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A generalization of Gottlieb polynomials in several variables

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ABSTRACT

Gottlieb polynomials were introduced and investigated in 1938, and then have been cited in several articles. Very recently, Khan and Akhlaq introduced and investigated Gottlieb polynomials in two and three variables to give their generating functions. Subsequently, Khan and Asif investigated the generating functions for the q-analogue of Gottlieb polynomials. In this sequel, by modifying Khan and Akhlaq's method, we show how to generalize the Gottlieb polynomials in m variables to present two generating functions of the generalized Gottlieb polynomials $\varphi_n^m(\cdot)$. Furthermore, it should be noted that, since one of the two generating functions is expressed in terms of the well-developed Lauricella series $F_D^{(m)}[\cdot]$, certain interesting and (potentially) useful identities for $\varphi_n^m(\cdot)$ and its reducible cases are shown to be easily found.

1. Introduction and preliminaries

Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. They are used in finding certain properties and formulas for numbers and polynomials in a wide variety of research subjects, indeed, in modern combinatorics. For a systematic introduction to, and several interesting (and useful) applications of the various methods of obtaining linear, bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of sequences of special functions (and polynomials) in one, two and more variables, among much abundant literature, we refer to the extensive work by Srivastava and Manocha [1]. While concerning some orthogonal polynomials on a finite or enumerable set of points, Gottlieb [2] developed the following interesting polynomials (see also [3,4]; [5, p. 303]; [1, pp. 185–186]):

$$\begin{aligned}\varphi_n(x;\lambda) &:= \mathrm{e}^{-n\lambda}\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\left(1-\mathrm{e}^{\lambda}\right)^k \\ &= \mathrm{e}^{-n\lambda}{}_2F_1\left(-n,-x;1;1-\mathrm{e}^{\lambda}\right),\end{aligned} \tag{1.1}$$

where ${}_2F_1$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of p numerator and q denominator parameters ($p, q \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, and $\mathbb{N}$ the set of positive integers) is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\quad z\right] &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z).\end{aligned} \tag{1.2}$$

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Here $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$
$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \tag{1.3}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function.

Gottlieb [2] presented many interesting identities for his polynomials $\varphi_n(x;\lambda)$, which is denoted by $l_n(x)$ in [2], including the following two generating functions (see also [3,4]; [5, p. 303]; [1, pp. 185–186]):

$$\sum_{n=0}^{\infty} \varphi_n(x;\lambda) t^n = (1-t)^x \left(1-t\mathrm{e}^{-\lambda}\right)^{-x-1} \quad (|t|<1); \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \varphi_n(x;\lambda) t^n = \left(1-t\mathrm{e}^{-\lambda}\right)^{-\mu} {}_2F_1\left[\begin{matrix} \mu, -x; \\ 1; \end{matrix} \quad \frac{\left(1-\mathrm{e}^{-\lambda}\right)t}{1-t\mathrm{e}^{-\lambda}}\right]. \tag{1.5}$$

Recently Khan and Akhlaq [3] introduced and investigated Gottlieb polynomials in two and three variables to give their generating functions. Subsequently, Khan and Asif [4] investigated the generating functions for the q-analogue of Gottlieb polynomials. In this sequel, by modifying Khan and Akhlaq's method [3], we show how to generalize the Gottlieb polynomials in several variables to present two generating functions of the generalized Gottlieb polynomials. Furthermore, it should be noted that, since one of the two generating functions is expressed in terms of the well-developed Lauricella series $F_D^{(m)}[\cdot]$ defined by (2.8), certain interesting and (potentially) useful identities for the $\varphi_n^m(\cdot)$ and its reducible cases can be easily found, for example, see Eq. (2.10).

2. Generalized Gottlieb polynomials and their generating functions

Here, we introduce a several variable analogue of the Gottlieb polynomials $\varphi_n(x;\lambda)$ to present their generating functions. To do this, we begin by defining a several variable analogue of the Gottlieb polynomials $\varphi_n(x;\lambda)$ as follows.

Definition. An extension of the Gottlieb polynomials $\varphi_n(x;\lambda)$ in m variables is defined by

$$\varphi_n^m(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m) = \exp(-n\sigma_m) \sum_{r_1=0}^{n} \sum_{r_2=0}^{n-r_1} \sum_{r_3=0}^{n-r_1-r_2} \cdots \sum_{r_m=0}^{n-r_1-r_2-\cdots-r_{m-1}}$$
$$\times \frac{(-n)_{\delta_m} \cdot \prod_{j=1}^{m} (-x_j)_{r_j} \cdot \prod_{j=1}^{m} \left(1-\mathrm{e}^{\lambda_j}\right)^{r_j}}{\prod_{j=1}^{m} r_j! \cdot \delta_m!} \quad (n, m \in \mathbb{N}), \tag{2.1}$$

where, for convenience,

$$\sigma_m := \sum_{j=1}^{m} \lambda_j \quad \text{and} \quad \delta_m := \sum_{j=1}^{m} r_j. \tag{2.2}$$

It is noted that the special case $m=1$ of (2.1) reduces immediately to the second one of the Gottlieb polynomials $\varphi_n(x;\lambda)$ in (1.1).

