Obstructions to Homotopy Equivalences

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An obstruction theory is developed to decide when an isomorphism of rational cohomology can be realized by a rational homotopy equivalence (either between rationally nilpotent spaces, or between commutative graded differential algebras). This is used to show that a cohomology isomorphism can be so realized whenever it can be realized over some field extension (a result obtained independently by Sullivan).

In particular an algorithmic method is given to decide when a c.g.d.a. has the same homotopy type as its cohomology (the c.g.d.a. is called formal in this case).

The chief technique is the construction of a canonically filtered model for a commutative graded differential algebra (over a field of characteristic zero) by perturbing the minimal model for the cohomology algebra. This filtered model is also used to give a simple construction of the Eilenberg-Moore spectral sequence arising from the bar construction. An example is given of a c.g.d.a. whose Eilenberg-Moore sequence collapses, yet which is not formal.

I. INTRODUCTION AND TOPOLOGICAL THEOREMS

A central problem in algebraic topology is to consider two path-connected topological spaces $S$ and $T$ with isomorphic integral homology, and ask

(1.1) When can a given isomorphism $H_\ast(S; \mathbb{Z}) \to H_\ast(T; \mathbb{Z})$ be realized by a continuous map from $S$ to $T$?

(Throughout this paper, homology and cohomology of spaces is singular.)

If $S$ and $T$ were simply connected CW-complexes, a solution to (1.1) would be a homotopy equivalence between $S$ and $T$; hence a method for solving (1.1) would enable us to decide if $S$ and $T$ had the same homotopy type. But it would yield more, since it would decide which isomorphisms of homology were "geometric."
One of the main results of this paper is to answer the analogue of (1.1) in the rational homotopy category; viz.,

(1.2) When can a given isomorphism \( f: H^*(S; Q) \cong H^*(T; Q) \) of rational cohomology algebras be realized by a rational homotopy equivalence between \( S \) and \( T \)?

We use the following definition of rational homotopy equivalence (cf. Quillen [27], Bousfield and Gugenheim [5]). Call an elementary equivalence between topological spaces \( S \) and \( T \) a continuous map either from \( S \) to \( T \) or from \( T \) to \( S \) which induces an isomorphism of rational cohomology.

Then define a rational homotopy equivalence between \( S \) and \( T \) to be a sequence of elementary equivalences \( S_i \approx S_{i+1} \) such that \( S_0 = S \) and \( S_{n+1} = T \). (If \( S \) and \( T \) are nilpotent spaces, this is the same as requiring the localizations \( S_Q \) and \( T_Q \) to have the same homotopy type—cf. [27, 29; Chap. III, 7, 22].)

Each rational homotopy equivalence induces a well defined isomorphism \( H^*(S; Q) \cong H^*(T; Q) \); the isomorphism is said to be realized by the equivalence. If there is at least one rational homotopy equivalence between \( S \) and \( T \), they are said to have the same rational homotopy type.

We shall answer (1.2) only for rationally nilpotent spaces. A space \( S \) is called rationally nilpotent if its Sullivan minimal model (see below and Section 2) has only finitely many generators in each degree. Such spaces have finite dimensional rational homology in each degree. On the other hand, nilpotent path connected spaces with finite dimensional rational homology in each degree are rationally nilpotent.

To solve (1.2), we construct a "sequence of obstructions" \( O_n(f) \) which are computable, and prove

1.3. Theorem. Assume \( S \) and \( T \) are rationally nilpotent. The isomorphism \( f \) can be realized by a rational homotopy equivalence if and only if the obstructions \( O_n(f) \) all vanish.

Now Sullivan's theory of minimal models, as described in his fundamental paper [30], gives a complete irredundant algebraic invariant for rational homotopy type: rationally nilpotent spaces have the same rational homotopy type if and only if their minimal models are isomorphic (cf. [30, p. 290]).

But if \( H^*(S; Q) \) and \( H^*(T; Q) \) are known a priori to be isomorphic then Theorem 1.3 provides a computational method for deciding whether or not \( S \) and \( T \) have the same rational homotopy type, and the calculation is frequently much faster than a direct attempt to determine if the minimal models are isomorphic (cf. Example 6.5).

In addition, the theorem enables us to decide which (if any) cohomology isomorphisms are geometric (again cf. Example 6.5).

Next observe that in the category of commutative graded differential algebras (c.g.d.a.'s) over a field \( k \) of characteristic zero, there is a completely analogous
notion of homotopy equivalence (again cf. Quillen [27]). If \((A, d_A)\) and \((B, d_B)\) are c.g.d.a.'s, call an elementary equivalence between them a homomorphism of c.g.d.a.'s (in either direction) inducing an isomorphism \(H(A) \approx H(B)\). Then a homotopy equivalence between \((A, d_A)\) and \((B, d_B)\) is a sequence of elementary equivalences \((A_i, d_i) \sim (A_{i+1}, d_{i+1})\), \(0 \leq i \leq n\), where \((A_0, d_0) = (A, d_A)\) and \((A_{n+1}, d_{n+1}) = (B, d_B)\). If there is at least one homotopy equivalence, \((A, d_A)\) and \((B, d_B)\) have the same homotopy type. (A homotopy equivalence is called a c-equivalence in [9; Sect. 0.10]).

A homotopy equivalence induces a well defined isomorphism \(H(A) \approx H(B)\), which is said to be realized by the equivalence. The isomorphism realized by an elementary equivalence \(\phi: (A, d_A) \to (B, d_B)\) is just \(\phi^*\).

Now consider a fixed isomorphism \(f: H(A) \cong H(B)\), and assume that \(H(A)\) is connected and has finite type.

5.10. Theorem. There is a sequence of computable obstructions, \(O_n(f)\), which vanish if and only if \(f\) can be realized by a homotopy equivalence.

We shall use the results of Sullivan (see [5] for detailed proofs) to show that Theorem 5.10 implies Theorem 1.3. In [30; p. 298] he defines a contravariant functor (which we denote by \(A_{PL}\)) from simplicial sets ([21]) to c.g.d.a.'s defined over \(Q\): if \(K\) is a simplicial set, an element of \(A_{PL}(K)\) is a function \(\Phi\) which assigns to each element of \(K^n (n = 0, 1, 2,...)\) a polynomial \(p\)-form on the standard \(n\)-simplex, such that \(\Phi\) commutes with the face and degeneracy operators. In [30; p. 300] he defines the adjoint contravariant functor \(A \sim A_{PL}\) from c.g.d.a.'s to simplicial sets.

If \(S\) is a topological space, let \(Sing S\) be the simplicial set of singular simplices on \(S\); we shall denote \(A_{PL}(Sing S)\) simply by \((A_{PL}(S), d_s)\). Integration provides a natural isomorphism \(\Pi(A_{PL}(S), d_s) \approx H^*(S; Q)\) (singular cohomology) cf. [30, p. 297]. It follows that \(A_{PL}\) converts a rational homotopy equivalence into a homotopy equivalence of c.g.d.a.'s; the induced isomorphisms of cohomology are identified by integration.

On the other hand assume \(S\) and \(T\) are rationally nilpotent spaces and an isomorphism \(f: H^*(S; Q) \leftrightarrow H^*(T; Q)\) can be realized by a homotopy equivalence between \(A_{PL}(T)\) and \(A_{PL}(S)\). By Proposition 2.9 it can then be realized by an equivalence of the form

\[
A_{PL}(S) \leftarrow m_S M_S \leftarrow M_T \rightarrow A_{PL}(T),
\]

where \(M_s\) and \(M_T\) are the minimal models.

The model homomorphisms \(m_s, m_T\) define simplicial maps \(Sing S \rightarrow \langle M_s \rangle\), \(Sing T \rightarrow \langle M_T \rangle\). It follows from [5; Section 10] that

\[
S \leftarrow Sing S \rightarrow \langle M_s \rangle \leftarrow \left(\langle M_T \rangle \right) \leftrightarrow Sing T \rightarrow T
\]

is a rational homotopy equivalence inducing \(f\). We have thus proved
1.4. **Proposition.** Let $S$ and $T$ be rationally nilpotent spaces. An isomorphism $f : H^*(S; \mathbb{Q}) \cong H^*(T; \mathbb{Q})$ can be realized by a rational homotopy equivalence if and only if the corresponding isomorphism $H(\text{APL}(S)) \cong H(\text{APL}(T))$ can be realized by a homotopy equivalence of c.g.d.a.'s.

In particular, Theorem 5.10 implies Theorem 1.3.

Theorem 5.10 also permits an effective study of formality. A c.g.d.a. $(A, d)$ is called formal ([30; p. 315], [6]; c-split in the terminology of [13; Section 0.10]) if it has the same homotopy type as $(H(A), 0)$. Because every automorphism of $H(A)$ is a homotopy equivalence of $(H(A), 0)$, $(A, d)$ is formal if and only if the identity isomorphism $H(A, d) = H(H(A), 0)$ can be realized as a homotopy equivalence between $(A, d)$ and $(H(A), 0)$. Thus Theorem 5.10 yields a sequence of obstructions which vanish if and only if $(A, d)$ is formal (cf. 6.1).

Notice that while for most c.g.d.a.'s, their homotopy type is a much finer invariant than their cohomology, formal c.g.d.a.'s are those whose "homotopy type is a formal consequence of their cohomology" [30; Section 12].

A topological space $S$ is called formal if $(\text{APL}(S), d_S)$ is formal; formal spaces abound in topology. They include spheres, projective spaces, Lie groups, symmetric spaces and indeed a wide class of homogeneous spaces (cf. [13; Theorem IV, Chap. XI] for a complete description). Compact Kahler manifolds are formal ([30; Section 12]). A bouquet of formal spaces is formal (Lemma 1.6 below). Finally, if for some $l > 1$, $H^p(S; \mathbb{Q}) = 0$ unless $p = 0$ or $l < p < 3l + 2$, then $S$ is formal (Corollary 5.16).

1.5. **Theorem.** Let $S$ be a rationally nilpotent space. Assume that $H^{2p}(S; \mathbb{Q}) = 0$, $p = 1, 2, \ldots$.

Then $S$ is formal, and has the rational homotopy type of a bouquet of odd spheres.

(This result is originally but independently due to Baues [1], cf. [33].)

**Proof.** Choose odd spheres $S^{8i}$ so that $H^*(S; \mathbb{Q}) \cong H^*(\bigvee_i S^{8i})$. By Lemma 1.6, $\bigvee_i S^{8i}$ is formal, while in Theorem 7.10 we show that $S$ is formal. Hence $\text{APL}(S)$ and $\text{APL}(\bigvee_i S^{8i})$ have the same homotopy type. Hence by Proposition 1.4 so do $S$ and $\bigvee_i S^{8i}$. Q.E.D.

Next, suppose $\{S_a\}_{a \in \mathcal{T}}$ is a collection of path connected, pointed topological spaces such that each inclusion $* \hookrightarrow S_a$ is a cofibration. Their bouquet $\bigvee_a S_a$ is the space obtained by identifying the base points.

1.6. **Lemma.** If each $S_a$ is formal, then so is $\bigvee_a S_a$.

**Proof.** For augmented c.g.d.a.'s $A_a \leftarrow k$, define $\bigvee A_a$ to be $k \oplus \prod_a \ker(A_a \rightarrow k)$. For path connected pointed spaces with $* \hookrightarrow S_a$ a cofibration, there is an equivalence $\text{APL}(\bigvee S_a) \rightarrow \bigvee \text{APL}(S_a)$. The minimal model for $\text{APL}(S_a)$ is (among
other things) a connected c.g.d.a. \((M_d, d_d)\) together with an equivalence \(m_\alpha: (M_d, d_d) \to (A_{PL}(S_d), d_{S_d})\) [30; p. 288]. Because \((A_{PL}(S_d), d_{S_d})\) is formal, there is also an equivalence \(n_\alpha: (M_d, d_d) \to (H(S_d), 0)\) (cf. Proposition 2.5). It follows that we have equivalences \(\forall A_{PL}(S_d) \leftarrow \forall M_d \to \forall H(S_d) = H(\forall S_d)\).

Q.E.D.

We also apply Theorem 5.10 to obtain an elementary proof of a theorem discovered independently by Sullivan [30; remark in Section 12] and proved by him via the theory of algebraic groups and torsors. The theorem reads

**6.8. Theorem.** Suppose \((A, d_A)\) and \((B, d_B)\) are c.g.d.a.'s defined over \(k\) with connected cohomology algebras of finite type. Let \(K\) be an extension field of \(k\).

Assume \(f: H(A) \to H(B)\) is an isomorphism such that \(f \otimes 1: H(A \otimes K) \to H(B \otimes K)\) can be realized by a homotopy equivalence between \((A \otimes K, d_A \otimes 1)\) and \((B \otimes K, d_B \otimes 1)\). Then \(f\) can be realized by a homotopy equivalence between \((A, d_A)\) and \((B, d_B)\).

**6.9. Corollary.** \((A, d_A)\) is formal if and only if \((A \otimes K, d_A \otimes 1)\) is formal.

Theorem 6.8 shows that the rational homotopy type of a \(C^\infty\)-manifold \(M\) is determined by the homotopy type of the c.g.d.a. \((A(M), d_M)\) of \(C^\infty\) differential forms on \(M\), together with the isomorphism \(H(A(M)) \cong H^*(M; \mathbb{Q}) \otimes R\), (cf. [30; Section 13]). The corollary, together with the principal theorem of [10], implies that compact Kähler manifolds are formal (cf. [30; Section 12]).

This paper is organized as follows: In Section 2 we recall definitions and collect together (without proof) the results from Sullivan’s theory of minimal models which will be needed in the sequel. (A minimal model of a c.g.d.a. \((A, d_A)\) is an elementary equivalence \((AX, D) \to (A, d_A)\) where \(AX\) is the free commutative graded algebra over a graded vector space \(X\) and \((AX, D)\) satisfies a nilpotence condition and a minimality condition, cf. 2.2 and 2.8 for a precise statement.)

In Section 3 we give a construction of the minimal model of a c.g.d.a. of the form \((H, 0)\) which exhibits it in the form \(\rho: (AZ, d) \to (H, 0)\) where \(Z\) carries a canonical second gradation. The first two steps of the construction, influenced by J. C. Moore, are implicit in an article of Tate [31; Section 6] who obtains there (Theorem 6) a special case of our Theorem 7.14.

The dimensions of the bigraded components of \(Z\) determine the dimensions of the spaces \(H^p\) (when \(H\) has finite type) via a “Hirsch formula” (Proposition 3.10); this generalizes a result of Koszul [18].

In Section 4 we construct a canonical filtered model \(\pi: (AZ, D) \to (A, d_A)\) for a \(c\)-connected c.g.d.a. by perturbing the differential in the bigraded model \((AZ, d)\) for \((H(A), 0)\). This is in the spirit of Gugenheim et al [14, 15] that differential homological algebra is obtained from homological algebra by perturbing bigraded objects to graded filtered objects. This also identifies the
set of homotopy types with the set of homology preserving deformations of \((AZ, d)\).

In general \((AZ, D)\) satisfies the nilpotence condition, but not always the minimality condition for a minimal model. Thus projecting \(A^+Z\) onto \(Z\) (with kernel \(A^+Z \cdot A^+Z\)) yields a differential \(D_\ast\) in \(Z\). \(D_\ast\) is zero if and only if \((AZ, D)\) is minimal. The filtered differential space \((Z, D_\ast)\) yields a spectral sequence converging form \(Z\) to \(H(Z, D_\ast)\).

If \((A, d_A) = (A_{PL}(S), d_s)\) where \(S\) is a 1-connected C.W. complex and \(H^\ast(S; \mathbb{Q})\) has finite type, then (cf. [30; Section 8]) we can write \(H(Z, D_\ast) = \text{space of generators for minimal model} = \text{Hom}_Z(\pi_\ast(S); \mathbb{Q})\). In this case our spectral sequence appears to be dual to a spectral sequence of Bousfield and Kan [6], established by them in a far more general setting, but traceable through Quillen [27] to Curtis [34].

In Section 5 we build the obstruction theory, using filtered models, and prove Theorem 5.10. Section 6 contains the applications for formality and field extensions.

In Section 7 we show that the Eilenberg–Moore spectral sequence arising from the bar construction for a c.g.d.a. can be computed directly from its filtered model, without passing to the bar construction, and that the cohomology of the construction can be calculated (trivially) from any nilpotent model. In particular, we show that the Eilenberg–Moore sequence collapses if and only if the filtered model is minimal.

If \((A, d_A) = (A_{PL}(S), d_s)\), where \(S\) is a 1-connected C.W. complex, then the bar construction has cohomology equal to the cohomology of the loop space \(ΩS\). This was shown (dually for the cobar construction and homology) by Adams [1] and generalized by Eilenberg and Moore [12]. A complete exposition for cohomology is given by Smith [28; part I]—except that for our purposes \(C^\ast\) has to be replaced by \(A_{PL}(S)\) everywhere in his paper.

(Historically the bar construction was invented for graded differential algebras where the differential had degree \(-1\), e.g. chains on an associative \(H\)-space. It was used to calculate the homology of \(K(\pi, n)\)'s and more general classifying spaces, cf. Eilenberg–Maclane [11; Chap. II], Cartan [9], Maclane [19; p. 306–p. 315]).

Finally, in Section 8 we discuss spherical cohomology and use examples to show that neither of the implications:

\[(A, d_A) \text{ formal} \Rightarrow \text{E.M.s.s. collapses} \Rightarrow H(A) \text{ spherically generated}\]

can be reversed. We end with the

**Problem.** Construct a filtered model for a homomorphism (Sullivan constructs the minimal model for a homomorphism) which will give a compatible version of the general Eilenberg–Moore spectral sequence.

