading, the pre-crack will either stay along the interface or deflect from the bonded interface into a softer matrix, see the dotted lines depicted in Fig. 1. Without any loss of generality, we assume material 2 is the more compliant one and take all propagating possibilities into account in what follows. $\Gamma_{in}(t)$ represents the upper and lower interfacial crack faces and it satisfies $\Gamma_{in}(t) \subset \partial B_{in}^{1,2}$. $\Gamma(t)$ represents the kinking crack faces, which is understood to appear only when the crack kinking happens.

2.1. Model description

The material parameters are given as follows: shear modulus $\mu_j$, Lamé constants $\lambda_j$; Poisson’s ratio $\nu_j$; mass density $\rho_j$; the shear and longitudinal wave velocities are $c_{T}^j = \sqrt{\mu_j/\rho_j}$ and $c_{L}^j = \sqrt{(\lambda_j + 2\mu_j)/\rho_j}$ with the superscript $j = 1$ and 2 representing material 1 and 2, respectively.

Both components satisfy the balance equations and the Hooke’s law:

$$\sigma_{ji}^{j} = \rho \ddot{u}_i^{j}$$
$$\sigma_{ji}^{j} = E^{j}_{i\beta\delta} u_{\beta}^{j}$$

where $E^{j}_{i\beta\delta}$ is the elasticity tensor; $\sigma_{ji}^{j}$ and $u_{ji}^{j}$ denote the stress and displacement components; a comma after a quantity designates spatial derivatives; while the dots stand for temporal derivatives. The conventional summation rule over double indices is applied with Greek indices $\alpha, \beta, \gamma, \delta = 1, 2$ for the present 2D problem.

The possible contact of closed crack faces is taken into account by using the penalty method developed by Seelig (1997). The contacting crack faces are assumed to be friction-free so that the traction vector on the crack faces is identified via the pressure $p$:

$$t_i(x, t) = p(x, t)n_i(x), \quad x \in \Gamma_{in}(t) \cup \Gamma(t).$$

During contact the kinematic condition of vanishing displacement jump in the normal direction:

$$\Delta u_i(x, t) = \Delta u_z(x, t)n_z(x) = 0,$$

must be fulfilled where $\Delta u_z(x, t)$ is the crack opening displacements (CODs) defined as:

$$\Delta u_z(x, t) = u_z(x \in \Gamma^+, \tau) - u_z(x \in \Gamma^-, \tau).$$

According to the penalty method in Seelig (1997), the contact pressure is considered to be proportional to the inadmissible material penetration:

$$p(x, t) = \frac{c_p}{2} [\Delta u_n(x, t) - |\Delta u_n(x, t)|],$$

where the penalty-stiffness $c_p$ has to be selected a factor 1000 times greater than the stiffness of the material. The contact pressure $p$ is, hence, negative for the closed crack ($\Delta u_n < 0$) and vanishes for the open crack ($\Delta u_n > 0$).

The interface conditions of the perfectly bonded interface are:

$$t_i^{in}(x, t) = -\ddot{t}_i^{in}(x, t), \quad u_i^{in}(x, t) = \ddot{u}_i^{in}(x, t), \quad x \in \partial B_{in}/\Gamma_{in}(t),$$

what means that the tractions and displacements are continuous across the interface. At the external boundary, we have:

$$t_i^{i}(x, t) = \ddot{t}_i^{i}(x, t), \quad x \in \partial B_i^i$$
$$u_i^{i}(x, t) = \ddot{u}_i^{i}(x, t), \quad x \in \partial B_u^i$$

Finally, the initial conditions are

$$u_i^{0}(x, 0) = \ddot{u}_i^{0}(x, 0) = 0.$$
2.2. Boundary integral equations

The time-domain BEM described by Zhang and Gross (1993) in conjunction with the multi-region technique as used by Lei et al. (2003, 2004) are adopted to treat this problem. This leads to a feasible application of non-hypersingular time-domain traction BIEs and traditional time-domain displacement BIEs. For the external boundaries $\partial B^1_{ex}$ and $\partial B^2_{ex}$, the traditional displacement BIEs are written as:

$$ c_{ab}(x)u^b_\mu(x, t) = \int_{\partial B^1_{ex}} \left\{ u_{1G}^{bG}(x, t; y, \tau) * r^1_\mu(y, \tau) - n^1_\mu(y) \sigma_{b\gamma}^{1G}(x, y, t - \tau) * u^1_\mu(y, \tau) \right\} ds(y) \quad x \in \partial B^1_{ex} \quad (10) $$

$$ c_{ab}(x)u^b_\mu(x, t) = \int_{\partial B^2_{ex}} \left\{ u_{2G}^{bG}(x, t; y, \tau) * r^2_\mu(y, \tau) - n^2_\mu(y) \sigma_{b\gamma}^{2G}(x, y, t) * u^2_\mu(y, \tau) \right\} ds(y) $$

