A high-dimensional test for the equality of the smallest eigenvalues of a covariance matrix

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Abstract

For the test of sphericity, Ledoit and Wolf [Ann. Statist. 30 (2002) 1081–1102] proposed a statistic which is robust against high dimensionality. In this paper, we consider a natural generalization of their statistic for the test that the smallest eigenvalues of a covariance matrix are equal. Some inequalities are obtained for sums of eigenvalues and sums of squared eigenvalues. These bounds permit us to obtain the asymptotic null distribution of our statistic, as the dimensionality and sample size go to infinity together, by using distributional results obtained by Ledoit and Wolf [Ann. Statist. 30 (2002) 1081–1102]. Some empirical results comparing our test with the likelihood ratio test are also given.

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1. Introduction

In this paper, we consider a hypothesis testing problem that is sometimes useful in the application of principal components analysis. Let $\Sigma$ represent the covariance matrix of a $m$-dimensional random vector, $x$, and suppose that $\lambda_1 \geq \cdots \geq \lambda_m$ are the ordered eigenvalues of $\Sigma$. We wish to test the hypothesis that the smallest $p$ eigenvalues of $\Sigma$ are equal, that is, the hypothesis

$$H_0: \lambda_{q+1} = \cdots = \lambda_m,$$

where $m = q + p$. It is common in principal components analysis to test this hypothesis repeatedly with different choices for $p$ in order to determine the number, $q$, of principal
components to retain for subsequent analyses. For instance, one could test (1) first with \( p = m - 1 \), then with \( p = m - 2 \), and continue until \( H_0^p \) is not rejected for some choice for \( p \).

Suppose we have a random sample, \( x_1, \ldots, x_{n+1} \), which is used to compute the usual unbiased sample covariance matrix \( S \). When sampling from a normal population, the likelihood ratio test rejects \( H_0^p \) for large values of the statistic

\[
T_p = c_n \left\{ p \ln \left( \sum_{i=q+1}^{m} \phi_i(S)/p \right) - \sum_{i=q+1}^{m} \ln(\phi_i(S)) \right\},
\]

where for any square matrix \( A \) we use the notation \( \phi_i(A) \) to denote its \( i \)th largest eigenvalue. For the constant \( c_n \), we can use the Bartlett-corrected multiplying factor [5]

\[
c_n = n - q - \frac{1}{6} \left( 2p + 1 + \frac{2}{p} \right) + \lambda^2 \sum_{i=1}^{q} \frac{1}{(\lambda_i - \lambda)^2},
\]

(2)

where \( \lambda \) represents the common value of the \( p \) smallest eigenvalues of \( \Sigma \). In practice, \( c_n \) is estimated by replacing \( \lambda_i \) by \( \phi_i(S) \) and \( \lambda \) by \( \sum_{i=q+1}^{m} \phi_i(S)/p \). Under \( H_0^p \), \( T_p \) converges in distribution, as \( n \) goes to infinity, to the chi-squared distribution with \( p(p+1)/2 - 1 \) degrees of freedom.

When the number of variables, \( m \), is large the chi-squared approximation may not be very accurate and, in fact, the likelihood ratio test is degenerate if \( m > n \) since in this case \( S \) is singular. In these situations, it would be better to use a test which is based on asymptotic theory which has both \( n \) and \( m \) going to infinity in such a way that the ratio \( m/n \) converges to a constant \( c \in (0, \infty) \). In principal components applications, one is usually most interested in values of \( q \) for which the first \( q \) principal components account for a large proportion of the total variability of the \( m \) variables so we hold this proportion fixed as \( n \) and \( m \) go to infinity. In particular, we will assume that for \( i = 1, \ldots, q \), \( \lambda_i/\text{tr}(\Sigma) \) converges to a constant \( \rho_i \in (0, 1) \) so that the proportion of variability explained by the first \( q \) principal components, \( \sum_{i=1}^{q} \lambda_i/\text{tr}(\Sigma) \), converges to \( \sum_{i=1}^{q} \rho_i \in (0, 1) \) as \( m \) goes to infinity. Thus, \( q \) is fixed while \( p \) goes to infinity as does \( \lambda_i \) for \( i = 1, \ldots, q \).

In Section 2, we propose an alternative statistic for testing \( H_{0^p} \) and obtain its asymptotic null distribution as \( n \) goes to infinity. Our derivation of the asymptotic null distribution of this statistic when both \( n \) and \( m \) go to infinity utilizes some inequalities for eigenvalues and these bounds are given in Section 3. The asymptotic null distribution of our test statistic is developed in Section 4. The two asymptotic distributions from Sections 2 and 4 are compared in Section 5, while Section 6 contains some empirical results for both our test and the likelihood ratio test.

### 2. A statistic for testing \( H_{0^p} \)

Much of the work in this paper could be viewed as an extension of some of the results of Ledoit and Wolf [6] who looked at a high-dimensional test for sphericity, that is, a test of
H$_0$._ Their test is based on the statistic

$$U_m = \frac{1}{m} \text{tr} \left\{ \left( \frac{S}{(1/m)\text{tr}(S)} - I_m \right)^2 \right\} = \frac{(1/m)\text{tr}(S^2)}{(1/(m)\text{tr}(S))^2} - 1.$$ 

Under H$_0$ and when sampling from a normal population, $nmU_m/2 \xrightarrow{d} \chi^2_d(m(m+1)/2-1)$ as $n$ goes to infinity, where $\chi^2_d$ denotes the chi-squared distribution with $d$ degrees of freedom.

We will consider testing H$_0$ using the natural generalization of $U_m$, that is, using

$$U_p = \frac{(1/p) \sum_{i=q+1}^{m} \phi_i^2(S)}{(1/p) \sum_{i=q+1}^{m} \phi_i(S)^2} - 1.$$ 

Since the eigenvalues of $S$ and $Q'SQ$ are the same for any orthogonal matrix $Q$, we will assume without loss of generality throughout the remainder of this paper that $\Sigma$ is diagonal.

In particular, under H$_0$, $\Sigma = \text{diag}(\lambda_1, \ldots, \lambda_q, \lambda, \ldots, \lambda) = \text{diag}(\Sigma_+, \lambda I_p)$. Since $U_p$ is unaffected by the multiplication of $S$ by a positive scalar, we can also assume that $\lambda = 1$.

Our first result shows that the asymptotic null distribution of $npU_p/2$, when $m$ is fixed and $n$ goes to infinity, is identical to that of the likelihood ratio statistic $T_p$.

**Theorem 1.** Under H$_0$ and when sampling from a normal distribution, as $n$ goes to infinity,

$$\frac{a_n p}{2} U_p \xrightarrow{d} \chi^2_{p(p+1)/2-1},$$

where $a_n$ represents any increasing sequence of constants for which $a_n/n \to 1$.

