Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 337 (2008) 144–168

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Global attractivity of an almost periodic N-species nonlinear ecological competitive model ☆

Yonghui Xia [a,*], Maoan Han [b], Zhenkun Huang [c]

[a] *College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350002, China*
[b] *The Institute of Mathematics, Shanghai Normal University, Shanghai 200234, China*
[c] *School of Sciences, Jimei University, Xiamen, Fujian 361021, China*

Received 11 October 2006

Available online 6 April 2007

Submitted by H.W. Broer

Abstract

By using comparison theorem and constructing suitable Lyapunov functional, we study the following almost periodic nonlinear N-species competitive Lotka–Volterra model:

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{N} a_{ij}(t)x_j^{\alpha_{ij}}(t) - \sum_{j=1}^{N} b_{ij}(t)x_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big) - \sum_{j=1}^{N} c_{ij}(t)x_i^{\alpha_{ii}}(t)x_j^{\alpha_{ij}}(t)\right].$$

A set of sufficient conditions is obtained for the existence and global attractivity of a unique positive almost periodic solution of the above model. As applications, some special competition models are studied again, our new results improve and generalize former results. Examples and their simulations show the feasibility of our main results.

Keywords: Competitive system; Lyapunov functional; Almost periodic solution; Global attractivity

☆ This work was supported by the National Natural Science Foundation of China under Grant (No. 10671127) and Shanghai Outstanding Discipline Leader Project (No. 06XD14034) and Shanghai Municipal Education Commission No. 06DZ002.

* Corresponding author.
E-mail addresses: yhxia@fzu.edu.cn, xiadoc@163.com (Y. Xia), mahan@shnu.edu.cn (M. Han).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.03.103

1. Introduction

In the present paper, we consider a generalized nonlinear N-species Gilpin–Ayala type competition system which takes the form

$$\dot{x}_i(t) = x_i(t)\Bigg[r_i(t) - \sum_{j=1}^{N} a_{ij}(t)x_j^{\alpha_{ij}}(t) - \sum_{j=1}^{N} b_{ij}(t)x_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big) \\ - \sum_{j=1}^{N} c_{ij}(t)x_i^{\alpha_{ii}}(t)x_j^{\alpha_{ij}}(t)\Bigg], \tag{1.1}$$

where $i = 1, 2, \dots, N$, $x_i(t)$ denotes the density of the ith specie X_i at time t; $r_i(t)$ is the intrinsic growth rate of the ith species X_i, $a_{ij}(t), b_{ij}(t)$ $(i \neq j)$ measures the amount of competition between the specie X_i and X_j, $\tau_{ij}(t)$ corresponds to the time delay; α_{ii} provides a nonlinear measure of intraspecific interference and α_{ij} $(i \neq j)$ provides a nonlinear measure of interspecific interference, α_{ij} is a positive constant; $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t), \tau_{ij}(t), i, j = 1, 2, \dots, N$, are all almost periodic functions defined on $(-\infty, +\infty)$; $a_{ij}(t), b_{ij}(t), c_{ij}(t), \tau_{ij}(t), i, j = 1, 2, \dots, N$, are all nonnegative; α_{ij} are positive constants; the intrinsic growth rate of the prey species $r_i(t)$ may be negative while $\lim_{T\to\infty} \frac{1}{T}\int_0^T r_i(t)\,dt > 0$; $a_{ii}, b_{ii}, i = 1, 2, \dots, N$, are positive. For the readers' convenience, the definition of almost periodic function is given here.

Definition 1.1. Let $f : \mathbf{R} \to \mathbf{R}$, $t \mapsto f(t)$ be a continuous function. We say that $f(t)$ is an almost periodic function on $\mathbf{R}$ if for all $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon)$ such that for all $t \in \mathbf{R}$,

$$\big\| f(t+\tau) - f(\tau) \big\| < \varepsilon.$$

The number τ is called an ε-translation number of $f(t)$.

The purpose of this article is to obtain some sufficient conditions for the global attractivity (see Definition 2.2) and existence of a positive almost periodic solution of the system (1.1). The remaining part of this paper is organized as follows. We introduce the background of our work in the rest of this section. In Section 2, we shall prove some preliminary results. Then, by using comparison theorem and the preliminary results, some sufficient conditions are obtained for the persistency (see Definition 2.1) of system (1.1). We then deduce the existence of a bounded solution of system (1.1) on $\mathbf{R}$. In Section 3, first, by constructing a suitable Lyapunov function, some sufficient conditions are obtained for the global attractivity of system (1.1). Then, we devote ourself to obtain some sufficient conditions for the existence of a unique almost periodic solution of system (1.1). The approach is based on the properties of almost periodic function and the choice of a suitable Lyapunov function. In Section 4, as applications, we deduce criteria for some well-known special cases of system (1.1) to illustrate the generality of our results which generalize former known results. Finally, a suitable example and their simulations show the feasibility of our results.

Traditional Lotka–Volterra competitive system is a rudimentary model on mathematical ecology which can be expressed as follows:

$$\dot{x}_i(t) = x_i(t)\Bigg[r_i(t) - \sum_{j=1}^{N} a_{ij}(t)x_j(t)\Bigg], \quad i = 1, 2, \dots, N. \tag{1.2}$$

The model has been investigated extensively (for example, see [1–18] and the references cited therein). Many interesting results concern with the persistency, extinction and global attractivity of periodic or almost periodic solutions.

On one hand, according to the culture figures that are obtained by Ayala on the studies of D. Psendoobscura and D. Serrata, the growth rate of some competitive species does not correspond with that of the Lotka–Volterra model (see Chen [1]). So the results obtained by Lotka–Volterra system are not applicable for every species. The reason is that ignoring nonlinear terms in the problem leads to neglecting many important factors, such as the effect of toxic (see Chattopadhyay [19] and Xia et al. [31]) or the age-structure of a population (see Cui et al. [20]). As these important factors are to be considered, we have to introduce more complex equations. For the above reasons, the following autonomous or nonautonomous system has been considered in [15,16], respectively,

$$\begin{cases} \dot{x}_1(t) = x_1(t)\big[r_1(t) - a_1(t)x_1(t) - b_1(t)x_2(t) - c_1(t)x_1(t)x_2(t)\big], \\ \dot{x}_2(t) = x_2(t)\big[r_2(t) - a_2(t)x_1(t) - b_2(t)x_2(t) - c_2(t)x_1(t)x_2(t)\big]. \end{cases}$$

On the other hand, in 1973, Ayala et al. [21] conducted experiments on fruit fly dynamics to test the validity of 10 models of competitions. One of the models accounting best for the experimental results is given by

$$\begin{cases} \dot{x}_1(t) = r_1 x_1(t)\left[1 - \left(\dfrac{x_1(t)}{K_1}\right)^{\theta_1} - \alpha_{12}\dfrac{x_2(t)}{K_2}\right], \\ \dot{x}_2(t) = r_2 x_2(t)\left[1 - \left(\dfrac{x_2(t)}{K_2}\right)^{\theta_2} - \alpha_{21}\dfrac{x_1(t)}{K_1}\right]. \end{cases} \tag{1.3}$$

In order to fit data in their experiments and to yield significantly more accurate results, Gilpin et al. [22] claimed that a slightly more complicated model was needed and proposed the following model:

$$\dot{x}_i(t) = r_i x_i(t)\left[1 - \left(\frac{x_i(t)}{K_i}\right)^{\theta_i} - \sum_{j=1,\, j\neq i}^{N} b_{ij}(t)\frac{x_j(t)}{K_j}\right], \quad i = 1, 2, \ldots, N, \tag{1.4}$$

where x_i is the population density of the ith species, r_i is the intrinsic exponential growth rate of the ith species, K_i is the environment carrying capacity of species i in the absence of competition, θ_i provides a nonlinear measure of intra-specific interference, and b_{ij} $(i \neq j)$ provides a measure of interspecific competition. Goh et al. [23] and Liao et al. [24] obtained sufficient conditions which guarantee the global asymptotic stability of the system (1.4). By means of Ahmad et al. [25] definitions of lower and upper averages of a function, Chen [26] studied the following generalized N-species nonautonomous Gilpin–Ayala type competitive model:

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{N} a_{ij}(t)x_j^{\alpha_{ij}}(t)\right], \quad i = 1, 2, \ldots, N. \tag{1.5}$$

Some sufficient conditions were obtained for the persistency and extinction of system (1.5). Also, Fan et al. [27] further incorporated time delays in the model (1.6) and they proposed the following delayed Gilpin–Ayala competitive model:

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{N} a_{ij}(t)x_j^{\alpha_{ij}}\big(t - \tau_{ij}(t)\big)\right], \quad i = 1, 2, \ldots, N. \tag{1.6}$$

By applying the coincidence degree theory (see Gaines et al. [37]), they obtained a set of easily verifiable sufficient conditions for the existence of at least one positive (componentwise) periodic solution of system (1.6).

It is well known that ecosystem in the real world are continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Due to the various seasonal effects of the environmental factors in real life situation (e.g., seasonal effects of weather, food supplies, mating habits, harvesting, etc.), it is rational and practical to study the ecosystem with periodic coefficients. A very basic and important ecological problem in the study of multi-species population dynamics concerns the existence and global attractivity of positive periodic solutions.

However, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. If we consider the effects of the environmental factors, the assumption of almost periodicity is more realistic, more important and more general. For more details about the significance of the almost periodicity, one could refer to [6,10,11,16,28–35] and references cited therein. Motivated by the above works, we propose system (1.1) which consider both the effect of toxic and Gilpin–Ayala effect. However, must of the existing works handle systems (from (1.2) to (1.6)) with the uniform persistence, global attractivity or existence of positive periodic solutions. To the best of the authors' knowledge, no study has concerned system (1.1) with the almost periodic coefficients so far. Our aim is to obtain sufficient conditions for the existence of a globally attractive positive almost periodic solution of the system (1.1). The method is based on the use of comparison theorem and constructing suitable Lyapunov functional. Our results generalize and improve those in Ahmad [6], Gopalsamy [9,10], Zhao [13], He [16], Zhao [18], Chen [26], Xia et al. [31].

Throughout this paper, we shall use the following notations:

- We always use $i, j = 1, \dots, N$, unless otherwise stated.
- If $f(t)$ is an almost periodic function defined on $(-\infty, +\infty)$, we set

$$\overline{f} = \sup_{t\in(-\infty,+\infty)} f(t), \qquad \underline{f} = \inf_{t\in(-\infty,+\infty)} f(t).$$

- Denote $p_i = (\frac{\bar{r}_i}{\underline{a}_{ii}})^{\frac{1}{\alpha_{ii}}}$, $q_i = [(\underline{r}_i - \sum_{j=1,\, j\neq i}^{N} \bar{a}_{ij} p_j^{\alpha_{ij}})/\bar{a}_{ii}]^{\frac{1}{\alpha_{ii}}}$.
- Let $\psi_{ij}(t) = t - \tau_{ij}(t)$. We assume that for all integers i, j this function is always invertible, i.e., with nonvanishing derivative. We denote by ψ_{ij}^{-1} its inverse.
- Denote mean value $m(f) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,dt$. When $f(t)$ is an ω-periodic function, then $m(f) = \frac{1}{\omega}\int_0^\omega f(t)\,dt$. Obviously, when $f(t)$ is an ω-periodic function, $m(f) > 0 \Leftrightarrow \int_0^\omega f(t)\,dt > 0$. We denote the hull of $f(t)$ by $H(f)$, where $H(f)$ is the set of real function $g(t)$ such that there exists a sequence t_n such that $\lim_{n\to+\infty} f(t+t_n) = g(t)$ uniformly on $\mathbf{R}$.
- Given $x(t) = (x_1(t), \dots, x_N(t)) \in \mathbf{R}^N$, $y(t) = (y_1(t), \dots, y_N(t)) \in \mathbf{R}^N$, we put $x(t) \geqslant y(t)$, if $x_i(t) \geqslant y_i(t)$, for all $i = 1, 2, \dots, N$.