$$ + \int_{\Gamma(t)} n^2_\mu(y) \sigma_{b\gamma}^{2G}(x, t; y, \tau) * \Delta u_\mu(y, \tau) ds(y) \quad x \in \partial B^2_{ex} \quad (11) $$

and for the kinking crack $\Gamma(t)$, the non-hypersingular time-domain traction BIE is written as:

$$ r^2_\gamma(x, t) = E_{stx0}n^2_\gamma(x) \int_{\partial B^2_{ex}} \left\{ e_{\delta\gamma}^G \sigma_{\beta\alpha}^{2G}(x, t; y, \tau) * \frac{\partial u_\beta^{bG}}{\partial s}(y, \tau) + \rho_2 u_{\beta\gamma}^{2G}(x, t; y, \tau) * \frac{\partial^2 u_\beta^{bG}}{\partial s^2}(y, \tau) \right\} ds(y) $$

$$ - n^2_\gamma(x) \int_{\partial B^2_{ex}} \sigma_{\delta\gamma}^{2G}(x, t; y, \tau) * r^1_\delta(y, \tau) ds(y) - E_{stx0}n^2_\gamma(x) $$

$$ \times \int_{\Gamma(t)} \left\{ e_{\delta\gamma}^G \sigma_{\beta\alpha}^{2G}(x, t; y, \tau) * \frac{\partial \Delta u_\beta}{\partial s}(y, \tau) + \rho_2 u_{\beta\gamma}^{2G}(x, t; y, \tau) * \Delta u_\beta(y, \tau) n^2_\gamma(y) \right\} ds(y) \quad x \in \Gamma(t) \quad (12) $$

where the quantities with the superscript $G$ are the 2-D elastodynamic Green’s functions for stresses or displacements (Seelig, 1997); $c_{ab}(x)$ is a constant matrix which depends on the position of the collocation point $x$ and reduces to $\delta_{ab}/2$ for a smooth boundary (where $\delta_{ab}$ is Kronecker delta); $e_{\delta\gamma}$ is the 2-D permutation tensor (Lei et al., 2003). It is understood that the integrals ‘$\int$’ appearing in Eqs. (10)–(12) are in the sense of Cauchy principal values.

2.3. Fracture criterion

Under a general dynamic loading, the pre-existing interfacial crack will either stay on the interface or deflect into a softer matrix. A dynamic fracture criterion which is generalized from HH criterion is proposed to control this dynamic fracture behavior (and thus it is named dynamic HH criterion). In addition, two additional criteria are needed to determine the crack growth along the interface and in the homogeneous matrix, respectively.

2.3.1. Dynamic HH criterion

Consider a semi-infinite crack lying on the interface between two semi-infinite blocks of isotropic elastic solids with different material properties as shown in Fig. 2. A straight crack segment of length $\Delta a$ and kink angle $\varphi$ (positive clockwise) kinks into material 2. The length $\Delta a$ is assumed to be small compared to the length

![Fig. 2. Propagation and kinking of an interfacial crack.](image-url)
of the parent interface segment of the crack, that is $\Delta a \to 0$. We now introduce the following material parameters: $I^d_{in}$ dynamic interface toughness; $I^d_1$ and $I^d_2$ the critical dynamic energy release rates of materials 1 and 2; $G(\Delta a, \varphi)$ the dynamic energy release rate when the crack is extended by a straight segment of length $\Delta a$ under an angle $\varphi$. We further denote:

$$G_{in} = G(\Delta a, 0^\circ), \quad G_m = G(\Delta a, \varphi_0) = \max(G(\Delta a, \varphi)), \quad \varphi \in (0^\circ, 180^\circ).$$

The dynamic HH criterion, used to assess whether an interfacial crack will tend to propagate along the interface or advance by kinking out of the interface, is stated as follows. If $\frac{G_{in}}{G_m} \geq \frac{I^d_1}{I^d_2}$, the crack will stay on the interface. Whether the crack will propagate along the interface or remain stationary is governed by the dynamic fracture criterion along the interface which will be introduced in Section 2.3.2. If $\frac{G_{in}}{G_m} < \frac{I^d_1}{I^d_2}$, the crack will tend to deflect out of the interface, but the kinking happens with the kink angle being $\varphi_0$ just as the inequality $G_m \geq I^d_2$ is satisfied. Otherwise, when $G_m < I^d_2$, the crack will stay in the current state (stationary or propagating along the interface). The kinking velocity (if kinking happens) is assumed to be equal to the current speed of the crack along the interface in the numerical simulation.