**Proof.** If we let $A = S - \Sigma$, then as $n$ goes to infinity, $n^{1/2} \text{vec}(A)$ converges in distribution to a normal random vector with zero mean vector and covariance matrix given by $\Psi = (I_m^2 + K_{mm})(\Sigma \otimes \Sigma)$, where $K_{mm}$ is a commutation matrix (see, for example, [7, Chapter 3]). A second-order expansion formula for $\sum_{i=q+1}^{m} \phi_i(S)/p$ is given by Schott [10, p. 370]

$$\sum_{i=q+1}^{m} \phi_i(S)/p = 1 + p^{-1}\text{tr}(AP) - p^{-1}\text{tr}(A\Lambda AP) + o_p(n^{-1}),$$

where $P$ is the block diagonal matrix $\text{diag}(0, I_p)$ and $\Lambda = (\Sigma - I_m)^{+}$. This then leads to

$$\left\{ \sum_{i=q+1}^{m} \phi_i(S)/p \right\}^{-2} = 1 - \frac{2}{p} \text{tr}(AP) + \frac{2}{p} \text{tr}(A\Lambda AP) + \frac{3}{p^2} [\text{tr}(AP)]^2 + o_p(n^{-1}).$$

(3)

Let $\hat{P} = \sum_{i=q+1}^{m} \gamma_i \gamma'_i$, where $\gamma_{q+1}, \ldots, \gamma_m$ is a set of orthonormal eigenvectors corresponding to the eigenvalues $\phi_{q+1}(S), \ldots, \phi_m(S)$. Using the identity

$$\sum_{i=q+1}^{m} \phi_i^2(S)/p = p^{-1}\text{tr}\{ (\Sigma + A)^2 \hat{P} \}$$
and the second-order expansion formula for $\hat{P}$ [10, p. 370], it is easy to obtain the second-order expansion formula

$$
\sum_{i=q+1}^{m} \phi_i^2(S)/p = 1 + p^{-1}\{2\text{tr}(AP) + \text{tr}(\Sigma^2AAPA\Lambda) - 2\text{tr}(A\Lambda AP) - 2\text{tr}(\Sigma A P A \Lambda) - \text{tr}(A\Lambda^2 AP) + \text{tr}(A^2 P)\} + o_p(n^{-1}).
$$

(4)

Multiplying (3) by (4) and then subtracting 1, we get

$$
Up = \frac{1}{p}\left\{\left[\text{tr}(\Sigma^2AAPA\Lambda) - 2\text{tr}(\Sigma A P A \Lambda) - \text{tr}(A\Lambda^2 AP) + \text{tr}(AP)\right] - p^{-1}(\text{tr}(AP))^2\right\} + o_p(n^{-1})
$$

$$
= \frac{1}{p}\left\{\left[\text{tr}(\Sigma^2AAPA\Lambda) - 2\text{tr}(\Sigma A P A \Lambda) - \text{tr}(A\Lambda^2 AP) + \text{tr}((I - P)AP A)\right] + \text{tr}(PAPA) - p^{-1}(\text{tr}(AP))^2\right\} + o_p(n^{-1})
$$

$$
= \frac{1}{p}\{\text{tr}(PAPA) - p^{-1}(\text{tr}(AP))^2\} + o_p(n^{-1}),
$$

where the last equality follows from the fact that the quantity within the brackets reduces to 0. Thus, using simple properties of the vec operator, we have

$$
\frac{np}{2} Up = \{n^{1/2}\text{vec}(A)\}' H \{n^{1/2}\text{vec}(A)\} + o_p(1),
$$

where $H = \frac{1}{2}(P \otimes P - p^{-1}\text{vec}(P)\text{vec}(P)')$. Since

$$
\Psi H \Psi = \Psi H \Psi = (I + K_{mn})(P \otimes P - p^{-1}\text{vec}(P)\text{vec}(P)')
$$

and $\text{tr}(H \Psi) = p(p + 1)/2 - 1$, the result then follows from well-known properties regarding the distribution of quadratic forms (see, for example, [10, p. 404]).

While the choice of $a_n$ may significantly influence the small sample performance of the chi-squared approximation, it has no affect on the asymptotic distribution of $a_n p U_p/2$ when $m$ is fixed. A natural choice for $a_n$ would be $n$. We will see later, however, that the choice of $a_n$, when using this chi-squared approximation in a high-dimensional setting, is critical since the asymptotic distribution when both $n$ and $m$ go to infinity depends on $a_n$.

Before deriving the asymptotic null distribution of $U_p$ as both $n$ and $m$ go to infinity, we will need to obtain some bounds for eigenvalues and sums of eigenvalues.

3. Inequalities for sums of eigenvalues

We begin with two inequalities relating the eigenvalues of two symmetric matrices to those of the sum of the matrices. Both of the results we present here are special cases of more general results. A proof of the first inequality, which is known as Weyl’s theorem, can be found in [4, p. 181], while a proof of the second is given by Wielandt [12].
Lemma 1. Let $T = R + S$, where $R$ and $S$ are $m \times m$ symmetric matrices. Then for $k = 1, \ldots, m$,

$$\phi_m(R) + \phi_k(S) \leq \phi_k(T) \leq \phi_1(R) + \phi_k(S) \quad (5)$$

and

$$\sum_{i=1}^k \phi_i(T) \leq \sum_{i=1}^k \phi_i(R) + \sum_{i=1}^k \phi_i(S). \quad (6)$$

Most of our remaining results in this section deal with eigenvalues of a partitioned matrix. Let $A$ be an $m \times m$ symmetric matrix partitioned as

$$
\begin{pmatrix}
B & C \\
C' & D
\end{pmatrix},
\quad (7)
$$

where $B$ is $q \times q$, $D$ is $p \times p$, and $C$ is $q \times p$. We will need several results relating the eigenvalues of $A$ to those of $B$ and $D$. Lemma 1 can be used to prove the following.

Corollary 1. Suppose $A$, as defined in (7), is nonnegative definite. Then for $k = 1, \ldots, m$

$$\phi_k(A) \leq \phi_1(B) + \phi_k(D) \quad (8)$$

and

$$\sum_{i=1}^k \phi_i(A) \leq \sum_{i=1}^k \phi_i(B) + \sum_{i=1}^k \phi_i(D), \quad (9)$$

where $\phi_i(B) = 0$ for $i > q$ and $\phi_i(D) = 0$, for $i > p$.