This has been solved by Micheline Vigué-Poirier in her thesis (Lille, 1978).
2. Algebraic Preliminaries and Minimal Models

2.1. Definition and Notation. Our field \( k \) will always be of characteristic zero and all vector spaces and algebras will be considered to be defined over \( k \) unless we explicitly say otherwise.

The identity map of any set will be denoted by \( 1 \).

A graded space \( X = \sum_p X^p \) is said to have finite type if each \( X^p \) has finite dimension. In this case the formal series \( \sum_p (\dim X^p) t^p \) is called the Poincaré series for \( X \).

A commutative graded algebra (c.g.a.) is an associative graded algebra \( H = \sum_{p \geq 0} H^p \) such that \( H^0 \) contains an identity \( 1 \) and such that \( ab = (-1)^{pq} ba \), \( a \in H^p, b \in H^q \). If \( A, B \subset H \) are subsets, \( A \cdot B \) denotes the subspace spanned by the elements \( ab, a \in A, b \in B \). \( H \) is connected if \( H^0 = k \cdot 1 \). We denote \( \sum_{p \geq 0} H^p \) by \( H^+ \). A derivation \( \Theta \) of degree \( p \) in \( H \) is a linear map of degree \( p \) such that \( \Theta(ab) = \Theta(a) \cdot b + (-1)^{pq} a \cdot \Theta(b) \).

If \( Z = \sum_{p \geq 0} Z^p \) is a graded space, we denote by \( \Lambda Z \) the free c.g.a. generated by \( Z \); it is the tensor product of the exterior algebra over \( Z^{\text{odd}} \) and the symmetric algebra over \( Z^{\text{even}} \) and is graded by the rule

\[
\deg(z_1 \wedge \cdots \wedge z_n) = \sum_{i=1}^n \deg z_i.
\]

Note that \( \"\Lambda\" \) denotes free, and not (as is often the case) exterior.

If \( Z \) is given a second gradation, we use exactly the same rule to extend the gradation to \( \Lambda Z \).

A commutative graded differential algebra (c.g.d.a.) is a pair \((A, d_A)\), where \( A = \sum_{p \geq 0} A^p \) is a c.g.a., \( d \) is a derivation of degree \( 1 \), and \( d^2 = 0 \). The cohomology of \((A, d_A)\) is the c.g.a. \( H(A) \) given by \( H(A) = \ker d / \text{im} d \); if \( H(A) \) is connected, \((A, d_A)\) is called \( c \)-connected. A homomorphism \( \varphi : (A, d_A) \to (B, d_B) \) is a degree zero homomorphism of algebras preserving the identity such that \( \varphi d_A = d_B \varphi \). It induces a homomorphism \( \varphi^* : H(A) \to H(B) \). Note that \( (\varphi \psi)^* = \varphi^* \psi^* \).

An augmented c.g.d.a. is a c.g.d.a. \((A, d_A)\) together with a homomorphism \( \varepsilon : (A, d_A) \to (k, 0) \).

If \( z \in (A, d_A) \) satisfies \( d_A z = 0 \) (resp. \( z = d_A a \)), then \( z \) is called a cocycle (resp. a coboundary). The element in \( H(A) \) represented by a cocycle \( z \) is called its cohomology class and denoted by \( \text{cl}(z) \).

2.2. Minimal models. We collect some results of Sullivan [30], although our terminology is different. A detailed exposition is given in [16]. A connected Koszul–Sullivan (K–S) complex is a c.g.d.a. of the form \((\Lambda X, D)\) where \( X = \sum_{p \geq 0} X^p \) is a strictly positive graded space and \( D \) satisfies the condition:

\[
(2.3) \quad (\text{"Nilpotence"}) \quad \text{There is a homogeneous basis} \{x_n\}_{n \in \mathbb{N}} \text{ for} \ X, \text{ indexed}
\]
by a well ordered set \( \zeta \), such that \( D x_a \) is a polynomial in those \( x_\alpha \) with \( \beta < \alpha \). The \( K-S \) complex is called minimal if it satisfies

\[
(2.4) \text{ (Minimality) } D(X) \subset (A^+X) \cdot (A^+X).
\]

Given a connected \( K-S \) complex \((AX, D)\), we denote by \((AX, D)'\) the c.g.d.a. \((AX \otimes AX \otimes AX, D)\) where

(i) \( \tilde{D} \mid AX = D \).

(ii) \( \bar{X} \) is the graded space defined by \( \bar{X}^p = X^{p+1} \). (We denote this isomorphism by \( \bar{x} \leftrightarrow x \).)

(iii) \( \hat{X} \) is a graded space isomorphic with \( X \). (We denote the isomorphism by \( \hat{x} \leftrightarrow x \).)

(iv) \( \hat{D}\bar{x} = \hat{x} ; \hat{D}\hat{x} = 0 \).

These conditions uniquely determine \( \tilde{D} \).

A derivation \( i \) of degree \(-1\) in \((AX, D)'\) is given by \( ix = \bar{x} , ixo = 0 = i\hat{x} , \ x \in X \). If we set \( \gamma = \hat{D}i + i\hat{D} \) then an automorphism \( \alpha \) of the c.g.d.a. \((AX, D)'\) is defined by

\[
\alpha(a) = \sum_{0}^{\infty} \frac{1}{n!} \gamma^n(a).
\]

(The right hand side is always a finite sum!) Let \( \lambda_0 : (AX, D) \rightarrow (AX, D)' \) be the inclusion. Set \( \lambda_1 = \alpha\lambda_0 \).

Now let \( \varphi_0 : (AX, D) \rightarrow (A, d_A) \) be a homomorphism of c.g.d.a.'s. A homotopy starting at \( \varphi_0 \) is a homomorphism of c.g.d.a.'s \( \Phi : (AX, D)' \rightarrow (A, d_A) \) such that \( \Phi \mid AX = \varphi_0 \). The homotopy is said to end at the homomorphism \( \varphi_1 : (AX, D) \rightarrow (A, d_A) \) defined by

\[
\varphi_1 a = \Phi\alpha(a) , \quad a \in AX ,
\]

and \( \varphi_0 \) and \( \varphi_1 \) are called homotopic. We write \( \varphi_0 \simeq \varphi_1 \). Note that \( \Phi \circ \lambda_i = \varphi_i \).

In [30; Corollary 3.4] it is shown that homotopy is an equivalence relation.

Next suppose \( \varphi : (A, d_A) \rightarrow (B, d_B) \) and \( \psi : (AX, D) \rightarrow (B, d_B) \) are homomorphisms of c.g.d.a.'s such that \( \varphi^* \) is an isomorphism, and \((AX, D)\) is connected and satisfies condition (2.3), then [30; Corollary 3.6] can be stated as

2.5. Proposition. There is a homomorphism \( \chi : (AX, D) \rightarrow (A, d_A) \), uniquely determined up to homotopy, such that \( \varphi \circ \chi \simeq \psi \).

Finally, [30; Theorem 5.1] reads

2.6. Theorem. Let \((A, d_A)\) be a c-connected c.g.d.a. Then there is a minimal
connected $K$–$S$ complex $(M_A, \delta_A)$ and a homomorphism $m_A^* : (M_A, \delta_A) \rightarrow (A, d_A)$ such that $m_A^*$ is an isomorphism.

If $m'_A : (M'_A, \delta'_A) \rightarrow (A, d_A)$ is another such, then there is an isomorphism $\varphi : (M'_A, \delta'_A) \cong (M_A, \delta_A)$, uniquely determined up to homotopy, such that $m'_A \circ \varphi \cong m_A$.

2.7. Corollary. A homomorphism between minimal connected $K$–$S$ complexes is an isomorphism if and only if it induces an isomorphism of cohomology.

2.8. Definition. $m_A : (M_A, \delta_A) \rightarrow (A, d_A)$ is called the minimal model for $(A, d_A)$.

2.9. Homotopy equivalences. Let

$$m_A : (M_A, \delta_A) \rightarrow (A, d_A) \quad \text{and} \quad m_B : (M_B, \delta_B) \rightarrow (B, d_B)$$

be the minimal models of $c$-connected c.g.d.a.'s $A$ and $B$. A special homotopy equivalence between $(A, d_A)$ and $(B, d_B)$ is a sequence of elementary equivalences of the form $m_A, \varphi, m_B$ where

$$\varphi : (M_A, \delta_A) \cong (M_B, \delta_B).$$

(cf. Section 1). The induced isomorphism is $m_B^* \circ \varphi^* \circ (m_A^*)^{-1}$.

It follows at once from Proposition 2.5 and Corollary 2.7 that an elementary equivalence $(A, d_A) \rightarrow (B, d_B)$ determines a special equivalence $m_A, \varphi, m_B$, where $m_B \circ \varphi \cong \psi \circ m_A$. Since $\varphi$ is an isomorphism, $m_B, \varphi^{-1}, m_A$ is also a special equivalence. This implies that any homotopy equivalence between $(A, d_A)$ and $(B, d_B)$ determines (uniquely up to homotopy) a special equivalence which realizes the same isomorphism of cohomology. In other words we have

2.10. Proposition. An isomorphism $f : H(A) \cong H(B)$ can be realized by a homotopy equivalence if and only if there is an isomorphism $\varphi : (M_A, \delta_A) \cong (M_B, \delta_B)$ such that $f = m_B^* \circ \varphi^* \circ (m_A^*)^{-1}$.

Finally we have for $c$-connected c.g.d.a.’s $(A, d_A)$, $(B, d_B)$ and $(C, d_C)$:

2.11. Proposition. (i) If $f : H(A) \cong H(B)$ and $g : H(B) \cong H(C)$ are isomorphisms of cohomology realizable by homotopy equivalences, then $g \circ f$ and $f^{-1}$ are realizable.

(ii) If $G(A, d_A)$ denotes the group of automorphisms of $H(A)$ realizable by homotopy equivalences and $f : H(A) \cong H(B)$ is realizable, then a group isomorphism $G(A, d_A) \cong G(B, d_B)$ is given by $g \mapsto f \circ g \circ f^{-1}$. 
3. The Bigraded Model for a C.G.A.

3.1. Construction. Let $H$ be a connected c.g.a. (later to be thought of as $H(A)$). We can regard $H$ as a c.g.d.a. by setting $d = 0$. As such we know it has a minimal model

$$\rho: (AZ, d) \to (H, 0).$$

In fact $(AZ, d)$ is a purely algebraic construct closely related to a resolution of $H$ by free commutative algebras, sometimes called the Tate resolution [31], (and extended to the graded case by Jozefia [35] as we learned after this paper was accepted). We construct $(AZ, d)$ from this point of view. As pointed out by the referee, the analogous construction resolving $H$ by free associative algebras is due to Lemaire [36, pp. 78-79]. In this construction we exhibit a second natural gradation carried by $AZ$, to be called the lower gradation. (The lower gradation appears in another form in [30; proof of Theorem 12.7] and dually for coalgebras in work of J. C. Moore [23, 24] and in the associative setting in Lemaire.)

In fact we construct graded spaces $Z_0, Z_1, Z_2, \ldots$ so that $Z = \sum_{n=0}^{\infty} Z_n$ and $d$ is homogeneous of lower degree $-1$ with respect to the induced lower gradation of $AZ$. The map $\rho$ will be defined on $Z_0$ (below) and extended to be zero on $Z_n$, $n \geq 1$.

We shall write

$$Z_{(n)} = Z_0 \oplus \cdots \oplus Z_n$$

then each $AZ_{(n)}$ will be $d$ stable, and the lower gradation of $AZ_{(n)}$ will induce a lower gradation in $H(AZ_{(n)}, d)$:

$$H(AZ_{(n)}, d) = \sum_{p \geq 0, k \geq 0} H_k^n(AZ_{(n)}, d).$$

The $Z_n$'s and $\rho$ and $d$ will be constructed so the following conditions hold:

(3.2)$_0$ $\rho: AZ_0 \to H$ is surjective.
(3.2)$_1$ $\rho^*: H_0(AZ_0, d) \to H$.
(3.2)$_n$ $\rho^*: H_0(AZ_n, d) \to H$, and $H_i(AZ_n, d) = 0$, $1 \leq i < n$, $n \geq 2$.

It will be convenient to write at times

$$Z_n^\rho := Z_{n-p-n}^\rho \quad \text{and} \quad (AZ)^\rho_n := (AZ)_{n-p-n}^\rho.$$

Now we give the construction of the $Z_n$, $\rho$ and $d$, and then prove uniqueness.
The space $Z_0$. Set $Z_0 = H^+/H+ \cdot H+; \text{it is the space of indecomposables for } H.$ Set $d = 0$ in $Z_0$. Define $\rho: AZ_0 \to H$ so its restriction to $Z_0$ splits the projection $H^+ \to Z_0,$ thus giving "a space of generators $Z_0$ for $H."$ Then $\rho$ is surjective; denote its kernel by $K.$ Clearly $K^0 = K^1 = 0.$

The space $Z_1$. Set $Z_1 = K/(K \cdot A^+Z_0)$ with a shift downward by one of degrees:

$$Z_1^p = (K/K \cdot A^+Z_0)^{p-1}.$$

$Z_1$ is the "space of generators for the relations in $H."$ Since $K^0 = K^1 = 0,$

$$Z_1 = \sum_{p \geq 1} Z_1^p.$$

Extend $d$ to $Z_1$ (and hence to $AZ(\Omega)$) by requiring that it be a linear map $Z_1 \to K$ splitting the projection. Then $d: Z_1 \to \Lambda Z_0$ and so $d$ is homogeneous of lower degree $-1$ in $AZ(\Omega).$ Extend $\rho$ to be zero on $Z_1.$ Clearly (3.2), holds. We have now to kill off the higher $H_i's.$

The spaces $Z_n$. Suppose $(AZ(\Omega), d)$ has been constructed for some $n > 1$ so that $d$ is homogeneous of lower degree $-1.$ Define $Z_{n+1}$ by

$$Z_{n+1}^p = (H_n(AZ(\Omega), d)/H_n(AZ(\Omega), d) \cdot H_0^+(AZ(\Omega), d))^{p+1}$$

and extend $d$ so that $d: Z_{n+1} \to (AZ(\Omega))_n \cap \ker d$ splits the projection of $(AZ(\Omega))_n \cap \ker d$ onto $Z_{n+1}.$ Extend $\rho$ to be zero on $Z_{n+1}.$

Now let $(AZ, d) \to^\rho (H, 0)$ be the homomorphism of c.g.d.a.'s constructed in this way, with $Z = \sum_{n=0}^\infty Z_n.$

3.4. Proposition. The c.g.d.a. $(AZ, d)$ has the following properties:

(i) $\rho^*: H_0(AZ, d) \xrightarrow{\sim} H.$

(ii) $H_*(AZ, d) = 0.$

(iii) $(AZ, d) \to^\rho (H, 0)$ is the minimal model.

Moreover, suppose $(C, d_\cdot) \to^\pi (H, 0)$ is a homomorphism of c.g.d.a.'s such that $C = \sum_{p \geq 0, n \geq 0} C_n^p$ is a bigraded algebra, $d_\cdot$ is homogeneous of lower degree $-1,$ $\pi(C_\cdot) = 0,$ and conditions (i) (iii) hold; then there is an isomorphism

$$\varphi: (AZ, d) \xrightarrow{\sim} (C, d_\cdot)$$

homogeneous of bidegree zero, and such that $\pi \varphi = \rho.$

3.5. Definition. $\rho: (AZ, d) \to (H, 0)$ is called the bigraded model for $H.$

Before proving the Proposition, we establish some preliminaries.

3.6. Lemma. (i) $(AZ)_n \cap \ker d \subset A^+Z \cdot A^+Z, n \geq 1.$

(ii) $Z_n = \sum_{p \geq 1} Z_n^p, n \geq 0.$

(iii) Conditions (3.2), hold, $n \geq 0.$
Proof. (i) We prove this for $n \geq 2$. (The case $n = 1$ is proved by a similar argument, which is left to the reader.) Suppose $n \geq 2$, let $u \in (AZ)_n \cap \ker d$, and write

$$u = u_0 + \sum_{i=1}^{m} u_i v_i + w,$$

where $u_i \in Z_n$ ($0 \leq i \leq m$), $v_i \in A^+ Z_0$ ($1 \leq i \leq m$), and $w \in (AZ)_+ \cdot (AZ)_+$. Then, in particular, $w \in A Z_{(n-1)}$.

Apply $d$ to obtain

$$0 = du_0 + \sum_{i=1}^{m} (du_i) v_i + dw.$$

Thus $du_0$ represents an element in $H_{n-1}(AZ_{(n-1)}) \cdot H_0^+(AZ_{(n-1)})$. Hence by definition $u_0 = 0$, and $u \in A^+ Z \cdot A^+ Z$.

(ii) This is already done for $n = 0, 1$. We induct on $n$. Assume it holds for some $n \geq 1$ and apply part (i) of the lemma to obtain

$$(AZ)_n \cap \ker d \subseteq (AZ)^+ \cdot (AZ)^+ \subseteq \sum_{p \geq 0} (AZ)^p.$$

Lowering degrees by one yields $Z_{n+1} = \sum_{p \geq 1} Z_{n+1}^p$.

(iii) We have already verified (3.2)$_0$ and (3.2)$_1$. Conditions (3.2)$_n$ ($n \geq 2$) follow from the definition of $d$ and (ii). Q.E.D.