2.3.2. Fracture criterion along the interface

The dynamic fracture criterion developed by Lo et al. (1994) is used to control the dynamically propagating interfacial crack. According to this criterion, continuous crack growth is possible if the driving force $G_{in}$ is at least equal to the dynamic interface toughness $I^d_{in}$. Physically, the dynamic fracture toughness $I^d_{in}$ of an interfacial crack may depend on several factors. Here we assume $I^d_{in}$ is known as a function of the current crack-tip velocity $\dot{a}$ and the phase angle $\psi$. Under these conditions, crack propagation takes place when:

$$Y_t = G_{in} - I^d_{in}(\dot{a}, \psi) \geq 0.$$  (13)

There is limited experimental work in this area, and no accurate data exist for the toughness of dynamic interfacial cracks. Therefore, the functional dependence of the toughness of a dynamic interfacial crack is postulated to be the same as proposed by Lo et al. (1994):

$$I^d_{in}(\dot{a}, \psi; \vartheta, \gamma) = \frac{I^*_{in}(1 - \dot{a}/v_{lim})^{-\gamma}}{1 + (\vartheta - 1) \sin^2 \psi},$$

where $I^*_{in}$ can be regarded as the dynamic crack initiation toughness when $\psi = 0^\circ$ and $v_{lim}$ is the upper limit of the crack-tip velocity. The parameters $\gamma$ and $\vartheta$ can be varied to adjust the function to the experimentally observed fracture toughness of a dynamically growing crack to allow the formula to cover a wide range of materials (Lo et al., 1994). Here we select $\vartheta = 0.3, \gamma = 0.2$ and $v_{lim}$ as the shear wave speed of the material #2 in our computation.

2.3.3. Fracture criterion in a matrix

As applied in Lei et al. (2004), Seelig (1997), the fracture criterion in a homogeneous matrix adopted here is similar to the maximum circumferential stress criterion developed by Rice (1988) for the quasi-static case, which states that crack advance will take place in the direction $\theta_0$ of the maximum circumferential stress $\sigma_{\theta\theta}$ when this stress reaches the same critical value as in pure mode-I fracture. The critical stress can be represented by the dynamic fracture toughness $K_{ID}(c)$ that has to be determined experimentally. It is noted that $\sigma_{\theta\theta}$ and the dynamic fracture toughness $K_{ID}$ are functions of the crack-tip speed $c$. Introducing

$$R_t = \max \sigma_{\theta\theta}(\theta; \dot{a}; K_1, K_1) - \sigma_{\theta\theta}(\theta = 0; \dot{a}; K_{ID}, 0),$$  (14)

where $K_1$ and $K_{ID}$ are the dynamic stress intensity factors (DSIFs), the fracture criterion states that the crack will grow once $R_t \geq 0$.

3. Numerical solution procedure

To solve Eqs. (10)–(12), discretization in both time and space with proper interpolation functions is required.
3.1. Discretization in time and space

In the current application, the system of Eqs. (10)–(12) is converted into a ‘discrete’ form by using discretization in both time and space with proper interpolation functions. Firstly, the time interval of interest \([0, t]\) is equally divided into \(m\) steps of the span \(\Delta t\). Constant-length boundary elements are employed to divide the external boundaries of the two materials, \(\partial B^{1/2}_{\text{ex}} \rightarrow \partial B^{1/2}_{\text{ex}} = \bigcup_{i=1}^{m} \Gamma_i\). The interfacial crack faces are a part of the interface and discretized, \(\Gamma_{in}(t) \rightarrow \Gamma_{in}(t) = \bigcup_{i=1}^{m} \Gamma_i\). The constant length of the interfacial crack elements is assumed to be \(\Delta y_{in}\). The crack growth along the interface is simulated by releasing the element nodes adjacent to the growing crack-tip. The kinking crack propagation in the matrix is simulated by adding new straight elements at the crack-tips, i.e., \(e_1\) and \(e_2\). The kinking crack vary with time \(t\).

3.2. Temporal and spatial interpolation functions

The unknown CODs, displacements and tractions along the boundaries are approximated by using the piecewise interpolation functions in both time and space similar to those in Gao and Tan (1992). Therefore only a brief description is presented here.

Within each time step the displacements are approximated by linear time interpolation functions and the tractions are piecewise constant. The time integrals are calculated analytically for these simple temporal shape functions. It should be noted that the singular stress oscillations characterized by the imaginary part of the complex stress intensity factor in the vicinity of the interfacial crack-tip are ignored. The same treatment can be found in Lee and Choi (1988), Gao and Tan (1992). Here we adopt \(g_e(y)\) as the spatial function to simulate the real part of the complex stress intensity factor of the interfacial crack and kinking crack. For the first elements at the crack-tips, i.e., \(e = 1\) and \(E_{in}(t)\) or \(E_{in}(t)\), the detailed expression of \(g_e(y)\) is \(g_e(y) = \sqrt{2r/\Delta y_{in}}\), for the elements next to the first elements, i.e., \(e = 2\) and \(E_{in}(t)\) or \(E_{in}(t)\) the interpolation function is chosen as \(g_e(y) = \sqrt{2r/3\Delta y_{in}}\); and for the remaining (inner) elements \(g_e(y) = 1\).

After the discretization in time and space, the system of Eqs. (10)–(12) in conjunction with the boundary conditions (4), (6)–(9) can be rewritten into a set of algebraic equations for the unknown coefficients, which can be solved by using a Gaussian elimination scheme. The reader is referred to Lei et al., 2004; Seelig, 1997 for details.