Proof. A proof of (8) is given in [3] and our proof of (9) is a simple modification of their proof. Let $A^{1/2}$ be the symmetric square root matrix of $A$ and partition it as $A^{1/2} = [F \ G]$, where $F$ is $m \times q$ and $G$ is $m \times p$. Since

$$A = (A^{1/2})'A^{1/2} = \begin{pmatrix} F'F & F'G \\ G'F & G'G \end{pmatrix},$$

we see that $B = F'F$ and $D = G'G$. But we also have

$$A = A^{1/2}(A^{1/2})' = FF' + GG'. \quad (10)$$

Since the nonzero eigenvalues of $B$ and $FF'$ are the same and the nonzero eigenvalues of $D$ and $GG'$ are the same, (9) follows by using (6) on (10). □

A proof of the following lemma, which is known as the Poincaré separation theorem, can be found in [10, p. 111].
**Lemma 2.** Let $A$ be an $m \times m$ symmetric matrix and $F$ be an $m \times f$ matrix satisfying $F'F = I_f$. Then, for $k = 1, \ldots, f$, it follows that

$$ \phi_{m-f+k}(A) \leq \phi_k(F'AF) \leq \phi_k(A). $$

In particular, if $A$ is partitioned as in (7), then

$$ \phi_{q+k}(A) \leq \phi_k(D) \leq \phi_k(A), $$

for $k = 1, \ldots, p$.

Our next result compares sums of eigenvalues of $A$ to those of $D$ and sums of squared eigenvalues of $A$ to those of $D$.

**Theorem 2.** Suppose $A$ is defined as in (7) and $\phi_1(B) = 0 < \phi_p(D)$. Then for $k = 1, \ldots, p$

$$ \sum_{i=1}^{k} (\phi_i(A) - \phi_i(D)) \leq \sum_{i=1}^{k} \phi_i(C'C)/\phi_k(D) $$

and

$$ \sum_{i=1}^{k} (\phi_i^2(A) - \phi_i^2(D)) \leq 2 \sum_{i=1}^{k} \phi_i(C'C). $$

**Proof.** The proof is very similar to the proof of Theorem 2.1 in [3]. Let $\hat{A}$ be the matrix obtained by replacing $B$ in (7) by the null matrix so that

$$ \hat{A}^2 = \begin{pmatrix} CC' & CD \\ DC' & D^2 + C'C \end{pmatrix}. $$

It follows from (5) that $\phi_i(\hat{A}) \geq \phi_i(A)$ for $i = 1, \ldots, m$ since $-B$ is nonnegative definite and $\hat{A} = A - \text{diag}(B, 0)$. It also follows that $\phi_i(\hat{A}^2) \geq \phi_i^2(\hat{A}) \geq \phi_i^2(A)$, for $i = 1, \ldots, p$ since using Lemma 2, $\phi_i(\hat{A}) \geq \phi_i(A) \geq \phi_i(D) > 0$ for these values of $i$. Now applying Corollary 1 to $\hat{A}^2$, we have

$$ \sum_{i=1}^{k} \phi_i(\hat{A}^2) \leq \sum_{i=1}^{k} \phi_i(C'C) + \sum_{i=1}^{k} \phi_i(D^2 + C'C). $$

(13)

Next, applying Lemma 1 to $D^2 + C'C$ yields

$$ \sum_{i=1}^{k} \phi_i(D^2 + C'C) \leq \sum_{i=1}^{k} \phi_i(D^2) + \sum_{i=1}^{k} \phi_i(C'C) $$

and when this is combined with (13) we get

$$ \sum_{i=1}^{k} \phi_i(\hat{A}^2) \leq \sum_{i=1}^{k} \phi_i^2(D) + 2 \sum_{i=1}^{k} \phi_i(C'C), $$

(12)
since the eigenvalues of $D$ are positive and the positive eigenvalues of $CC'$ and $C'C$ are the same. Next, using the fact that $\phi_i^2(A) \leq \phi_i^2(A^2)$ for $i = 1, \ldots, p$, we have

$$\sum_{i=1}^{k} \phi_i^2(A) \leq \sum_{i=1}^{k} \phi_i^2(D) + 2 \sum_{i=1}^{k} \phi_i(C'C)$$

or equivalently

$$\sum_{i=1}^{k} \{\phi_i^2(A) - \phi_i^2(D)\} = \sum_{i=1}^{k} \{\phi_i(A) + \phi_i(D)\} \{\phi_i(A) - \phi_i(D)\} \leq 2 \sum_{i=1}^{k} \phi_i(C'C), \quad (14)$$

which establishes (12). Continuing from (14), we also have

$$\{\phi_k(A) + \phi_k(D)\} \sum_{i=1}^{k} \{\phi_i(A) - \phi_i(D)\} \leq \sum_{i=1}^{k} \{\phi_i(A) + \phi_i(D)\} \{\phi_i(A) - \phi_i(D)\} \leq 2 \sum_{i=1}^{k} \phi_i(C'C),$$

so that

$$\sum_{i=1}^{k} \{\phi_i(A) - \phi_i(D)\} \leq 2 \sum_{i=1}^{k} \phi_i(C'C)/\phi_k(A) + \phi_k(D) \leq \sum_{i=1}^{k} \phi_i(C'C)/\phi_k(D),$$

which proves (11). ☐

Theorem 2 can be generalized to the case in which $\phi_1(B) > 0$ as follows.

**Corollary 2.** Suppose that $A$ is defined as in (7) and $0 < \phi_1(B) < \phi_p(D)$. Then for $k = 1, \ldots, p$,

$$\sum_{i=1}^{k} \{\phi_i(A) - \phi_i(D)\} \leq \sum_{i=1}^{k} \phi_i(C'C)/\{\phi_k(D) - \phi_1(B)\} \quad (15)$$

and

$$\sum_{i=1}^{k} \{\phi_i^2(A) - \phi_i^2(D)\} \leq 2 \left\{1 + \frac{\phi_1(B)}{\phi_k(D) - \phi_1(B)}\right\} \sum_{i=1}^{k} \phi_i(C'C). \quad (16)$$
Proof. Applying (11) to $\tilde{A} = A - \phi_1(B) I_m$ immediately leads to (15), while the application of (12) yields

$$\sum_{i=1}^{k} \{\phi_i^2(A) - \phi_i^2(D)\} \leq 2 \left\{ \sum_{i=1}^{k} \phi_i(C'C) + \phi_1(B) \sum_{i=1}^{k} \{\phi_i(A) - \phi_i(D)\} \right\},$$

and when this is combined with (15), we get (16). □

Our main result of this section, which relates the eigenvalues of $A$ to those of $D - C' B^{-1} C$ when $A$ is positive definite, is next.