Throughout this paper, we suppose that the following conditions are satisfied:

(H$_1$) $r_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$, $c_{ij}(t)$ are nonnegative almost periodic functions defined for $t \in (-\infty, +\infty)$ and $\inf_{t\in\mathbf{R}} a_{ii}(t) > 0$, $\inf_{t\in\mathbf{R}} b_{ii}(t) > 0$; $\alpha_{ij} > 0$.

(H$_2$) $\tau_{ij}(t)$ are nonnegative, continuously differentiable and almost periodic functions on $t \in \mathbf{R}$, $\psi_{ij}(t) = t - \tau_{ij}(t)$ are invertible. Moreover, $\dot{\tau}_{ij}(t)$ are all uniformly continuous on $\mathbf{R}$ with $\inf_{t\in\mathbf{R}}\{1 - \dot{\tau}_{ij}(t)\} > 0$.

(H$_3$) $m(r_i) > 0$.

2. Existence of bounded solutions

Since we are interested in the positive solutions of system (1.1), we assume the system (1.1) to be supplemented with initial conditions of the form

$$x_i(s) = \phi_i(s) \in \mathbf{C}\big([-\tau, 0], \mathbf{R}_+\big), \qquad \phi_i(0) > 0, \tag{2.1}$$

where $\mathbf{C}$ denotes the set of continuous functions, $\phi_i \in \mathbf{C}$ is the initial value function and $\tau = \sup_{t\in\mathbf{R}}\{\tau_{ij}(t),\ i, j = 1, 2, \ldots, N\}$, $\mathbf{R}_+ = (0, +\infty)$. Integrating (1.1) over $[t_0, t]$ leads to

$$x_i(t) = x_i(t_0)\exp\Biggl\{\int_{t_0}^{t}\Biggl[r_i(s) - \sum_{j=1}^{N} a_{ij}(s)x_j^{\alpha_{ij}}(s) - \sum_{j=1}^{N} b_{ij}(s)x_j^{\alpha_{ij}}\bigl(s - \tau_{ij}(s)\bigr) - \sum_{j=1}^{N} c_{ij}(s)x_i^{\alpha_{ii}}(s)x_j^{\alpha_{ij}}(s)\Biggr]ds\Biggr\}.$$

Hence, any solution $x(t) = (x_1(t), \ldots, x_N(t))$ of the initial value problem (1.1)–(2.1) exists and satisfies $x_i(t) > 0$, for all $i = 1, 2, \ldots, N$ and $t \geqslant t_0$, i.e., $x(t) > 0$ for $t \geqslant t_0$.

Definition 2.1. The initial value problem (1.1)–(2.1) is said to be *persistent*, if for any positive solution of the initial value problem (1.1)–(2.1) there exists positives constants m, M such that for all solution $x(t)$ there exists $T > 0$ such that $m \leqslant x_i(t) \leqslant M$, for all $t \geqslant T$. The solution of the initial value problem is also called *ultimately bounded above and below*.

Definition 2.2. A positive bounded solution $x(t) = (x_1(t), \ldots, x_N(t))$ of the initial value problem (1.1)–(2.1) is said to be *globally attractive*, if for any other positive solution $y(t) = (y_1(t), \ldots, y_N(t))$ of the initial value problem (1.1)–(2.1), we have

$$\lim_{t\to+\infty}\bigl|x_i(t) - y_i(t)\bigr| = 0, \quad i = 1, 2, \ldots, N.$$

In the following we will prove a preliminary result which will be used in the proof of our main results.

Now we consider the generalized almost periodic logistic equation

$$\dot{x}(t) = x(t)\bigl[r(t) - a(t)x^{\alpha}(t)\bigr], \tag{2.2}$$

where $r(t)$ and $a(t)$ are continuous almost periodic functions with $\alpha > 0, \underline{a} > 0$ and $m(r) > 0$. We now state the following lemma:

Lemma 2.1. *System* (2.2) *has a unique positive globally attractive almost periodic solution* $\tilde{x}(t)$ *with* $\tilde{x}(t) \leqslant (\bar{r}_i/\underline{a}_{ii})^{\frac{1}{\alpha}}$. *Let* $\tilde{x}_i$ $(i = 1, 2)$ *be the unique solution of* (2.2) *when replacing* $r(t)$ *by* $r_i(t)$, $a(t)$ *by* $a_i(t)$ $(i = 1, 2)$, *respectively. If* $r_2(t) \geqslant r_1(t)$ *and* $a_2(t) \geqslant a_1(t)$, *then* $\tilde{x}_2(t) \geqslant \tilde{x}_1(t)$.

Proof. By the transformation of variable

$$z(t)=\frac{1}{x^{\alpha}(t)}, \tag{2.3}$$

then Eq. (2.2) becomes

$$\dot z(t)=\alpha a(t)-\alpha r(t)z(t). \tag{2.4}$$

Since $\underline{a}>0$ and $m(r)>0$, obviously, system (2.4) has a unique almost periodic solution represented as follows:

$$\tilde z(t)=\alpha\int_{-\infty}^{t}\exp\left(-\alpha\int_{s}^{t}r(u)\,du\right)a(s)\,ds, \tag{2.5}$$

with $\tilde z(t)\geqslant \underline{a}/\bar r>0$ and $\tilde z(t)$ is bounded for $t\in R$. Therefore, $\tilde x(t)=(\frac{1}{\tilde z(t)})^{\frac{1}{\alpha}}$ is the unique positive almost periodic solution of (2.2) with $0<\tilde x(t)\leqslant(\bar r/\underline{a})^{\frac{1}{\alpha}}$.

Next we will show that the unique positive almost periodic solution of (2.2) is globally attractive. In fact, let $\tilde z(t)$ be given by (2.5) and $z(t)=z(t,0,z_0)$ ($t_0=0$) be any other solution of system (2.4) with any initial value $(0,z_0)$, then by using the variation of constants formula, we have

$$z(t)=\left[z_0+\alpha\int_0^t a(s)\exp\left(\int_0^s\alpha r(u)\,du\right)ds\right]\exp\left(-\int_0^t\alpha r(u)\,du\right). \tag{2.6}$$

Simple computation shows that $z(t)\geqslant \underline{a}/\bar r[1+(\bar r/\underline{a}z_0-1)e^{-\alpha\bar r t}]$. This implies that there exists $T>0$ such that $z(t)\geqslant\underline{a}/\bar r>0$, for all $t\geqslant T$. Also, it follows from (2.5) and (2.6) that

$$\begin{aligned}\left|z(t)-\tilde z(t)\right|&=\left|z_0-\alpha\int_{-\infty}^{0}a(s)\exp\left(\int_s^0\alpha r(u)\,du\right)ds\right|\exp\left(-\int_0^t\alpha r(u)\,du\right)\\&=\left|z_0-\tilde z(0)\right|\exp\left(-\int_0^t\alpha r(u)\,du\right).\end{aligned}$$

Since $m(r)>0$, we have $\int_0^t\alpha r(u)\,du\to+\infty$ as $t\to+\infty$. Thus, $z(t)-\tilde z(t)\to 0$ as $t\to+\infty$. Let $\tilde x(t)=(1/\tilde z(t))^{\frac{1}{\alpha}}$ and $x(t)=(\frac{1}{z(t)})^{\frac{1}{\alpha}}$ and note that the function $y=x^a$ is nonincreasing as $a\leqslant 0$. By the mean value theorem, together with $\tilde z(t), z(t)\geqslant\underline{a}/\bar r>0$, for all $t\geqslant T$, we have

$$\begin{aligned}\left|x(t)-\tilde x(t)\right|&=\left|\left(\frac{1}{\tilde z(t)}\right)^{\frac{1}{\alpha}}-\left(\frac{1}{z(t)}\right)^{\frac{1}{\alpha}}\right|=\left|\tilde z^{-\frac{1}{\alpha}}(t)-z^{-\frac{1}{\alpha}}(t)\right|\\&\leqslant\frac{1}{\alpha}(\underline{a}/\bar r)^{-\frac{1}{\alpha}-1}\left|z(t)-\tilde z(t)\right|,\quad\text{for all } t\geqslant T.\end{aligned}$$

Therefore, $x(t)-\tilde x(t)\to 0$ as $t\to+\infty$, i.e., $\tilde x(t)$ is the unique positive almost periodic solution of system (2.2) which is globally attractive. By using (2.3) and (2.5), the remaining part of Lemma 2.1 follows and this completes the proof. □

Remark 2.1. Take $\alpha=1$, then Lemma 2.1 reduces to Lemma 2.2 in Xia et al. [31].

If (H_1) and (H_3) hold, it follows from Lemma 2.1 that the following system:

$$\dot{x}_i(t) = x_i(t)\big[r_i(t) - a_{ii}(t)x_i^{\alpha_{ii}}(t)\big] \tag{2.7}$$

has a unique globally attractive positive almost periodic solution, denoted by $X_i(t)$.

Lemma 2.2. *If* (H_1)–(H_3) *hold, then*

$$m\left(r_i(t) - \sum_{j=1,\, j\neq i}^{N} \big[a_{ij}(t)X_j^{\alpha_{ij}}(t) + b_{ij}(t)X_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big)\big]\right)$$
$$= m\left(r_i(t) - \sum_{j=1,\, j\neq i}^{N} \left[a_{ij}(t)X_j^{\alpha_{ij}}(t) + \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}\right]X_j^{\alpha_{ij}}(t)\right).$$

Proof. The proof of Lemma 2.2 is similar to the proof of Lemma 2.2 in Xia et al. [33]. From (H_1), (H_2) and the properties of almost periodic functions, we note that $\tau_{ij}(t)$ and $\frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}X_j^{\alpha_{ij}}(t)$ are all almost periodic functions. By the boundedness of the almost periodic functions, we can verify that $\int_T^{T-\tau_{ij}} \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}X_j^{\alpha_{ij}}(t)\,dt$ is bounded. Then, we have

$$\begin{aligned}
&m\big(b_{ij}(t)X_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big)\big)\\
&= \lim_{T\to\infty} T^{-1}\int_0^T b_{ij}(t)X_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big)\,dt\\
&= \lim_{T\to\infty} T^{-1}\int_{-\tau_{ij}(0)}^{T-\tau_{ij}(T)} \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}X_j^{\alpha_{ij}}(t)\,dt\\
&= \lim_{T\to\infty} T^{-1}\left[\int_0^T \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}X_j^{\alpha_{ij}}(t)\,dt + \int_{-\tau_{ij}(0)}^{0} \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}X_j^{\alpha_{ij}}(t)\,dt\right.\\
&\qquad\left. + \int_T^{T-\tau_{ij}(T)} \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}X_j^{\alpha_{ij}}(t)\,dt\right]\\
&= \lim_{T\to\infty} T^{-1}\int_0^T \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}X_j^{\alpha_{ij}}(t)\,dt = m\left(\frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}X_j^{\alpha_{ij}}(t)\right). \qquad \square
\end{aligned}$$

Theorem 2.1. *In addition to* (H_1)–(H_3), *if further assume that*

(H_4) $m(r_i(t) - \sum_{j=1,\, j\neq i}^{N}[a_{ij}(t) + \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}]X_j^{\alpha_{ij}}(t)) > 0$, *where* $X_j(t)$ *is the unique globally attractive positive almost periodic solution of system* (2.7). *Then the initial value problem* (1.1)–(2.1) *is persistent.*

