Bivariate quartic spline spaces and quasi-interpolation operators

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Received 31 August 2004

Dedicated to Professor Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

In this paper, we study two bivariate quartic spline spaces \( S^{3,2}_{4} (A_{mn}^{(2)}) \) and \( S^{2,3}_{4} (A_{mn}^{(2)}) \), and present two classes of quasi-interpolation operators in the two spaces, respectively. Some results on the operators are given.

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Keywords: Bivariate spline space; Type-2 triangulation; Quasi-interpolation operator

1. Introduction

Let \( D = [0, m] \times [0, n] \), where \( m \) and \( n \) are positive integers, and partition \( D \) with a four-directional mesh (or type-2 triangulation) by using the grid lines:

\[ x - i = 0, \quad y - i = 0, \quad x - y - i = 0, \quad x + y - i = 0, \]

where \( i = \ldots, -1, 0, 1, \ldots \). The type-2 triangulation (denoted by \( A_{mn}^{(2)} \)) is a cross-cut partition.
The space of all polynomials of total degree \( k \) in two variables is denoted by \( \mathbb{P}_k \). Let \( \mu \) be a nonnegative integer, and \( S^\mu_k(\Delta_c) \) denote the space of all spline functions in \( C^\mu(D) \) whose restrictions on the cells in \( D \) determined by any cross-cut partition \( \Delta_c \) are polynomials in \( \mathbb{P}_k \).

The dimension of \( S^\mu_k(\Delta_c) \) was presented in [1,4] as the following theorem:

**Theorem 1.1.** Let \( D \) be a simply connected domain in \( \mathbb{R}^2 \) and \( \Delta_c \) a cross-cut partition of \( D \) with \( L \) cross-cuts and \( V \) interior grid points \( A_1, \ldots, A_V \) in \( D \), such that \( N_i \) cross-cuts intersect at \( A_i \), \( i = 1, \ldots, V \).

Then, the dimension of the bivariate spline space \( S^\mu_k(\Delta_c) \), \( 0 \leq \mu \leq k - 1 \), is

\[
\dim S^\mu_k(\Delta_c) = \left( \frac{k + 2}{2} \right) + L \cdot \left( \frac{k - \mu + 1}{2} \right) + \sum_{i=1}^{V} d^\mu_k(N_i),
\]

where

\[
d^\mu_k(n) := \frac{1}{2} \left( k - \mu - \left\lfloor \frac{\mu + 1}{n - 1} \right\rfloor \right) + \left( n - 1 \right) k - (n + 1) \mu + (n - 3) + (n - 1) \left\lfloor \frac{\mu + 1}{n - 1} \right\rfloor,
\]

[x] denotes the largest integer \( \leq x, u_+ = \max(0, u) \).

By Theorem 1.1, a necessary condition for the existence of a nontrivial locally supported spline in \( S^\mu_k(\Delta_c) \) is that \( N_i \) must satisfy the basic inequality [1,4]

\[
N_i > \frac{k + 1}{k - \mu}.
\]

Since every interior grid point in \( D \) is the intersection of exactly four lines from the grid partition \( \Delta_{mn}^{(2)} \), the degree \( k \) and the smoothness \( \mu \) must satisfy

\[
\mu < \frac{3k - 1}{4}.
\]

It is easy to see that \( \mu \leq 2 \) when \( k = 4 \). Therefore, the quartic spline space with the highest possible uniform smoothness is \( S^2_4(\Delta_{mn}^{(2)}) \).

In order to raise the smoothness, we consider different smoothness on different grid segments. Let the smoothness on the rectangle grid segments and the diagonal grid segments be \( \mu_1 \) and \( \mu_2 \), respectively. The spline space with degree \( k \) and this kind of smoothness (denoted by \( C^{\mu_1,\mu_2} \)) on the type-2 triangulation is denoted by \( S^{\mu_1,\mu_2}_k(\Delta_{mn}^{(2)}) \). For quartic spline, there are two spaces \( S^{3,2}_4(\Delta_{mn}^{(2)}) \) and \( S^{2,3}_4(\Delta_{mn}^{(2)}) \). We discuss the two quartic spline spaces and corresponding quasi-interpolation operators in the following sections, respectively.

