JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 24, 290–295 (1968)

An Extension of a Theorem of A. M. Lyapunov to Semi-groups of Operators

R. DATKO

McGill University, Montréal, Canada

A well-known theorem of A. M. Lyapunov (see e.g. [1]) is the following: Let A be an $n \times n$ complex matrix. Then a necessary and sufficient condition that A have all its characteristic roots with real parts negative is that given any positive definite Hermitian matrix H there exists a unique positive definite Hermitian matrix B satisfying the equation $A^*B + BA = -H$ ($*$ denotes the conjugate transpose of a matrix). In this note the above result is extended in a natural manner to a general class of strongly continuous semigroups of operators defined on a Banach space X. The only restriction placed on X will be the requirement that it possess some continuous positive definite Hermitian forms.

NOTATION AND STATEMENT OF PRELIMINARY PROPERTIES

1. X will henceforth denote a complex Banach space. The norm on X and on the space of continuous endomorphisms from X into itself will be denoted by $\|\cdot\|$. It will be assumed that there exist continuous positive nondegenerate Hermitian forms from $X \times X \to C$ (complex numbers). This means that there exists at least one mapping $H : X \times X \to C$ such that if x, y and z are any vectors in X and λ is any complex number:

(i) $H(x + y, z) = H(x, z) + H(y, z)$,

(ii) $H(x, y) = \overline{H(y, x)}$,

(iii) $H(\lambda x, y) = \lambda H(x, y)$,

(iv) $H(x, \lambda y) = \bar{\lambda} H(x, y)$

(v) $H(x, x) > 0$ if $x \neq 0$ and

(vi) there exists a constant $0 < M < +\infty$ such that

$$| H(x, y) | \leqslant M \| x \| \| y \|$$

for all x, y in X. We denote the norm of H by $\| H \|$ and the set of all such continuous nondegenerate positive bilinear mappings by the symbol $\mathscr{H}$.

2. $\{T(t)\}$, $0 \leqslant t < \infty$, will denote a strongly continuous semigroup of endomorphisms on X such that $T(0) = I$ (the identity). Properties of such mappings can be found in [2] or [3].

3. Associated with the operators $\{T(t)\}$ is a closed linear operator A with dense domain $\mathscr{D}(A)$ in X. This operator is defined by the relation

$$\lim_{t\to 0} \frac{(T(t) - I)}{t} x = Ax$$

whenever the limit exists. A is called the infinitesimal generator of the semi-group.

4. The symbols $P\sigma(Q)$ and $\sigma(Q)$ will denote respectively the point spectrum and the spectrum of an endomorphism Q which is defined on X.

The following properties of $\{T(t)\}$ and A will be needed in the sequel and the proofs, with the exception of property 5, can be found in either [2] or [3].

PROPERTY 1 (See [2] p. 619). Let A be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}$. Then if $x \in \mathscr{D}(A)$:

(a) $T(t)\, x \in \mathscr{D}(A)$ for $0 \leqslant t < \infty$,
(b) $(d/dt)\,(T(t)\,x) = AT(t)\,x = T(t)\,Ax$ and
(c) $[T(t) - T(s)]\,x = \int_s^t T(u)\,Ax\,du$ for $0 \leqslant s \leqslant t$.

PROPERTY 2 (See [3] p. 467). Let $T(t)$ be a strongly continuous semi-group of operators with infinitesimal generator A. Then

$$P\sigma(T(t) = \exp tP\sigma(A),$$

plus, possibly the point $\lambda = 0$. If μ is in $P\sigma(T(t))$ for some fixed $t > 0$ where $\mu \neq 0$ and if α_n is the set of roots of $e^{\alpha t} = \mu$, then at least one of the points α_n lies in $P\sigma(A)$. The null manifold $\mathscr{N}[\mu I - T(t)]$ is the closed extension of the linearly independent null manifolds $\mathscr{N}[\alpha_n I - A]$, where n ranges over all α_n in $P\sigma(A)$.

PROPERTY 3 (See p. 182 [3]). Let T be a bounded compact endomorphism on a Banach space X. Aside from $\lambda = 0$ the points of the spectrum of T belong to $P\sigma(T)$ and are isolated.

PROPERTY 4 (See [2] or [3]). The spectral radius $r(T)$ of a bounded endomorphism T defined on a Banach space X is given by the relation

$$r(T) = \lim_{n\to\infty} \| T^n \|^{1/n}.$$

PROPERTY 5 (See [4]). Let $T(t)$ be a strongly continuous semigroup of operators. If for some $t_0 > 0$ the spectral radius $r(T(t_0))$ is $\neq 0$ and $Bt_0 = \ell n\, r(T(t_0))$, then for any $\nu > 0$, there is a finite constant $\kappa(\nu) \geqslant 1$ such that $\| T(t) \| \leqslant \kappa(\nu)\, e^{(\beta+\nu)t}$ for all $t \geqslant 0$.