Proof. Proof of Theorem 2.1 is motivated by Teng [17] and Xia et al. [33]. First, we show that any positive solution of system (1.1) are ultimately bounded above by some positive constant. Let $x(t)=(x_1(t),\dots,x_N(t))^T$ be any other solution of system (1.1). It follows from (1.1) that

$$\begin{aligned}\dot{x}_i(t)=x_i(t)\Bigg[&r_i(t)-\sum_{j=1}^{N}a_{ij}(t)x_j^{\alpha_{ij}}(t)-\sum_{j=1}^{N}b_{ij}(t)x_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big)\\&-\sum_{j=1}^{N}c_{ij}(t)x_i^{\alpha_{ii}}(t)x_j^{\alpha_{ij}}(t)\Bigg]\\\leqslant{}& x_i(t)\big[r_i(t)-a_{ii}(t)x_i^{\alpha_{ii}}(t)\big],\end{aligned}$$

for all $t\geqslant t_0$. By using the comparison theorem, we have

$$x_i(t)\leqslant X_i(t),\quad \text{for all } t\geqslant t_0, \tag{2.8}$$

where $X_i(t)$ is the unique globally attractive positive almost periodic solution of system (2.7) which satisfies the initial condition $x_i(t_0)\leqslant X_i(t_0)$. From Lemma 2.1 and (2.8), it is not difficult to obtain that the following two inequalities hold:

$$\limsup_{t\to+\infty}x_i(t)\leqslant\left(\frac{\bar{r}_i}{\underline{a}_{ii}}\right)^{\frac{1}{\alpha_{ii}}}:=p_i,\quad \text{for all } t\in\mathbf{R},$$

and there exists $\tilde{T}_i>t_0$ such that

$$x_i(t)\leqslant p_i,\quad \text{for all } t\geqslant\tilde{T}_i. \tag{2.9}$$

We choose $M=\max_i p_i$, then $x_i(t)\leqslant M$, for all $t\geqslant\max_i\tilde{T}_i$.

Second, we shall show that any positive solution of system (1.1) is ultimately bounded below by some positive constant. To this end, we proceed with two steps.

Step 1: We show that there exists $\delta_0>0$ such that $\limsup_{t\to+\infty}x_i(t)\geqslant\delta_0$, for all i. In fact, it follows from (2.8) that for any constant $\epsilon>0$, there exists $T(\epsilon)\geqslant t_0$ such that

$$x_i(t)\leqslant X_i(t)+\epsilon,\quad \text{for all }\ t\geqslant T(\epsilon),\ i=1,2,\dots,N. \tag{2.10}$$

Denote

$$\begin{aligned}B_i(t)&=r_i(t)-\sum_{j=1,\,j\neq i}^{N}\big[a_{ij}(t)X_j^{\alpha_{ij}}(t)+b_{ij}(t)X_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big)\big],\\B_i(t,\epsilon)&=r_i(t)-\sum_{j=1,\,j\neq i}^{N}\big[a_{ij}(t)\big(X_j(t)+\epsilon\big)^{\alpha_{ij}}+b_{ij}(t)\big(X_j\big(t-\tau_{ij}(t)\big)+\epsilon\big)^{\alpha_{ij}}\big],\\A_i(t,\epsilon)&=a_{ii}(t)+b_{ii}(t)+\sum_{j=1}^{N}c_{ij}(t)\big(X_j(t)+\epsilon\big)^{\alpha_{ij}},\quad \text{for all }\ t\geqslant T(\epsilon).\end{aligned}$$

It follows from (H_4) and Lemma 2.2 that

$$\lim_{T\to\infty}\int_s^{T+s}B_i(t)\,dt=m\left(r_i(t)-\sum_{j=1,\,j\neq i}^{N}\left[a_{ij}(t)+\frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}\right]X_j^{\alpha_{ij}}(t)\right)>0,$$

uniformly for $s \in \mathbf{R}$. Therefore, there exist positive constants λ and δ such that

$$\int_t^{t+\lambda} B_i(u)\,du \geqslant \delta, \quad \text{for all } t \in \mathbf{R}. \tag{2.11}$$

From (2.11), we can choose sufficient small positive constants $\epsilon_0, \delta_0, \gamma_0$ such that

$$\int_t^{t+\lambda} \left[B_i(u,\epsilon_0) - \delta_0^{\alpha_{ii}} A_i(u,\epsilon_0)\right] du > \gamma_0, \quad \text{for all } t \in \mathbf{R}. \tag{2.12}$$

Now we claim that the following inequality holds:

$$\limsup_{t\to+\infty} x_i(t) \geqslant \delta_0. \tag{2.13}$$

By way of contradiction, suppose that $\limsup_{t\to+\infty} x_p(t) < \delta_0$ for a certain $p \in \{1,2,\ldots,N\}$, then there exists $T_2 > T_1 = T(\epsilon_0)$ such that $x_p(t) < \delta_0$, for all $t \geqslant T_2$. This, together with (2.10), gives

$$\begin{aligned}
\dot{x}_p(t) &= x_p(t)\Bigg[r_p(t) - \sum_{j=1}^N a_{pj}(t) x_j^{\alpha_{pj}}(t) - \sum_{j=1}^N b_{pj}(t) x_j^{\alpha_{pj}}\big(t-\tau_{pj}(t)\big) \\
&\quad - \sum_{j=1}^N c_{pj}(t) x_p^{\alpha_{pp}}(t) x_j^{\alpha_{pj}}(t)\Bigg] \\
&\geqslant x_p(t)\Bigg[r_p(t) - a_{pp}(t)\delta_0^{\alpha_{pp}} - \sum_{j=1,\, j\neq p}^N a_{pj}(t)\big(X_j(t)+\epsilon_0\big)^{\alpha_{pj}} - b_{pp}(t)\delta_0^{\alpha_{pp}} \\
&\quad - \sum_{j=1,\, j\neq p}^N b_{pj}(t)\big(X_j\big(t-\tau_{pj}(t)\big)+\epsilon_0\big)^{\alpha_{pj}} - \sum_{j=1}^N c_{pj}(t)\big(X_j(t)+\epsilon_0\big)^{\alpha_{pj}}\delta_0^{\alpha_{pp}}\Bigg] \\
&= x_p(t)\big[B_p(t,\epsilon_0) - \delta_0^{\alpha_{pp}} A_p(t,\epsilon_0)\big],
\end{aligned} \tag{2.14}$$

for all $t \geqslant T_2$. An integration of (2.14) over $[T_2, t]$ leads to

$$x_p(t) \geqslant x_p(T_2) \exp \int_{T_2}^t \left[B_p(s,\epsilon_0) - \delta_0^{\alpha_{pp}} A_p(s,\epsilon_0)\right] ds.$$

Obviously, it follows from (2.12) that $x_p(t) \to +\infty$ as $t \to +\infty$, which contradicts $x_i(t) \leqslant p_i$, for all $t \geqslant \tilde{T}_i$ in (2.9). Hence, the inequality (2.13) is correct.

Step 2: We show that for any solution $x(t) = (x_1(t), \ldots, x_N(t))$, there exists a positive constant $m > 0$ such that

$$\liminf_{t\to+\infty} x_i(t) \geqslant m, \quad \text{for all } i = 1,2,\ldots,n. \tag{2.15}$$

In fact, suppose the contrary, there exist a certain $p \in \{1,2,\ldots,N\}$ and a sequence of solution $x_p^{(n)}(t)$, $n = 1,2,\ldots,$ such that

$$\limsup_{t\to+\infty} x_p^{(n)}(t) < \frac{1}{n^2}, \quad \text{for all } n = 1, 2, \ldots. \tag{2.16}$$

Then there exist two time sequences $\{s_q^{(n)}\}$ and $\{t_q^{(n)}\}$ such that

$$\begin{aligned} &0 < s_1^{(n)} < t_1^{(n)} < s_2^{(n)} < t_2^{(n)} < \cdots < s_q^{(n)} < t_q^{(n)} < \cdots, \quad \text{for all } n = 1, 2, \ldots, \\ &s_q^{(n)} \to \infty, \quad t_q^{(n)} \to \infty, \quad \text{as } q \to \infty, \qquad x_p^{(n)}\big(t_q^{(n)}\big) = \frac{1}{n^2}, \quad x_p^{(n)}\big(s_q^{(n)}\big) = \frac{1}{n}, \end{aligned} \tag{2.17}$$

and

$$\frac{1}{n^2} < x_p^{(n)}(t) < \frac{1}{n}, \quad \text{for all } t \in \big(s_q^{(n)}, t_q^{(n)}\big). \tag{2.18}$$

It follows from (2.10) that there exists $T_2^{(n)} > T_1$ such that

$$x_i^{(n)}(t) \leqslant X_i(t) + \epsilon_0, \quad t \geqslant T_2^{(n)}.$$

Clearly, there exists an integer sequence $N_1^{(n)} > 0$ such that $s_q^{(n)} > T_2^{(n)}$, for all $q \geqslant N_1^{(n)}$ and all $n = 1, 2, \ldots$. Hence, for any $t \in [s_q^{(n)}, t_q^{(n)}]$ and $q \geqslant N_1^{(n)}$, we have

$$\begin{aligned} \dot{x}_p^{(n)}(t) &= x_p^{(n)}(t)\Bigg[r_p(t) - \sum_{j=1}^N a_{pj}(t)\big(x_j^{(n)}(t)\big)^{\alpha_{pj}} - \sum_{j=1}^N b_{pj}(t)\big(x_j^{(n)}\big(t - \tau_{pj}(t)\big)\big)^{\alpha_{pj}} \\ &\quad - \sum_{j=1}^N c_{pj}(t)\big(x_p^{(n)}(t)\big)^{\alpha_{pp}}\big(x_j^{(n)}(t)\big)^{\alpha_{pj}}\Bigg] \\ &\geqslant x_p^{(n)}(t)\Bigg[r_p(t) - \sum_{j=1}^N a_{pj}(t)\big(X_j^{(n)}(t) + \epsilon_0\big)^{\alpha_{pj}} \\ &\quad - \sum_{j=1}^N b_{pj}(t)\big(X_j^{(n)}\big(t - \tau_{pj}(t)\big) + \epsilon_0\big)^{\alpha_{pj}} \\ &\quad - \sum_{j=1}^N c_{pj}(t)\big(X_j^{(n)}(t) + \epsilon_0\big)^{\alpha_{pj}}\big(X_p^{(n)}(t) + \epsilon_0\big)^{\alpha_{pp}}\Bigg] \\ &\geqslant -k_0 x_p^{(n)}(t), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} k_0 = \sup_{t\in\mathbf{R}}\Bigg\{&\sum_{j=1}^N a_{pj}(t)\big(X_j^{(n)}(t) + \epsilon_0\big)^{\alpha_{pj}} + \sum_{j=1}^N b_{pj}(t)\big(X_j^{(n)}\big(t - \tau_{pj}(t)\big) + \epsilon_0\big)^{\alpha_{pj}} \\ &+ \sum_{j=1}^N c_{pj}(t)\big(X_j^{(n)}(t) + \epsilon_0\big)^{\alpha_{pj}}\big(X_p^{(n)}(t) + \epsilon_0\big)^{\alpha_{pp}}\Bigg\}. \end{aligned}$$

An integration of (2.19) over $[s_q^{(n)}, t_q^{(n)}]$ leads to

$$\frac{1}{n^2} = x_p^{(n)}\big(t_q^{(n)}\big) \geqslant x_p^{(n)}\big(s_q^{(n)}\big)\exp\big[-k_0\big(t_q^{(n)} - s_q^{(n)}\big)\big] = \frac{1}{n}\exp\big[-k_0\big(t_q^{(n)} - s_q^{(n)}\big)\big],$$

for all $q \geqslant N_1^{(n)}$ and all $n = 1, 2, \dots$, which implies

$$t_q^{(n)} - s_q^{(n)} \geqslant \frac{\ln n}{k_0}, \quad \text{for all } q \geqslant N_1^{(n)},\ n = 1, 2, \dots. \tag{2.20}$$