2. **The bivariate spline space \( S^{3,2}_4(\Delta_{mn}^{(2)}) \)**

2.1. **The dimension of \( S^{3,2}_4(\Delta_{mn}^{(2)}) \)**

A spline \( s \in S^{3,2}_4(\Delta_{mn}^{(2)}) \) is a piecewise polynomial of degree 4 with the following two continuous conditions:

(a) \( s \) is \( C^3 \) continuous on the rectangle grid segments: \( x - i = 0, y - i = 0; \)
(b) $s$ is $C^2$ continuous on the diagonal grid segments: $x - y - i = 0, x + y - i = 0$, where $i = \ldots, \ldots, -1, 0, 1, \ldots$.

According to formula (1), denote the corresponding last term in the dimension formula of $S_{4,2,4}^{3,2}(A_{mn}^{(2)})$ by $d_{4}^{3,2}(4)$. It equals the dimension of the conformality condition $[3, 4]$ at one grid point of $A_{mn}^{(2)}$ in $D$ as follows:

$$\sum_{i=1}^{2} a_i (x + x_i y)^4 + \sum_{i=3}^{4} (a_i x + b_i y + c_i) (x + x_i y)^3 \equiv 0,$$

where $x + x_i y$ denote the four intersecting grid lines, $x_i \in \mathbb{R}$ ($i = 1, 2, 3, 4$) are different from each other, and $a_1, a_2, a_3 x + b_3 y + c_3, a_4 x + b_4 y + c_4$ denote $C^3$ and $C^2$ smoothing cofactors of the four grid lines, respectively. The conformality condition is equivalent to the following system of homogeneous linear equations with $a_1, a_2, a_3, a_4, b_3, b_4, c_3, c_4$ as unknown quantities:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
4x_1 & 4x_2 & 3x_3 & 3x_4 & 1 & 1 & 0 & 0 \\
6x_1 & 6x_2 & 3x_3 & 3x_4 & 3x_2 & 3x_4 & 0 & 0 \\
4x_1 & 4x_2 & x_3 & x_4 & 3x_2 & 3x_4 & 0 & 0 \\
x_1^4 & x_2^4 & 0 & 0 & x_3^3 & x_4^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 3x_3 & 3x_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 3x_3^2 & 3x_4^2 \\
0 & 0 & 0 & 0 & 0 & 0 & x_3^3 & x_4^3
\end{pmatrix} \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
b_3 \\
b_4 \\
c_3 \\
c_4
\end{pmatrix} = 0.$$

It is easy to see that the rank of the coefficient matrix is 7, and hence $d_{4}^{3,2}(4) = 1$. By formula (1), the dimension of the bivariate quartic spline space $S_{4,2,4}^{3,2}(A_{mn}^{(2)})$ is

$$\dim S_{4,2,4}^{3,2}(A_{mn}^{(2)}) = \left(\begin{array}{c}4 + 2 \\ 2\end{array}\right) + (m - 1) + (n - 1) + 2(m + n - 1) \cdot 3 + (m - 1)(n - 1)d_{4}^{3,2}(4)$$

$$= mn + 6m + 6n + 8. \quad (5)$$

2.2. The basis of $S_{4,2,4}^{3,2}(A_{mn}^{(2)})$

By using the smoothing cofactor-conformality method $[3, 4]$, we obtain a spline denoted by $A(x, y)$ whose support and Bézier representation are shown in Fig. 1, where the center of the support is at $(1/2, 1/2)$. Here, the Bézier coefficients (B-net) which are not shown are either zero or can be easily obtained by symmetry, and in order to avoid fractions, we have multiplied the coefficients by 384. The local support of $A(x, y)$ is minimal.