THEOREM 1. *Suppose $T(t)$ is a strongly continuous semigroup of operators and that there exist constants $\kappa > 0$ and $\mu > 0$ such that $\| T(t) \| \leqslant \kappa e^{-\mu t}$ for all $t \geqslant 0$. Then given any $H \in \mathscr{K}$ there exists a unique B in $\mathscr{K}$ which satisfies the relations*

$$B(x, y) = \int_0^\infty H(T(t)\, x, T(t)\, y)\, dt \tag{1}$$

and

$$\begin{aligned} \frac{dB}{dt}(T(t)\, x, T(t)\, y) &= B(AT(t)\, x, T(t)\, y) + B(T(t)\, x, AT(t)\, y) \\ &= -H(T(t)\, x, T(t)\, y) \end{aligned} \tag{2}$$

for all x, y in $\mathscr{D}(A)$ and $t > 0$.

PROOF. Since, by hypothesis,

$$| H(T(t)\, x, T(t)\, y) | \leqslant \| H \|\, K^2 e^{-2\mu t} \| x \| \, \| y \|$$

it follows that B is well defined on $\mathscr{D}(A) \times \mathscr{D}(A)$ and hence on $X \times X$ since $\mathscr{D}(A)$ is dense on X. Moreover from (1) we obtain the estimate

$$| B(x, y) | \leqslant \frac{\| H \|\, K^2 \, \| x \| \, \| y \|}{2\mu}.$$

Hence B is continuous from $X \times X \to C$ and is a nondegenerate positive Hermitian form since H is. Thus $B \in \mathscr{K}$.

Equation (2) follows from the fact that if x and y are in $\mathscr{D}(A)$

$$\begin{aligned} \frac{d}{dt}[B(T(t)\, x, T(t)\, y)] &= \frac{d}{dt} \int_0^\infty H(T(t+s)\, x, T(t+s)\, y)\, ds \\ &= \int_0^\infty \frac{d}{ds} H(T(t+s)\, x, T(t+s)\, y)\, ds \\ &= H[T(t+s)\, x, T(t+s)\, y] \Big|_{s=0}^{s=\infty} \end{aligned} \tag{3}$$

The expression on the extreme right is a consequence of Property 1, and the bilinearity of H where the upper limit is allowed to tend to infinity. Expanding the extreme right and left sides of (3) and noting that by assumption $\lim_{t \to \infty} T(t) = 0$ for all x we obtain the relation (2).

THEOREM 2. *Assume $T(t)$ is a semigroup of operators which satisfies Eq.* (2) *where B and H are in $\mathscr{K}$. Then $P\sigma(A)$ lies in the left half plane* $\operatorname{Re}\lambda < 0$. *Moreover given any $x \in \mathscr{D}(A)$ the relation*

$$\int_0^\infty H(T(t)\,x,\, T(t)\,x)\,dt < +\infty \tag{4}$$

is valid.

PROOF. Let λ be in $P\sigma(A)$ and x a nonzero characteristic vector associated with λ. Then $T(t)\,x = e^{\lambda t}x$ for all $t \geqslant 0$. Define the scalar function

$$V(t) = B(T(t)\,x,\, T(t)\,x) = e^{2\operatorname{Re}\lambda t}B(x, x) = e^{2\operatorname{Re}\lambda t}V(0). \tag{5}$$

Since $B \in \mathscr{K}$ it follows that $V(t) > 0$ for all t if $x \neq 0$. Differentiating V and using equation (2) we obtain

$$\frac{dV}{dt} = 2\operatorname{Re}\lambda V(t) = -\,e^{2\operatorname{Re}\lambda t}H(x, x) < 0. \tag{6}$$

Hence $\operatorname{Re}\lambda < 0$ which proves the first part of the theorem.

Next assume $x \in \mathscr{D}(A)$ and Eq. (2) is satisfied. Then

$$B(T(t)\,x,\, T(t)\,x) = B(x, x) - \int_0^t H(T(s)\,x,\, T(s)\,x)\,ds. \tag{7}$$

But $B(x, x) \geqslant 0$ for all $x \in \mathscr{D}(A)$. Hence from (7) we obtain

$$\lim_{t\to\infty} \int_0^t H(T(s)\,x,\, T(s)\,x)\,ds \leqslant B(x, x),$$

which proves the second part of the theorem.

Theorem 2 can be looked upon as a stability result in the sense that if we renorm the space X into a pre-Hilbert space with norm $\|x\| = [H(x, x)]^{1/2}$ then the trajectories $T(t)\,x$ have square integrable norms in this space. The following two corrollaries to theorem 2 indicate how stability results can be obtained in the original norm provided we specialize the conditions on the semi-group $T(t)$.