3.3. Calculations of some fracture mechanics parameters

3.3.1. The propagating interfacial crack

There are several methods using BEM to determine the complex DSIF \(K\) of an interfacial crack such as the displacement extrapolation technique (Zhang et al., 1999), the singular crack-tip element method (Gao and Tan, 1992) and the contour integral method (Miyazaki et al., 1993). In this paper, we directly extend the displacement extrapolating formulae for a stationary interfacial crack introduced by Gao and Tan (1992) to the dynamic growing interfacial crack using the definition of the complex COD to compute the complex dynamic stress intensity factor. The complex COD, \(\delta\), can be defined as \(\delta = \delta_1 + i\delta_2\), where \(\delta_1\) and \(\delta_2\) are the tangential and normal COD. Considering only the \(K\)-dominant terms in the series solution for an interfacial crack, the CODs of the crack faces can be represented as:

\[
\delta_1(r) = \frac{H_{22}}{\cosh(\pi e)} \sqrt{\frac{2r}{\pi \eta \sqrt{1 + 4e^2}}} \sin(\phi + e \ln r - \tan^{-1}(2e)),
\]

\[
\delta_2(r) = \frac{H_{22}}{\cosh(\pi e)} \sqrt{\frac{2r}{\pi \sqrt{1 + 4e^2}}} \cos(\phi + e \ln r - \tan^{-1}(2e)),
\]

where \(r\) is the distance from the crack-tip; \(\phi = \ln(r/2a)\); \(a\) is the length of the interfacial crack; \(\eta\) is a traction resolution factor given by \(\eta = \sqrt{H_{22}/H_{11}}\); and \(e\) is the oscillatory index and defined by Dundurs’ parameters \(\beta\):
where $H_0$ is the component of the following $2 \times 2$ complex variable matrix $H$,

$$H = \begin{bmatrix}
\frac{\bar{z}_0(1-z_0^2)}{p_1b_1} + \frac{\bar{z}_0(1-z_0^2)}{p_2b_2} & i\left(\frac{1+z_0^2-2z_0^2}{p_1b_1} - \frac{1+z_0^2-2z_0^2}{p_2b_2}\right) \\
-i\left(\frac{1+z_0^2-2z_0^2}{p_1b_1} - \frac{1+z_0^2-2z_0^2}{p_2b_2}\right) & \frac{\bar{z}_0(1-z_0^2)}{p_1b_1} + \frac{\bar{z}_0(1-z_0^2)}{p_2b_2}
\end{bmatrix}$$

(18)

with $\bar{z}_0 = \sqrt{1 - (a/c)^2}$, $\bar{z}_x = \sqrt{1 - (a/c)^2}$ and $D_x = 4\bar{z}_0\bar{z}_x - (1 + \bar{z}_0^2)^2$.

Then the CODs can be related to the magnitude of the complex stress intensity factor through

$$|K| = \frac{\cosh(\pi \varepsilon)}{H_{22}} \lim_{r \to 0} \left\{ \frac{\pi(1 + 4\varepsilon^2)}{2r} \sqrt{\eta^2 \delta_1^2 + \delta_2^2} \right\}.$$

(19)

In this formulation, the small region of oscillations in the crack face displacements predicted by the linear elastic theory is ignored based on Rice’s small-scale model (Rice, 1988). For $\dot{a} = 0$, Eq. (19) can be simplified as

$$|K| = \frac{4\pi \sqrt{1 + 4\varepsilon^2} \cosh(\pi \varepsilon)}{c_1 + c_2} \lim_{r \to 0} \frac{\sqrt{\delta_1^2 + \delta_2^2}}{\sqrt{2\pi r}}.$$  

(20)

where

$$c_x = (\kappa_x + 1)/\mu_x, \quad \varepsilon = \frac{1}{2\pi} \ln \left(\frac{\kappa_1/\mu_1 + 1/\mu_2}{\kappa_2/\mu_2 + 1/\mu_1}\right),$$

with $\kappa_x = 3 - 4\nu_x$ for the plane strain state and $\kappa_x = \frac{1 - \nu_x}{1 + \nu_x}$ for the plane stress state.

The crack-tip mode mixity or phase angle is the function of the distance from the crack-tip:

$$\psi(L) = \tan^{-1} \left\{ \frac{\text{Im}[KL^m]}{\text{Re}[KL^m]} \right\}. \quad (21)$$

Here we determine the crack-tip mode mixity $\psi$ for $L \to 0$, that is

$$\psi = \tan^{-1} \left[ \frac{K_{II}}{K_1} \right], \quad (22)$$

where $K_{II}/K_1$ can be computed from:

$$\frac{K_{II}}{K_1} = \lim_{r \to 0} \left[ \frac{1 - \delta_2}{\delta_1} \frac{\tan[\alpha \ln(r/a)] - 2\varepsilon}{1 + 2\alpha \tan[\alpha \ln(r/a)]} - \frac{\delta_2}{\delta_1} + \frac{\tan[\alpha \ln(r/a)] - 2\varepsilon}{1 + 2\alpha \tan[\alpha \ln(r/a)]} \right].$$