It follows from (2.20) that there exists a sufficient large integer n_0 such that

$$\delta_0 > \frac{1}{n}, \quad t_q^{(n)} - s_q^{(n)} \geqslant \lambda, \quad \text{for all } n \geqslant n_0,\ q \geqslant N_1^{(n)}.$$

Therefore, for any $n \geqslant n_0$, $q \geqslant N_1^{(n)}$ and $t \in [s_q^{(n)}, t_q^{(n)}]$, it follows from (2.17) and (2.18) that

$$\begin{aligned}
\dot{x}_p^{(n)}(t) &= x_p(t)\Bigg[r_p(t) - \sum_{j=1}^{N} a_{pj}(t)\big(x_j^{(n)}(t)\big)^{\alpha_{pj}} - \sum_{j=1}^{N} b_{pj}(t)\big(x_j^{(n)}\big(t-\tau_{pj}(t)\big)\big)^{\alpha_{pj}} \\
&\quad - \sum_{j=1}^{N} c_{pj}(t)\big(x_p^{(n)}(t)\big)^{\alpha_{pp}}\big(x_j^{(n)}(t)\big)^{\alpha_{pj}}\Bigg] \\
&\geqslant x_p(t)\Bigg[r_p(t) - a_{pp}(t)\left(\frac{1}{n}\right)^{\alpha_{pp}} - \sum_{j=1,\, j\neq p}^{N} a_{pj}(t)\big(X_j^{(n)}(t)+\epsilon_0\big)^{\alpha_{pj}} \\
&\quad - b_{pp}(t)\left(\frac{1}{n}\right)^{\alpha_{pp}} - \sum_{j=1,\, j\neq p}^{N} b_{pj}(t)\big(X_j^{(n)}\big(t-\tau_{pj}(t)\big)+\epsilon_0\big)^{\alpha_{pj}} \\
&\quad - \sum_{j=1}^{N} c_{pj}(t)\big(X_j^{(n)}(t)+\epsilon_0\big)^{\alpha_{pj}}\left(\frac{1}{n}\right)^{\alpha_{pp}}\Bigg] \\
&\geqslant x_p(t)\Bigg[r_p(t) - a_{pp}(t)\delta_0^{\alpha_{pp}} - \sum_{j=1,\, j\neq p}^{N} a_{pj}(t)\big(X_j^{(n)}(t)+\epsilon_0\big)^{\alpha_{pj}} - b_{pp}(t)\delta_0^{\alpha_{pp}} \\
&\quad - \sum_{j=1,\, j\neq p}^{N} b_{pj}(t)\big(X_j^{(n)}\big(t-\tau_{pj}(t)\big)+\epsilon_0\big)^{\alpha_{pj}} - \sum_{j=1}^{N} c_{pj}(t)\big(X_j^{(n)}(t)+\epsilon_0\big)^{\alpha_{pj}}\delta_0^{\alpha_{pp}}\Bigg] \\
&= x_p(t)\big[B_p(t,\epsilon_0) - \delta_0^{\alpha_{pp}} A_p(t,\epsilon_0)\big].
\end{aligned} \tag{2.21}$$

Together with (2.12), (2.17) and (2.18), an integration of (2.21) over $[t_q^{(n)} - \lambda, t_q^{(n)}]$ leads to

$$\frac{1}{n^2} = x_p^{(n)}\big(t_q^{(n)}\big) \geqslant x_p^{(n)}\big(t_q^{(n)} - \lambda\big) \exp \int_{t_q^{(n)}-\lambda}^{t_q^{(n)}} \big[B_p(t,\epsilon_0) - \delta_0^{\alpha_{pp}} A_p(t,\epsilon_0)\big]\,dt > \frac{1}{n^2} e^{\gamma_0} > \frac{1}{n^2}.$$

This is a contradiction, thus the inequality (2.15) is correct. That is to say, any positive solution $x(t)$ of the initial value problem (1.1)–(2.1) is ultimately bounded below by a positive constant m. From Definition 2.1, the proof of Theorem 2.1 is complete. □

Denote

$$K = \big\{x(t) = \big(x_1(t), \dots, x_N(t)\big) \,\big|\, m \leqslant x_i(t) \leqslant M,\ \text{for all } t \in \mathbf{R},\ i = 1, 2, \dots, N\big\}.$$

Theorem 2.2. *If* (H$_1$)–(H$_4$) *hold, then the initial value problem* (1.1)–(2.1) *has at least one positive* (*componentwise*) *solution on all of* **R** *belonging to* K.

Proof. Theorem 2.2 implies that there exists $T^0 \geqslant t_0$ such that system (1.1) has at least one positive solution $x(t)$ satisfying $0 < m \leqslant x_i(t) \leqslant M$ for $t \geqslant T^0$. In what follows, we will prove that system (1.1) has at least one positive solution $v(t) = (v_1(t), \dots, v_N(t))$ defined on **R** such that $m \leqslant v_i(t) \leqslant M$, for all $t \in \mathbf{R}$.

Since $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t), \tau_{ij}(t)$ are almost periodic, there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $r_i(t+t_n) \to r_i(t)$, $a_{ij}(t+t_n) \to a_{ij}(t)$, $b_{ij}(t+t_n) \to b_{ij}(t)$, $c_{ij}(t+t_n) \to c_{ij}(t)$, $\tau_{ij}(t+t_n) \to \tau_{ij}(t)$ uniformly for all $t \in \mathbf{R}$ as $n \to \infty$.

We claim that the sequence $\{x(t+t_n)\}$ is uniformly bounded and equi-continuous on any bounded interval in **R**.

In fact, for any bounded interval $[t_-, t_+] \subset \mathbf{R}$, as n is large enough, $t_n + \beta \geqslant T^0$. So $m \leqslant x_i(t+t_n) \leqslant M$ for $t \in [t_-, t_+]$, which implies that the sequence $\{x(t+t_n)\}$ is uniformly bounded. On the other hand, $\forall t_1, t_2 \in [t_-, t_+]$, from the elementary mean value theorem of differential calculus, we have

$$\left|x_i(t_1+t_n) - x_i(t_2+t_n)\right| \leqslant M\left[\bar{r}_i + \sum_{j=1}^{N}\left(\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij}M^{\alpha_{ii}}\right)M^{\alpha_{ij}}\right]|t_1 - t_2|.$$

The above inequality shows that the sequence $\{x(t+t_n)\}$ is equi-continuous on $[t_-, t_+]$, the claim follows.

By Ascoli–Arzela theorem, there exist a subsequence of $\{t_n\}$ (we still denote it as $\{t_n\}$) and a continuous function $v(t)$ such that

$$x_i(t+t_n) \to v_i(t), \quad i = 1, \dots, N, \text{ as } n \to \infty,$$

uniformly in t on any bounded interval in **R**. Let $\sigma \in \mathbf{R}$ be given. We may assume that $t_n + \sigma \geqslant 0$, for all n. For $t \geqslant t_0$, an integration of (1.1) over $[t_n + \sigma, t + t_n + \sigma]$ leads to

$$\begin{aligned}
&x_i(t+t_n+\sigma) - x_i(t_n+\sigma) \\
&\quad = \int_{t_n+\sigma}^{t+t_n+\sigma} x_i(s)\left[r_i(s) - \sum_{j=1}^{N} a_{ij}(s)x_j^{\alpha_{ij}}(s) - \sum_{j=1}^{N} b_{ij}(s)x_j^{\alpha_{ij}}\left(s - \tau_{ij}(s)\right)\right. \\
&\qquad \left. - \sum_{j=1}^{N} c_{ij}(s)x_i^{\alpha_{ii}}(s)x_j^{\alpha_{ij}}(s)\right] ds \\
&\quad = \int_{\sigma}^{t+\sigma} x_i(s+t_n)\left[r_i(s+t_n) - \sum_{j=1}^{N} a_{ij}(s+t_n)x_j^{\alpha_{ij}}(s+t_n)\right. \\
&\qquad - \sum_{j=1}^{N} b_{ij}(s+t_n)x_j^{\alpha_{ij}}\left(s+t_n - \tau_{ij}(s+t_n)\right) \\
&\qquad \left. - \sum_{j=1}^{N} c_{ij}(s+t_n)x_i^{\alpha_{ii}}(s+t_n)x_j^{\alpha_{ij}}(s+t_n)\right] ds.
\end{aligned}$$

Using the Lebesgue's dominated convergence theorem, one has

$$v_i(t+\sigma)-v_i(\sigma)=\int_{\sigma}^{t+\sigma} v_i(s)\Bigg[r_i(s)-\sum_{j=1}^{N}a_{ij}(s)v_j^{\alpha_{ij}}(s)-\sum_{j=1}^{N}b_{ij}(s)v_j^{\alpha_{ij}}\big(s-\tau_{ij}(s)\big) -\sum_{j=1}^{N}c_{ij}(s)v_i^{\alpha_{ii}}(s)v_j^{\alpha_{ij}}(s)\Bigg]\,ds.$$

This means that $v(t)=(v_1(t),\dots,v_N(t))$ is a solution of system (1.1), and by the arbitrary of σ, $v(t)$ is a solution of system (1.1) on $\mathbf{R}$ with $m\leqslant v_i(t)\leqslant M$. Therefore, Theorem 2.2 is valid. □

3. Existence of a unique almost periodic solution

In what follows, we will give some lemmas which will be used in the proving of Theorems 3.1 and 3.2.

In order to investigate the global attractivity of the bounded positive (componentwise) solution of (1.1), we shall make some preparations. Let d be any positive constant such that $0<d<m$, making the change of variable $u_i=\frac{x_i}{d}$, then system (1.1) change to

$$\dot u_i(t)=u_i(t)\Bigg[r_i(t)-\sum_{j=1}^{N}a_{ij}(t)d^{\alpha_{ij}}u_j^{\alpha_{ij}}(t)-\sum_{j=1}^{N}b_{ij}(t)d^{\alpha_{ij}}u_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big) -\sum_{j=1}^{N}c_{ij}(t)d^{\alpha_{ii}+\alpha_{ij}}u_i^{\alpha_{ii}}(t)u_j^{\alpha_{ij}}(t)\Bigg]. \tag{3.1}$$

Obviously, system (1.1) is equivalent to system (3.1). In what follows, we will investigate some basic results of system (3.1).

Lemma 3.1. *If all conditions in Theorem* 2.1 *hold, then system* (3.1) *is persistent.*

Proof. Let $U_i(t)$ be the unique globally attractive almost periodic solution of the following system (3.2)

$$\dot u_i(t)=u_i(t)\big[r_i(t)-a_{ii}(t)d^{\alpha_{ii}}u_i^{\alpha_{ii}}(t)\big]. \tag{3.2}$$

From Theorem 2.2, if $m(r_i(t)-\sum_{j=1,\,j\neq i}^{N}[a_{ij}(t)+\frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot\tau_{ij}(\psi_{ij}^{-1}(t))}]d^{\alpha_{ij}}U_j^{\alpha_{ij}}(t))>0$, then system (3.1) is persistent and $\frac{m}{d}\leqslant\liminf_{t\to+\infty}u_i(t)\leqslant\limsup_{t\to+\infty}u_i(t)\leqslant\frac{M}{d}$. It is easy to see that system (3.2) is equivalent to system (2.7) by making change of variables $U_i(t)=\frac{X_i(t)}{d}$. Therefore, we have

$$m\left(r_i(t)-\sum_{j=1,\,j\neq i}^{N}\left[a_{ij}(t)+\frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot\tau_{ij}(\psi_{ij}^{-1}(t))}\right]X_j^{\alpha_{ij}}(t)\right) =m\left(r_i(t)-\sum_{j=1,\,j\neq i}^{N}\left[a_{ij}(t)+\frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot\tau_{ij}(\psi_{ij}^{-1}(t))}\right]d^{\alpha_{ij}}U_j^{\alpha_{ij}}(t)\right)>0.$$

The assertion of Lemma 3.1 follows immediately. □

Denote

$$K_1 = \left\{ u(t) = \big(u_1(t), \dots, u_N(t)\big) \;\Big|\; \frac{m}{d} \leqslant x_i(t) \leqslant \frac{M}{d}, \text{ for all } t \in \mathbf{R},\ i = 1, 2, \dots, N \right\}.$$

Similar to the proof of Theorem 2.3, the following lemma follows.