COROLLARY 1. *Suppose there exist B and H in $\mathscr{K}$ which satisfy Eq.* (2) *and in addition there exists a constant $M > 0$ such that $B(x, x) \leqslant MH(x, x)$ for all x in X. Then $P\sigma(A)$ lies in the half plane* $\operatorname{Re}\lambda \leqslant -1/2M$.

PROOF. Let x be a nonzero characteristic vector associated with λ in $P\sigma(A)$. Then from Eq. (2) we obtain

$$2\operatorname{Re}\lambda B(x, x) = -\,H(x, x) \leqslant -\frac{B(x, x)}{M}$$

from which the conclusion of the corollary follows.

COROLLARY 2. *Suppose $T(t_0)$ is a compact operator for some $t_0 > 0$ and there exist H and B in $\mathscr{K}$ which satisfy Eq.* (2). *Then there exist constants $\kappa > 0$ and $\mu > 0$ such that $\| T(t) \| \leqslant \kappa e^{-\mu t}$ for all $t \geqslant 0$.*

PROOF. From property 3 it follows that the spectrum of $T(t_0)$ consists only of isolated points of $P\sigma(T(t_0))$, plus, possibly the point 0. Hence the spectral radius of $T(t_0)$ is determined by the point spectrum of $T(t_0)$. By Theorem 2 Re $\lambda < 0$ for all λ in $P\sigma(A)$ and by property 2 this implies, since $\tau = e^{\lambda t_0}$ for some $\lambda \in P\sigma(A)$ if $\tau \in P\sigma(T(t_0))$, that $| \tau | < 1$ for all τ in $\sigma(T(t_0))$. Hence, since $\sigma(T(t_0))$ can have an accumulation point only at $\lambda = 0$, it follows that there exists an α with $0 < \alpha < 1$ such that the spectrum of $T(t_0)$ lies in the interior of the circle $| \lambda | \leqslant \alpha$. Thus the spectral radius

$$r(T(t_0)) \leqslant \alpha = e^{-\beta t_0} < 1.$$

Choosing $V > 0$ and such that $-\beta + V < 0$ and applying property 5, where we let $\mu = -(-\beta + V)$ and $\kappa = K(V)$, yields the result.

An immediate consequence of the second corollary to Theorem 2 and Theorem 1 is:

THEOREM 3. *Let X be a Banach space over the complex numbers on which there exist nondegenerate positive Hermitian forms. Let $T(t)$ be a strongly continuous semigroup of operators on X and suppose $T(t_0)$ is compact for some $t_0 > 0$. Then $T(t)$ will satisfy the inequality*

$$\| T(t) \| \leqslant \kappa e^{-\mu t} \qquad \text{for all} \qquad t \geqslant 0,$$

where κ and μ are positive, if and only if given any $H \in \mathscr{K}$ there exists a $B \in \mathscr{K}$ which satisfies Eq. (2).

PROOF. From Theorem 1 it follows that if $\| T(t) \| \leqslant \kappa e^{-\mu t}$ then given any H in $\mathscr{K}$ there exists a unique B in $\mathscr{K}$ which satisfies Eqs. (1) and (2).

Conversely, if for any $H \in \mathscr{K}$ there exists a $B \in \mathscr{K}$ which satisfies Eq. (2) we know from the conclusion of Corollary 2 to Theorem 2 that $\| T(t) \| \leqslant \kappa e^{-\mu t}$. We then apply Theorem 1 to H to obtain a unique B in $\mathscr{K}$ satisfying Eqs. (1) and (2).

REMARK. Theorem 3 gives a necessary and sufficient, condition for the asymptotic stability (i.e., the condition that $\| T(t) \| \leqslant \kappa e^{-\mu t}$) of linear differential difference equations of the type studied by Hale [5] and Krasovskii [6]. There the underlying space is a real Banach space, but this presents no problem since it can be extended to a complex Banach space on which there exist positive non-degenerate Hermitian forms.

References

1. R. Bellman. "Introduction to Matrix Analysis". McGraw-Hill, New York, 1960.
2. N. Dunford and J. T. Schwartz. "Linear Operators," Part I. Wiley (Interscience), New York, 1958.
3. E. Hille and P. S. Phillips. "Functional Analysis and Semi-Groups." A. M. S. Colloquium Publications Vol. XXXI, 1957.
4. A. Stokes. A Floquet theory for functional differential equations. *Proc. Nat. Acad. Sci.* **48** (1962), 1330–1334.
5. J. K. Hale. Linear functional equations with constant coefficients. *Cont. Diff. Eqs.* **1** (1962), 291–317.
6. N. N. Krasovskii. "Stability of Motion" Stanford Univ. Press, Stanford, 1963.