Now, we will present two generating functions for $\varphi_n^m(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m)$.

Theorem 1. *The following generating function for $\varphi_n^m(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m)$ holds true:*

$$\sum_{n=0}^{\infty} \varphi_n^m(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m) t^n = \left(1-t\mathrm{e}^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m} x_j\right)-1} \cdot \prod_{j=1}^{m} \left(1-t\mathrm{e}^{-\lambda_j-\sigma_m}\right)^{x_j}, \tag{2.3}$$

where $m \in \mathbb{N}$ and σ_m is given in (2.2).

Proof. We begin by recalling a formal manipulation of a double series:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n+k}. \tag{2.4}$$

Let L_n^m be the left hand side of (2.3). If we apply (2.4) to the expression of taking $\sum_{n=0}^{\infty}$ on the right hand side of (2.1), and then multiplying by t^n with $k = r_1$, we find

$$L_n^m = \sum_{r_1=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r_2=0}^{n}\sum_{r_3=0}^{n-r_2}\cdots\sum_{r_m=0}^{n-r_2-\cdots-r_{m-1}} \exp\left[-(n+r_1)\sigma_m\right]\cdot\frac{(-n-r_1)_{\delta_m}\cdot\prod_{j=1}^{m}(-x_j)_{r_j}\cdot\prod_{j=1}^{m}\left(1-\mathrm{e}^{\lambda_j}\right)^{r_j}\cdot t^{n+r_1}}{\prod_{j=1}^{m}r_j!\cdot\delta_m!}.$$

If we apply (2.4) to the resulting identity with $k = r_2$, we obtain

$$\begin{aligned} L_n^m &= \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r_3=0}^{n}\cdots\sum_{r_m=0}^{n-r_3-\cdots-r_{m-1}} \exp\left[-(n+r_1+r_2)\sigma_m\right] \\ &\quad\times\frac{(-n-r_1-r_2)_{\delta_m}\cdot\prod_{j=1}^{m}(-x_j)_{r_j}\cdot\prod_{j=1}^{m}\left(1-\mathrm{e}^{\lambda_j}\right)^{r_j}\cdot t^{n+r_1+r_2}}{\prod_{j=1}^{m}r_j!\cdot\delta_m!}. \end{aligned}$$

By making a repeated application of (2.4) to the consecutive resulting equations with $k = r_3, \ldots, r_m$, we finally have

$$L_n^m = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}\sum_{n=0}^{\infty}\exp\left[-(n+\delta_m)\sigma_m\right]\cdot\frac{(-n-\delta_m)_{\delta_m}\cdot\prod_{j=1}^{m}(-x_j)_{r_j}\cdot\prod_{j=1}^{m}\left(1-\mathrm{e}^{\lambda_j}\right)^{r_j}\cdot t^{n+\delta_m}}{\prod_{j=1}^{m}r_j!\cdot\delta_m!}.$$

If we apply the following identity

$$(-n-s)_s = (-1)^s\,\frac{(n+s)!}{n!} = \frac{(-1)^s s!(s+1)_n}{n!} \quad (n, s \in \mathbb{N}_0) \tag{2.5}$$

to the first factor of the denominator of the last fraction, we get

$$L_n^m = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}(-1)^{\delta_m}\prod_{j=1}^{m}(-x_j)_{r_j}\cdot\prod_{j=1}^{m}\left(1-\mathrm{e}^{\lambda_j}\right)^{r_j}\exp\left(-\delta_m\sigma_m\right)\,\frac{1}{\prod_{j=1}^{m}r_j!}\cdot t^{\delta_m}\sum_{n=0}^{\infty}\frac{(\delta_m+1)_n}{n!}\left(t\mathrm{e}^{-\sigma_m}\right)^n,$$

where δ_m is given in (2.2).