Lemma 3.2. *If* (H_1)–(H_4) *hold, then system* (3.1) *has at least one positive solution on R belonging to K_1.*

We now state the following lemmas.

Lemma 3.3. *(See [36].) Let f be a nonnegative function defined on* $[0, +\infty)$ *such that f is integrable on* $[0, +\infty)$ *and is uniformly continuous on* $[0, +\infty)$. *Then* $\lim_{t\to+\infty} f(t) = 0$.

Lemma 3.4. *(See Fink [28, Theorem 10.1], He [29, Theorem 3.2].) Consider system $\dot{x} = f(t, x)$, suppose $f(t, x)$ is almost periodic in t uniformly in $x \subset K$, K compact in $\mathbf{R}^N$. If each equation $\dot{x} = g(t, x)$, $g \in H(f)$ (where $H(f)$ is the hull of f) has a unique solution on $\mathbf{R}$ belonging to K, then these solutions are almost periodic.*

Assume $\alpha(t)$ is an almost periodic function and $\beta(t)$ is a continuously differentiable almost periodic function. Furthermore, $\dot{\beta}(t)$ is uniformly continuous on $\mathbf{R}$ with $\inf_{t\in R}\{1 - \dot{\beta}(t)\} > 0$. Obviously, $\dot{\beta}(t)$ is also a continuously almost periodic function.

Lemma 3.5. *(See [17] or [29].) Suppose $(\alpha^*(t), \beta^*(t)) \in H(\alpha(t), \beta(t))$ and $\sigma^{-1}(t), \sigma^{*-1}(t)$ is the inverse of $\sigma(t) = t - \beta(t), \sigma^*(t) = t - \beta^*(t)$, respectively. Then we have*

(a) *If there exists a sequence $\{t_n\}$ such that $\alpha(t + t_n) \to \alpha^*(t)$, $\beta(t + t_n) \to \beta^*(t)$ uniformly on R as $n \to \infty$, then $\alpha(\sigma^{-1}(t + t_n)) \to \alpha^*(\sigma^{*-1}(t))$ uniformly on R.*
(b) *$\alpha(\sigma^{-1}(t)), \alpha^*(\sigma^{*-1}(t))$ are all almost periodic and $\alpha^*(\sigma^{*-1}(t)) \in H(\alpha(\sigma^{-1}(t)))$.*

Theorem 3.1. *In addition to* (H_1)–(H_4), *if the initial value problem* (1.1)–(2.1) *also satisfies the following conditions*:

(H_5) $\alpha_{ii} \geqslant \max_j\{\alpha_{ji}\}$.
(H_6) *There exist positive constants θ_i $(i = 1, 2, \dots, N)$, d $(0 < d < m)$ and ς such that $\min_{t\in R}\{\varphi_i(t),\ i = 1, 2, \dots, N\} > \varsigma$, where*

$$\varphi_i(t) = \theta_i \left(d^{\alpha_{ii}} a_{ii}(t) + \sum_{j=1}^{N} c_{ij}(t) d^{\alpha_{ii}} m^{\alpha_{ij}} \right) - \left(\sum_{j=1,\, j\neq i}^{N} \theta_j d^{\alpha_{ji}} a_{ji}(t) + \sum_{j=1}^{N} d^{\alpha_{ji}} \frac{b_{ji}(\psi_{ji}^{-1}(t))}{1 - \dot{\tau}_{ji}(\psi_{ji}^{-1}(t))} + \sum_{j=1}^{N} c_{ji}(t) d^{\alpha_{ji}} M^{\alpha_{jj}} \right).$$

Then the initial value problem (1.1)–(2.1) *has a unique positive bounded solution $x(t)$ which is globally attractive.*

Proof. In order to show the global attractivity of the bounded solution $x(t)$ of system (1.1), we shall show that the bounded solution $u(t) = (u_1(t), \ldots, u_N(t))$ of system (3.1) is globally attractive. Let $v(t) = (v_1(t), \ldots, v_N(t))$ be any other solution of system (3.1). By Lemma 3.2 and $0 < d < m$, there exists $T^0 > 0$ such that $1 < \frac{m}{d} \leqslant u_i(t), v_i(t) \leqslant \frac{M}{d}$, for all $t \geqslant T^0$. Consider the following Lyapunov functional:

$$W(t) = \sum_{i=1}^{N} \theta_i \left[\left| \ln u_i(t) - \ln v_i(t) \right| + \sum_{i=1}^{N} \int_{t-\tau_{ij}(t)}^{t} \frac{b_{ij}(\psi_{ij}^{-1}(s))}{1 - \dot{\tau}_{ij}(\psi_{ij}^{-1}(s))} \left| u_i^{\alpha_{ij}}(s) - v_i^{\alpha_{ij}}(s) \right| ds \right]. \tag{3.3}$$

Calculating the upper right derivative $D^+W(t)$ of $W(t)$ along the solution of (3.1), by simplifying, we get

$$\begin{aligned}
D^+W(t) \leqslant{}& -\sum_{i=1}^{N} \theta_i a_{ii}(t) d^{\alpha_{ii}} \left| u_i^{\alpha_{ii}}(t) - v_i^{\alpha_{ii}}(t) \right| \\
&+ \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \theta_j a_{ij}(t) d^{\alpha_{ij}} \left| u_j^{\alpha_{ij}}(t) - v_j^{\alpha_{ij}}(t) \right| \\
&+ \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \theta_j d^{\alpha_{ij}} \frac{b_{ij}(\psi_{ij}^{-1}(t))}{1 - \dot{\tau}_{ij}(\psi_{ij}^{-1}(t))} \left| u_j^{\alpha_{ij}}(t) - v_j^{\alpha_{ij}}(t) \right| \\
&- \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_j c_{ij}(t) d^{\alpha_{ii}} m^{\alpha_{ij}} \left| u_i^{\alpha_{ii}}(t) - v_i^{\alpha_{ii}}(t) \right| \\
&+ \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_j c_{ij}(t) d^{\alpha_{ij}} M^{\alpha_{ii}} \left| u_j^{\alpha_{ij}}(t) - v_j^{\alpha_{ij}}(t) \right| \\
={}& -\sum_{i=1}^{N} \Bigg\{ \theta_i \left(d^{\alpha_{ii}} a_{ii}(t) + \sum_{j=1}^{N} c_{ij}(t) d^{\alpha_{ii}} m^{\alpha_{ij}} \right) \left| u_i^{\alpha_{ii}}(t) - v_i^{\alpha_{ii}}(t) \right| \\
&- \Bigg(\sum_{j=1,\, j\neq i}^{N} \theta_j d^{\alpha_{ji}} a_{ji}(t) + \sum_{j=1}^{N} d^{\alpha_{ji}} \frac{b_{ji}(\psi_{ji}^{-1}(t))}{1 - \dot{\tau}_{ji}(\psi_{ji}^{-1}(t))} \\
&+ \sum_{j=1}^{N} c_{ji}(t) d^{\alpha_{ji}} M^{\alpha_{jj}} \Bigg) \left| u_i^{\alpha_{ji}}(t) - v_i^{\alpha_{ji}}(t) \right| \Bigg\}.
\end{aligned} \tag{3.4}$$

Note that the function $y = |a^x - b^x|$ is increasing when $a, b > 1$. It follows from Lemma 3.2 and $0 < d < m$ that $u_i(t), v_i(t) \geqslant \frac{m}{d} > 1$, for all $t \geqslant T^0$. This, together with (H_5), implies that

$$\left| u_i^{\alpha_{ji}}(t) - v_i^{\alpha_{ji}}(t) \right| \leqslant \left| u_i^{\alpha_{ii}}(t) - v_i^{\alpha_{ii}}(t) \right|, \quad \text{for all } t \geqslant T^0. \tag{3.5}$$

Therefore, for $t \geqslant T^0$, it follows from (3.4) and (3.5) that

$$D^+W(t) \leqslant -\sum_{i=1}^{N} \Bigg\{ \Bigg[\theta_i \left(d^{\alpha_{ii}} a_{ii}(t) + \sum_{j=1}^{N} c_{ij}(t) d^{\alpha_{ii}} m^{\alpha_{ij}} \right) \left| u_i^{\alpha_{ii}}(t) - v_i^{\alpha_{ii}}(t) \right|$$

$$- \left(\sum_{j=1,\, j\neq i}^{N} \theta_j d^{\alpha_{ji}} a_{ji}(t) + \sum_{j=1}^{N} d^{\alpha_{ji}} \frac{b_{ji}(\psi_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\psi_{ji}^{-1}(t))} + \sum_{j=1}^{N} c_{ji}(t) d^{\alpha_{ji}} M^{\alpha_{jj}} \right) \Bigg] \left| u_i^{\alpha_{ii}}(t) - v_i^{\alpha_{ii}}(t) \right| \Bigg\}$$

$$\leqslant -\varsigma \sum_{i=1}^{N} \left| u_i^{\alpha_{ii}}(t) - v_i^{\alpha_{ii}}(t) \right| < 0. \tag{3.6}$$

An integration of (3.6) over $[T^0, t]$, we obtain that

$$\int_{T^0}^{t} \varsigma \left[\sum_{i=1}^{N} \left| u_i^{\alpha_{ii}}(s) - v_i^{\alpha_{ii}}(s) \right| \right] ds < W(T^0) - W(t), \quad \text{for all } t \geqslant T^0.$$

Therefore, we have

$$\limsup_{t\to+\infty} \int_{T^0}^{t} \left[\sum_{i=1}^{N} \left| u_i^{\alpha_{ii}}(s) - v_i^{\alpha_{ii}}(s) \right| \right] ds < \frac{W(T^0)}{\varsigma} < +\infty, \quad \text{for all } t \geqslant T^0. \tag{3.7}$$

By Lemma 3.3, from (3.7), one can easily deduce that

$$\lim_{t\to\infty} \left| u_i(t) - v_i(t) \right| \to 0, \quad i = 1, 2, \ldots, N,$$

which implies the global attractivity of system (3.1). By using the equivalence between (3.1) and (1.1), it follows that the bounded solution $x(t)$ of system (1.1) is also globally attractive. This completes the proof of Theorem 3.1. □

Now consider the hull system

$$\dot{x}_i(t) = x_i(t) \Bigg[r_i^*(t) - \sum_{j=1}^{N} a_{ij}^*(t) x_j^{\alpha_{ij}}(t) - \sum_{j=1}^{N} b_{ij}^*(t) x_j^{\alpha_{ij}}\big(t - \tau_{ij}^*(t)\big) - \sum_{j=1}^{N} c_{ij}^*(t) x_i^{\alpha_{ii}}(t) x_j^{\alpha_{ij}}(t) \Bigg], \quad i = 1, 2, \ldots, N, \tag{3.8}$$

where, for some sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$,

$$r_i(t+t_n) \to r_i^*(t), \quad a_{ij}(t+t_n) \to a_{ij}^*(t), \quad b_{ij}(t+t_n) \to b_{ij}^*(t), \quad c_{ij}(t+t_n) \to c_{ij}^*(t),$$
$$\tau_{ij}(t+t_n) \to \tau_{ij}^*(t), \quad X_i^*(t+t_n) \to X_i^*(t), \quad \text{uniformly for all } t \in \mathbf{R} \text{ as } n \to \infty.$$