3.7. Proof of Proposition 3.4. (i) and (ii) follow from part (iii) of the lemma. Since $d$ is homogeneous of lower degree $-1$, condition (2.3) is satisfied. Part (i) of the lemma implies that $(AZ, d)$ is minimal.

To prove uniqueness, note that because $\pi^*: H_0(C) \to H$, there is a linear map $\varphi: Z_0 \to C_0$ with $\varphi = \rho$. Extend $\varphi$ to $AZ_0$. Then $\varphi \cdot \rho = \rho$ on $AZ_0$ hence

$$\varphi(dZ_1) \subseteq \varphi(ker \rho) \subseteq ker \pi.$$

Since $\pi^*$ is an isomorphism, $\varphi$ extends to a linear map $Z_1 \to C_1$ so that $\varphi dz = d_\varphi \varphi z$, $z \in Z_1$. Extend $\varphi$ to $AZ_1$, and continue in this way to get a homomorphism $\varphi: (AZ, d) \to (C, d_C)$, homogeneous of bidegree zero. Since $\pi(C_+) = 0$ we have at once that $\varphi \cdot \rho = \rho$. Since $\pi^*$ and $\rho^*$ are isomorphisms, so is $\varphi^*$. But because $(AZ, d)$ and $(C, d_C)$ are both minimal models, this implies that $\varphi$ is an isomorphism (cf. Theorem 2.6). Q.E.D.

3.8. Remarks. (1) $\cdots \to^d (AZ)_{n+1} \to^d (AZ)_n \to^d \cdots \to^d (AZ)_0 \to^d H$ is a resolution of $H$ by free $AZ_0$-modules (note $(AZ)_0 = AZ_0^!$). Thus it may be used to calculate $\text{Tor}_{AZ_0}(H, \cdot)$. 


(2) $H$ is finitely generated as an algebra if and only if each $Z_n$ has finite dimension. Indeed, since $\rho(Z_0)$ generates $H$, if $\dim Z_0 < \infty$ then $H$ is finitely generated. Conversely, if $H$ is finitely generated, then $\dim Z_0 < \infty$, and so $\ker \rho$, as an ideal in the noetherian algebra $AZ_0$, is finitely generated. Hence $\dim Z_1 < \infty$.

If $\dim Z_{(n)} < \infty$, then each $(AZ_{(n)})_l$ is a finitely generated $AZ_0$-module, and so the same holds for $H_l(AZ_{(n)})$. In particular, $Z_{n+1}$, being the generating space for $H_n(AZ_{(n)})$, is finite dimensional. Thus $\dim Z_n < \infty$, all $n$.

(3) $H$ has finite type (as a graded vector space) if and only if each $Z_k$ has finite type (by the same argument as in the previous remark).

(4) If $H^p = 0$, $1 \leq p \leq l$ and $H^0 = k$, then

$$Z_{n}^{p} = 0, \quad 0 \leq p \leq (n + 1) l.$$ 

Indeed, this is obvious for $n = 0$. Assume it is proved for $Z_k$, $k \leq n$. Then clearly $(AZ_{(n)})_l = 0, 1 \leq p \leq (i + 1) l$. Hence

$$[A^+ Z_{(n)} \cdot A^+ Z_{(n)}]_l = 0, \quad p < (n + 2) l \leq 2.$$

But by Lemma 3.6(i), $d(Z_{n+1}) \subseteq A^+ Z_{(n)} \cdot A^+ Z_{(n)}$. Since $d$ has degree 1, this yields $Z_{n+1} = 0, 0 \leq p \leq (n + 2) l$.

3.9. Poincaré series. Suppose that $H$ has finite type. We shall show how to calculate $\dim H^p$ from the integers $\dim Z_n^p$.

Recall from [13; p. 671] that the Poincaré-Koszul series of a bigraded space $C := \sum_{n \geq 0, \, m \geq 0} C_{n,m}$ (with $\dim C_{n,m} < \infty$) is given by

$$U_C(t) = \sum_{p=0}^{\infty} \left( \sum_{n=0}^{p} (-1)^n \dim C_{n,m}^{p-a} \right) t^n.$$

Moreover (cf. [13; Lemma III, p. 671]) $U_{C \oplus C'} = U_C + U_{C'}$ and $U_{C \otimes C'} = U_C U_{C'}$. If $C = A(x)$ with $x \in C_{n,m}$ ($C$ is exterior or symmetric as $p$ is odd or even) then

$$U_C = (1 - (-t)^{p+n})^{(-1)^{p+1}}.$$

Now write $\dim Z_{n,m}^{p} = b_{n,m}^{p}$. Then

$$U_{Z_{n,m}}^{p} = \begin{cases} (1 - (-t)^{p+n})^{b_{n,m}^{p}}, & p \text{ odd} \\ (1 - (-t)^{p+n})^{-b_{n,m}^{p}}, & p \text{ even.} \end{cases}$$
It follows that

\[ U_{AZ} = \frac{\prod_{n=0}^{\infty} \prod_{p=0}^{\infty} (1 - (-t)^{2p+1+n})^{b^p_{n+1}}}{\prod_{n=0}^{\infty} \prod_{p=1}^{\infty} (1 - (-t)^{2p+n})^{b^p_{n}}} \]

On the other hand, because \( d \) is homogeneous of degree 1 and lower degree -1, the spaces \( \sum_n (AZ)^{p-n}_n \) are \( d \)-stable, for each \( p \), and

\[ \sum_{n=0}^{p} (-1)^n \dim (AZ)^{p-n}_n = \sum_{n=0}^{p} (-1)^n \dim H^{p-n}(AZ, d) = \dim H^p. \]

We have thus proved

3.10. \textbf{Proposition.} \textit{The Poincaré series for} \( H \) \textit{is given by}

\[ \sum_{p=0}^{\infty} (\dim H^p) t^p = \frac{\prod_{n=0}^{\infty} \prod_{p=0}^{\infty} (1 - (-t)^{2p+n+1})^{b^p_{n+1}}}{\prod_{n=0}^{\infty} \prod_{p=1}^{\infty} (1 - (-t)^{2p+n})^{b^p_{n}}} \]

3.11. \textbf{Remark.} Proposition 3.10 generalizes the "Hirsch formula" of Cartan-Koszul (cf. [8, 18, 13; Corollary III, p. 69]) for the cohomology of certain homogeneous spaces. Moreover the argument above is essentially the same as their proof of the special case.

As noted by the referee, Poincaré series of a related nature appear in Lemaire [36, (A.2.4), p. 130] and in Quillen [37, (11.3), p. 86].

4. \textbf{The Filtered Model for a c.g.d.a.}

4.1. \textbf{Introduction.} Let \( (A, d_A) \) be a \( c \)-connected c.g.d.a. We shall construct a canonical "filtered model" for \( (A, d_A) \) by perturbing the bigraded model \( (AZ, d) \rightarrow (H(A), 0) \) of Section 3.

Filter \( AZ \) by

\[ F_n(AZ) = \sum_{m \leq n} (AZ)_m \quad n = \ldots, -2, -1, 0, 1, \ldots \]

This is an increasing filtration with \( F_n(AZ) = 0 \) for \( n \lesssim -1 \).

A linear map \( \varphi: AZ \rightarrow AZ \) will be called \textit{filtration decreasing} if

\[ \varphi(F_n(AZ)) \subseteq F_{n-1}(AZ) \quad \text{for each} \ n. \]

If \( \varphi \) is a derivation, this is equivalent to

\[ \varphi(Z_n) \subseteq F_{n-1}(AZ) \]
Our idea is to model \((A, d_A)\) by a c.g.d.a. \((AZ, D)\) such that

\[(D - d): Z_n \to F_{n-2}(AZ),\]

so that \(D\) is genuinely a perturbation of \(d\) in the sense of Gugenheim et al. \[14, 15\]. Said differently, \(D\) can be written as \(D = d_1 + d_2 + d_3 + \cdots\), where \(d_i \mid Z_n \subseteq (AZ)_{n-i}\). Before proceeding with the general theory, we give an example of such a perturbation.

**4.3. Example.** Consider the algebra \(H = A(x_1, x_2, x_3, x_4)/I\), where \(|x_1| = |x_2| = |x_3| = 3\), \(|x_4| = 5\), and \(I\) is the ideal generated by \(x_1x_2, x_1x_3x_4, x_2x_3x_4\). (\(|x|\) denotes the degree of \(x\).) Let \((AZ, d) \to^o (H, 0)\) be the bigraded model.

A simple calculation shows that \(Z_0\), \(Z_1\) and \(Z_2\) have bases given respectively by:

\[Z_0: x_1, x_2, x_3, x_4; \quad Z_1: y_1, y_2, y_3; \quad Z_2: z_1, \ldots, z_{10},\]

with

\[

dy_1 = x_1x_2 \quad \text{and} \quad dy_2 = x_1x_3x_4 \\
zy_1 = y_1x_2 \quad \text{and} \quad dy_3 = x_2x_3x_4
\]

We now perturb \((AZ, d)\) to a c.g.d.a. \((AZ, D)\) so that \(D - d: Z_n \to F_{n-2}(AZ)\), all \(n\). Indeed we set \(D = d\) on \(Z_0\) and on \(Z_1\) and define \(D\) on \(Z_2\) by

\[
Dz_1 = y_1x_1 + z_2x_4 \quad \text{and} \quad Dz_i = dz_i, \quad i \geq 2.
\]

Then \(D^2 = 0\) on \(AZ_{(2)}\).

We then use induction on \(n\) to show that \(D\) can be extended to all of \(AZ\). Suppose \(D\) has been extended to \(Z_{(n)}\), some \(n \geq 2\). For \(u \in Z_{n-1}\),
$D(du) \in F_{n-2}(AZ)$. Hence Lemma 4.5 below, applied to the $D$-cocycle $D(du)$ in $(AZ, D)$, shows that

$$D(du) = Dv + \eta(\alpha)$$

for some $\alpha \in H$ and some $v \in F_{n-1}(AZ)$, where $v$ may be chosen to depend linearly on $u$. (The map $\eta : H(A) \to AZ_0$ satisfies $\rho \eta = \iota$.)

Since $H^1 = H^2 = 0$, Remark 3.8.4 applies, and shows that $Z^n = 0$, $p \leq 2(n + 2)$. Since $n + 1 \geq 3$, we have $|u| \geq 9$ and so $|\alpha| = |Ddu| \geq 11$. Thus $H$ contains only elements of degrees 0, 3, 5, 6 and 8; hence $\alpha = 0$.

Now we have $D(du - v) = 0$ and so we may extend $D$ to $Z_{n+1}$ by setting $D(u) = du - v$, so that $D^2 = 0$ and $(D - d) : Z_{n+1} \to F_{n-1}(AZ)$.

Next observe that $Dv$ is a $D$-cocycle of degree 10 such that $Dv - x_1 x_1 \in F_3(AZ)$. The elements of degree 10 in $F_3(AZ)$ have $z_4, z_5, z_6$ as basis; hence

$$Dv = z_4 x_1 + \lambda_1 z_5 x_4 + \lambda_2 z_6 x_2 + \lambda_3 z_6 x_3 x_4.$$ 

Thus

$$0 = D^2 = (y_1 x_1 + x_3 x_4) x_1 + \lambda_1 x_1 x_2 x_4 + \lambda_2 x_2 x_3 x_4 + \lambda_3 x_2 x_3 x_4.$$ 

It follows that $\lambda_1 = \lambda_2 = 0$ and $\lambda_2 = -1$; i.e.,

$$Dv = z_4 x_1 - y_2.$$ 

This equation shows that the perturbation $(AZ, D)$ is no longer minimal: $D(Z) \notin A^+ Z \cdot A^+ Z$. Hence it is not possibly isomorphic with $(AZ, d)$.

We now return to the general theory of 4.1.

4.4. Theorem. Let $(A, d_A)$ be a $c$-connected c.g.d.a. and let $p : (AZ, d) \to (H(A), 0)$ be the bigraded model for $H(A)$. Then there is a c.g.d.a. $(AZ, D)$ and a homomorphism $\pi : (AZ, D) \to (A, d_A)$ such that

- $(E_1) \quad (D - d) : Z_n \to F_{n-2}(AZ), n \geq 0.$
- $(E_2) \quad \text{cl}(\pi z) = \rho z, z \in AZ_0.$ (Note that $(E_1)$ implies that $Dz = 0$.)
- $(E_3) \quad \pi^* \text{ is an isomorphism}.$

Moreover, suppose $\pi' : (AZ, D') \to (A, d_A)$ also satisfies these conditions. Then there is an isomorphism $\varphi : (AZ, D) \cong (AZ, D')$ such that

- $(U_1) \quad (\varphi - \iota)$ is filtration decreasing,
- $(U_2) \quad \pi' \varphi \cong \pi : (AZ, D) \to (A, d_A).$
4.5. Lemma. Let \((AZ, d) \to (H, 0)\) be the bigraded model for a connected c.g.a. \(H\), and let \(\eta: H \to AZ_0\) be a linear map such that \(p\eta = c\). Suppose \((AZ(n), D)\) is a c.g.d.a. such that \((D - d): Z_1 \to F_{i-2}(AZ)_l, 0 \leq l \leq n\).

Assume \(u \in F_{n-1}(AZ)\) satisfies \(Du = 0\). Then for some \(v \in F_{n}(AZ)\) and some \(\alpha \in H\), we have

\[ u = Dv + \eta(\alpha). \]

Remark. If \(H = k\), this is the usual statement of acyclicity.

Proof. By induction on \(n\). If \(n = 1\), then \(u \in AZ_0\) and we can write \(u = dv + \eta(\alpha) = Dv + \eta(\alpha)\) as follows from (3.2).

Suppose the lemma holds for \(n - 1\) \(\geq 1\). Write \(u = \sum_{j=0}^{n-1} u_j, u_j \in (AZ)_j\).

Then \(du_{n-1} = 0\). Hence by formula (3.2), \(u_{n-1} = dv_{n-1}\), some \(v_{n-1} \in (AZ)_n\).

Applying the induction hypothesis to the \(D\)-cocycle

\[ u - Dv_n \in F_{n-1}(AZ) \]

we obtain \(v' \in F_{n-1}(AZ), \alpha \in H(A)\) so that

\[ u - Dv_n = Dv' + \eta(\alpha). \]

Now set \(v = v_n + v'\) to obtain \(v\) as asserted.

Q.E.D.

4.6. Proof of Theorem 4.4. Existence. Fix a linear map \(\eta: H(A) \to AZ_0\) such that \(p\eta = c\). We define \(D\) and \(\pi\) inductively on \(Z_0, Z_1, \ldots\). The argument is a modification of the usual comparison of resolutions construction. Set \(D = 0\) in \(AZ_0\). Define \(\pi\) on \(Z_0\) so that \(d_A\pi(Z_0) = 0\) and so \(cl(\pi z) = p\xi, \xi \in Z_0\).

Then extend \(\pi\) to a homomorphism \(\pi: (AZ_0, D) \to (A, d_A)\); it necessarily satisfies \(E_2\).

Set \(D = d\) on \(Z_1\) (and hence on \(AZ_{(1)}\)). Then for \(z \in Z_1,\)

\[ cl(\pi Dz) = cl(\pi dz) = \rho dz = 0. \]

Thus \(\pi\) extends to a degree zero linear map \(\pi: Z_1 \to A\) such that \(d_A\pi z = \pi dz, z \in Z_1\). This defines a homomorphism \(\pi: (AZ_{(1)}, D) \to (A, d_A)\).

Suppose \(z \in Z_2\). Then \(dz \in AZ_{(1)}\), and so \(D dz - D^2 z = 0\). Hence \(d_A\pi dz = 0\).

Extend \(D\) to \(Z_2\) by setting

\[ Dz = dz - \eta(cl(\pi dz)), z \in Z_2, \]

and extend this \(D\) to a derivation (of degree one) in \(AZ_{(2)}\). Since \(D^2 z = d Dz = 0, z \in Z_2\), we have \(D^2 = 0\).
Since $\text{cl}(\pi \alpha) = \rho \eta \alpha = \alpha$, $\alpha \in H(A)$, we have

$$\text{cl}(\pi Dz) = \text{cl}(\pi dx) - \text{cl}(\pi \eta(\text{cl} \pi dx)) = 0, \quad z \in Z_2.$$

Thus we can extend $\pi$ to a degree zero linear map $Z_0 \to A$ so that $d_\pi z = \pi Dz$, $z \in Z_2$. $\pi$ then extends to a homomorphism $\pi: (AZ_0, D) \to (A, d_A)$.

Next, suppose $D$ and $\pi$ have been extended to $AZ_n$ (some $n \geq 2$) so that $E_1$ holds. For $z \in Z_{n+1}$, $D dz$ is a $D$-cocycle in $F_{n-2}(AZ)$. Hence by Lemma 4.5 we can find $w \in F_{n-2}(AZ)$ and $\alpha \in H(A)$ so that

$$D(dz) = Dw + \eta(\alpha).$$

Applying $\pi$ yields $d_\pi \pi dz = d_\pi \pi w + \pi \eta \alpha$ or

$$0 = \text{cl}(\pi \eta(\alpha)) = \alpha;$$

i.e., $D(dz - w) = 0$.

Evidently $w$ may be chosen to depend linearly on $z$. Extend $D$ to $Z_{n+1}$ by setting

$$Dz = dz - w - \eta(\text{cl} \pi(dx - w)), \quad z \in Z_{n+1}.$$

Then $\text{cl}(\pi Dz) = 0$, and so we may extend $\pi$ to a homomorphism $\pi: (AZ_{n+1}, D) \to (A, d_A)$.