By applying the generalized binomial theorem

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n \quad (|z| < 1) \tag{2.6}$$

to the last resulting equation, we obtain

$$\begin{aligned} L_n^m &= \left(1-t\mathrm{e}^{-\sigma_m}\right)^{-1}\cdot\prod_{k=1}^{m}\left\{\sum_{r_k=0}^{\infty}\frac{(-x_k)_{r_k}}{r_k!}\left(\frac{-t\left(1-\mathrm{e}^{\lambda_k}\right)\mathrm{e}^{-\sigma_m}}{1-t\mathrm{e}^{-\sigma_m}}\right)^{r_k}\right\} \\ &= \left(1-t\mathrm{e}^{-\sigma_m}\right)^{-1}\prod_{k=1}^{m}\left(\frac{1-t\mathrm{e}^{-\lambda_k-\sigma_m}}{1-t\mathrm{e}^{-\sigma_m}}\right)^{x_k}, \end{aligned}$$

which is easily seen to be equal to the right hand side of (2.3). This completes the proof. □

Theorem 2. *The following generating function for $\varphi_n^m(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m)$ holds true:*

$$\begin{aligned} \sum_{n=0}^{\infty}(\mu)_n\varphi_n^m(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m)\frac{t^n}{n!} &= \left(1-t\mathrm{e}^{-\sigma_m}\right)^{-\mu} \\ &\quad\times F_D^{(m)}\left[\mu, -x_1, \ldots, -x_m; 1; \frac{t\left(\mathrm{e}^{\lambda_1}-1\right)\mathrm{e}^{-\sigma_m}}{1-t\mathrm{e}^{-\sigma_m}}, \ldots, \frac{t\left(\mathrm{e}^{\lambda_m}-1\right)\mathrm{e}^{-\sigma_m}}{1-t\mathrm{e}^{-\sigma_m}}\right], \end{aligned} \tag{2.7}$$

where $F_D^{(m)}[\cdot]$ denotes one of the Lauricella series in m variables (see [6, p. 33, Eq. (4)]) defined by

$$F_D^{(m)}[a, b_1, \ldots, b_m; c; x_1, \ldots, x_m] = \sum_{r_1=0,\ldots,r_m=0}^{\infty} \frac{(a)_{\delta_m}(b_1)_{r_1}\cdots(b_m)_{r_m}}{(c)_{\delta_m}} \frac{x_1^{r_1}}{r_1!}\cdots\frac{x_m^{r_m}}{r_m!} \ (\max\{|x_1|, \ldots, |x_m|\} < 1), \tag{2.8}$$

and σ_m, δ_m are given in (2.2).

Proof. The same argument as in the proof of Theorem 1 will establish Theorem 2. □

We conclude this paper by giving some comments on the results in Theorems 1 and 2.

Remark. (a) It is easily seen that Eq. (2.3) in Theorem 1 and Eq. (2.7) in Theorem 2 when $m = 1$ reduce immediately to Eqs. (1.4) and (1.5), respectively.

(b) Lauricella [7] generalized the four Appell series F_j ($j = 1, 2, 3, 4$) (see [6, pp. 22–23]) to the series in m variables and defined his multiple hypergeometric series as $F_A^{(m)}[\cdot]$, $F_B^{(m)}[\cdot]$, $F_C^{(m)}[\cdot]$, including $F_D^{(m)}[\cdot]$ in (2.8). Lauricella [7] presented several elementary properties of these series including, for example, integral representations of the Eulerian type, transformations and reducible cases, and the systems of partial differential equations associated with them. A summary of Lauricella's work is given by Appell and Kampé de Fériet [8, Chapter VII, pp. 114–120]. A result of particular interest is the following reduction formula:

$$F_D^{(m)}[a, b_1, \ldots, b_m; c; x, \ldots, x] = {}_2F_1(a, b_1 + \cdots + b_m; c; x), \tag{2.9}$$

which is due to Lauricella himself [7, p. 150] (see also [8, p. 116, Eq. (11)] and [6, p. 34, Eq. (6)]). Using Eq. (2.9), the following interesting special case of (2.8) is easily seen to be given as

$$\sum_{n=0}^{\infty}(\mu)_n \varphi_n^m(x_1, x_2, \ldots, x_m; \lambda, \ldots, \lambda)\frac{t^n}{n!} = \left(1 - t\mathrm{e}^{-m\lambda}\right)^{-\mu} {}_2F_1\left(\mu, -\sum_{j=1}^{m} x_j; 1; \frac{t\left(\mathrm{e}^{\lambda}-1\right)}{\mathrm{e}^{m\lambda}-t}\right). \tag{2.10}$$

For another example, since $F_D^{(2)} = F_1$ one of the Appell series (see [6, p. 33, Eq. (5)]), the left hand side of Eq. (2.7) when $m = 2$ can be easily expressed in terms of the Appell series F_1.

(c) We may find certain other interesting and (potentially) useful identities for $\varphi_n^m(\cdot)$ by using already-developed formulas and properties for the $F_D^{(m)}[\cdot]$ and its reducible cases, as noted in (b).

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