**Theorem 3.** Suppose $A$ in (7) is positive definite and let $\hat{B} = B - CD^{-1} C', \hat{D} = D - C'B^{-1} C$, and $\hat{C} = -B^{-1} C \hat{D}^{-1}$. Then if $\phi_1(\hat{D}) < \phi_q(\hat{B})$,

$$0 \leq \sum_{i=1}^{k} \{\phi_{p-i+1}(\hat{D}) - \phi_{m-i+1}(A)\} \leq \frac{\phi_{p-k+1}^2(\hat{D})}{\phi_{p-k+1}(\hat{D}) - \phi_{q-1}(\hat{B})} \sum_{i=1}^{k} \phi_i(C'C)$$

(17)

and

$$0 \leq \sum_{i=1}^{k} \{\phi_{p-i+1}^2(\hat{D}) - \phi_{m-i+1}^2(A)\} \leq 2\phi_{p-k+1}^4(\hat{D}) \left\{1 + \frac{\phi_{q-1}(\hat{B})}{\phi_{p-k+1}(\hat{D}) - \phi_{q-1}(\hat{B})} \right\} \sum_{i=1}^{k} \phi_i(C'C)$$

(18)

for $k = 1, \ldots, p$.

Proof. The inverse matrix of $A$ can be expressed as (see, for example, [10, p. 256])

$$A^{-1} = \begin{pmatrix} B^{-1} & \hat{C} \\ \hat{C}' & \hat{D}^{-1} \end{pmatrix}.$$  

Applying Lemma 2 to $A^{-1}$, we have for $i = 1, \ldots, p$, $\phi_i(\hat{D}^{-1}) \leq \phi_i(A^{-1})$ so that $\phi_{p-i+1}^{-1}(\hat{D}) \leq \phi_{m-i+1}^{-1}(A)$ or, equivalently, $\phi_{p-i+1}(\hat{D}) > \phi_{m-i+1}(A)$. This proves the lower bound in both (17) and (18). Since $\phi_p(D^{-1}) = \phi_1^{-1}(\hat{D}) > \phi_q^{-1}(\hat{B}) = \phi_1(\hat{B}^{-1})$, we can apply (15) of Corollary 2 to $A^{-1}$, which leads to

$$\sum_{i=1}^{k} \{\phi_{m-i+1}^{-1}(A) - \phi_{p-i+1}^{-1}(\hat{D})\} \leq \sum_{i=1}^{k} \phi_i(C'C)/([\phi_{p-k+1}^{-1}(\hat{D}) - \phi_{q-1}(\hat{B})]).$$
But
\[
\sum_{i=1}^{k} \{ \phi_{m-i+1}^{-1}(A) - \phi_{p-i+1}^{-1}(\hat{D}) \} = \sum_{i=1}^{k} \frac{\phi_{p-i+1}(\hat{D}) - \phi_{m-i+1}(A)}{\phi_{m-i+1}(A) \phi_{p-i+1}(\hat{D})} \\
\geq \phi_{m-k+1}(A) \phi_{p-k+1}(\hat{D}) \\
\times \sum_{i=1}^{k} \{ \phi_{p-i+1}(\hat{D}) - \phi_{m-i+1}(A) \}
\]

so
\[
\sum_{i=1}^{k} \{ \phi_{p-i+1}(\hat{D}) - \phi_{m-i+1}(A) \} \leq \frac{\phi_{m-k+1}(A) \phi_{p-k+1}(\hat{D})}{\{ \phi_{p-k+1}(\hat{D}) - \phi_{q-1}(\hat{B}) \}} \sum_{i=1}^{k} \phi_{i}(\hat{C}'\hat{C}) \\
\leq \frac{\phi_{p-k+1}(\hat{D})}{\{ \phi_{p-k+1}(\hat{D}) - \phi_{q-1}(\hat{B}) \}} \sum_{i=1}^{r} \phi_{i}(\hat{C}'\hat{C})
\]

thereby establishing (17). Applying (16) of Corollary 2 to \(A^{-1}\), we get
\[
\sum_{i=1}^{k} \{ \phi_{m-i+1}^{-2}(A) - \phi_{p-i+1}^{-2}(\hat{D}) \} \leq 2 \left\{ 1 + \frac{\phi_{q-1}^{-1}(\hat{B})}{\phi_{p-k+1}^{-1}(\hat{D}) - \phi_{q-1}^{-1}(\hat{B})} \right\} \sum_{i=1}^{k} \phi_{i}(\hat{C}'\hat{C}). \Box
\]

Now combining this with the fact that
\[
\sum_{i=1}^{k} \{ \phi_{m-i+1}^{-2}(A) - \phi_{p-i+1}^{-2}(\hat{D}) \} = \sum_{i=1}^{k} \frac{\phi_{p-i+1}^{-2}(\hat{D}) - \phi_{m-i+1}^{-2}(A)}{\phi_{m-i+1}^{-2}(A) \phi_{p-i+1}^{-2}(\hat{D})} \\
\geq \phi_{m-k+1}^{-2}(A) \phi_{p-k+1}^{-2}(\hat{D}) \\
\times \sum_{i=1}^{k} \{ \phi_{p-i+1}^{-2}(\hat{D}) - \phi_{m-i+1}^{-2}(A) \}
\]

leads to
\[
\sum_{i=1}^{k} \{ \phi_{p-i+1}^{-2}(\hat{D}) - \phi_{m-i+1}^{-2}(A) \} \\
\leq 2 \phi_{m-k+1}^{-2}(A) \phi_{p-k+1}^{-2}(\hat{D}) \left\{ 1 + \frac{\phi_{q-1}^{-1}(\hat{B})}{\phi_{p-k+1}^{-1}(\hat{D}) - \phi_{q-1}^{-1}(\hat{B})} \right\} \sum_{i=1}^{k} \phi_{i}(\hat{C}'\hat{C}) \\
\leq 2 \phi_{p-k+1}^{-4}(\hat{D}) \left\{ 1 + \frac{\phi_{q-1}^{-1}(\hat{B})}{\phi_{p-k+1}^{-1}(\hat{D}) - \phi_{q-1}^{-1}(\hat{B})} \right\} \sum_{i=1}^{k} \phi_{i}(\hat{C}'\hat{C})
\]

and so the proof is complete.

Theorem 3 can be generalized to the case in which \(A\) is singular as follows.