(23)

The dynamic energy release rate for a propagating interfacial crack is given by Yang et al. (1991) as:

$$G = \frac{H_{22}}{4\cosh^2(\pi \varepsilon)} |K|^2.$$

3.3.2. The propagating kinked matrix crack

The DSIFs at a running kinking crack-tip can be calculated from Lei et al. (2004):

$$\begin{cases}
K_1(t, \dot{a}) = B_1(\dot{a}) \mu \sqrt{2\pi} \lim_{r \to 0} \frac{\delta_1(t/r)}{\sqrt{r}}, \\
K_{II}(t, \dot{a}) = B_2(\dot{a}) \mu \sqrt{2\pi} \lim_{r \to 0} \frac{\delta_2(t/r)}{\sqrt{r}},
\end{cases}$$

(24)
where

\[ B_{1/2}(\dot{a}) = \frac{4\beta_1\beta_2 - (1 + \beta_2^2)^2}{4\beta_1(1 - \beta_2^2)}, \]

with \( \beta_{1/2} = \sqrt{1 - (\dot{a}/c_L)^2} \) for \( \dot{a} \neq 0 \) and simplified as \([4(1 - \nu)]^{-1}\) for \( \dot{a} = 0 \).

The dynamic energy release rate for a propagating kinked crack can be achieved from Freund (1972) as:

\[ G(t; \dot{a}) = \frac{1 - \beta_2^2}{4\beta_1\beta_2 - (1 + \beta_2^2)^2} \frac{\beta_1K_1^2 + \beta_2K_{II}^2}{2\mu}. \]  
(25)

### 3.4. Discrete modelling of crack propagation

Assume the interfacial crack is stationary initially. Under dynamic loading, it will either start to propagate along the interface or kink out off the interface. The crack deflecting behavior is numerically simulated through the following steps.

1. Compute the driving forces \( G_{in} \) and \( G_m \).
2. If the inequality \( G_m \geq I_{cr}^2 \) is satisfied, then employ the dynamic HH criterion to estimate whether the crack will stay on the interface or deflect.
3. If the crack deflects, then apply the fracture criterion in a matrix to control the kinked crack; if not, employ the fracture criterion along the interface to control the propagation along the interface, then repeat the steps from (1) to (3) until crack deflection occurs or the whole considered time interval is passed.

Similar to the approach developed by Lo et al. (1994), the interfacial crack growth is modeled by releasing the nodes ahead of the moving crack-tip whenever condition Eq. (13) is satisfied. This will take place only after several time steps, say at discrete times \( t_{m-1}, t_m, \ldots \), because of the limitation of the crack element length,

\[ \dot{a} < c_R^2 < c_L^2 \leq \Delta y_{in}/\Delta t. \]

It should be noted that the nodes will not be released until the facture criterion is satisfied at the next consecutive time step. The numerical crack-tip speed is computed as the averaged value over the whole interval between two instants of discrete crack advance

\[ \dot{a}(t) \approx \frac{\Delta y_{in}}{t_m - t_{m-1}} = \frac{\Delta y_{in}}{\Delta t(m_k - m_{k-1})}, \quad t \in [t_{m-1}, t_m]. \]  
(26)

Assume \( t_m \) is the current time and \( t_{m-1} \) is the last instant of the discrete crack advance. After the onset of crack growth (\( \dot{a} > 0 \)), the SIFs are averaged as

\[ K(t_m) = \frac{1}{t_m - t_{m-1}} \int_{t_{m-1}}^{t_m} K(t)dt. \]

Three stages of the crack propagation along the interface should be simulated: crack initiation, crack extension and crack arrest.

For a stationary crack, once the condition \( Y_t \geq 0 \) (see Eq. (13)) is firstly satisfied at \( t_{in} \), then the instant \( t_{in} \) denotes the theoretical crack initiation time instead of the numerical crack initiation time because \( \Delta y_{in}/\Delta t < c_T^2 \). Here we define that the first new node will be released until the facture criterion is satisfied at the two consecutive time steps, i.e., \( Y_{m-1} \geq 0 \) for \( \dot{a}(t_{m-1}) = 0 \) and \( Y_{m} \geq 0 \) for \( \dot{a}(t_{m}) = \Delta y_{in}/(t_m - t_n) \). Thus the numerical crack initiation time is \( t_m \). The modeling of crack extension is similar to that for the crack initiation. But if, at \( t_m \), the time passed since the last crack-tip jump becomes longer than that corresponding to the value currently in use for \( c, \) the crack-tip speed is made topical according to \( \dot{a}(t) = \min\{\dot{a}(t), \Delta y_{in}/[\Delta t(m - m_k)]\}. \) This will take place when the crack slows down.
In the process of crack propagation, if $Y_{tm-1} \geq 0$ and $Y_{tm} < 0$ appear, no new elements will be added, i.e., $E_{TM}(m) = E_{TM}(m-1)$, then we set the next crack-tip speed $\dot{a}(t_{m+1}) = 0$. However, in this case, two situations should be considered. One case is $Y_{tm+1} < 0$ for $\dot{a}(t_{m+1}) = 0$, which means that the crack will arrest. The other is $Y_{tm+1} \geq 0$ which means that the crack-tip speed should be in the interval $[0, \dot{a}(t_m)]$. Then we use an iterative scheme to gradually decrease the crack-tip speed, that is, calculate the crack-tip speed with $\dot{a}(t_{m+1}) = \Delta y_{in}/[(m_k - m_{k-1} + q)\Delta t]$ iteratively by taking $q = 1, 2, 3, \ldots$ If $Y_{tm+1} > 0$ for $q = j$ and $Y_{tm+1} < 0$ for $q = j + 1$, then $\dot{a}(t_{m+1}) = \Delta y_{in}/[(m_k - m_{k-1} + j)\Delta t]$. So the crack slows down in this situation.