From Lemma 3.5, it follows that

$$\lim_{k\to\infty} \left\{ r_i(t+t_n) - \sum_{j=1,\, j\neq i}^{N} \left[a_{ij}(t+t_n) + \frac{b_{ij}(\psi_{ij}^{-1}(t+t_n))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t+t_n))} \right] X_j^{\alpha_{ij}}(t+t_n) \right\}$$
$$= r_i^*(t) - \sum_{j=1,\, j\neq i}^{N} \left[a_{ij}^*(t) + \frac{b_{ij^*}(\psi_{ij}^{*-1}(t))}{1-\dot{\tau}_{ij}^*(\psi_{ij}^{*-1}(t))} \right] X_j^{*\alpha_{ij}}(t); \tag{3.9}$$

$$\begin{aligned}
\lim_{n\to\infty}\varphi_i(t+t_n) &= \lim_{n\to\infty}\Biggl\{\theta_i\Biggl(d^{\alpha_{ii}}a_{ii}(t+t_n)+\sum_{j=1}^{N}c_{ij}(t)d^{\alpha_{ii}}m^{\alpha_{ij}}\Biggr)\\
&\quad-\Biggl(\sum_{j=1,\,j\neq i}^{N}\theta_j d^{\alpha_{ji}}a_{ji}(t+t_n)+\sum_{j=1}^{N}d^{\alpha_{ji}}\frac{b_{ji}(\psi_{ji}^{-1}(t+t_n))}{1-\dot{\tau}_{ji}(\psi_{ji}^{-1}(t+t_n))}\\
&\quad+\sum_{j=1}^{N}c_{ji}(t+t_n)d^{\alpha_{ji}}M^{\alpha_{jj}}\Biggr)\Biggr\}\\
&=\theta_i\Biggl(d^{\alpha_{ii}}a_{ii}^*(t)+\sum_{j=1}^{N}c_{ij}^*(t)d^{\alpha_{ii}}m^{\alpha_{ij}}\Biggr)\\
&\quad-\Biggl(\sum_{j=1,\,j\neq i}^{N}\theta_j d^{\alpha_{ji}}a_{ji}^*(t)+\sum_{j=1}^{N}d^{\alpha_{ji}}\frac{b_{ji}^*(\psi_{ji}^{*-1}(t))}{1-\dot{\tau}_{ji}^*(\psi_{ji}^{*-1}(t))}\\
&\quad+\sum_{j=1}^{N}c_{ji}^*(t)d^{\alpha_{ji}}M^{\alpha_{jj}}\Biggr):=\varphi_i^*(t). \qquad (3.10)
\end{aligned}$$

Note that $r_i^*(t), a_{ij}^*(t), b_{ij}^*(t), c_{ij}^*(t), \tau_{ij}^*(t), X_i^*(t)$ and $r_i^*(t)-\sum_{j=1,\,j\neq i}^{N}[a_{ij}^*(t)+\frac{b_{ij*}(\psi_{ij}^{*-1}(t))}{1-\dot{\tau}_{ij}^*(\psi_{ij}^{*-1}(t))}]$, $X_j^{*\alpha_{ij}}(t)$ are also almost periodic in t.

Lemma 3.6. *Suppose the conditions in Theorem* 3.1 *hold, then the hull system* (3.8) *has a unique bounded solution* $x^*(t)=(x_1^*(t),x_2^*(t),\dots,x_N^*(t))\subset K$ *on* **R**, *which is globally attractive.*

Proof. By the definition of mean value and the assumptions (H_1)–(H_2), together with (3.9) and (3.10), it is easy to prove that $m(r_i^*(t))=m(r_i(t))>0$,

$$\begin{aligned}
&m\Biggl(r_i^*(t)-\sum_{j=1,\,j\neq i}^{N}\biggl[a_{ij}^*(t)+\frac{b_{ij}^*(\psi_{ij}^{*-1}(t))}{1-\dot{\tau}_{ij}^*(\psi_{ij}^{*-1}(t))}\biggr]X_j^{*\alpha_{ij}}(t)\Biggr)\\
&\quad=m\Biggl(r_i(t)-\sum_{j=1,\,j\neq i}^{N}\biggl[a_{ij}(t)+\frac{b_{ij}(\psi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\psi_{ij}^{-1}(t))}\biggr]X_j^{\alpha_{ij}}(t)\Biggr)>0
\end{aligned}$$

and $\min_{t\in R}\{\varphi_i^*(t),\ i=1,2,\dots,N\}>\tilde{\varsigma}$ for some positive constant $\tilde{\varsigma}\leqslant\varsigma$. These imply that corresponding to (3.8), all the requirements in Theorem 3.1 are satisfied. Then applying Theorem 3.1 to the hull system (3.8), we obtain that system (3.8) has a unique positive bounded solution $x^*(t)\subset K$ on **R**, which is globally attractive. This completes the proof Lemma 3.6. □

By Lemma 3.6, it follows that for each $g(t,X)\in H(f(t,X))$, the hull equation

$$\dot{x}=h(t,X)$$

has a unique bounded solution on **R** with value in K. Hence, from Lemma 3.4, this unique solutions are all almost periodic. By the global attractivity, $x(t)$ is the unique almost periodic solution of system (1.1) contained in K. Thus, our main results follows

Theorem 3.2. *Suppose the conditions in Theorem* 3.1 *hold, then the initial value problem* (1.1)–(2.1) *has a unique positive* (*componentwise*) *almost periodic solution* $x(t) = (x_1(t), \dots, x_N(t)) \subset K$ *on* $\mathbf{R}$*, which is globally attractive.*

When we consider system (1.1) in periodic environment, i.e., $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t), \tau_{ij}(t)$ are all ω-periodic. We have

Theorem 3.3. *Suppose the coefficients of system* (1.1) *are all* ω*-periodic and all the conditions in Theorem* 3.1 *hold, then the initial value problem* (1.1)–(2.1) *has a unique positive* (*componentwise*) ω*-periodic solution* $x(t) = (x_1(t), \dots, x_N(t)) \subset K$ *on* $\mathbf{R}$*, which is globally attractive.*

Proof. Let $x(t)$ be the unique positive almost periodic solution of (1.1)–(2.1), but in the periodic case, $r_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t), \tau_{ij}(t)$ are all ω-periodic, therefore, $x(t+\omega)$ is also an almost periodic solution of (1.1)–(2.1). By the uniqueness of almost periodic solution, it follows that $x(t) = x(t+\omega)$, for all $t \in \mathbf{R}$. This completes the proof of Theorem 3.3. □

4. Applications

To illustrate the generality of our results, we shall apply the results (Theorems 2.2, 3.2, 3.3) obtained in Sections 2 and 3 to some particular competition system, which have been studied extensively in the existed literature. The following two applications will show that our easily verifiable sufficient conditions are more general and weaker than those available in the literature.

Application 4.1. Consider the competition system without delays, that is,

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{N} a_{ij}(t) x_j^{\alpha_{ij}}(t)\right], \quad i = 1, 2, \dots, N, \tag{4.1}$$

where system (4.1) is supplemented with the initial condition $x_i(0) = x_{i0}$.

Applying Theorems 2.1, 3.1–3.3 to system (4.1), we have

Theorem 4.1. *In addition to* (H_1)*,* (H_3)*, if further assume that*

(H_7) $m(r_i(t) - \sum_{j=1,\, j\neq i}^{N} a_{ij}(t) X_j^{\alpha_{ij}}(t)) > 0$*, where* $X_j(t)$ *is defined in Theorem* 2.2*.*

Then the system (4.1) *is persistent.*

Theorem 4.2. *In addition to* (H_1)*,* (H_3)*,* (H_5) *and* (H_7)*, if system* (4.1) *also satisfies the following conditions*:

(H_8) *There exist positive constants* θ_i $(i = 1, 2, \dots, N)$*,* d $(0 < d < m)$ *and* ς *such that* $\min_{t\in R}\{\psi_i(t),\ i = 1, 2, \dots, N\} > \varsigma$*, where*

$$\psi_i(t) = \theta_i d^{\alpha_{ii}} a_{ii}(t) - \sum_{j=1,\, j\neq i}^{N} \theta_j d^{\alpha_{ji}} a_{ji}(t).$$

Then system (4.1) *has a unique positive bounded solution* $x(t)$ *which is globally attractive.*

Theorem 4.3. *Suppose the conditions in Theorem* 4.2 *hold, then system* (4.1) *has a unique positive* (*componentwise*) *almost periodic solution* $x(t) = (x_1(t), \dots, x_N(t)) \subset K$ *on* $\mathbf{R}$, *which is globally attractive.*

Theorem 4.4. *Suppose the coefficients of system* (4.1) *are all* ω*-periodic and all the conditions in Theorem* 4.2 *hold, then system* (4.1) *has a unique positive* (*componentwise*) ω*-periodic solution* $x(t) = (x_1(t), \dots, x_N(t)) \subset K$ *on* $\mathbf{R}$, *which is globally attractive.*

Remark 4.1. Obviously, Theorems 4.1 and 4.2 generalize the results of Theorem 2.1 in [26]. Take $\alpha_{ij} = 1$, we also generalize the results of Theorem 2.1 in [13].

In order to obtain more easily verified results, we need the following lemmas.

Lemma 4.1. *(See [34, Lemma 2.2].) If* $a > 0, b > 0$ *and* $\frac{dx(t)}{dt} \leqslant x(t)[b - ax(t)]$, *when* $t \geqslant 0$ *and* $x(0) > 0$, *we have* $x(t) \leqslant \frac{b}{a}[1 + (\frac{b}{ax(0)} - 1)e^{-bt}]^{-1}$. *Moreover, if* $0 < x(0) \leqslant \frac{b}{a}$, *then* $0 < x(t) \leqslant \frac{b}{a}$.

To the contrary, if $a > 0, b > 0$ *and* $\frac{dx(t)}{dt} \geqslant x(t)[b - ax(t)]$, $t \geqslant 0$, *we have* $x(t) \geqslant \frac{b}{a}[1 + (\frac{b}{ax(0)} - 1)e^{-bt}]^{-1}$. *Moreover, if* $x(0) \geqslant \frac{b}{a} > 0$, *then* $x(t) \geqslant \frac{b}{a} > 0$.

Lemma 4.2. *If* $a > 0, b > 0$ *and* $\frac{dx(t)}{dt} \leqslant x(t)[b - ax^{\alpha}(t)]$, *when* $t \geqslant 0$ *and* $x(0) > 0$, *we have* $x(t) \leqslant (\frac{b}{a})^{\frac{1}{\alpha}}[1 + (\frac{b}{ax^{-\alpha}(0)} - 1)e^{-b\alpha t}]^{-\frac{1}{\alpha}}$. *Moreover, if* $0 < x(0) \leqslant (\frac{b}{a})^{\frac{1}{\alpha}}$, *then* $0 < x(t) \leqslant (\frac{b}{a})^{\frac{1}{\alpha}}$.