**Corollary 3.** Suppose \(A\) in (7) is nonnegative definite with \(B\) being positive definite while \(\text{rank}(\hat{D}) = r\), where \(\hat{D} = D - C'B^{-1}C\). Let \(Q\) be any \(p \times r\) matrix satisfying \(Q'Q = I_r\)
and $\hat{D} = Q\Delta Q'$, where $\Delta$ is a diagonal matrix with the positive eigenvalues of $\hat{D}$ as its diagonal elements. Define $\hat{B}_* = B - CQ(Q'DQ)^{-1}Q'C'$ and $\hat{C}_* = -B^{-1}CQ\Delta^{-1}$. Then if $\phi_1(\hat{D}) < \phi_q(\hat{B}_*)$,

$$0 \leq \sum_{i=1}^{p-r+k} \{\phi_{p-i+1}(\hat{D}) - \phi_{m-i+1}(A)\}$$

$$\leq \frac{\phi_r^{2}(\hat{D})}{\{\phi_{r-k+1}(\hat{D}) - \phi_{q}^{-1}(\hat{B}_*)\}} \sum_{i=1}^{k} \phi_i(\hat{C}_*\hat{C}_*)$$

(19)

and

$$0 \leq \sum_{i=1}^{p-r+k} \{\phi_{p-i+1}^{2}(\hat{D}) - \phi_{m-i+1}^{2}(A)\}$$

$$\leq 2\phi_r^{4}(\hat{D}) \left\{1 + \frac{\phi_{q}^{-1}(\hat{B}_*)}{\phi_{r-k+1}^{-1}(\hat{D}) - \phi_{q}^{-1}(\hat{B}_*)}\right\} \sum_{i=1}^{k} \phi_i(\hat{C}_*\hat{C}_*)$$

(20)

for $k = 1, \ldots, r$.

**Proof.** Since rank$(A) = \text{rank}(B) + \text{rank}(\hat{D})$ (see, for example, [9]), it follows that $\phi_{p-i+1}(\hat{D}) = \phi_{m-i+1}(A) = 0$, for $i = 1, \ldots, p - r$. Also, Smith [11] has shown that under our conditions for $A$, $\phi_{p-i+1}(\hat{D}) > \phi_{m-i+1}(A)$ for $i = p - r + 1, \ldots, p$. This establishes the lower bound in both (19) and (20). Let

$$A_* = \begin{pmatrix} I_q & 0 \\ 0 & Q' \end{pmatrix} A \begin{pmatrix} I_q & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} B & CQ \\ Q'C' & Q'DQ \end{pmatrix} = \begin{pmatrix} B & C_* \\ C_* & D_* \end{pmatrix}.$$ 

Note that $A_*$ is positive definite and $\hat{D}_* = D_* - C_*B^{-1}C_* = \Delta$ so that the positive eigenvalues of $\hat{D}_*$ and $\hat{D}$ are the same and, in particular, $\phi_1(\hat{D}_*) < \phi_q(\hat{B}_*)$. Thus, we can apply (17) of Theorem 3 to $A_*$ to get

$$\sum_{i=1}^{p-r+k} \{\phi_{p-i+1}(\hat{D}) - \phi_{m-i+1}(A)\} = \sum_{i=p-r+1}^{p-r+k} \{\phi_{p-i+1}(\hat{D}) - \phi_{m-i+1}(A)\}$$

$$\leq \sum_{i=1}^{k} \{\phi_{r-i+1}(\hat{D}_*) - \phi_{q+r-i+1}(A_*)\}$$

$$\leq \frac{\phi_r^{2}(\hat{D}_*)}{\{\phi_{r-k+1}(\hat{D}_*) - \phi_{q}^{-1}(\hat{B}_*)\}} \sum_{i=1}^{k} \phi_i(\hat{C}_*\hat{C}_*)$$

$$= \frac{\phi_r^{2}(\hat{D})}{\{\phi_{r-k+1}(\hat{D}) - \phi_{q}^{-1}(\hat{B}_*)\}} \sum_{i=1}^{k} \phi_i(\hat{C}_*\hat{C}_*).$$

The first inequality follows from the fact that $\phi_i(A_*) \leq \phi_i(A)$ for $i = 1, \ldots, q + r$ which is a consequence of Lemma 2. The upper bound in (20) is established in a similar fashion. □
Our final result of this section gives bounds for the eigenvalues of a matrix product. A proof of this result can be found in Anderson and Das Gupta [1].

**Lemma 3.** Let $A$ be an $m \times m$ nonnegative definite matrix and $B$, an $m \times m$ positive definite matrix. Then, for $i = 1, \ldots, m$,

$$\phi_i(A)\phi_m(B) \leq \phi_i(AB) \leq \phi_i(A)\phi_1(B).$$

4. Asymptotic null distribution of $U_p$ when both $n$ and $m$ go to infinity

The development of the asymptotic properties of $U_p$ as both $n$ and $m$ go to infinity will utilize the following conditions.

**Condition 1.** Both $p = p_k$ and $n = n_k$ are increasing functions of an index $k = 1, 2, \ldots$ such that $\lim_{k \to \infty} p_k = \infty$, $\lim_{k \to \infty} n_k = \infty$, and $\lim_{k \to \infty} p_k/n_k = c \in (0, \infty)$. The quantity $q$ does not depend on $k$ and so $m_k = q + p_k$ has the same limiting properties as does $p_k$.

**Condition 2.** For each $k$, the sample covariance matrix can be expressed as $S_k = n_k^{-1}X'_k(1_{nk+1} - (n_k+1)^{-1}1_{nk+1}1'_{nk+1})X_k$, where $1_{nk+1}$ is the $(n_k+1) \times 1$ vector of 1’s and the rows of the $(n_k+1) \times m_k$ matrix $X_k$ are independently and identically distributed normal random vectors with mean vector $\mu_k$ and covariance matrix $\Sigma_k$. The eigenvalues $\lambda_{1,k}, \ldots, \lambda_{m_k,k}$ of $\Sigma_k$ are such that $x = \sum_{i=q+1}^{m_k} \lambda_{i,k}/p_k > 0$ and $\delta^2 = \sum_{i=q+1}^{m_k} (\lambda_{i,k} - x)^2/p_k$ are independent of $k$. Further, for $i = 1, \ldots, q$, $\lambda_{i,k}$ is an increasing function of $k$ such that $\lim_{k \to \infty} \lambda_{i,k} = \infty$ and $\lim_{k \to \infty} \lambda_{i,k}/\text{tr}(\Sigma_k) = \rho_i \in (0, 1)$, where $\sum_{i=1}^{q} \rho_i \in (0, 1)$.

**Condition 3.** For $j = 3, 4$

$$\lim_{k \to \infty} \sum_{i=q+1}^{m_k} \frac{(\lambda_{i,k})^j}{p_k} = b_j \in (0, \infty).$$

For notational convenience, the dependence of $n$, $p$, $\Sigma$ and $S$ on $k$ will be suppressed throughout the remainder of the paper. Our next result gives the asymptotic null distribution of $U_p$ when both $n$ and $p$ go to infinity.

**Theorem 4.** Under conditions 1–3, if $\delta^2 = 0$, then

$$(n - q)U_p - p \to^d N(1, 4),$$

where $N(\mu, \sigma^2)$ denotes the normal distribution with mean $\mu$ and variance $\sigma^2$.