This completes the construction of $(AZ, D)$ and of $\pi$. $E_1$ and $E_2$ are satisfied by definition. Lemma 4.5 implies that $\eta^*: H(A) \to H(AZ, D)$ is surjective. Since $E_2$ implies that $\pi^* \eta^* = 1$, $\eta^*$ must be an isomorphism and $\pi^*$ the inverse isomorphism. This proves $E_3$.

**Uniqueness.** Recall from Section 2 that a homotopy starting at $\pi$: $(AZ, D) \to (A, d_A)$ is a homomorphism

$$\Phi: (AZ \otimes AZ \otimes A DZ, D) \to (A, d_A)$$

such that $\Phi | AZ = \pi$. The correspondence $\Phi \to \Phi | Z$ defines a bijection between homotopies starting at $\pi$ and degree zero linear maps $Z \to A$.

We must therefore construct $\varphi: (AZ, D) \to (AZ, D')$ to satisfy $U_1$ ($\varphi$ is then automatically an isomorphism) and a linear map $\Phi: Z \to A$ whose unique extension to a homotopy $\Phi$ starting at $\pi$ satisfies $\Phi \circ \lambda_1 = \pi' \circ \varphi$.

Before beginning the construction we note (as immediate from the definitions) that the linear map $\theta: Z \to AZ \otimes AZ \otimes A DZ$ defined by

$$\lambda_1 z - z = Dz + \theta(x), \quad z \in Z \tag{4.7}$$
satisfies $\theta(Z_n) \subset AZ_{(n-1)} \otimes AZ_{(n-1)} \otimes ADZ_{(n-1)}$, $n \geq 0$. Applying $D$, we find
\[
\lambda_1 Dz - Dz \to D\theta(z), \quad z \in Z.
\] (4.8)

Continue to fix $\eta: H(A) \to AZ_0$ as at the start of the existence proof. We construct $p$ and $\Phi$ inductively. Set $\Phi = \pi$ on $AZ$. Set $\eta = \iota$ on $Z_0$. Then for $z \in Z_0$, $cl(\pi'\varphi z - \pi z) = g \varphi z - g z = 0$. Extend $\Phi$ to $AZ \otimes AZ_0 \otimes ADZ_0$ so that $d_\Phi \varphi z = \pi'\varphi z - \pi z$, $z \in Z_0$. Then $\Phi_1(z) = \pi'\varphi(z)$, $z \in Z_0$.

Next, let $z \in Z_1$. Then since $D = D' = d$ on $Z_1$,
\[
d\pi'z - \pi z - \Phi\theta(z) = \pi' D'z - \pi Dz - \Phi D\theta(z)
\] (4.8)
as follows from (4.8). Thus we may write
\[
\pi' z - \pi z - \Phi\theta(z) = \pi'\eta(\alpha) + d\omega, \quad z \in Z_1, \quad \text{where } \alpha \in H(A) \quad (4.9)
\]
and $\omega \in A$ depends linearly on $z$.

Now extend $\varphi$ to $AZ_{(1)}$ and $\Phi$ to $AZ \otimes AZ_{(1)} \otimes ADZ_{(1)}$ by setting $\varphi z = z - \eta(\alpha)$ and $\Phi z = \omega$ and of course $\Phi D\varphi z = d\omega$, $z \in Z_1$. Then $D'\varphi z = d(z - \varphi(\alpha)) = \varphi Dz$, while
\[
(\Phi \circ \lambda_1) z = \Phi z + \Phi D\varphi z + \Phi\theta(z) \quad \text{by (4.7)}
\] (4.7)
\[
= \pi z + d\omega + \Phi\theta(z)
\] (4.7)
\[
= \pi'\varphi z, \quad \text{by (4.9), } \quad z \in Z_1.
\]

Finally, suppose $\varphi$ is extended to $AZ_{(n)}$ and $\Phi$ is extended to $AZ \otimes AZ_{(n)} \otimes ADZ_{(n)}$, some $n \geq 1$. For $z \in Z_{n+1}$,
\[
D'\varphi Dz = \varphi D^2z = 0.
\]

Thus $\varphi Dz - D'z$ is a $D'$ cocycle in $F_{n-1}(AZ)$.

Hence Lemma 4.5 shows that for some $\nu \in F_n(AZ)$ and some $\beta \in H(A)$,
\[
\varphi Dz = D'z + D'\nu + \eta(\beta)
\]
where $\nu$ can be chosen to depend linearly on $z \in Z_{n+1}$. Applying $\pi'$ at this level where $\pi' = \Phi \circ \lambda_1$ already, we obtain
\[
\pi'\eta(\beta) + d\lambda(\pi'z + \pi'\nu) = \pi'\varphi Dz - \Phi\lambda_1 Dz
\] (4.8)
\[
= \Phi Dz + \Phi D\theta(z) \quad \text{by (4.8)}
\] (4.8)
\[
= d\lambda\pi z + d\Phi\theta(z).
\]
It follows that \( \beta = \text{cl}(\pi'\gamma(\beta)) = 0 \); hence

\[ \varphi Dz = D'(z + v), \]

and

\[ d_A(\pi'z + \pi'v - \pi z - \Phi \theta(z)) = 0, \quad z \in Z_{n+1}. \quad (4.10) \]

Thus we may write

\[ \pi'z + \pi'v - \pi z - \Phi \theta(z) = \pi'\gamma(x) + d_Aw, \]

where \( x \in H(A) \) and \( w \in A \) depend linearly on \( z \in Z_{n+1} \).

Now extend \( \varphi \) to \( AZ_{(n+1)} \) and extend \( \Phi \) to \( AZ \otimes AZ_{(n+1)} \otimes A DZ_{(n+1)} \) by setting

\[ \varphi z = z + v - \gamma(x), \quad \Phi z = w, \quad \text{and} \quad \Phi Dz = d_Aw, \quad z \in Z_{n+1}. \]

It is easy to verify that \( \varphi D = D'\varphi \) and \( \Phi \circ \lambda_1 = \pi' \circ \varphi \). This completes the inductive step, and the proof of the theorem. Q.E.D.

4.11. Definition. \( (AZ, D) \rightarrow (A, d_A) \) is called the filtered model for \( (A, d_A) \). (The differential \( D \) respects filtration but is not in general bihomogeneous.)

4.12. Remarks. (1) Condition \( E_1 \) shows that \( D(AZ_0) = 0 \), while \( d((AZ)_1) = d((AZ)_1) \). Thus the inclusion \( \lambda: AZ_0 \rightarrow AZ \) induces a homomorphism \( \lambda^*: \Lambda Z_0 \rightarrow \Lambda Z_0/\text{d)((AZ)_1} \rightarrow H(AZ, D) \). Condition \( E_2 \) implies that the diagram

\[
\begin{array}{ccc}
(AZ_0)/d((AZ)_1) & \rightarrow & H(AZ, D) \\
\lambda^* \downarrow & & \pi^* \\
H(A) & \rightarrow & H(A)
\end{array}
\]

commutes. Hence \( \lambda^* \) is an isomorphism.

(2) Condition \( E_1 \) implies that \( (AZ, D) \) satisfies condition (2.3). It need not, however, be minimal as is shown in Example 4.3. Indeed it is minimal if and only if the Eilenberg–Moore spectral sequence for \( (A, d_A) \) collapses, as we show in Theorem 7.16.

4.14. The homotopy spectral sequence. The usual conventions for diagramming a spectral sequence associated to a bigraded object assume the total differential lowers (or raises) both indices. To conform to this convention and since our underlying topology leads us to consider cohomology, when we filter
(AZ, D) and related objects we will define $Z^p$ to have total degree $p$ and define $E_r^{p,q}$ so $E_0^{n,p-q}(AZ) = (AZ)_n^p$ when we use filtration (4.1). Then the $E_1$ term of the spectral sequence is $(AZ, d)$, and so the $E_2$-term is given by

$$E_2^{p,q} = H^p(A), \quad E_2^{*,*} = 0$$

(cf. formula (3.2)). Thus the spectral sequence collapses at the $E_2$-term.

On the other hand the projection $\zeta: A^+Z \rightarrow Z$ with kernel $A^+Z \rightarrow Z$ determines a differential $D_\zeta$ in $Z$ via $D_\zeta \circ \zeta = \zeta \circ D$. Filtering $Z$ by the spaces $F_n = \sum_{m<n} Z_m$, we obtain, with the convention above, a second quadrant spectral sequence. Since $(AZ, d)$ is minimal (Proposition 3.4), $\zeta \circ d = 0$. Thus the $E_2$-term of this spectral sequence is simply $Z$.

According to [30; Section 8], the space of generators for the minimal model of $(A, d_A)$ can be identified with $H(Z, D_\zeta)$ (because $(AZ, D)$ satisfies (2.3)). In particular, $(AZ, D)$ is minimal if and only if $D_\zeta = 0$.

Now the space of generators of the minimal model for $(A, d_A)$ is called the pseudo dual homotopy of $(A, d_A)$ and written $\pi^*_0(A, d_A)$. Thus

$$Z = \pi^*_0(H(A), 0) \quad \text{and} \quad H(Z, D_\zeta) = \pi^*_0(A, d_A).$$

Hence the spectral sequence above starts at $\pi^*_0(H(A), 0)$ and converges to $\pi^*_0(A, d_A)$, collapsing if and only if $(AZ, D)$ is minimal. It will be called the homotopy spectral sequence and will be denoted by $\varepsilon_{i,j}$. [The pseudo and $\psi$ may often be omitted; $\pi^*_0(A_{\rho^j}(S), d_{\rho^j}) \approx \text{Hom}(\pi_*(S), Q)$ if $S$ is a simply connected C.W.-complex of finite type rationally.]

5. Obstruction Theory

5.1. In this section, we consider a fixed isomorphism of graded algebras

$$f: H(A) \rightarrow H(B),$$

where $(A, d_A)$ and $(B, d_B)$ are $c$-connected $c.g.d.a.'s$, and develop an obstruction theory to decide when $f$ is realizable by a homotopy equivalence (Section 1).

Let $\rho_A: (AZ, d) \rightarrow (H(A), 0)$ be the bigraded model (Definition 3.5). Then $f \circ \rho_A: (AZ, d) \rightarrow (H(B), 0)$ satisfies the conditions of Proposition 3.4; hence we may (and do) choose it as the bigraded model for $H(B)$, writing $f \circ \rho_A = \rho_B$. Fix linear maps $\eta_A: H(A) \rightarrow AZ_0$ and $\eta_B: H(B) \rightarrow AZ_0$ (of degree zero) so that $\rho_A \circ \eta_A = \eta$ and $\rho_B \circ \eta_B = \eta$.

These bigraded models may be perturbed (Theorem 4.4) to filtered models for $(A, d_A)$ and $(B, d_B)$; these will be denoted by

$$\pi_A: (AZ, D_A) \rightarrow (A, d_A) \quad \text{and} \quad \pi_B: (AZ, D_B) \rightarrow (B, d_B).$$

(5.2)
All of this notation is now fixed for the rest of the section. In particular, the filtration (4.2) of $AZ$ is the same, whether we work with $(A, d_A)$, $(H(A), 0)$, $(B, d_B)$ or $(H(B), 0)$.

**5.3. Theorem.** $f$ can be realized by a homotopy equivalence if and only if there is an isomorphism $\varphi: (AZ, D_A) \to (AZ, D_B)$ such that $\varphi - \iota$ decreases filtrations.

**Proof.** Given $\varphi$, then the sequence $\pi_A$, $\varphi$, $\pi_B$ is a homotopy equivalence. If $\alpha \in H(A)$ then $\eta_A\alpha \in AZ_0$ has the property that $\text{cl} (\pi_A \eta_A \alpha) = \alpha$ (cf. condition $E_2$ of Theorem 4.4). By hypothesis, $\varphi$ is the identity on $AZ_0$. Hence

$$\pi_B^* \circ \varphi^* \circ (\pi_A^*)^{-1} (\alpha) = \text{cl} \pi_B (\eta_A \alpha) = \rho_B \eta_A \alpha = f \rho_A \eta_A \alpha = f (\alpha).$$

Thus the equivalence $\pi_A$, $\varphi$, $\pi_B$ realizes $f$.

Conversely, suppose $f$ can be realized by a homotopy equivalence. According to Proposition 2.10, there is an isomorphism

$$\psi: (M_A, \delta_A) \to (M_B, \delta_B)$$

between the minimal models, such that $m_B^* \circ \psi^* \circ (m_A^*)^{-1} = f$. Because $(AZ, D_A)$ satisfies (2.3), Proposition 2.5 implies that there is a homomorphism $\gamma: (AZ, D_A) \to (M_A, \delta_A)$ such that $m_A \circ \gamma \simeq \pi_A$.

Now consider $m_B \circ \psi \circ \gamma: (AZ, D_A) \to (B, d_B)$. We have $m_B^* \circ \psi^* \circ \gamma^* = f \circ \pi_A^*$. Hence for $z \in Z_0$, $\text{cl} (m_B \circ \psi \circ \gamma (z)) = f (\text{cl} \pi_A z) = f \circ \rho_A (z) = \rho_B z$. It follows that $m_B \circ \psi \circ \gamma: (AZ, D_A) \to (B, d_B)$ satisfies conditions $E_1$, $E_2$ and $E_3$ of Theorem 4.4. Hence, by that theorem, there is an isomorphism $\varphi: (AZ, D_A) \xrightarrow{\sim} (AZ, D_B)$ such that $\varphi - \iota$ decreases filtrations. Q.E.D.

Theorem 5.3 motivates:

**5.4. Definition.** The isomorphism $f: H(A) \xrightarrow{\sim} H(B)$ is $n$-realizable if there is an isomorphism $\varphi: (AZ_{(n+1)}, D_A) \to (AZ_{(n+1)}, D_B)$ such that $\varphi - \iota$ decreases filtrations. Such a $\varphi$ will be called an $n$-realizer for $f$.

If $\varphi$ is an $n$-realizer for $f$, the degree 1 linear map

$$o(\varphi): Z_{n+2} \to H(B)$$

given by $o(\varphi) z = \text{cl} (\pi_B \varphi D_A z)$, $z \in Z_{n+2}$, will be called the obstruction element determined by $\varphi$. We shall write

$$O_{n+1}(f) = \{o(\varphi) \mid \varphi \text{ is an } n\text{-realizer for } f\}.$$
5.5. Remarks. (1) Theorem 5.3 shows that if $f$ can be realized by a homotopy equivalence, then $f$ is $n$-realizable for all $n$. The converse (although not obvious) is true if $H(A)$ has finite type, as we show in Theorem 5.10.

(2) $f$ is always 0-realizable by the identity map of $AZ_f$, since $D_A = D_B = d$ on $AZ_f$.

(3) $O_{n+1}(f)$ is a subset of $\text{Hom}^1(\mathbb{Z}_{n+2}; H(B))$, where the superscript denotes maps of degree 1. If $H(A)$ has finite type so does each $\mathbb{Z}_k$ (cf. Remark 3.3.4) and we may regard $O_{n+1}(f)$ as a subset of $\sum_p (\mathbb{Z}_p^n)^* \otimes H^{p+1}(B)$. Recall from 4.14 that $Z^p = \pi_p^*(H(A), 0)$, and write $\pi_p^*(H(A)) = \sum_n (\mathbb{Z}_n^p)^*$. Then we have

$$O_{n+1}(f) \subset \sum_{n=0}^{\infty} H^{p+1}(B) \otimes \pi_p^*(H(A)) \subset H^p(B; \pi_p^*(H(A))).$$

(4) Problem: Find a topological description of $n$-realizability.

Let $M_n$ be the space of filtration decreasing derivations, $\theta$, of degree zero in $AZ(n)$ which satisfy $D_B \theta = \theta D_B$. Define a linear map $\gamma: M_n \to \text{Hom}^1(\mathbb{Z}_{n+1}; H(B))$ by

$$\gamma(\theta)(x) = \text{cl}(\pi_B \theta D_B x), \quad x \in \mathbb{Z}_{n+1}, \quad \theta \in M_n.$$ 

The following two propositions set up our obstruction theory.

5.6. Proposition. Suppose $\varphi$ is some $(n - 1)$ realizer for $f$. Then

$$O_n(f) = o(\varphi) + \gamma(M_n).$$

5.7. Proposition. An $(n - 1)$ realizer, $\varphi$, for $f$ extends to an $n$-realizer if and only if $o(\varphi) = 0$.

5.8. Corollary. Assume $f$ is $(n - 1)$-realizeable. Then $f$ is $n$-realizeable if and only if

$$O_n(f) = \gamma(M_n).$$

5.9. Remark. Proposition 5.6 shows that $O_n(f)$ may be regarded as a single element $O_n(f)$ in the quotient space $\text{Hom}^1(\mathbb{Z}_{n+1}; H(B))/\gamma(M_n)$. We call $O_n(f)$ the $n$th obstruction class to realizeability. Corollary 5.8 asserts that $O_n(f) = 0$ if and only if $f$ is $n$-realizable, in which case, of course, $O_{n+1}(f)$ is defined.

We thus have a "sequence of obstructions" to realizeability. Our obstruction theory is completed by
5.10. **Theorem.** Assume $H(A)$ has finite type, then $f$ can be realized by a homotopy equivalence if and only if all the obstruction classes $O_n(f)$ vanish.

We proceed as follows. We first prove Proposition 5.6 and Proposition 5.7. Then, as a corollary, we establish a special case of Theorem 5.10 (Theorem 5.15) in which the conclusion is in fact stronger. Finally, we prove Theorem 5.10.