To the contrary, if $a > 0, b > 0$ *and* $\frac{dx(t)}{dt} \geqslant x(t)[b - ax^{\alpha}(t)]$, $t \geqslant 0$ *and* $x(0) > 0$, *we have* $x(t) \geqslant (\frac{b}{a})^{\frac{1}{\alpha}}[1 + (\frac{b}{ax^{-\alpha}(0)} - 1)e^{-b\alpha t}]^{-\frac{1}{\alpha}}$. *Moreover, if* $x(0) \geqslant (\frac{b}{a})^{\frac{1}{\alpha}} > 0$, *then* $x(t) \geqslant (\frac{b}{a})^{\frac{1}{\alpha}} > 0$.

Proof. We give a proof of Lemma 4.2. From $\frac{dx(t)}{dt} \leqslant x(t)[b - ax^{\alpha}(t)]$, we have

$$\frac{dx^{-\alpha}(t)}{dt} \geqslant a\alpha - b\alpha x^{-\alpha}(t).$$

From Lemma 4.1 and the above inequality, we have

$$x^{-\alpha}(t) \geqslant \frac{a}{b}\left[1 + \left(\frac{b}{ax^{-\alpha}(0)} - 1\right)e^{-b\alpha t}\right]^{-1}, \quad t \geqslant 0.$$

Therefore, we obtain

$$x(t) \leqslant \left(\frac{b}{a}\right)^{\frac{1}{\alpha}}\left[1 + \left(\frac{b}{ax^{-\alpha}(0)} - 1\right)e^{-b\alpha t}\right]^{-\frac{1}{\alpha}}, \quad t \geqslant 0.$$

Then it is not difficult to derive that if $0 < x(0) \leqslant (\frac{b}{a})^{\frac{1}{\alpha}}$, then $0 < x(t) \leqslant (\frac{b}{a})^{\frac{1}{\alpha}}$. This completes the proof of Lemma 4.2. □

Applying Lemma 4.2 to system (4.1), it is easy to obtain that the values $m = \min_i q_i$, $M = \max_i p_i$. Then, we have the following four theorems which are much easier to be verified.

Theorem 4.5. *In addition to* (H_1), *if further assume that*

(H_9) $\underline{r}_i > \frac{\underline{r}_i - \sum_{j=1, j\neq i} \bar{a}_{ij} p_j^{\alpha_{ij}}}{\bar{a}_{ii}}$.

Then the system (4.1) *is persistent.*

Proof. Since $m(r_i) \geqslant \underline{r}_i > 0$, from the discussion in Section 2, $X_i(t) \leqslant p_i$, $t \geqslant 0$. This implies

$$m\left(r_i(t) - \sum_{j=1, j\neq i}^{N} a_{ij}(t) X_j^{\alpha_{ij}}(t)\right) \geqslant \underline{r}_i - \frac{\underline{r}_i - \sum_{j=1, j\neq i} \bar{a}_{ij} p_j^{\alpha_{ij}}}{\bar{a}_{ii}} > 0.$$

By Theorem 4.1, Theorem 4.5 holds. □

Theorem 4.6. *If system* (4.1) *satisfies* (H_1), (H_8) *and* (H_9), *then system* (4.1) *has a unique positive bounded solution* $x(t)$ *which is globally attractive.*

Theorem 4.7. *Suppose the conditions in Theorem* 4.6 *hold, then system* (4.1) *has a unique positive* (*componentwise*) *almost periodic solution* $x(t) = (x_1(t), \ldots, x_N(t)) \subset K$ *on* **R**, *which is globally attractive.*

Theorem 4.8. *Suppose the coefficients of system* (4.1) *are all* ω*-periodic and all the conditions in Theorem* 4.6 *hold, then system* (4.1) *has a unique positive* (*componentwise*) ω*-periodic solution* $x(t) = (x_1(t), \ldots, x_N(t)) \subset K$ *on* **R**, *which is globally attractive.*

Application 4.2. Take $\alpha_{ij} = 1$, then system (4.1) reduces to the famous Lotka–Volterra competition system [9,10,13,14]

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{N} a_{ij}(t) x_j(t)\right], \quad i = 1, \ldots, N, \tag{4.2}$$

where $b_i(t), a_{ij}(t) \in C(R, [0, +\infty))$ are all almost periodic for all $t \in R$ with $\underline{a}_{ii} > 0$.

Now apply the results obtained in Application 4.1 to system (4.2), we recover the following results.

Corollary 4.1. *(See Xia et al. [31, Theorem 4.1].) If system* (4.2) *satisfies*

(i) $m(r_i) > 0$ *and* $m(r_i(t) - \sum_{j=1, j\neq i}^{N} a_{ij}(t) X_j(t)) > 0$ ($X_i(t)$ *defined in Section* 2);
(ii) *there exist strictly positive constants* s_i *and* ε *such that*

$$s_i a_{ii}(t) > \sum_{j=1, j\neq i}^{N} s_j a_{ji}(t) + \varepsilon,$$

then system (4.2) *has a unique positive almost periodic solution which is globally attractive.*

In Corollary 4.1, take $s_i \equiv 1$, $i = 1, \ldots, n$, we have

Corollary 4.2. *(See He [16, Theorem 4].) If system* (4.1) *satisfies*

(iii) $m(r_i) > 0$ *and* $m(r_i(t) - \sum_{j=1,\, j\neq i}^{N} a_{ij}(t)X_j(t)) > 0$;
(iv) *there exist strictly positive constants* s_i *and* ε *such that*

$$a_{ii}(t) > \sum_{j=1,\, j\neq i}^{N} a_{ji}(t) + \varepsilon,$$

then system (4.2) *has a unique positive almost periodic solution which is globally attractive.*

Corollary 4.3. *(See Xia et al. [31, Theorem 4.2].) Suppose* $r_i(t), a_{ij}(t)$ *are nonnegative* ω*-periodic functions defined for* $t \in [0, \omega]$. *If system* (4.2) *satisfies*

(v) $\int_0^{\omega} b_i(t)\,dt > 0$, $\int_0^{\omega}(b_i(t) - \sum_{j=1,\, j\neq i}^{N} a_{ij}(t)X_j(t))\,dt > 0$;
(vi) *there exist strictly positive constants* s_i *such that*

$$s_i a_{ii}(t) > \sum_{j=1,\, j\neq i}^{N} s_j a_{ji}(t),$$

then system (4.2) *has a unique positive* ω*-periodic solution which is globally attractive.*

Take $s_i \equiv 1$, we have

Corollary 4.4. *(See Zhao [18, Theorem 3].) Suppose* $r_i(t), a_{ij}(t)$ *are nonnegative* ω*-periodic functions defined for* $t \in [0, \omega]$. *In addition to* (v)*, if system* (4.2) *satisfies*

$$a_{ii}(t) > \sum_{j=1,\, j\neq i}^{N} a_{ji}(t),$$

then system (4.2) *has a unique positive* ω*-periodic solution which is globally attractive.*

Through the similar proof as Theorem 4.4, one can obtain

Corollary 4.5. *(See Gopalsamy [10, Theorem 3.1].) Assume that*

$$\underline{r}_i > 0, \quad \underline{a}_{ii} > 0, \quad \underline{r}_i > \sum_{j=1,\, j\neq i}^{N} \bar{a}_{ij}\frac{\bar{r}_j}{\underline{a}_{jj}} \quad \text{and} \quad \min_{t\in\mathbf{R}} a_{ii}(t) > \sum_{j=1,\, j\neq i}^{N} \max_{t\in\mathbf{R}} a_{ji}(t).$$

Then system (4.2) *has a unique positive almost periodic solution which is globally attractive.*

Remark 4.2. When $N = 2$, Corollary 4.5 reduces to Theorem 2 in Ahmad [6].

Corollary 4.6. *(See Gopalsamy [9, Theorem 3.1].) Suppose* $r_i(t), a_{ij}(t)$ *are nonnegative* ω*-periodic functions defined on* $t \in [0, \omega]$. *If the conditions in Corollary* 4.5 *hold, then system* (4.2) *has a unique positive* ω*-periodic solution which is globally attractive.*

Remark 4.3. Obviously, the results in [6,9,10,16,18,31] are special cases of Theorems 4.2 and 4.3. So our results are fresh and more general. Hence, we generalize and improve the main results in [6,9,10,16,18,31].

5. Example and simulations

Example 5.1. Consider a two-species competition system with almost periodic coefficients:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\big[4 - (2+\cos\sqrt{2}t)x_1^2(t) - x_2(t)\big], \\ \dot{x}_2(t) = x_2(t)\big[3 - x_1^{\frac{2}{3}}(t) - (2+\sin\sqrt{3}t)x_2(t)\big]. \end{cases} \tag{5.1}$$

In this case, $r_1(t) = 4$, $r_2(t) = 3a_{11}(t) = 2 + \cos\sqrt{2}t$, $a_{22}(t) = 2 + \sin\sqrt{3}t$, $a_{12}(t) = a_{21}(t) = 1$, $\alpha_{11} = 2$, $\alpha_{12} = 1$, $\alpha_{21} = \frac{2}{3}$, $\alpha_{22} = 1$. The corresponding generalized Logistic equations of (5.1) is as follows:

$$\dot{x}_1(t) = x_1(t)\big[4 - (2+\cos\sqrt{2}t)x_1^2(t)\big], \tag{5.2}$$

$$\dot{x}_2(t) = x_2(t)\big[3 - (2+\sin\sqrt{3}t)x_2(t)\big]. \tag{5.3}$$

Obviously, $m(r_1) = 4 > 0$ and $m(r_2) = 3 > 0$. By using (2.3), (2.5) and Theorem 2.1, we know that system (5.2) has a unique globally attractive positive almost periodic solution which can be represented as

$$X_1(t) = \left[\int_{-\infty}^{t} 8\exp\left(-\int_{s}^{t} 2(2+\cos\sqrt{2}u)\,du\right)ds\right]^{-\frac{1}{2}}. \tag{5.4}$$

Also we know that system (5.3) has a unique globally attractive positive almost periodic solution which can be represented as

$$X_2(t) = \left[\int_{-\infty}^{t} 3\exp\left(-\int_{s}^{t} (2+\sin\sqrt{3}u)\,du\right)ds\right]^{-1}. \tag{5.5}$$

By amplifying, it follows from (5.5) and (5.6) that $X_1(t) \leqslant \frac{\sqrt{3}}{2}$ and $X_2(t) \leqslant 1$. This leads to

$$m\big(r_2(t) - a_{21}(t)X_1^{\frac{2}{3}}(t)\big) = m\big(3 - X_1^{\frac{2}{3}}(t)\big) \geqslant 3 - \left(\frac{\sqrt{3}}{2}\right)^{\frac{2}{3}} > 0 \tag{5.6}$$

and

$$m\big(r_1(t) - a_{12}(t)X_2(t)\big) = m\big(4 - X_2(t)\big) \geqslant 3 - 1 > 0. \tag{5.7}$$

The above inequalities show that conditions (H_4) of Theorem 2.2 hold, thus, system (5.1) is persistent. Figure 1 shows the dynamics behavior of system (5.1).