**Proof.** Partition $S$ as

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{pmatrix} = \begin{pmatrix} \Sigma_{1/2}^{1/2} & W_{11} \Sigma_{1/2}^{1/2} & \Sigma_{1/2}^{1/2} W_{12} \\ W_{12} \Sigma_{1/2}^{1/2} & \Sigma_{1/2}^{1/2} W_{22} \end{pmatrix} = \Sigma^{1/2} W \Sigma^{1/2},$$
where $S_{11}$ is $q \times q$ and $S_{22}$ is $p \times p$, so that the matrix $W$ has a central Wishart distribution with covariance matrix $I_m$. The $p \times p$ matrix $S_{22} = S_{22} - S_{12} S_{11}^{-1} S_{12}$ has a central Wishart distribution with covariance matrix $I_p$ and $n - q$ degrees of freedom (see, for example, [8, p. 93]). It follows from Proposition 2 of Ledoit and Wolf [6] that under our conditions

$$ (n - q) \begin{pmatrix} \frac{1}{p} \operatorname{tr}(S_{22.1}) - 1 \\ \frac{1}{p} \operatorname{tr}(S_{22.1}^2) - (n-q)^p + 1 \end{pmatrix} $$

converges in distribution to a bivariate normal with mean vector 0 and covariance matrix as given in [6]. This result was then used by Ledoit and Wolf [6] in their Proposition 3 to show that

$$ (n - q) \begin{pmatrix} (1/p)\operatorname{tr}(S_{22.1}^2) \\ [(1/p)\operatorname{tr}(S_{22.1})]^2 - 1 \end{pmatrix} - p \rightarrow^d N(1, 4). $$

Our proof will then be complete if we can show that

$$ \begin{pmatrix} \frac{1}{p} \operatorname{tr}(S_{22.1}) - \frac{1}{p} \sum_{i=q+1}^m \phi_i(S) \\ \frac{1}{p} \operatorname{tr}(S_{22.1}^2) - \frac{1}{p} \sum_{i=q+1}^m \phi_i^2(S) \end{pmatrix} = o_p(n^{-1}). \quad (21) $$

Let $r = \text{rank}(S_{22.1}) = \min(p, n-q)$ and let $Q$ be any $p \times r$ matrix for which $Q'Q = I_r$ and $Q'S_{22.1}Q = \Delta$ is diagonal with positive diagonal elements. Consider the set $C = \{S : \phi_1(S_{22.1}) < \phi_q(\hat{S}_{11})\}$, where $\hat{S}_{11} = S_{11} - S_{12} Q (Q'S_{22} Q)^{-1} Q'S_{12}$. Now $\phi_q^{-1}(\hat{S}_{11})$ is the largest eigenvalue of $\hat{S}_{11}^{-1} = S_{11}^{-1} + S_{11}^{-1} S_{12} Q \Sigma^{-1} Q'S_{12} S_{11}^{-1}$ and so

$$ \phi_q^{-1}(\hat{S}_{11}) \leq \phi_1(S_{11}^{-1}) + \phi_1(S_{11}^{-1} S_{12} Q \Sigma^{-1} Q'S_{12} S_{11}^{-1}) $$

$$ = \phi_1(\Sigma_s^{-1/2} W_{11}^{-1} \Sigma_s^{-1/2}) + \phi_1(\Sigma_s^{-1/2} W_{11}^{-1} W_{12} Q \Sigma^{-1} Q'W_{12} W_{11}^{-1} \Sigma_s^{-1/2}) $$

$$ \leq \phi_1(\Sigma_s^{-1}) \phi_1(W_{11}^{-1}) + \phi_1(\Sigma_s^{-1}) \phi_1^2(W_{11}^{-1}) \phi_1(\Sigma_2^{-1}) \phi_1(Q'W_{12} W_{12}^{-1} Q) $$

$$ \leq \lambda_q^{-1} \phi_q^{-1}(W_{11}) + \lambda_q^{-1} \phi_q^{-2}(W_{11}) \phi_q^{-1}(S_{22.1}) \operatorname{tr}(W_{12} W_{12}^{-1}). \quad (22) $$

Here we have used Lemma 1 to get the first inequality, Lemma 3 for the second inequality, and Lemma 2 for the third inequality. It follows from Bai and Yin [2] that $\phi_q^{-1}(S_{22.1})$ converges in probability to $(1 - c^{1/2})^{-2}$. It also follows from Proposition 1 of Ledoit and Wolf [6] that both $m^{-1} \operatorname{tr}(W^2)$ and $p^{-1} \operatorname{tr}(W_2^2)$ converge in probability to $(1 + c)$. This implies that $m^{-1} \operatorname{tr}(W_{12} W_{12}^{-1})$ converges in probability to 0 since $\operatorname{tr}(W^2) = \operatorname{tr}(W_{11}^2) + \operatorname{tr}(W_{22}^2) + 2 \operatorname{tr}(W_{12} W_{12}^{-1})$, $p/m \rightarrow 1$, and $m^{-1} \operatorname{tr}(W_2^2)$ converges in probability to 0. Thus, since $\lambda_q = O(n) = O(m)$ and $\phi_q(W_{11}) \rightarrow^p 1$, which follows from $W_{11} \rightarrow^p I_q$ and the continuity of eigenvalues, we have shown that the right-hand side of (22), and hence also $\phi_q^{-1}(\hat{S}_{11})$, converges in probability to 0. Since $\phi_1(S_{22.1})$ converges in probability to $(1 + c^{1/2})^2$ [13], we have established that $P(C) \rightarrow 1$ and so attention can be restricted to
Wishart matrix $S$ this set. For $S \in C$, Corollary 3 implies that
\[ 0 \leq \text{tr}(S_{22,1}) - \sum_{i=q+1}^{m} \phi_i(S) \leq \frac{\phi_i^2(S_{22,1})}{\{\phi_1^{-1}(S_{22,1}) - \phi_q^{-1}(\hat{S}_{11})\}} \times \text{tr}(\Delta^{-1} Q' S_{12}^{-2} S_{12} Q \Delta^{-1}) \] (23)

and
\[ 0 \leq \text{tr}(S_{22,1}^2) - \sum_{i=q+1}^{m} \phi_i^2(S) \leq 2\phi_1^2(S_{22,1}) \left(1 + \frac{\phi_q^{-1}(\hat{S}_{11})}{\phi_1^{-1}(S_{22,1}) - \phi_q^{-1}(\hat{S}_{11})}\right) \text{tr}(\Delta^{-1} Q' S_{12}^{-2} S_{12} Q \Delta^{-1}). \] (24)