5.11. **Proof of Proposition 5.6.** The $(n - 1)$ realizers for $f$ are the isomorphisms $(AZ_{(n)} , D_A) \rightarrow (AZ_{(n)} , D_B)$ of the form $\phi_i = \psi \psi_i$, where $\psi$ is an automorphism of $(AZ_{(n)} , D_B)$ such that $\psi - \iota$ decreases filtrations.

Given such a $\psi$ define $\log \psi$ by

$$
(\log \psi)(u) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} (\psi - \iota)^n(u), \quad u \in AZ_{(n)}.
$$


(Because $\psi - \iota$ is filtration decreasing, the right hand side is always a finite sum!) It is easy to check that $\log \psi \in M_n$. Conversely, if $\theta \in M_n$ then $e^\theta = \sum_{n=0}^{\infty} (1/p!) \theta^n$ is an automorphism of $(AZ_{(n)} , D_B)$, and $e^\theta - \iota$ decreases filtrations. Since $e^{\log \psi} = \psi$ and $\log e^\theta = \theta$, it follows that the $(n - 1)$ realizers of $f$ are the isomorphisms of the form $e^\theta \psi$, $\theta \in M_n$.

Thus $O_n(f) = \{ o(e^\theta \psi) \mid \theta \in M_n \}$, and the proposition will follow from the formula

$$
(5.12) \quad o(e^\theta \psi) = o(\psi) + \gamma(\theta), \quad \theta \in M_n.
$$

To prove (5.12), note first that for $z \in Z_{n+1}$, $\varphi D_A z - D_B z$ is a $D_B$-cocycle in $F_{n-1}(AZ)$. Hence by Lemma 4.5

$$
\varphi D_A z - D_B z = D_B w + \eta_\beta(\alpha)
$$

for some $w \in F_n(AZ)$ and $\alpha \in H(B)$. It follows at once from the definition that $\alpha = o(\varphi) z$; thus

$$
(5.13) \quad \varphi D_A z = D_B (z + w) + \eta_\beta(\gamma(\theta) z).
$$

Next observe that $\theta D_B z$ is a $D_B$-cocycle in $F_{n-1}(AZ)$. Hence, again by Lemma 4.5,

$$
\theta D_B z = D_B v + \eta_\beta(\beta)
$$

for some $\beta \in H(B)$ and $v \in F_n(AZ)$. By definition $\beta = \gamma(\theta)(z)$ so

$$
\theta D_B z = D_B v + \eta_\beta(\gamma(\theta) z).
$$
Since $\theta$ decreases filtrations, this equation yields
\[ e^\theta D_B z = D_B z + \sum_{k=0}^\infty \frac{\theta^k}{(k+1)!} \theta D_B z \]
\[ = D_B z + D_B \left( \sum_{k=0}^\infty \frac{\theta^k}{(k+1)!} \right) + \eta_B(\gamma(\theta) z). \]

Substituting this in (5.13), we find
\[ e^\theta \phi D_A z = D_B z + D_R \left( \sum_{k=0}^\infty \frac{\theta^k}{(k+1)!} \right) + \eta_B(\gamma(\theta) z) + D_R e^\theta w + \eta_B(o(\phi) z). \]

It follows at once that
\[ \text{cl}(\pi_B e^\theta \phi D_A z) = o(\phi) z + \gamma(\theta) z, \quad z \in Z_{n+1}, \]
whence (5.12).

5.14. Proof of Proposition 5.7. Suppose $\varphi$ extends to an $n$-realizer, $\bar{\varphi}$. Then for $z \in Z_{n+1}$, $\varphi D_A z = \bar{\varphi} D_A z = D_B \bar{\varphi} z$. Thus $\text{cl}(\pi_B \varphi D_A z) = 0$, $z \in Z_{n+1}$; i.e., $o(\varphi) = 0$.

Conversely, assume $o(\varphi) = 0$. Then (5.13) reads
\[ \varphi D_A z = D_B (z + w), \quad z \in Z_{n+1}, \]
where $w \in F_n(AZ)$ can be made to depend linearly on $z$. Extend $\varphi$ to $\bar{\varphi}: (AZ_{(n+1)} : D_A) \cong (AZ_{(n+1)} : D_H)$ by setting $\bar{\varphi} z = z + w, z \in Z_{n+1}$. Q.E.D.

5.15. Theorem. Assume $H^p(A) = 0$ for $1 \leq p \leq l$ and for $p > m$. Then $f$ is realizable by a homotopy equivalence if and only if
\[ O_n(f) = 0, \quad 1 < n < \frac{m-2}{l} - 2. \]

5.16. Corollary. Let $H$ be a connected c.g.a. such that $H^p = 0$ for $1 \leq p \leq l$ and for $p > 3l + 1$. Suppose $g: H(C) \to H$ is an isomorphism of c.g.a.'s for some c.g.d.a. $(C, d_C)$. Then $g$ can be realized by a homotopy equivalence between $(C, d_C)$ and $(H, 0)$.

In particular, $H$ is realized as the cohomology of one and only one homotopy type.
5.17. Proof of Theorem 5.15. Recall first from Remark 3.8.4 that since $H^p(A) = 0$, $1 \leq p \leq l$,

\[(5.18) \quad Z^p_n = 0, \quad 1 \leq p \leq (n + 1) l.\]

Now if $f$ is realizable, then it is $n$-realizable for all $n$ (Theorem 5.3) and hence $O_n(f) = 0$ for all $n$ (Remark 5.9). Conversely, suppose $O_n(f) = 0$, $n \leq (m - 2)/l - 2$. Then for each such $n$ there is an $n$-realizer. In particular, let $\varphi$ be an $N$-realizer for $f$, where $N$ is the largest integer such that $N \leq (m - 2)/l - 2$.

Then $O(\varphi) \in \text{Hom}^l(Z_{N+1}^n ; H(B))$. But $(N + 1) > (m - 2)/l - 2$, and (5.18) implies that $\text{Hom}^l(Z_{N+1}^n , H(B)) = 0$, $n > (m - 2)/l - 2$, because $H^p(B) \approx H^p(A) = 0$, $p > m$. Hence $O(\varphi) = 0$. Thus Proposition 5.7 shows that $\varphi$ extends to an $(N + 1)$-realizer $\varphi_1$.

Iterating this argument yields a sequence $(\varphi_p)_{p=1,2,\ldots}$ such that $\varphi_p$ is an $(N + p)$-realizer for $f$ extending $\varphi_{p-1}$. Define $\varphi: (AZ, D_A) \to (AZ, D_B)$ by $\varphi(u) = \varphi_p(u)$, $u \in AZ_{(n+p+1)}$. Theorem 5.3 now shows that $f$ can be realized by a homotopy equivalence. Q.E.D.

Notice that Theorem 5.15 asserts that if $H^1(A) = 0$ and $H^p(A) = 0$, $p > m$ (some $m$) then the realizability of $f$ is a finite problem; i.e., there are only finitely many obstructions that have to vanish.

This is not the case in the general situation of Theorem 5.10, to which we now turn our attention.

Before proving the theorem, we establish some notation (always assuming $H(A)$ has finite type). Let $M^p_{n,n+m} \subseteq \text{Hom}^n(Z_{(n)}^{<p}, AZ_{(n)})$ denote the space of those degree zero linear maps $Z_{(n)}^{<p} \to AZ_{(n)}$ which extend to elements of $M^p_{n,n+m}$. Clearly

\[(5.19) \quad M^p_{n,n} \supseteq M^p_{n,n+1} \supseteq \cdots \supseteq M^p_{n,n+m} \supseteq \cdots.\]

Since $H(A)$ has finite type, so does each $Z_n$ (cf. Remark 3.8.3), and so $\dim \text{Hom}^n(Z_{(n)}^{<p}, AZ_{(n)}) < \infty$. Hence each $M^p_{n,n+m}$ has finite dimension. In particular, the sequence (5.19) is finite: for some integer $N(p, n) > n$,

\[M^p_{n,l} = M^p_{n,N(p,n)}, \quad l \geq N(p, n)\]

5.20. Lemma. Suppose $\varphi$ is an $N(p, n)$-realizer for $f$ and that $f$ is $l$-realizable for some $l \geq N(p, n)$. Then $f$ has an $l$-realizer $\bar{\varphi}$ such that

\[\bar{\varphi} |_{\Lambda(Z_{(n)}^{<p})} = \varphi |_{\Lambda(Z_{(n)}^{<p})}.\]

Proof. Denote $N(p, n)$ simply by $N$. Let $\bar{\varphi}$ be an $l$-realizer for $f$; denote
its restriction to $AZ(N)$ by $\psi$. Then $\psi \circ \varphi^{-1}$ is an automorphism of $(AZ(N), D_B)$, and $(\psi \circ \varphi^{-1}) - \iota$ decreases filtrations. Hence (cf. Proof of Proposition 5.6)

$$\psi \circ \varphi^{-1} = e^\theta,$$

some $\theta \in M_N$.

Since (by definition) $M_{n,N}^{m} = M_{n,1}^{m}$, we can find $\bar{\theta} \in M_1$ such that

$$\bar{\theta} |_{Z_{(m)}^{\leq \theta}} = \theta |_{Z_{(m)}^{\leq \theta}}.$$  

(5.21)

Set $\bar{\varphi} = e^{-\bar{\theta}} \varphi$.

Then $\bar{\varphi}$ is an $I$-realizer for $f$. Moreover, on $AZ(N)$ we have

$$\bar{\varphi} = e^{-\bar{\theta}} (\psi \varphi^{-1}) \varphi = e^{-\bar{\theta}} e^{\theta} \varphi.$$  

Now (5.21) implies that $\varphi$ coincides with $\varphi$ on $A(Z_{(n)}^{\infty})$. Q.E.D.

5.22. Proof of Theorem 5.10. Suppose $f$ is realizeable by a homotopy equivalence. Then $f$ is $n$-realizeable for all $n$ and so all the obstruction classes vanish (cf. Remark 5.9).

Conversely if all the obstruction classes vanish then $f$ is $n$-realizeable for all $n$. Let $m_1 \ll m_2 \ll \cdots$ satisfy $m_n \geq N(n, n)$. Suppose $\varphi_n$ is an $m_n$-realizer for $f$.

Since $m_n \geq N(n, n)$ and since $f$ is $m_{n+1}$-realizeable, Lemma 5.20 yields an $m_{n+1}$-realizer, $\varphi_{n+1}$, such that

$$\varphi_{n+1} |_{A(Z_{(n)}^{\infty})} = \varphi_n |_{A(Z_{(n)}^{\infty})}.$$  

(5.23)

In this way we obtain a sequence $\varphi_1, \ldots, \varphi_n, \ldots$ of $m_n$-realizers satisfying (5.23) for all $n$.

Set $\varphi = \lim \varphi_n$; it is a well defined isomorphism $(AZ, D_A) \to (AZ, D_B)$. Clearly $\varphi - \iota$ decreases filtrations. Thus Theorem 5.3 implies that $f$ is realizeable by a homotopy equivalence. Q.E.D.

6. Applications

6.1. Formality. Recall from Section 1 that a $c$-connected c.g.d.a. $(A, d_A)$ is formal if it has the same homotopy type as $(H(A), 0)$. A connected c.g.a. $H$ is called intrinsically formal if all c.g.d.a.'s with cohomology algebras isomorphic with $H$ are formal, i.e., if $H$ is realized by only one homotopy type.

In Section 1 we observed that $(A, d_A)$ was formal if and only if the identity map

$$f: H(A, d_A) \to H(H(A), 0)$$

could be realized by a homotopy equivalence. We can thus apply Section 5
to $f$, with $(H(A), 0)$ playing the role of $(B, d_B)$. This yields (cf. Theorem 5.10) a sequence of obstructions to formality, which vanish if and only if $(A, d_A)$ is formal, assuming $H(A)$ has finite type. Notice that $D_B = d$ and $\nu_A = \nu_B = \nu_B$. We write $\rho_A = \rho$.

Recall that the $n$th obstruction class for the realizeability of an isomorphism $f: H(A) \to H(B)$ is an element $O_n(f)$ in $\text{Hom}^1(Z_{n+1}; H(B))/\gamma(M_n)$, where $\gamma(M_n)$ is quite complicated to understand. The obstructions to formality, however, live in a simple space.

In fact let $(A, d_A)$ be a $c$-connected c.g.d.a., with bigraded model $(\Lambda Z, d)$ for $H(A)$. Define a linear map $\lambda: \text{Hom}^0(Z_n; H(A)) \to \text{Hom}^1(Z_{n+1}; H(A))$ as follows. Let $\eta: H(A) \to \Lambda Z_0$ be a linear map such that $\rho \eta = \iota$. Given an element $\psi \in \text{Hom}^0(Z_n; H(A))$, let $\theta_\psi$ be the derivation of $\Lambda Z_0$ defined by

$$\theta_\psi(z) = \begin{cases} 0, & z \in Z_{(n-1)} \\ \eta \psi(z), & z \in Z_n \end{cases}$$

Then $d \theta_\psi = \theta_\psi d$, and so we can form $\gamma(\theta_\psi) \in \text{Hom}^1(Z_{n+1}; H(A))$:

$$\gamma(\theta_\psi)(z) = c(\rho \theta_\psi \, dz), \quad z \in Z_{n+1}.$$ 

Define $\lambda$ by

$$\lambda(\psi) = \gamma(\theta_\psi), \quad \psi \in \text{Hom}^0(Z_n; H(A)).$$

6.2. Proposition. With the notation above, $\text{Im} \lambda = \text{Im} \gamma$, and so the $n$th obstruction to formality is an element (if defined) $O_n \in \text{Hom}^1(Z_{n+1}; H(A))/\text{Im} \lambda$.

Proof. Let $\theta \in M_n$; thus $\theta$ is a filtration decreasing derivation in $\Lambda Z_0$ such that $d \theta = \theta d$. Define $\psi \in \text{Hom}^0(Z_n; H(A))$ by $\psi z = \rho \theta z$, $z \in Z_n$. We show that

$$\rho \theta(u) = \rho \theta_\psi(u), \quad u \in (\Lambda Z_0)_n.$$ 

In fact write $u = \sum u_i x_i + \sum v_j w_j$, where $u_i \in \Lambda Z_0$, $x_i \in Z_n$, $v_j$, $w_j \in (\Lambda Z)_+$. Then $\rho v_j = \rho w_j = \rho x_i = 0$, and so, since $\theta$ is a derivation,

$$\rho \theta(u) = \sum_i \rho u_i \theta z_i = \sum_i \rho u_i \rho \theta_\psi z_i = \rho \theta_\psi u.$$ 

This implies that for $z \in Z_{n+1}$

$$\gamma(\theta)(z) = \rho \theta \, dz = \rho \theta_\psi \, dz = \gamma(\theta_\psi)(z) = \lambda(\psi)(z).$$

Hence $\gamma(\theta) = \gamma(\theta_\psi) = \lambda(\psi)$, and $\text{Im} \gamma = \text{Im} \lambda$. Q.E.D.
6.4. Example. Let $H = H^*(S^3 = S^2; k)$ then $H$ has elements $1, x_1, x_2$ as basis (with $|x_1| = |x_2| = 2$) and $x_1^2 = x_1x_2 = x_2^2 = 0$. It follows from Corollary 5.16 that $H$ is intrinsically formal (cf. also Lemma 1.6).

6.5. Example. Let $H = A(x_1, x_2, x_3)/I$, where $|x_1| = |x_2| = 2, |x_3| = 3$, and $I$ is the ideal generated by $x_1x_2, x_1^2$ and $x_2^2$. Thus $H = H^*(S^3 = (S^2 = S^2); k) = H^*(S^3; k) \otimes H^*(S^2 = S^2; k)$.

We shall show that there are exactly two homotopy types with cohomology $H$ (so that $H$, although the tensor product of intrinsically formal c.g.a.'s, is not intrinsically formal).

First, let $(AZ, d)$ be the bigraded model for $H$. Then $Z_0, Z_1, Z_2$ have bases given by

$$Z_0: x_1, x_2, x_3; \
Z_1: y_1, y_2, y_3; \
Z_2: z_1, z_2,$$

where

$$dy_1 = x_1^2, \\ dz_1 = x_2y_1 - x_1y_3, \\ dy_2 = x_2^2, \\ dz_2 = x_1y_2 - x_2y_3, \\ dy_3 = x_1x_2,$$

The degrees are given by

$$|x_1| = |x_2| = 2, \\ |x_3| = |y_1| = |y_2| = 3, \\ |z_1| = 4.$$  

Since $H^1 = 0$, Remark 3.8.4 yields

$$Z^p_a = 0, \quad p \leq n + 1.$$  

Now we present one way to perturb $d$ to obtain a c.g.d.a. $(AZ, D)$ with $D = d: Z_n \rightarrow F_{n-2}(AZ)$. (It follows that the inclusion $AZ_0 \rightarrow AZ$ induces an isomorphism

$$H = AZ_0 \otimes((AZ)_0) \xrightarrow{\approx} H(AZ, D).$$

$D$ is constructed as follows. We set $D = d$ on $AZ_0$ and define

$$Dx_1 = dx_1 + x_2x_3, \\ Dx_2 = dx_2.$$  

The argument of Example 4.3 shows that we can extend $D$ to all of $AZ$. 
To see that \((AZ, d)\) and \((AZ, D)\) do not have the same homotopy type, we have only to show that the identity isomorphism

\[
f: H(AZ, d) = H = H(AZ, D)
\]
cannot be realized (cf. 6.1). We show that \(O_{1}(f) \neq 0\).