Now further take $\theta_1 = (\frac{2}{3})^{-\frac{5}{3}}$, $\theta_2 = 1$, $\varsigma < \min\{(\frac{2}{3})^{\frac{1}{3}} - (\frac{2}{3})^{\frac{2}{3}}, (\frac{2}{3})^{-\frac{2}{3}} - (\frac{2}{3})^2\}$ and $d = \frac{2}{3}$, it is easy to verify that

$$\begin{cases} \psi_1(t) = \theta_1 d^2 a_{11}(t) - \theta_2 d^{\frac{2}{3}} a_{21}(t) \geqslant \left(\frac{2}{3}\right)^{\frac{1}{3}} - \left(\frac{2}{3}\right)^{\frac{2}{3}} > \varsigma, \\ \psi_2(t) = \theta_2 d a_{22}(t) - \theta_1 d^2 a_{12}(t) \geqslant \left(\frac{2}{3}\right)^{-\frac{2}{3}} - \left(\frac{2}{3}\right)^{2} > \varsigma. \end{cases}$$

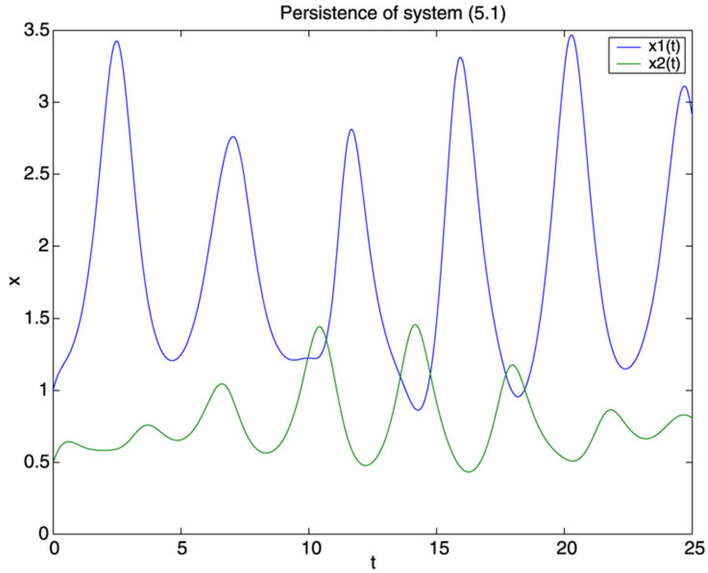

Fig. 1. Persistence of system (5.1) with initial values $(x_1(0), x_2(0)) = (1, 0.5)$ and $t \in [0, 25]$. The blue curve denotes the species of x_1 and the green one denotes the species of x_2. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

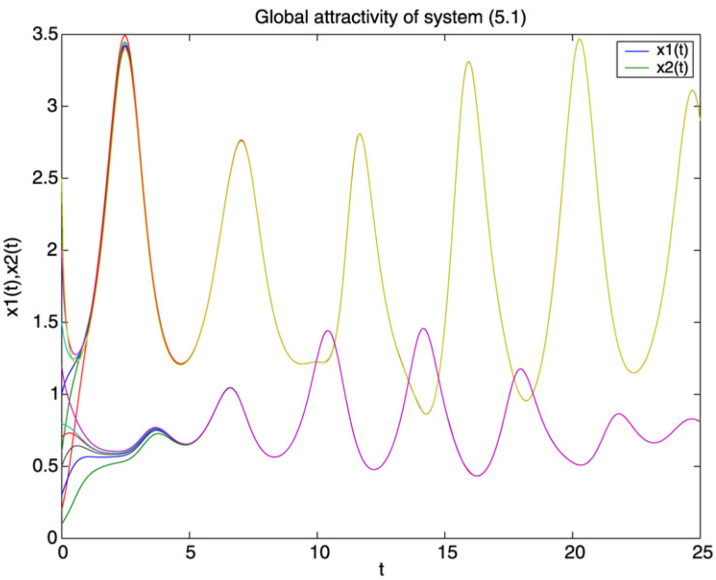

Fig. 2. Dynamics of system (5.1) with initial values $(x_1(0), x_2(0)) = (0.2, 0.1)$, $(0.6, 0.3)$, $(1, 0.5)$, $(1.5, 0.7)$, $(2, 0.8)$, $(2.5, 1.2)$, respectively, and $t \in [0, 25]$. The curves in different colors denote different initial value conditions, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The above inequality shows that conditions (H_8) of Theorem 4.2 hold. Moreover, the inequalities (5.7) and (5.8) show that the other conditions of Theorem 4.2 also hold. Therefore, by

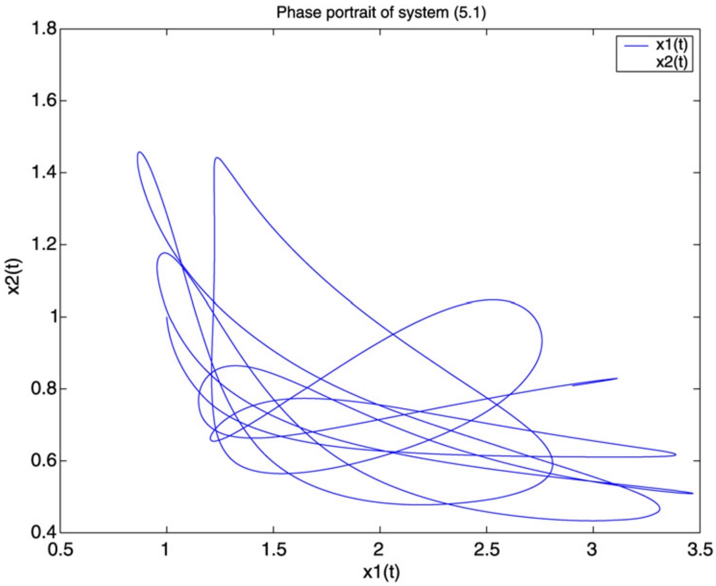

Fig. 3. 'Phase portrait' of system (5.1) with initial values $(x_1(0), x_2(0)) = (1, 0.5)$ and $t \in [0, 25]$. The blue curve denotes the 'Phase portrait' of system (5.1). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Theorem 4.3, then system (5.1) has a unique positive (componentwise) almost periodic solution $x^0(t) = (x_1^0(t), x_2^0(t))$ on R, which is globally attractive. Figure 2 shows that for different initial conditions $x(t) = (x_1(t), x_2(t))$ globally asymptotic to $x^0(t) = (x_1^0(t), x_2^0(t))$; Fig. 3 shows the 'phase portrait' of system (5.1).

Acknowledgments

The authors would like to express their gratitude to the reviewers and editor for their valuable comments and careful reading.

References

[1] L. Chen, Mathematical Models and Methods in Ecology, Science Press, Beijing, 1988 (in Chinese).
[2] P. de Mottoni, A. Schiaffino, Competition system with periodic coefficients: A geometric approach, J. Math. Biol. 11 (1981) 319–335.
[3] J.M. Cushing, Two species competition in a periodic environment, J. Math. Biol. 24 (1986) 381–403.
[4] J.M. Cushing, Periodic Lotka–Volterra competition equations, J. Math. Biol. 10 (1980) 385–400.
[5] S. Ahmad, Convergence and ultimate bounds of solutions of nonautonomous Volterra–Lotka competition equations, J. Math. Anal. Appl. 127 (2) (1987) 377–387.
[6] S. Ahmad, On almost periodic solutions of the competing species problems, Proc. Amer. Math. Soc. 102 (4) (1988) 855–865.
[7] S. Ahmad, On nonautonomous Volterra–Lotka competition equations, Proc. Amer. Math. Soc. 177 (1) (1993) 199–204.
[8] K. Gopalsamy, Exchange of equilibria in two species Lotka–Volterra competition models, J. Aust. Math. Soc. Ser. B 24 (1982) 160–170.
[9] K. Gopalsamy, Global asymptotic stability in a periodic Lotka–Volterra system, J. Aust. Math. Soc. Ser. B 27 (1985) 66–72.

[10] K. Gopalsamy, Global asymptotic stability in a almost periodic Lotka–Volterra system, J. Aust. Math. Soc. Ser. B 27 (1986) 346–360.
[11] K. Gopalsamy, X.Z. He, Oscillations and convergence in an almost periodic competition system, Acta Appl. Math. 46 (1977) 247–266.
[12] C. Alvarez, A.C. Lazer, An application of topological degree to the periodic competing species problem, J. Aust. Math. Soc. Ser. B 28 (1986) 202–219.
[13] J.D. Zhao, J.F. Jiang, A.C. Lazer, The permanence and global attractivity in a nonautonomous Lotka–Volterra system, Nonlinear Anal. Real World Appl. 5 (2004) 265–276.
[14] A. Tineo, C. Alvarez, A different consideration about the globally asymptotically stable solution of the periodic n-competing species problem, J. Math. Anal. Appl. 159 (1991) 44–45.
[15] A. Tineo, On the asymptotic behavior of some population model, J. Math. Anal. Appl. 167 (1992) 516–529.
[16] C. He, On almost periodic solutions of Lotka–Volterra almost periodic competition systems, Ann. Differential Equations 9 (1) (1993) 26–36.
[17] Z. Teng, Permanence and stability of Lotka–Volterra type n-species competitive systems, Acta Math. Sinica 45 (5) (2002) 905–918.
[18] X. Zhao, The qualitative analysis of n-species Lotka–Volterra periodic competition systems, Math. Comput. Modelling 15 (11) (1991) 3–8.
[19] J. Chattopadhyay, Effect of toxic substance on a two-species competitive system, Ecol. Model 84 (1996) 287–289.
[20] J. Cui, L. Chen, Asymptotic behavior of the solution for a class of time-dependent competitive system with feedback controls, Ann. Differential Equations 9 (1) (1993) 11–17.
[21] F.J. Ayala, M.E. Gilpin, J.G. Eherenfeld, Competition between species: Theoretical models and experimental tests, Theor. Popul. Biol. 4 (1973) 331–356.
[22] M.E. Gilpin, F.J. Ayala, Global models of growth and competition, Proc. Natl. Acad. Sci. USA 70 (1973) 3590–3593.
[23] B.S. Goh, T.T. Agnew, Stability in Gilpin and Ayalas model of competition, J. Math. Biol. 4 (1977) 275–279.
[24] X.X. Liao, J. Li, Stability in Gilpin–Ayala competition models with diffusion, Nonlinear Anal. 28 (1997) 1751–1758.
[25] S. Ahmad, A.C. Lazer, Average conditions for global asymptotic stability in a nonautonomous Lotka–Volterra system, Nonlinear Anal. 40 (2000) 37–49.
[26] F.D. Chen, Average conditions for permanence and extinction in nonautonomous Gilpin–Ayala competition model, Nonlinear Anal. Real World Appl. 4 (2006) 895–915.
[27] M. Fan, K. Wang, Global periodic solutions of a generalized n-species Gilpin–Ayala competition model, Comput. Math. Appl. 40 (2000) 1141–1151.
[28] A.M. Fink, Almost Periodic Differential Equations, Lecture Notes in Math., vol. 3777, Springer-Verlag, Berlin, 1974.
[29] C. He, Almost Periodic Differential Equations, Higher Education Press, 1992 (in Chinese).
[30] Y.H. Xia, F.D. Chen, A. Chen, J. Cao, Existence and global attractivity of an almost periodic ecological model, Appl. Math. Comput. 157 (2) (2004) 449–475.
[31] Y.H. Xia, J. Cao, H. Zhang, F.D. Chen, Almost periodic solutions of n-species competitive system with feedback controls, J. Math. Anal. Appl. 294 (2) (2004) 503–522.
[32] Y.H. Xia, M. Lin, J. Cao, The existence of almost periodic solutions of certain perturbation systems, J. Math. Anal. Appl. 310 (1) (2005) 81–96.
[33] Y.H. Xia, J. Cao, Almost periodicity in an ecological model with M-predators and N-preys by "pure-delay type" system, Nonlinear Dynam. 39 (3) (2005) 275–304.
[34] Y.H. Xia, J. Cao, Almost periodic solutions for an ecological model with infinite delays, Proc. Edinb. Math. Soc. 50 (1) (2007) 229–249.
[35] M. Fan, Y. Kuang, Dynamics of a nonautonomous predator prey system with the Beddington–DeAngelis functional response, J. Math. Anal. Appl. 295 (2004) 15–39.
[36] I. Barbălat, Systems d'equations differentielle d'oscillations nonlineaires, Rev. Roumaine Math. Pures Appl. 4 (1959) 267–270.
[37] R.E. Gaines, J.L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Springer, Berlin, 1977.