4. Numerical results and discussion

The above described scheme is applied to study some typical examples. The DSIFs for a stationary interfacial crack and phase angle and those for a running crack at a constant velocity are chosen to verify the program. In further examples the crack growth trajectories for kinking of an interfacial crack are computed for different material combinations.

4.1. Verifying samples

4.1.1. Stationary interfacial crack

Two square plates with the same dimension $20 \text{ mm} \times 20 \text{ mm}$ are bonded together and a center crack lies on the interface, see Fig. 3. The length of the crack is $5 \text{ mm}$. The whole bi-material system is subjected to a uniformly distributed impact load $\sigma_2 H(t)$ at the upper and lower sides of the plates. For the computation, the crack is divided into four elements and the external boundaries of the two square plates are divided into 64 elements. The time interval is chosen as $\Delta t = 0.1635 \mu\text{s}$. The material parameters of the two materials are set to be equal as $v_1 = v_2 = 0.3$, $\rho_1 = \rho_2 = 5000 \text{ kg m}^{-3}$ and $\mu_1 = \mu_2 = 76923 \text{ MPa}$. In this case, a center interfacial crack in a bi-material plate with the same material properties is in theory identical to a center matrix crack in a homogeneous rectangular plate. This special case can be treated by two methods. One is to use the method developed in this paper and to consider the crack as an interfacial crack. The magnitude of the complex stress intensity factor $|K|$ is computed by Eq. (24). The other is to apply the method presented by

![Fig. 3. A center interfacial crack in a bi-material plate.](image-url)
Zhang and Gross (1993) and to consider a crack as in a homogeneous medium and directly determine the DSIF $K_1$. Because of the symmetry in both configuration and loading, the second DSIF $K_{II} = 0$. So we have $|K| = |K_1|$. Fig. 4 shows the curves of $|K|$ versus time $t$ from the two methods. They are basically overlapped which partly verifies the correctness of our program. A slight difference appears here which may be resulted from the numerical disposal with the different BIEs used in the two methods. As stated in Section 2.2, the present method applies the mixed BIEs involving the traditional displacement BIEs, while Zhang and Gross (1993) used the single non-hypersingular traction BIEs.

4.1.2. Running interfacial crack with a constant velocity

A dynamically moving crack with its initial length $2a_0$ along the bi-material interface of two infinitely large isotropic solids is considered by using time-domain BEM. This problem has been studied by Lo et al. (1994) using FEM. The whole system is under a remote pure tensile load. The Poisson’s ratios and mass densities are fixed as $\nu_1 = \nu_2 = 0.3$ and $\rho_1 = \rho_2 = 1.3461$ g cm$^{-3}$. The ratio of Young’s moduli is selected as (1) $E_2/E_1 = 1.0$ (homogeneous case) and (2) $E_2/E_1 = 5.0$, respectively. Our interest is to investigate how the phase angle changes as the crack propagates and to compare ours with Lo’s results. Three different values of crack velocity, $c/c_1 = 0.8$, 0.6 and 0.4, are chosen for analysis. The present results (the solid lines) and the corresponding results by Lo et al. (1994) (the scattered squares) are plotted and compared in Fig. 5. Good agreement can be observed especially for lower crack-tip speeds. The slight difference shown in the figure is due to the fact that the phase angle is taken for the case of $L = 0$ in the present paper but for the case of $L \to a_0$ in Lo et al. (1994). A somewhat notable difference (the error is up to 12%) appears for the case of $c/c_1 = 0.8$. This may be the result of the inadaptable selection of the growing crack-element length. For coherence and comparability of the computed results, we take the same crack-element length. And as known, longer crack-elements may yield more notable numerical errors for faster crack growth than for slower crack growth.

![Fig. 4. Comparison of the results of two methods.](image-url)

![Fig. 5. Phase angles for different crack-tip velocities compared with Lo’s results (Lo et al., 1994).](image-url)
4.2. Dynamic kinking of an interfacial crack

Recently, Kimberley and Lambros (2004) have experimentally analyzed the dynamic crack kinking from a bi-material interface applying the optical technique of coherent gradient sensing (CGS) coupled with a high-speed camera. This problem will be numerically studied by using BEM developed above and the results will be compared with the experimental ones.