But
\[ \text{tr}(\Delta^{-1} Q' S_{12}^{-2} S_{12} Q \Delta^{-1}) = \text{tr}(\Delta^{-1} Q' W_{12} W_{11}^{-1} \Sigma_s^{-1} W_{11}^{-1} W_{12} Q \Delta^{-1}) \leq \lambda_q^{-1} \phi_1^{-2}(W_{11}) \phi_q^{-2}(S_{22,1}) \text{tr}(W_{12} W_{12}'). \]

Since $\phi_q(W_{11}) \to_p 1$, $\phi_r(S_{22,1}) \to_p (1 - c^{1/2})^2$, $m^{-1}\text{tr}(W_{12} W_{12}') \to_p 0$, and $m/\lambda_q = O(1)$, it follows that $\text{tr}(\Delta^{-1} Q' S_{12}^{-2} S_{12} Q \Delta^{-1}) = o_p(1)$. Using this and the fact that $\phi_1(S_{22,1}) = O_p(1)$ and $\phi_q^{-1}(\hat{S}_{11}) = o_p(1)$, we find that the right-hand side of both (23) and (24) are $o_p(1)$. Since $p^{-1} = O(n^{-1})$, this establishes (21). \qed

In view of the proof of Theorem 4, it might be conjectured that (21) also holds for the nonnull case; that is, if $\Sigma = \text{diag}(\Sigma_s, \Sigma_\#)$, where $\Sigma_s$ is as previously defined but now $\Sigma_\# = \text{diag}(\lambda_{q+1}, \ldots, \lambda_m) \neq I_p$, then (21) would hold with the covariance matrix of the Wishart matrix $S_{22,1}$ now being $\Sigma_\#$. A simple modification of the proof of Theorem 4 can partially prove this conjecture. We need $\phi_1(S_{22,1}) = O_p(1)$ and $\phi_q^{-1}(S_{22,1}) = O_p(1)$ and this can be guaranteed if $\lambda_{q+1} \to b_1 < \infty$ and $\lambda_{q+r} \to b_2 > 0$ as $p$ goes to infinity.

5. Comparing the two asymptotic distributions

A statistic $t$ for testing a hypothesis $H_0$ is robust against high dimensionality if the asymptotic null distribution obtained by first letting $n$ go to infinity while $m$ is fixed, and then letting $m$ go to infinity is in agreement with its asymptotic null distribution as both $n$ and $m$ go to infinity. In this section, we show that $U_p$ has this property. Confirmation of this will follow directly from our next result which is a special case of a more general result given in [6].

Lemma 4. Suppose that $Y_p \sim \chi^2_{p(p+1)/2-1}$. Then
\[ \frac{2}{p} Y_p - p \to_d N(1, 4) \]
as $p$ goes to infinity.
From Theorem 1
\[ a_n U_p - p \overset{d}{\to} \frac{2}{p} \chi^2_{p(p+1)/2-1} - p \]  
(25)
as \( n \) goes to infinity, while an application of Lemma 4 to the right-hand side of (25) gives
\[ \frac{2}{p} \chi^2_{p(p+1)/2-1} - p \to^d N(1, 4) \]as \( p \) goes to infinity. These two results are in agreement with the one asymptotic result given in Theorem 4 if we use 
\[ a_n = n - q. \]The implication is that in practice, a high-dimensional test of \( H_0 \) can be based simply on the asymptotic result in Theorem 1 by rejecting \( H_0 \) if
\[ (n - q)U_p > \chi^2_{p(p+1)/2-1,1-\alpha}, \]where \( \alpha \) denotes the desired significance level and \( \chi^2_{p(p+1)/2-1,1-\alpha} \) is the \( 1 - \alpha \) quantile from the chi-squared distribution with \( p(p + 1)/2 - 1 \) degrees of freedom.

6. Some simulation results

Some simulation results were obtained so as to assess the effectiveness of the asymptotic chi-squared distribution in approximating the actual null distributions of \( U_p \) and \( T_p \) for finite sample sizes. In all of our simulations, the common smallest eigenvalue was taken to be 1, while for \( i = 1, \ldots, q \), \( \lambda_i = \rho_i p / (1 - \sum_{j=1}^{q} \rho_j) \). We considered both \( q = 1 \) and 2, and used \( \rho_1 = 0.75 \) for \( q = 1 \), while \( \rho_1 = 0.56 \) and \( \rho_2 = 0.24 \) were used for \( q = 2 \). Both \( m \) and \( n \) ranged over the values 4, 8, 16, 32, 64, 128, and 256, and for each setting the significance level was estimated from 5000 simulations. The nominal significance level used was 0.05.

Tables 1 and 2 present the results for \( U_p \) when \( q = 1 \) and 2, respectively. In both tables, we see that the chi-squared approximation works quite well except for situations in which both \( m \) and \( n \) are very small. Corresponding results for the likelihood ratio statistic, \( T_p \), are given in Tables 3 and 4. In these tables, we see that for fixed \( m \), as \( n \) increases the estimated significance level generally gets closer to the nominal significance level as is expected. However, the approximation worsens as \( m \) increases and is particularly bad when \( m = n \). Upon comparing the results in Tables 3 and 4 with the tabulated values given by Ledoit and Wolf [6] for \( T_m \) in the test of sphericity, we find that our results are slightly better. This reflects the fact that we used the Bartlett-corrected multiplying factor given in (2), while apparently Ledoit and Wolf [6] used the uncorrected factor \( c_n = n \) in their simulations.

The requirement in Condition 2 that \( \lim_{k \to \infty} \lambda_{i,k} = \infty \) for \( i = 1, \ldots, q \) was an essential part of the development of the asymptotic results of Section 4. This and the assumption that \( \lim_{k \to \infty} \sum_{i=1}^{q} \lambda_{i,k} / \text{tr}(\Sigma_k) = \sum_{i=1}^{q} \rho_i \in (0, 1) \) were accommodated by holding \( q \) fixed. However, in practice, as the number of variables under consideration, \( m \), increases, one would expect \( q \) to increase also if the fraction of the total variability explained is to be held constant. For this reason, we conducted a second simulation study in which both \( \tau_1 = q/m \) and \( \tau_2 = \sum_{i=1}^{q} \lambda_i / \text{tr}(\Sigma) \) were held fixed. Thus, \( q \to \infty \) as \( m \to \infty \). In these simulations,
Table 1
Estimated significance levels for \( U_p \) when \( q = 1 \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.000</td>
<td>0.016</td>
<td>0.024</td>
<td>0.041</td>
<td>0.046</td>
<td>0.048</td>
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</tr>
<tr>
<td>8</td>
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<td>0.037</td>
<td>0.040</td>
<td>0.041</td>
<td>0.049</td>
<td>0.048</td>
<td>0.051</td>
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</tr>
<tr>
<td>16</td>
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<td>0.047</td>
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<td>0.049</td>
<td>0.047</td>
<td></td>
</tr>
<tr>
<td>32</td>
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<td>0.048</td>
<td>0.047</td>
<td>0.049</td>
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</tr>
<tr>
<td>64</td>
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<td>0.051</td>
<td>0.063</td>
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<td>0.048</td>
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<td></td>
</tr>
<tr>
<td>128</td>
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<td>0.052</td>
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<td>0.052</td>
<td>0.052</td>
<td>0.051</td>
<td></td>
</tr>
<tr>
<td>256</td>
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<td>0.050</td>
<td>0.047</td>
<td>0.052</td>
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</tr>
</tbody>
</table>