In fact, identify \(\text{Hom}^{1}(Z_{2} : H)\) with \(k^{4}\) via \(\alpha \leftrightarrow (\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4})\), where

\[
\alpha z_{1} = \alpha_{1}x_{1}x_{3} + \alpha_{2}x_{2}x_{3}
\]
and

\[
\alpha z_{2} = \alpha_{3}x_{1}x_{3} + \alpha_{4}x_{2}x_{3}.
\]

Identity \(\text{Hom}^{0}(Z_{1} ; H)\) with \(k^{3}\) via \(\beta \leftrightarrow (\beta_{1}, \beta_{2}, \beta_{3})\), where

\[
\beta y_{i} = \beta_{i}x_{3}, \quad i = 1, 2, 3.
\]

Then the linear map \(\lambda: \text{Hom}^{0}(Z_{1} ; H) \rightarrow \text{Hom}^{1}(Z_{2} ; H)\) defined in 6.1 is given, in terms of these identifications, by

\[
\lambda(\beta_{1}, \beta_{2}, \beta_{3}) = (-\beta_{3}, \beta_{1}, \beta_{2}, -\beta_{3}).
\]

Hence \(\text{Im} \lambda = \{(\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}) \mid \alpha_{1} = \alpha_{4}\}\).

Finally, let \(\varphi_{0} = \iota: (AZ(1), D) \rightarrow (AZ(1), d)\); it is a 0-realizer for \(f\). Calculating, we find that

\[
\rho \varphi_{0} Dz_{1} = x_{1}x_{3} \quad \text{and} \quad \rho \varphi_{0} Dz_{2} = 0.
\]

Thus \(o(\varphi_{0}) \in \text{Hom}^{1}(Z_{2}, H)\) is given under the identification by \(o(\varphi_{0}) = (1, 0, 0, 0)\). Hence \(o(\varphi_{0}) \notin \text{Im}(\lambda)\) and so \(O_{1}(f) \neq 0\) (Proposition 6.2). Thus \(f\) is not realizable and \((AZ, d)\) and \((AZ, D)\) have distinct homotopy types.

Next, consider automorphisms, \(\varphi\), of \(H\). Each automorphism \(\varphi\) must satisfy

\[
\begin{align*}
\varphi x_{1} &= \alpha_{4}x_{1} + \alpha_{2}x_{2} \\
\varphi x_{2} &= \beta_{3}x_{1} + \beta_{3}x_{3} \\
\varphi x_{3} &= \gamma x_{3}, \quad \gamma \neq 0, \quad \alpha_{i} \beta_{i} - \alpha_{2} \beta_{1} \neq 0; \quad \gamma, \alpha_{i}, \beta_{i} \in k,
\end{align*}
\]

and each such set of equations determines a unique automorphism, \(\varphi\).

Because \((AZ, d)\) is formal, any automorphism of \(H\) can be realized by a homotopy equivalence of \((AZ, d)\), (cf. Section 1). On the other hand an argument
identical with that given above shows that \( \varphi \) can be realized by a homotopy equivalence of \((AZ, D)\) if and only if

\[
\gamma = x_1\beta_2 - x_2\beta_1.
\]

Finally we show that any c.g.d.a. \((A, d_A)\) with \(H(A) \approx H\) has the homotopy type of either \((AZ, d)\) or \((AZ, D)\). Since \(H(A) \approx H\), there is a filtered model \((AZ, D')\) for \((A, d_A)\) such that \((D' - d): Z_n \to F_{n-2}(AZ)\). In particular, \((AZ, D')\) and \((A, d_A)\) have the same homotopy type.

Note that \((D' - d): Z_2 \to (AZ_0)^5\) and that the pair of elements \(x_1x_3, x_2x_3\) is a basis for \((AZ_0)^5\). Thus

\[
D'x_1 = dx_1 + \mu_1 x_1x_3 + \mu_2 x_2x_3
\]

and

\[
D'x_2 = dx_2 + \sigma_1 x_1x_3 + \sigma_2 x_2x_3,
\]

\(\mu_i, \sigma_i \in k\).

We distinguish two cases:

**Case 1.** \(\mu_1 - \sigma_2 \neq 0\). In this case \((AZ, D')\) has the same homotopy type as \((AZ, D)\). In fact, define

\[
\varphi: (AZ_0, D') \to (AZ_0, D)
\]

by

\[
\varphi x_1 = x_1 \quad \varphi y_1 = y_1 - \frac{\mu_2}{\mu_1 - \sigma_2} x_3 \quad \varphi x_1 = z_1
\]

\[
\varphi x_2 = x_2 \quad \varphi y_2 = y_2 - \frac{\sigma_1}{\mu_1 - \sigma_2} x_3 \quad \varphi x_2 = z_2.
\]

\[
\varphi x_3 = \frac{1}{\mu_1 - \sigma_2} x_3 \quad \varphi y_3 = y_3 + \frac{\sigma_2}{\mu_1 - \sigma_2} x_3
\]

Extend \(\varphi\) inductively (on \(n\)) to a homomorphism \(\varphi: (AZ_0, D') \to (AZ_0, D)\) as follows: if \(\varphi\) is defined in \(AZ_0\) and \(x \in Z_{n+1}\) then \(\varphi D'x\) is a \(D\)-cocycle. Moreover

\[
|\varphi D'x| = |x| + 1 \geq 6,
\]

since \(Z_{n+1}^p = 0, \ p \leq n + 2, \) and here \(n \geq 2\). But \(H^p(AZ, D) \approx H^p = 0, \ p \geq 6\). It follows that \(\varphi D'x = Du\); extend \(\varphi\) by setting \(\varphi z = u\).

In the limit we obtain a homomorphism \(\varphi: (AZ, D') \to (AZ, D)\). It induces an isomorphism of \(AZ_0/dl((AZ)_1)\); hence \(\varphi^*\) is an isomorphism. Thus \((AZ, D')\) and \((AZ, D)\) have the same homotopy type.
Case II. \( \mu_1 = \sigma_2 = 0 \). Define an isomorphism \( f: H(\mathcal{A}Z, D') \to H \) by
\[
H(\mathcal{A}Z, D) \xrightarrow{\omega} \mathcal{A}Z_0[d(\mathcal{A}Z_1)] \xrightarrow{\varphi} H.
\]
The same argument as given above to show that \((\mathcal{A}Z, d)\) and \((\mathcal{A}Z, D)\) had distinct homotopy types, now shows that \( \omega(f) = 0 \). Hence Theorem 5.15 (with \( l = 1 \) and \( m = 5 \)) shows that \( f \) can be realized by a homotopy equivalence between \((\mathcal{A}Z, D')\) and \((\mathcal{A}Z, d)\).

6.6. Example. Let \( H = H^*(S^2 \vee S^2 \vee S^5; k) \) with basis \( x_1, x_2, x_3 \) of degrees 2, 2 and 5, respectively. Again there are exactly two homotopy types realizing \( H \). Up through dimension 5, \( Z_0, Z_1, Z_2 \) have bases given by

\[
Z_0: x_1, x_2, x_3 \quad Z_1: y_1, y_2, y_3 \quad Z_2: z_1, z_2
\]
with
\[
\begin{align*}
dy_1 &= x_1^2 & dx_1 &= x_1 y_2 - x_2 y_1 \\
dy_2 &= x_1 x_2 & dx_2 &= x_1 y_3 - x_2 y_2 \\
dy_3 &= x_2^2 & dz_1 &= z_1 y_1 - x_1 y_1 \\
dz_2 &= z_2 y_2 & dz_3 &= z_2 y_3 - x_2 y_2
\end{align*}
\]
The degrees are: \( |y_i| = 3, |z_i| = 4 \).

First observe that any perturbation, \( D \), of \( d \) satisfies
\[
Dz_i = dz_i + \mu_i x_3 \quad \text{for some coefficients } \mu_i, \quad i = 1, 2.
\]
If \( (\mu_1, \mu_2) \neq 0 \) and \( (\sigma_1, \sigma_2) \) is also \( \neq 0 \) we will show that \((\mathcal{A}Z, D)\) has the same homotopy type as \((\mathcal{A}Z, D')\), where \( D' \) is a perturbation of \( d \) with
\[
D'z_i = dz_i + \sigma_i x_3, \quad i = 1, 2.
\]
Indeed, choose a \( 2 \times 2 \) matrix \((b_{ij})\) of determinant \( \neq 0 \) such that \( \sum b_{ij} y_i - \mu_i, \quad i = 1, 2 \). Define an automorphism \( \psi \) of \( H \) by
\[
\psi x_i = \sum b_{ij} x_j \quad i = 1, 2 \quad \text{and } \psi x_3 = x_3.
\]
Extend \( \psi \) to an automorphism of the bigraded c.g.d.a. \((\mathcal{A}Z, d)\) (cf. Proposition 3.4). Thus \( D' = \psi D \psi^{-1} \) satisfies the above equation.

On the other hand, suppose \( D_1 \) and \( D_2 \) are two perturbations which agree on \( \mathcal{A}Z_0 \) and \( D_1 z_i = D_2 z_i, \quad i = 1, 2 \). Let \( \varphi = \nu: (\mathcal{A}Z_0, D_1) \to (\mathcal{A}Z_0, D_2) \). Then \( o(\varphi) z_i = 0 \) by hypothesis, while if \( x \in \mathcal{A}Z_0^p (p \geq 5) \) then \( o(\varphi) x \in H^{p+q} = 0 \). Hence \( o(\varphi) = 0 \); i.e., the identity isomorphism of \( H \) is 1-realizable as a homotopy equivalence between \((\mathcal{A}Z, D_1)\) and \((\mathcal{A}Z, D_2)\).
Now Theorem 5.15 applies (with $l = 1$ and $m = 5$) and thus the identity is realizable; i.e., $(AZ, D_1)$ and $(AZ, D_2)$ have the same homotopy type.

On the other hand, if $\theta \in M_1$ then $\theta(y_i) \in (AZ_0)^{y_i} = 0$, $i = 1, 2, 3$. It follows that $\gamma(\theta)(x_1) = \gamma(\theta) x_2 = 0$, and so $\gamma(M_1) = 0$. (Note that $M_1$ is independent of the perturbation.)

Hence, the obstruction to formality of $(AZ, D)$ is given by

$$z_1 \rightarrow \text{cl}(\mu_1 x_3)$$

$$z_2 \rightarrow \text{cl}(\mu_2 x_3)$$

which for $(\mu_1, \mu_2) \neq (0, 0)$ is non-trivial. (The indeterminacy $\lambda M_n$ is zero in the relevant dimensions because $M_n$ is.)

Of course, in the non-formal case, the homotopy type is represented by $S^2 \vee S^2 \vee e^6$ attached by, e.g., $[t_1, [t_2, t_3]]$ so that the 5-cell carries the nontrivial Massey product $<x_1, x_1, x_2>$. Since Massey products are invariants of homotopy type (modulo indeterminacy which here is 0), the Massey product distinguishes the two rational homotopy types. It also reminds us how much more subtle the integral problem will be; cf. [4, final paragraph].

6.7. Field extensions. Let $(A, d_A)$ and $(B, d_B)$ be $c$-connected c.g.d.a.’s whose cohomology has finite type, and suppose $f: H(A) \rightarrow H(B)$ is an isomorphism of graded algebras. Let $K$ be a field extension of $k$.

6.8. Theorem. Assume $f \otimes 1: H(A) \otimes K \rightarrow H(B) \otimes K$ can be realized by a homotopy equivalence between $(A \otimes K, d_A \otimes 1)$ and $(B \otimes K, d_B \otimes 1)$, regarded as c.g.d.a.’s over $K$.

Then $f$ can be realized by a homotopy equivalence between $(A, d_A)$ and $(B, d_B)$.

6.9. Corollary. $(A, d_A)$ is formal if and only if $(A \otimes K, d_A \otimes 1)$ is formal.

As we observed in Section 1, Theorem 6.8 has been obtained independently by Sullivan using techniques from algebraic groups. An independent proof along those lines in the simply connected case has been obtained by Neyendorf [25].

6.10. Proof of Theorem 6.8. Consider the filtered models defined in 5.1: $\pi_A: (AZ, D_A) \rightarrow (A, d_A)$ and $\pi_B: (AZ, D_B) \rightarrow (B, d_B)$. Tensoring with $K$ gives the filtered models for $(A \otimes K, d_A \otimes 1)$ and $(B \otimes K, d_B \otimes 1)$.

Recall the standard $k$-linear inclusion

$$i: \text{Hom}^l(Z_{n+1}; H(B)) \rightarrow \text{Hom}^l(Z_{n+1} \otimes K; H(B) \otimes K),$$

where the right hand side denotes $K$-linear maps. A simple calculation shows that

$$(6.11)\quad i(\gamma(M_n)) = (\text{Im } i) \cap \gamma(M_n),$$
where $M_n$ (resp. $\bar{M}_n$) is the space of filtration decreasing derivations of $\Lambda Z_{(n)}$ (resp. $\Lambda Z_{(n)} \otimes K$) commuting with $D_B$ (resp. $D_B \otimes 1$).

Finally, we show by induction on $n$ that $f$ is $n$-realizable for all $n$; in view of Theorem 5.10 it will then follow that $f$ is realizable by a homotopy equivalence.

Assume $f$ is $(n-1)$-realizable, and let $\varphi$ be an $(n-1)$-realizer. Then $\varphi \otimes 1$ is an $(n-1)$-realizer for $f \otimes 1$. Hence by Corollary 5.8, (since $f \otimes 1$ is $n$-realizable),

$$o(\varphi \otimes 1) \in \gamma(M_n).$$

On the other hand, clearly

$$o(\varphi \otimes 1) = i(o(\varphi)).$$

Thus $o(\varphi \otimes 1) \in \gamma(M_n) \cap \text{Im } i$. Now formula (6.11) shows that $o(\varphi \otimes 1) = i(\psi)$, $\psi \in \gamma(M_n)$. Hence $i(\psi) = i(o(\varphi))$; since $i$ is injective, $\psi = o(\varphi)$. Thus $o(\varphi) \in \gamma(M_n)$.

Now Proposition 5.6 implies that $O_n(f) = \gamma(M_n)$ and so Corollary 5.8 shows that $f$ is $n$-realizable. This completes the induction, and the proof. Q.E.D.

### 7. The Eilenberg–Moore Spectral Sequence

#### 7.1. Definition.

Let $\epsilon : (A, d_A) \to k$ be an augmented, $c$-connected c.g.d.a. We recall the definition of the bar construction $(BA, V)$ and the Eilenberg–Moore spectral sequence (E.M.s.s.) converging from $\text{Tor}_{H(A)}(k, k)$ to $H(BA, V)$, as given in [28; part 1].

Let $I = \ker \epsilon$ and set $BA = \sum_{n \geq 0} \otimes^n I$. If $a_i \in I$ has degree $p_i$ , denote $a_1 \otimes \cdots \otimes a_n$ by $[a_1 | \cdots | a_n]$ and bigrade $BA$ so this element has bidegree $(-n, \sum p_i)$, and total degree $(\sum p_i - n) = (\sum (p_i - 1))$.

The usual bar construction is for d.g.a.'s $(A, d_A)$ with $d_A$ of degree $-1$. In that context, $n$ rather than $-n$ appears in the above formulas. The d.g. module $BA$ is always and naturally a d.g. coalgebra with diagonal

$$[a_1 | \cdots | a_m] \to \sum_{p=0}^{m} [a_1 | \cdots | a_p] \otimes [a_{p+1} | \cdots | a_m]$$

For our present purposes, this structure is unimportant. We focus instead on the fact that if $A$ is commutative, $BA$ inherits the structure of a c.g.a. with

$$[a_1 | \cdots | a_m] \cdot [a_{m+1} | \cdots | a_{m+n}] = \sum_{\sigma} \epsilon_0(a_{\sigma(1)} | \cdots | a_{\sigma(m+n)}),$$

where the sum is over all shuffles (permutations $\sigma$ such that $\sigma^{-1}(1) < \cdots < \sigma^{-1}(m)$).
and $\sigma(m + 1) < \cdots < \sigma(m + n)$. The map $\sigma \to \epsilon_\sigma$ is the $\mathbb{Z}_2$ representation of the permutation group which assigns to the transposition $j \to j + 1$ the sign $(-1)^{(p_j - 1)(p_{j + 1} - 1)}$.

Set $s(i) = \sum_{j=1}^{i} (p_j - 1)$ and define two differentials $d_A$ (the internal differential) and $\delta$ (the combinatorial differential) in $BA$ by

$$d_A[a_1 | \cdots | a_n] = \sum_{i=1}^{n} (-1)^{s(i-1)} [a_1 | \cdots | d_A a_i | \cdots | a_n],$$

and

$$\delta[a_1 | \cdots | a_n] = \sum_{i=2}^{n} (-1)^{s(i-1)} [a_1 | \cdots | a_{i-1} a_i | \cdots | a_n].$$

They are derivations of bidegrees $(0, 1)$ and $(1, 0)$ and satisfy $d_A \delta + \delta d_A = 0$. Hence a c.g.d.a. $(BA, \delta)$ is defined by $\delta = d_A + \delta_i$; it is called the bar construction for $(A, d_A)$ and sometimes written $(B(A, d_A), \delta)$. Note: If $A$ is not connected, $(BA, \delta)$ will contain elements of negative degrees!