The experimental specimen (see Fig. 6) was generated by bonding a plate of polymethylmethacrylate (PMMA) to a plate of Homalite-100 (a transparent polymer with mechanical properties similar to those of PMMA) and their material properties are given in Table 1. The PMMA and Homalite constituents are all 127 mm x 152 mm rectangular plates. A sharp crack of 20 mm length is introduced on the prospective interface. The two plates are then polymerized together and thus do not introduce a third material into the specimen. The dynamic interface toughness has been determined as \( \Gamma_m = 150 \text{ J/m}^2 \). In their experiments, to generate a range of near tip mode mixity and to achieve different kink angles, specimens were impact-loaded using a high-speed gas gun in two different loading configurations, represented schematically in Fig. 6, where \( \delta_l \) and \( \delta_r \) indicate the impact location of the centerline of the projectile (50 mm diameter, 75 mm length) with respect to the interface.

In our computation, the exterior boundaries of the two materials are divided into 122 elements. The interface has 32 elements and the initial crack has five elements, i.e., \( N_C = 5 \). The time interval is chosen as \( \Delta \tau = 1.8 \mu s \). The impact load by gas gun is simulated by a linear function \( F(t) = At \) with \( A = 4.5 \times 10^6 \text{ J/m}^2 \).

![Fig. 6. Impact loading configurations used in the experimental study with different crack-tip mode mixity being generated by different value of \( \delta_l \) or \( \delta_r \).](image)

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<th>Material parameters of some materials</th>
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<td>( E ) (GPa)</td>
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<td>( K_{IC} ) (MPa/\text{\sqrt{m}})</td>
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<td>( c_L ) (m/s)</td>
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<td>( \rho ) (kg m(^{-3}))</td>
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\(^a\) Note: the data of PMMA in the table above or under ‘/’ are for the material combination aluminum/PMMA from Lo et al. (1994) and PMMA/Homalite-100 from Kimberley and Lambros (2004), respectively.
10^6 MN s^{-1}. The dynamic fracture toughness $K_{ID}$ as a function of the crack-tip velocity $v$ is computed by using the empirical formula (Seelig, 1997):

$$K_{ID} = K_{IC} \times \left[1.0 + 2.5 \times \left(\frac{a}{c_T}\right)^2 + 3.9 \times 10^4 \times \left(\frac{a}{c_T}\right)^{10}\right].$$

It should be noted that the static critical energy release rate $\Gamma_C$ can be derived from $\Gamma_C = K_{IC}^2(1 - v^2)/E$ for the plane strain state and from $\Gamma_C = K_{IC}^2/E$ for the plane stress state. The dynamic critical energy release rate $\Gamma_d (\Gamma_d^1$ or $\Gamma_d^2$) can be determined by $\Gamma_d^i (\phi, \gamma) = \Gamma_{id}^i \left(1 - \frac{a}{c_{id}}\right)^{-\gamma}$ (where $\Gamma_{id}$ denotes the initial toughness under mode-I loading) (Rosakis et al., 1984) or $\Gamma_d = K_{ID}^2/(8\mu B_1)$. Here the former is employed. Additionally, $G_m$ is the maximum value of $G_2(\phi)$ with $\phi$ varying from 1° to 179° which is computed one by one. Unfortunately, this brings enormous computational complexity. To avoid this, we replace $G_m \geq G_2^i$ by $\eta G_m \geq G_2^i$. According to Table 1, the Dundurs’ parameters for PMMA/Homalite-100 are $\alpha \approx 0.0275$ and $\beta \approx 0.00635$. So it is in proximity to homogeneous medium. Based on the quasi-static relationship between $G_m/G_2$ and the kink angle $\phi$ for various crack-tip mixities $\psi$ presented in He and Hutchinson (1989), it can be found that $G_m \approx 1.0 \sim 2.0 G_m$ for smaller values of $\psi$. Therefore we take $\eta = 1.0 \sim 2.0$ in the computations.

We first consider the case of $d_r = 25$ mm, which employs impact on the right (uncracked) side of the sample and results in the shear opening of the crack faces. We select $\eta = 1.0, 1.5$ and 2.0 in our computations. The numerical results show that different choice of $\eta$ has little influence on the crack advance trajectory but obviously effects the initiation time of the crack kinking (The initiation times for $\eta = 1.0, 1.5$ and 2.0 are 91.0 $\mu$s, 84.1 $\mu$s and 79.0 $\mu$s, respectively). Since we are interested in the crack growth path, we choose $\eta = 1.0$ in the following calculation for simplicity. Fig. 7 shows the numerically simulated crack advance trajectory. The curve with $\Delta a/a_0 = 0.2$ is for $N_C = 5$ (i.e., five elements for the initial interfacial crack). The crack kinks into Homalite and the angles of the first five elements are $-5.0^\circ, -58.0^\circ, -41.9^\circ, -34.0^\circ$ and $-30.1^\circ$. For comparison, the experimental result taken from the interferograms by Kimberley and Lambros (2004) is also plotted by the dotted line in the figure. The kink angle in their experiment is $-37^\circ$. It is found that our results agree well with the experimental results except for the first element. This is due to the fact that the condition of dynamic HH criterion $\Delta a \rightarrow 0$ is not well satisfied in our numerical simulation ($\Delta a = 4$ mm). Therefore we selected a smaller crack-element length $\Delta a = 2$ mm ($\Delta a/a_0 = 0.1$) and recomputed this case. The results are also plotted in Fig. 7. Compared with the former results for $\Delta a/a_0 = 0.2$, we can find that the crack kinking
is closer to the initial crack-tip. So we could expect that the crack will directly kink into Homalite at about \(-50^\circ\) (closer to the experimental results) if a sufficiently small value of \(\Delta a\) is selected.