Table 2
Estimated significance levels for \( U_p \) when \( q = 2 \)

<table>
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<tr>
<th>( m )</th>
<th>( n )</th>
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<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
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<td>0.000</td>
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<td>0.047</td>
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<tr>
<td>8</td>
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<td>0.041</td>
<td>0.046</td>
<td>0.049</td>
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</tr>
<tr>
<td>16</td>
<td>0.016</td>
<td>0.040</td>
<td>0.047</td>
<td>0.049</td>
<td>0.043</td>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
<td>256</td>
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<td>0.052</td>
<td>0.049</td>
<td>0.054</td>
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</tr>
</tbody>
</table>

Table 3
Estimated significance levels for \( T_p \) when \( q = 1 \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.104</td>
<td>0.057</td>
<td>0.046</td>
<td>0.053</td>
<td>0.050</td>
<td>0.049</td>
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<td>0.306</td>
<td>0.060</td>
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<td>0.052</td>
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</tr>
<tr>
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<td>0.052</td>
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<tr>
<td>32</td>
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<tr>
<td>64</td>
<td>1.000</td>
<td>1.000</td>
<td>0.180</td>
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<tr>
<td>128</td>
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<tr>
<td>256</td>
<td>1.000</td>
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<td></td>
<td></td>
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</tr>
</tbody>
</table>

we had \( \lambda_1 = \cdots = \lambda_q = (1 - \tau_1)\tau_2/\{\tau_1(1 - \tau_2)\} \), which is constant as \( m \) increases, and again the common smallest eigenvalue was 1. For simplicity, we only considered the case in which \( m = n \).

Some estimated significance levels are given in Table 5. The eigenvalues, \( \lambda_1 = \cdots = \lambda_q \), for the four rows of Table 5 are 27, 9, 4.5, and 10.5, respectively. We see that the chi-squared approximation is conservative, but is reasonably accurate if \( \lambda_q \) is substantially larger than the common smallest eigenvalue of 1. If \( \lambda_q \) is not far enough above the smallest eigenvalue, the test becomes excessively conservative as illustrated when \( (\tau_1, \tau_2) = (0.25, 0.6) \). Some estimates of the power of the test based on \( U_p \), under the same settings as the simulations in Table 5, are given in Table 6. For the simulations with \( \tau_1 = 0.25 \), \( p/3 \) of the eigenvalues
Table 4
Estimated significance levels for $T_p$ when $q = 2$

<table>
<thead>
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<th>$m$</th>
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<th>16</th>
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<th>128</th>
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</thead>
<tbody>
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<td>0.049</td>
<td>0.049</td>
<td>0.049</td>
<td>0.049</td>
</tr>
<tr>
<td>16</td>
<td>0.638</td>
<td>0.069</td>
<td>0.050</td>
<td>0.053</td>
<td>0.053</td>
<td>0.053</td>
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<td>32</td>
<td>0.984</td>
<td>0.096</td>
<td>0.052</td>
<td>0.051</td>
<td>0.051</td>
<td>0.051</td>
<td>0.051</td>
</tr>
<tr>
<td>64</td>
<td>1.000</td>
<td>0.173</td>
<td>0.065</td>
<td>0.065</td>
<td>0.065</td>
<td>0.065</td>
<td>0.065</td>
</tr>
<tr>
<td>128</td>
<td>1.000</td>
<td>0.412</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
</tr>
<tr>
<td>256</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 5
Estimated significance levels for $U_p$ when $\tau_1 = q / m$ and $\tau_2 = \sum_{i=1}^{q} \lambda_i / \text{tr}(\Sigma)$ are fixed and $m = n$

<table>
<thead>
<tr>
<th>$(\tau_1, \tau_2)$</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.250,0.90)</td>
<td>0.000</td>
<td>0.025</td>
<td>0.050</td>
<td>0.051</td>
<td>0.048</td>
<td>0.045</td>
<td>0.041</td>
</tr>
<tr>
<td>(0.250,0.75)</td>
<td>0.000</td>
<td>0.025</td>
<td>0.038</td>
<td>0.038</td>
<td>0.030</td>
<td>0.022</td>
<td>0.008</td>
</tr>
<tr>
<td>(0.250,0.60)</td>
<td>0.000</td>
<td>0.018</td>
<td>0.018</td>
<td>0.009</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>(0.125,0.60)</td>
<td>0.031</td>
<td>0.042</td>
<td>0.051</td>
<td>0.045</td>
<td>0.040</td>
<td>0.031</td>
<td>0.031</td>
</tr>
</tbody>
</table>

Table 6
Power of $U_p$ when $\tau_1 = q / m$ and $\tau_2 = \sum_{i=1}^{q} \lambda_i / \text{tr}(\Sigma)$ are fixed and $m = n$

<table>
<thead>
<tr>
<th>$(\tau_1, \tau_2)$</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.250,0.90)</td>
<td>0.000</td>
<td>0.079</td>
<td>0.210</td>
<td>0.524</td>
<td>0.963</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>(0.250,0.75)</td>
<td>0.000</td>
<td>0.059</td>
<td>0.168</td>
<td>0.449</td>
<td>0.928</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>(0.250,0.60)</td>
<td>0.000</td>
<td>0.043</td>
<td>0.091</td>
<td>0.166</td>
<td>0.483</td>
<td>0.964</td>
<td>1.000</td>
</tr>
<tr>
<td>(0.125,0.60)</td>
<td>0.113</td>
<td>0.322</td>
<td>0.806</td>
<td>0.999</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

of $\Sigma$ were 1.25, $p/3$ of them were 1, and $p/3$ of them were 0.75. The simulations with $\tau_1 = 0.125$ had $3p/7$ of the eigenvalues at 1.25, $3p/7$ of them at 0.75, and $p/7$ of them were equal to 1. In all cases, the power increases as $m = n$ increases. As expected, the power approaches 1 at a slower rate when the test is more conservative such as for the case in which $(\tau_1, \tau_2) = (0.25, 0.6)$.

References