Filtering $BA$ by the left degree ($\sum_{i\geq n} BA^{i,*}$), we obtain a second quadrant spectral sequence, called the Eilenberg–Moore spectral sequence. Because it is a second quadrant sequence, the $E_2$ term is defined ($E_{2,0} = \lim_{i \to \infty} E_{i,i}$) and equal to the associated bigraded algebra of $H(B(A,d_A),\delta)$ although the filtration is in general infinite. The $E_2$ term is given by

$$E_2 = H^* (B(H(A),0),\delta) = \text{Tor}_{H(A)}(k,k).$$

If $\varphi: (A,d_A) \to (B,d_B)$ is a homomorphism of augmented c.g.d.a.'s then $\varphi$ extends to the homomorphism $B(\varphi): BA \to BB$ defined by $B(\varphi)([a_1 | \cdots | a_n]) = [\varphi a_1 | \cdots | \varphi a_n]$. Thus $B(\varphi)$ induces a homomorphism of spectral sequences, $B(\varphi)_i$, with $B(\varphi)_1 = B(\varphi^*)$. If $\varphi^*$ is an isomorphism, so is $B(\varphi)_1$ and hence so is $B(\varphi)^*$.  

7.2. The bar construction for $(AX,D)$. Suppose $(AX,D)$ is a connected c.g.d.a. satisfying condition (2.3). We will obtain a simple form of $H(B(AX),D)$.

Define the suspension of $X$ to be the graded space $\tilde{X}$ given by $\tilde{X}^p = X^{p+1}$ and denote the (identity) degree $-1$ isomorphism by $\tilde{=} : \tilde{X} \to X$. Recall from 4.14 that the projection $\zeta: A^+X \to X$ with kernel $A^+X \cdot A^+X$ determines a differential $D_\zeta$ in $X$ via $D_\zeta \circ \zeta = \zeta \circ D_\zeta$. $D_\zeta$ is the linear part of $D$. Define $\tilde{D}$ in $\tilde{X}$ by $\tilde{D} = \zeta \circ \zeta \circ D_\zeta$.

Extend $\tilde{D}$ to a derivation in $AX$ (necessarily $\tilde{D}^2 = 0$) and define the squish homomorphism (of c.g.d.a.'s), 

$$\sigma: (B(AX,D),\delta) \to (AX,\tilde{D})$$

by $\sigma([a_1 | \cdots | a_n]) = \tilde{\sigma} a_1 \wedge \cdots \wedge \tilde{\sigma} a_n/\lvert n! \rvert$, $a_i \in A^+X$, $\tilde{\sigma} = \tilde{\sigma}_\zeta$. 
7.3. Theorem. With the notation and hypotheses above, \( \sigma \) induces an isomorphism of cohomology:

\[
\sigma^* : H(B(AX), \nabla) \longrightarrow H(AX, D).
\]

7.4. Corollary. \( H(B(AX), \nabla) \approx \Lambda H(X, D) \). In particular, if \((AX, D)\) is minimal (so that \(D = 0\)), then \(H(B(AX)) \approx \Lambda X\).

7.5. Corollary. Let \((A, d_A)\) be a c-connected augmented c.g.d.a. with minimal model \((AX, D) \rightarrow (A, d_A)\). Then \(H(BA, \nabla) \approx \Lambda X\).

7.6. Lemma. Theorem 7.3 is correct when \(D \rightarrow 0\).

Proof. In this case the differential \(\nabla\) in \(B(AX)\) is just \(\delta\). Define a homomorphism \(\lambda : (AX, 0) \rightarrow (B(AX), \delta)\) by \(\lambda(x) = [x \longmapsto \bar{x}],\ x \in X\). Clearly \(\sigma \circ \lambda = \iota\) and so we need only show that \(\lambda^*\) is an isomorphism.

Two right \(AX\)-free resolutions of \(k\) (the bar and Koszul resolutions) are given by

\[
\cdots \longrightarrow B^{-n, \ast}(AX) \otimes AX \xrightarrow{\delta_B} B^{-n+1, \ast}(AX) \otimes AX \xrightarrow{\delta_B} \cdots \quad \text{(bar)}
\]

and

\[
\cdots \longrightarrow A_{n\bar{x}} \otimes AX \xrightarrow{\delta_{\bar{x}}} A_{n-1\bar{x}} \otimes AX \xrightarrow{\delta_{\bar{x}}} \cdots \quad \text{(Koszul)},
\]

where

\[
\delta_B[a_1 | \cdots | a_n] \otimes a = \delta[a_1 | \cdots | a_n] \otimes a + (-1)^{i(n)} [a_1 | \cdots | a_{n-1}] \otimes a_n \land a
\]

and

\[
\delta_{\bar{x}}(\bar{x}_1 \wedge \cdots \wedge \bar{x}_n \otimes a) = \sum_{i=1}^n \epsilon_i \bar{x}_1 \wedge \cdots \wedge \mathring{x}_i \wedge \cdots \wedge \bar{x}_n \otimes x_i \land a,
\]

\(a, a_i \in AX, \ \bar{x}_j \in \bar{X}\).

(Here \(\bar{s}^{-1} : \bar{X} \longrightarrow X\) is denoted by \(x \rightarrow \mathring{x}: s(n)\) is as defined in 7.1; \(\epsilon_i\) is the sign of the permutation \(\bar{x}_1, \ldots, \bar{x}_n \rightarrow \bar{x}_1, \ldots, \mathring{x}_i, \ldots, \bar{x}_n, \ \bar{x}_i\) defined as in 7.1).

A straightforward check shows that \(\lambda \otimes \iota : AX \otimes AX \rightarrow B(AX) \otimes AX\) is a morphism of resolutions. Hence (by standard homological algebra, cf. eg. [14]) if we tensor by \(k\) (over \(AX\)) on the right and pass to homology, \(\lambda \otimes \iota\) induces an isomorphism. This isomorphism is precisely \(\lambda^* : AX \longrightarrow H(B(AX))\). Q.E.D.

7.7. Proof of Theorem 7.3. Since \(\sigma\) is surjective, we have to prove that \(H(\ker \sigma, \nabla) = 0\); Lemma 7.6 shows that \(H(\ker \sigma; \delta) = 0\). Now condition (2.3) implies that \((AX, D) = \lim(AX_n, D)\) where the limit is over finite dimensional
graded subspaces $X_a \subset X$ such that $AX_a$ is $D$-stable. It is thus sufficient to consider the case $\dim X$ finite, say $\dim X = m$. Fix a homogeneous basis $x_1, ..., x_m$ of $X$ such that $Dx_i : A(x_1, ..., x_{i-1}), \ i = 1, ..., m$.

Multigrade $AX$ by assigning $x_1^{k_1} \wedge ... \wedge x_m^{k_m}$ the multidegree $k = (k_1, ..., k_m)$. Multigrade $B(AX)$ by assigning $[a_1 | ... | a_{mn}]$ the multidegree $k_1 + ... + k_m$ if $a_i$ has multidegree $k_i$. Finally, multigrade $AX$ by assigning $x_1^{k_1} \wedge ... \wedge x_m^{k_m}$ the multidegree $k = (k_1, ..., k_m)$. Note that $\sigma$ is homogeneous of multidegree zero, hence $\ker \sigma$ is multigraded. Moreover $\sigma : B(AX)^{n,q} \to (\Lambda^n X)^{n,q}$.

Thus

$$\ker \sigma = \sum_{k_{i,q}} (\ker \sigma)_{k_{i,q}}$$

where $\Phi \in (\ker \sigma)_{k_{i,q}}$ if $\Phi \in B(AX)^{n,q}$ and has multidegree $k$. Since $\delta$ is homogeneous of multidegree zero and of bidegree $(1, 0)$, we obtain from the fact that $H(\ker \sigma, \delta) = 0$ the equations

$$H((\ker \sigma)_{k_{i,q}}, \delta) = 0, \ \text{all } q \text{ and } k.$$

Next well order the multidegrees (lexicographic ordering) by setting $1 < k$ if for some $p, l_p < k_p$ and $l_{p+q} = k_{p+q}$, $i \geq 1$. Set

$$(\ker \sigma)_{k_{i,q}} = \sum_{l_{i,q}} (\ker \sigma)_{l_{i,q}} \quad \text{and} \quad (\ker \sigma)_{k_{i,q}} = \sum_{l_{i,q}} (\ker \sigma)_{l_{i,q}}.$$

We shall show that any $\nabla$-cocycle $\Phi \in (\ker \sigma)_{k_{i,q}}$ is cohomologous to a cocycle in $(\ker \sigma)_{k_{i,q}}$; this clearly establishes the theorem.

To prove this, write $\Phi = \Phi_{\sigma} + \cdots + \Phi_{p+q}$, where $\Phi_{i} \in B(AX)^{-i,s+i}$ ($s = \deg \Phi$). Then for each $i$

$$\Phi_{i} \in (\ker \sigma)_{k_{i,q}}$$

and $\delta \Phi_{i} = 0$. It follows from (7.8) that $\Phi_{p} = \delta \Psi$, where

$$\Psi \in (\ker \sigma)_{k_{i,q}}$$

and

$$(7.9) \quad D \Psi \in (\ker \sigma)_{k_{i,q}}$$

since $D$ decreases multidegrees.

Thus $\Phi - \nabla \Psi = (\Phi_{p+1} - D \Psi) + \cdots + \Phi_{p+q}$ is again in $(\ker \sigma)_{k_{i,q}}$. Proceeding in this way (decreasing $q$), we reduce to the case $\Phi = \Phi_{p}$. But then $\Phi - \nabla \Psi = -D \Psi \in (\ker \sigma)_{k_{i,q}}$. This completes the proof. Q.E.D.

7.10. Theorem. Let $H$ be a connected c.g.a. of finite type such that $H^{2p} = 0, \ p \geq 1$. Then $H$ is intrinsically formal and $\sigma^{2p}_{\delta}(H, 0) = 0, \ p = 1, 2, ...$.
Proof. Observe first that $BH$ is evenly graded: $[(BH)^p = 0, \text{ even}]$ and so $H(BH, V) \approx BH$ is also evenly graded. Let $\rho: (\mathcal{A}Z, d) \to (H, 0)$ be the bigraded model (Definition 3.5). Since it is minimal (Proposition 3.4), Corollary 7.5 shows that $\Lambda Z \approx BH$. Hence $Z$ is evenly graded, and so $Z^{2p} = 0, p = 1, 2, \ldots$. This proves that $\pi_0^2(H, O) = 0$.

Let $(\mathcal{A}Z, D)$ be a c.g.d.a. such that $(D - d): 2, \to F_{-2}(\mathcal{A}Z)$. Let $f: H(\mathcal{A}Z, D) \to H$ be the isomorphism

$$H(\mathcal{A}Z, D) \leftarrow \Lambda Z^0(d((\mathcal{A}Z)_{\text{odd}}) = H.$$  

The obstructions to realizing $f$ by a homotopy equivalence between $(\mathcal{A}Z, D)$ and $(\mathcal{A}Z, d)$ lie in quotient spaces of the spaces $\text{Hom}^1(Z_{n+1}; H)$. Since $Z^{2p} = 0, p$ even, and $H^{2p} = 0, p$ even, it follows that $\text{Hom}^1(Z_{n+1}; H) = 0, n = 0, 1, 2, \ldots$. Hence all the obstructions are zero and so Theorem 5.10 (or Proposition 5.7 applied directly) implies that $(\mathcal{A}Z, D)$ and $(\mathcal{A}Z, d)$ have the same homotopy type.

Finally, suppose $(\mathcal{A}, d_A)$ satisfies $H(\mathcal{A}) \approx H$. By Theorem 4.4, $(\mathcal{A}, d_A)$ has the homotopy type of some $(\mathcal{A}Z, D)$; by the argument above it has the homotopy type of $(\mathcal{A}Z, d)$ and hence of $(H, 0)$. Thus $H$ is intrinsically formal. Q.E.D.

7.11. Remark. If $H$ is any connected c.g.a. with trivial multiplication, $(H^+ \cdot H^+ = 0)$ then clearly the differential in $B(H, 0)$ is zero and so $BH \approx \Lambda Z$, where $(\mathcal{A}Z, d)$ is the minimal model for $(H, 0)$.

An explicit description of the generators $Z \subset BH$ is given by Hilton in [17], for $H$ can be realized by a wedge of spheres.

7.12. Filtered models. Let $(\mathcal{A}, d_A)$ be an augmented, c-connected c.g.d.a. Let $(\Lambda \mathcal{A}Z, d) \to (H(\mathcal{A}), 0)$ be the bigraded model (3.5) and let $(\Lambda \mathcal{A}Z, D) \to (\mathcal{A}, d_A)$ be the filtered model (4.11). We have just shown that $H(\Lambda \mathcal{A}Z, D) \approx H(B(\Lambda \mathcal{A}Z, D), V) \approx H(B(\mathcal{A}), V)$. Now we show how to use the extra filtered structure on $(\Lambda \mathcal{A}Z, D)$ to recover the E.M.s.s. for $(\mathcal{A}, d_A)$. (A related result on commutative coalgebras has been known to J. C. Moore [23, 24].)

First, bigrade $Z$ by setting $Z_n^p = Z_n^{p+1}$ and extend this lower gradation to a gradation of the algebra $\Lambda Z$. In order to make our indexing conform to that of Eilenberg–Moore, we write

$$(\Lambda Z)^{-n,a} = (\Lambda Z)^{-n+a}_{\text{even}}.$$  

In particular

$$Z^{-n,a} = Z^{-n+a} = Z_{n-1}^{n+a} = Z^{-n+1,a}.$$  

Note that $D: Z_{-n \cdot a} \to Z_{-n+1, a}$ (see 4.14).

Our old filtration $F_{-n}(\Lambda Z)$, with corresponding change of notation is $F_{-n}(\Lambda Z) = \sum_{i \geq n} (\Lambda Z)^{i, a}$, which makes $(\Lambda Z, D)$ into a filtered c.g.d.a., whose
spectral sequence will be denoted by $(E_i, d_i)$. It will be shown to be isomorphic with the E.M.s.s.

7.13. Remarks. (1) Since $F^n(\Lambda Z) = \Lambda^n = k$, while $F^{n+1}(\Lambda Z) = 0$, $n > 0$, $(E_i, d_i)$ is a second quadrant spectral sequence.

(2) Recall (Theorem 4.4) that $(D - d): Z \to F^{n+1}(\Lambda Z)$, and (Proposition 3.4) that $d(Z) \subseteq \Lambda^1 Z \cdot \Lambda^2 Z$. It follows that $D \circ \sigma = \sigma \circ D$ and so

$$D(Z_n) = \sigma \circ (D - d)(Z_{n-1})$$

$$\subseteq \sigma(F^{n+1}(\Lambda Z))$$

$$\subseteq F^{n+2}(\Lambda Z).$$

It follows that $D: F^l(\Lambda Z) \to F^{l+2}(\Lambda Z)$, all $l$, whence

$$E_0 = E_1 = E_2 = \Lambda Z.$$

(3) Since $\bar{D}$ maps $\bar{Z}$ into itself, $(\Lambda Z, \bar{D})$ is the direct sum of the filtered differential spaces $(\Lambda^1 \bar{Z}, \bar{D})$. Thus $(E_i, d_i)$ is the direct sum of the corresponding spectral sequences. In particular the spectral sequence for $(\bar{Z}, \bar{D})$ is included into the spectral sequence $(E_i, d_i)$.

On the other hand, the isomorphism $\bar{\sigma}: Z \cong Z$ of 7.2 is in fact an isomorphism $\bar{\sigma}: (Z, D) \cong (\bar{Z}, \bar{D})$ which maps $F^n(Z) \approx F^{n+1}(\bar{Z})$, cf. 4.14. Thus $\bar{\sigma}$ identifies the homotopy spectral sequence with the spectral sequence for $(\bar{Z}, \bar{D})$, with a shift in degrees. Hence we have an inclusion of spectral sequences:

$$(\pi E_i, \bar{d}_i) \hookrightarrow (E_i, d_i)$$

which restricts to inclusions $\pi E_i^{n, \cdot} \hookrightarrow E_i^{n-1, \cdot}$. When $n = 0$ these are (obviously) isomorphisms

$$E_i^{n, \cdot} \xrightarrow{\approx} E_i^{n-1, \cdot}.$$

7.14. Theorem. Let $(A, d_A)$ be an augmented c-connected c.g.d.a., with filtered model $(\Lambda Z, D)$. Then the E.M.s.s. for $(A, d_A)$ can be naturally identified (as a spectral sequence of bigraded algebras) with the spectral sequence $(E_i, d_i)$, for $i \geq 2$.

There is, moreover, an isomorphism of graded algebras

$$H(B(A, d_A), \nabla) \to H(\Lambda Z, \bar{D}).$$

7.15. Corollary. The homotopy spectral sequence can be identified (with a degree shift) with a sub spectral sequence of the E.M.s.s.
7.16. COROLLARY. There is an isomorphism of bigraded algebras, $\text{Tor}_H(k, k) \cong \Lambda Z$.

Proof. Observe that $\text{Tor}_H(k, k)$ is the $E_2$-term of the E.M.s.s., while by Remark 7.13.2, $E_2 = \Lambda Z$.

7.17. COROLLARY. The structure of $\text{Tor}_H(k, k)$ as a bigraded algebra determines the "Betti numbers" $\dim H^n(A)$, if $A$ has finite type.

Proof. By Corollary 7.16, if we know $\text{Tor}_H(k, k)$ as a bigraded algebra, we also know $\Lambda Z$ as a bigraded algebra. In particular, we know the numbers

$$\dim Z_n^p = \dim Z_{n+1}^p = \dim(A^{-1}Z/A^0Z \cdot A^1Z)^{p-1}.$$ 

Now Proposition 3.10 shows that the Betti numbers for $(A, d_A)$ are determined. Q.E.D.