Next we consider the case of \(\delta_1 = +25\) mm. The numerical and experimental results (Kimberley and Lambros, 2004) are given in Fig. 8. Kimberley and Lambros got the experimental kink angle as \(36^\circ\) (Kimberley and Lambros, 2004). Our results say that the crack kinks into PMMA and the angles of the first five elements are \(3.0^\circ, 64.6^\circ, 49.5^\circ, 42.8^\circ\) and \(41.6^\circ\). Again we can find that our results agree well with the experimental results except for the first element.

In the following example the case of \(\delta_1 = 25\) mm is considered, where the impact appears on the cracked side which causes the crack faces to close since a compressive stress state is generated in the crack-tip region. Here we introduced the penalty method developed by Seelig (1997) to deal with the contact problem. Fig. 9 displays the numerical results for the crack growth trajectory in comparison with the experimental result (Kimberley and Lambros, 2004). The experimental kink angle was \(-86^\circ\). The numerical result shows the crack will kink into Homalite and the angles of the first five elements are \(-68.0^\circ, -53.9^\circ, -84.9^\circ, -65.8^\circ\) and \(-76.3^\circ\). Again good agreement between the numerical experimental results is observed.

It can be found from Figs. 7–9 that the differences between the numerical results and the experimental ones are gradually increasing after a stage of the crack increment. The differences perhaps result from the fact that the ideal boundary conditions adopted in our numerical simulations are not very strictly agreed with the experimental conditions. This may becomes more notable when the crack grows near to the boundary. Also the exact numerical simulation of the practical loading cases is almost impossible.

The averaged crack-tip speed after crack kinking was found by applying a linear fit to the crack-tip position data in Kimberley and Lambros (2004). In most cases the averaged crack-tip speed achieved after kinking was around \(0.5c_R\) (\(c_R\) is the Rayleigh wave speed in Homalite) (Kimberley and Lambros, 2004). The numerically calculated crack speed is about \(0.47c_R\). Considering the roughness of the element meshing and speed simulation, this result is basically in accordance with the experimental result.

Comparison of the so far presented numerical results with the experimental results shows that the developed numerical method is efficient. Therefore we will use the same numerical method to study some further examples for a uniformly distributed impact load \(\sigma_0 H(t)\), see Fig. 10. The geometry of the plate is the same as shown in Fig. 6. Two material combinations, Al/PMMA and PMMA/Homalite are considered. The material parameters are also listed in Table 1.
Fig. 11 shows the crack growth path in an Al/PMMA bi-material plate for $r_1 = 0$ and $r_2 = 80$ MPa, i.e., the case of uniaxial loading. The interfacial crack directly kinks out of the interface with the kink angle $\phi \approx -75^\circ$ and then grows along a path nearly parallel to the interface.

In Fig. 12 the results for the material combination of PMMA/Homalite under uniaxial loading are displayed. The difference to Fig. 11 for Al/PMMA is that the crack propagates along the interface (two crack
elements), and then kinks into Homalite at an angle of about $-7^\circ$, and subsequently deflects gradually away from the interface.

Fig. 13 shows the crack growth path in a PMMA/Homalite bi-material plate under bi-axial loading $\sigma_1 = \sigma_2 = 80$ MPa. The crack directly kinks away from the interface with $\varphi \approx -67^\circ$ and grows in this direction.
5. Conclusion

In this paper a dynamic HH criterion was generalized from the quasi static one. Coupled with the hybrid time-domain BEM, the dynamic kinking behaviors of an interfacial crack has been numerically simulated under different impact loads. A spatial function with the shape as \( g(y) \sim \sqrt{r} \) was introduced to avoid the singular stress oscillatory in the vicinity of the interfacial crack-tips. A penalty method was adopted to deal with the contact of the crack faces under some loading conditions.

Some simple examples were studied to verify the correctness of the presented scheme. The kinking of an interfacial crack has been numerically studied using BEM and the results were compared with the experimental interferograms given by Kimberley and Lambros (2004). Good agreement states that the numerical method and dynamic HH criterion are sufficiently effective and can be employed to simulate the dynamic kinking of an interfacial crack. Some differences between our results and the experiments, we believe, are caused by the roughness of the element meshing and simulation of the dynamic load.

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