7.18. Proof of Theorem 7.14. Because $\pi^n: H(\Lambda Z, D) \rightarrow H(A), B(\pi)$ induces an isomorphism of Eilenberg-Moore spectral sequences, $(i \geq 2)$, and of cohomology. We are thus reduced to the case

$$(A, d_A) = (\Lambda Z, D) \quad \text{and} \quad \pi = \iota;$$

we assume this henceforth. Thus in particular Theorem 7.3 shows that $\sigma: (B(\Lambda Z), V) \rightarrow (\Lambda Z, D)$ induces a cohomology isomorphism.

Denote the E.M.s.s. by $(\tilde{E}_i, \check{d}_i)$. We shall define a new filtration, $\check{F}^n$, of $B(\Lambda Z)$ such that

$$\sigma: \check{F}^n \rightarrow F^n(\Lambda Z) \quad \text{and} \quad \check{F}^n \subset \tilde{F}^n,$$

where $\check{F}^n$ is the Eilenberg-Moore filtration. The proof will be completed by showing that the induced homomorphisms of spectral sequences,

$$\tilde{E}_i \rightarrow E_i \quad \text{and} \quad \check{E}_i \rightarrow \tilde{E}_i, \quad i \geq 2,$$

are isomorphisms. It is, of course, sufficient to prove this for $i = 2$. Trigrate $B(\Lambda Z)$ by setting

$$B(\Lambda Z) = \sum_{l \geq 0, \, n \geq 0, \, s \geq 0} B(\Lambda Z)^{-l, -, n, s},$$

where if $\Phi_i \in (\Lambda^1 Z)^{-n_i, l}$, then $[\Phi_1 \cdots \Phi_i]$ is assigned tridegree $(-l, \Sigma_i - n_i, \Sigma_i q_i)$. Then the Eilenberg-Moore filtration is defined by

$$\hat{F}^l = \sum_{j \geq l} B(\Lambda Z)^{l, *, *}. $$
Now define the filtration $\hat{F}^m$ by

$$\hat{F}^m = \sum_{j+n\geq m} B(AZ)^{j,n,*}.$$ 

(Then $\hat{F}^m = 0$, $m > 0$). Observe that $(E_0, d_0) = (B(AZ), 0)$ and $(E_1, d_1) = (B(AZ), d + \delta)$, where $d$ is the extension of $d$ from $AZ$ to $B(AZ)$. In particular $(E_1, d_1)$ is the bar construction for the c.g.d.a. $(AZ, d)$.

It is obvious that $\sigma: B(AZ) \to AZ$ is filtration preserving: $\sigma: E^m \to F^m(AZ)$. If $\sigma: E_i \to E_i$ is the induced homomorphism of spectral sequences, then

$$\sigma_i : (B(AZ), d + \delta) \to (AZ, 0)$$

is precisely the squish homomorphism for the c.g.d.a. $(AZ, d)$. Thus Theorem 7.3 shows that $\sigma_i^*$ is an isomorphism; i.e.,

$$\sigma_i : E_i \to E_i, \quad i \geq 2.$$

Since $B(AZ)^{*,n,*} = 0$, $n > 0$, it follows that $F^1 \subset F^l$, all $l$. Thus the identity map of $B(AZ)$ induces a homomorphism

$$\varphi_i : E_i \to E_i$$

of spectral sequences. We complete the proof of this theorem by showing that $\varphi_2$ is an isomorphism.

Let $\psi: (AZ, 0) \to (AZ, D)$ be the homomorphism defined by first projecting onto $AZ_0$ (with kernel $(AZ)_+$) and then including into $AZ$. Then $\varphi_0 : (E_0, d_0) \to (E_0, d_0)$ can be identified as $\varphi_0 = B(\psi): (B(AZ), 0) \to (B(AZ), D)$.

Next, denote $H(AZ, D)$ simply by $H$. Then $\psi^* = \rho: AZ \to H$. It follows that

$$\varphi_0^* = B(\rho): B(AZ) \to B(H).$$

Hence we may identify

$$\varphi_1 = B(\rho): (B(AZ), d + \delta) \to (B(H), \delta);$$

i.e., $\varphi_1 = B(\rho): (B(AZ, d), \nabla) \to (B(H, 0), \delta)$. Since $\rho^*$ is an isomorphism so is $B(\rho)^*$. Thus $\varphi_2 = \varphi_1^*$ is an isomorphism. Q.E.D.

7.19. **Collapse of the E.M.s.s.** We show how the (potential) failure of the filtered model to be minimal exactly explains the (potential) failure of the E.M.s.s. to collapse at the $E_2$-term.

7.20. **Theorem.** Let $(A, d_A)$ be an augmented, c-connected c.g.d.a. Then the following conditions are equivalent:

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The filtered model $(\Delta Z, D) \to (A, d_A)$ is minimal.

(ii) The E.M.s.s. collapses at the $E_2$-term.

(iii) The homotopy spectral sequence collapses at the $E_2$-term.

(iv) $D_\xi = 0$ (cf. 7.2).

If, moreover, $\pi_\delta(A, d_A)$ has finite type, these conditions are equivalent to

(v) $\dim \pi_\delta^p(A, d_A) = \dim \pi_\delta^p(H(A), 0)$, $p \geq 1$.

Proof. Recall from 7.2 that $\delta: (Z, D_\xi) \to (Z, D)$. Now apply the results of 4.14 and Theorem 7.14. Q.E.D.

8. Spherical Cohomology

8.1. Topology. Let $S$ be a simply connected C.W. complex. The spherical cohomology of $S$ is the cohomology "which pulls back non-trivially to spheres." It is the image of the linear map $H^+(S; \mathbb{Q}) \to \text{Hom}_\mathbb{Z}(\pi_\#(S); \mathbb{Q})$ dual to the Hurewicz homomorphism.

On the other hand, the cohomology suspension $\Omega$ for $S$ is the degree $-1$ linear map $H^+(S) \to H^*(\Omega S)$ given by

$$H^+(S) \xrightarrow{\pi^*} H(PS, \Omega S) \xrightarrow{\delta} H(\Omega S),$$

where $\pi: PS \to S$ is the path space fibration, and $\Omega S$ is the loop space. It is well known (and easy to check) that

$$H^+(S; \mathbb{Q}) \xrightarrow{\text{cohomology suspension}} H^*(\Omega S; \mathbb{Q}) \xrightarrow{\text{Hurewicz}} \text{Hom}_\mathbb{Z}(\pi_\#(S); \mathbb{Q}),$$

commutes, and that the Hurewicz and cohomology suspension maps have the same kernels. [Here $\delta$ denotes the standard isomorphism $\pi_\delta(S) \cong \pi_{n-1}(\Omega S).$]

Now let $(\Delta Z, D)$ be the filtered model for $A_{pl}(S)$. Then (cf. 4.14) we write $\pi_\delta^*(\Delta Z; D) = H(Z, D_\xi)$. If $(\Delta X, d_x)$ is the minimal model for $A_{pl}(S)$ then there is a commutative diagram

$$H^+(\Delta Z, D) \xrightarrow{\pi^*} H(Z, D_\xi) \xrightarrow{\sim} H^+(\Delta X, d_x) \xrightarrow{\xi^*} X.$$ 

Hence (cf. [30; Section 8]), $H(Z, D_\xi)$ is identified with $\text{Hom}_\mathbb{Z}(\pi_\#(S); \mathbb{Q})$ and $\xi^*$ with the Hurewicz map.
On the other hand [28; Proposition 4.5] the cohomology suspension for \( S \) is identified with the linear map.

\[
\Sigma^\omega_{APL}(S) : H^\ast(APL(S)) \to H(BAPL(S))
\]
given by \( \Sigma^\omega_{APL}(a) = [a] \). Using the equivalence \( \pi : (AZ, D) \to APL(S) \), we can identify this with

\[
\Sigma^\ast : H^\ast(AZ, D) \to H(BAZ), \nabla
\]

(again \( \Sigma(z) = [z], z \in AZ \)).

Finally, recall the isomorphism \( \sigma : (Z, D_z) \simeq (\bar{Z}, \bar{D}) \) and the squish homomorphism \( \sigma \), defined in 7.2. Then the following diagram (cf. Theorem 7.3) commutes by definition:

\[
\begin{array}{ccc}
H^\ast(AZ, D) & \xrightarrow{\sigma^\ast} & H(BAZ), \nabla \\
\downarrow \sigma & & \downarrow \sigma^\ast \\
H(Z, D_z) & \xrightarrow{\sigma^\ast} & H(AZ, D);
\end{array}
\tag{8.3}
\]

it is almost the algebraic analogue of (8.2). Since \( \sigma^\ast \) maps \( H(Z, D_z) \) isomorphically onto \( H(AZ, D) \subset H(AZ, D) \), we obtain in fact that \( \ker \xi^\ast = \ker \Sigma^\ast \).

8.4. Remarks. (1) These constructions and conclusions still hold if \( APL(S) \) is replaced by any augmented c-connected c.g.d.a. \((A, d_A)\) over \( k \).

(2) According to [20; p. 533] \( \ker \Sigma^\ast_d \) is generated linearly by matrix Massey products (including ordinary ones) of arbitrary orders in \( H^+(A) \).

8.5. Spherical cohomology. Let \((AZ, D)\) be the filtered model for a c-connected augmented c.g.d.a. \((A, d_A)\) over \( k \). As in [30], we call \( \xi^\ast \) the Hurewicz map and \( \text{Im} \xi^\ast \) the spherical cohomology.

It is immediate that \( H^+(AZ) \cdot H^+(AZ) \subset \ker \xi^\ast \). Choose graded spaces \( K \subset \ker \xi^\ast \) and \( L \subset H^+(AZ) \) so that \( K \) complements \( H^+(AZ) \cdot H^+(AZ) \) in \( \ker \xi^\ast \), and \( \xi^\ast \) maps \( L \) isomorphically onto \( \text{Im} \xi^\ast \). Then

\[
H^+(AZ) = H^+(AZ) \cdot H^+(AZ) \oplus K \oplus L
\tag{8.6}
\]

and \( K \oplus L \) is a minimal set of generators for \( H(AZ) \). Hence \( L \) generates \( H(AZ) \) if and only if

\[
\ker \xi^\ast = H^+(AZ) \cdot H^+(AZ).
\tag{8.7}
\]

8.8. Definition. \( H(AZ) \) is called spherically generated if (8.7) holds.

Next, recall (3.1) that \( Z_0 = H^+(A)/H^+(A) \cdot H^+(A) \), and so we have a canonical
projection $H^+(A) \rightarrow Z_0$. Using the commutative diagram (4.13), we find that

$$
\begin{array}{c}
\xymatrix{
H^+(A) \ar[d]^-{\pi^*} & H^+(AZ, D) \ar[d]^-{\iota^*} \\
Z_0 & H(Z, D_c)
}
\end{array}
$$

(8.9)

commutes, where $i: Z_0 \rightarrow Z$ is the inclusion. This shows that $\text{Im} \, \xi^* = \text{Im} \, \iota^*$, and so we have the canonical identifications:

$$
\text{Spherical cohomology} = \text{Im} \, i^* = \pi E^{0,*}_\infty
$$

(because $Z_0 = \pi E^{0,*}_2$ and $i$ is the edge homomorphism, cf. 4.14).

This yields a result of Thom [32; Theorem 51 (see also [28; Corollary 4.7]).

8.11. PROPOSITION. Suppose $H^p(A) = 0, 1 \leq p \leq l$. Then

$$
(\ker \Sigma^* A)^m = [H^+(A) \cdot H^+(A)]^m, \quad 1 \leq m \leq 3l + 1.
$$

Proof. Since $\ker \Sigma^* = \ker \xi^*$, (in terms of (8.6)) we have to show that $K^m = 0, 1 \leq m \leq 3l + 1$ (see 8.6). But $K$ can be identified easily with $\ker \iota^*$; hence we must show $\iota^*$ is injective in degrees $\leq 3l + 1$.

Now $D_\xi(\iota Z) = \iota (dZ_1) = 0$. On the other hand, for $n \geq 2$, Remark 3.8.4 shows that $Z_2^p = 0, p \leq 3l$. Thus $\text{Im} \, D_\xi \subset \sum_{j \geq 3l+2} Z_j^p$, and so $\iota^*$ is injective in degrees $\leq 3l + 1$.

Q.E.D.

8.12. THEOREM. The following are equivalent conditions on a $c$-connected, augmented c.g.d.a. $(A, d_A)$ with filtered model $(AZ, D)$:

(i) $H^+(AZ, D)$ is spherically generated.

(ii) $\ker \Sigma^* A = H^-(A) \cdot H^+(A)$.

(iii) The map $i^*: Z_0 \rightarrow H(Z, D_c)$ is injective.

(iv) $\pi E^2_* = \pi E^0_*$. 

(v) In the E.M.s.s. for $(A, d_A), E_{-1}^{-1,*} = E_{-1}^{-1,*}$.

Proof. (i) $\iff$ (ii) by definition; (ii) $\iff$ (iii) by (8.3) and (8.9); (iii) $\iff$ (iv) by (8.10); (v) restates (iv) by Remark 7.13.3. Q.E.D.

8.13. EXAMPLES. Let $(A, d_A)$ be a $c$-connected c.g.d.a. with filtered model $(AZ, D)$. We have just seen (cf. Theorem 8.12(v)) that

$(A, d_A)$ formal $\Rightarrow$ the E.M.s.s. collapses at the $E_0$ term $\Rightarrow H^+(AZ, D)$ is spherically generated.

Now we shall see that neither of these implications can be reversed.
First consider the c.g.d.a. of Example 4.3, \((AZ, D)\). We observed there that \((AZ, D)\) was not minimal; hence by Theorem 7.20(i), the E.M.s.s. does not collapse. [Indeed, it is easy, using Theorem 7.14 to find the non-trivial differential.] On the other hand \(Z^{p}_{n+1} = 0, p \leq 2(n + 2)\). Since \(Z^{p}_{0} = 0, p \geq 6\) and since \(D_{c}: Z \to Z\) raises degrees by 1, \(D_{c}(Z) \cap Z_{0} = 0\). Thus \(i^{*}: Z_{0} \to H(Z, D_{c})\) is injective, and so Theorem 8.12 shows that \(H(AZ, D)\) is spherically generated.

Next recall the c.g.d.a. \((AZ, D)\) of Example 6.5. We showed in Example 6.5 that \((AZ, D)\) was not formal. Now we show that the E.M.s.s. collapses at the \(E_{2}\)-term.

In fact, let \(H_{1} = H^{*}(S^{2} \vee S^{2}; k)\) and \(H_{2} = H^{*}(S^{n}; k) = \Lambda(x_{3})\). Then as we remarked in Example 6.5,

\[
H(AZ, D) = H(AZ, d) = H_{1} \otimes H_{2}.
\]

It follows that

\[
(AZ, d) = (AZ \otimes \Lambda(x_{3}), \tilde{d} \otimes 1),
\]

where \((AZ, \tilde{d})\) is the bigraded model for \(H_{1}\).

Thus with this identification we may write (noting that \(Dx_{3} = 0\))

\[
D(\Phi) = \tilde{d}\Phi \otimes 1 + \theta\Phi \otimes x_{3}, \quad \Phi \in A\tilde{Z},
\]

where \((A\tilde{Z}, \tilde{d})\) is a c.g.d.a., and \(\theta\) is a degree \(-2\) derivation of \(A\tilde{Z}\) such that \(\tilde{d}\theta = D\theta.\) Moreover

\[
(\tilde{d} - d)_{*}: Z_{n} \to F_{n-2}(A\tilde{Z}).
\]

By Example 6.4, \(H_{1}\) is intrinsically formal. Since \(\tilde{d}\) is a perturbation of \(d\), \(H(A\tilde{Z}, \tilde{d}) \approx H(A\tilde{Z}, \tilde{d}) \approx H_{1}\). Hence \((A\tilde{Z}, \tilde{d})\) is formal, and its E.M.s.s. collapses. In particular Theorem 7.20 shows that \(\tilde{d}(Z) \subseteq A^{+}Z \cdot A^{+}Z\).

On the other hand, since \(Z_{n}^{p} = 0, p \leq n + 1, x_{1}, x_{2}\) are the only elements of degree 2 in \(Z\), since \(Dx_{1} = 0, \theta(x_{1}) = 0\). Thus \(\tilde{Z} \to A^{1}\tilde{Z}\). Equation (8.14) now shows that \(D(Z) \subseteq A^{+}Z \cdot A^{+}Z\) and so Theorem 7.20 implies that the E.M.s.s. collapses at the \(E_{2}\)-term.

These examples involve ordinary Massey products \(\langle x, y, z \rangle\). In Example 6.5, \(\langle x_{1}, x_{1}, x_{2} \rangle\) contains \(x_{1}x_{3}\) and so vanishes as a coset although the obvious representative \(x_{2}y_{1} - x_{1}y_{3}\) is not cohomologous to zero. This first hint of subtlety in the statement "all Massey products vanish" is reflected in Sullivan's statement [10; p. 247]: "a minimal model is a formal consequence of its cohomology ring if and only if all the higher order products vanish in a uniform
way." It indicates why we have found our present obstruction theory much more useful than a version closer to the folk statement: \((A, d_A)\) is formal if and only if all matrix Massey products vanish.

**References**


34. E. B. Curtis, Some relations between homotopy and homology, *Ann. of Math.* 83 (1965), 386–413.