Denote

$$A_{i,j} = A_{i,j}(x, y) = A(x - i, y - j),$$
where \( i, j = \ldots, -1, 0, 1, \ldots \). It is clear that the index set for which the functions \( A_{i,j} \) do not vanish identically on \( D \) is

\[
E = \{ (i, j) = (x, \beta) : i \neq (-2, -2), (m + 1, -2), (-2, n + 1), (m + 1, n + 1), \\
\quad -2 \leq x \leq m + 1, -2 \leq \beta \leq n + 1 \}.
\]

By checking the sums of the appropriate Bézier coefficients, we have the following identities:

**Theorem 2.1.** For all \((x, y) \in D\),

\[
\sum_{(i, j) \in E} (-1)^{i+j} A_{i,j}(x, y) = 0, \tag{6}
\]

\[
\sum_{(i, j) \in E} A_{i,j}(x, y) = 1. \tag{7}
\]

Because the number of \( A_{i,j} \) (equals to the cardinality of \( E \)), namely

\[mn + 4m + 4n + 12\]

is less than the dimension of \( S_4^{3,2}(A_{mn}^{(2)})\), \( \{ A_{i,j} : (i, j) \in E \} \) cannot be a spanning set of all of \( S_4^{3,2}(A_{mn}^{(2)})\).

It is the same as the case of \( S_4^2(A_{mn}^{(1)}) \) in paper [2]. We add \( 2(m + n) \) truncated powers \((x + y - r)_+^4, (x - y - s)_+^4, r = 0, \ldots, m + n - 1; s = -n, \ldots, m - 1\) to construct the basis of \( S_4^{3,2}(A_{mn}^{(2)})\). We have the following result:

**Theorem 2.2.** Let \((x_0, \beta_0), -1 \leq x_0 \leq m and -1 \leq \beta_0 \leq n, be arbitrarily chosen, and set

\[
E_1 = E \setminus \{(x_0, \beta_0), (x_0, \beta_0 + 1), (x_0, \beta_0 - 1), (x_0 - 1, \beta_0)\},
\]

\[
E_2 = E \setminus \{(x_0, \beta_0), (x_0, \beta_0 + 1), (x_0, \beta_0 - 1), (x_0 + 1, \beta_0)\},
\]

\[
E_3 = E \setminus \{(x_0, \beta_0), (x_0 + 1, \beta_0), (x_0 - 1, \beta_0), (x_0, \beta_0 + 1)\},
\]
\[ E_4 = E \setminus \{(x_0, \beta_0), (x_0 + 1, \beta_0), (x_0 - 1, \beta_0), (x_0, \beta_0 - 1)\}, \]
\[ E_5 = E \setminus \{(x_0 - 1, \beta_0), (x_0 + 1, \beta_0), (x_0, \beta_0 - 1), (x_0, \beta_0 + 1)\}. \]

Then, each of the five collections
\[ \mathcal{A}_k = \{A_{i,j}, (x + y - r)^q_x, (x - y - s)^q_y : (i, j) \in E_k, r = 0, \ldots, m + n - 1; s = -n, \ldots, m - 1\}, \]
k = 1, 2, 3, 4, 5 is a basis of \( S^{3,2}_4(A_{mn}^{(2)}) \).

Since the cardinality of each \( \mathcal{A}_k \) is the same as the dimension of \( S^{3,2}_4(A_{mn}^{(2)}) \), it is sufficient to prove that each \( \mathcal{A}_k \) is a linearly independent set on \( D \). This can be done by following the proof of Theorem 3.1 in paper [2].

Furthermore, we can prove that there are no other locally supported splines that can be used in place of some of the basis elements \( (x + y - r)^q_x, (x - y - s)^q_y \) in the basis \( \mathcal{A}_k \) of \( S^{3,2}_4(A_{mn}^{(2)}) \).

To see this, we consider the following proper subspace:
\[ lS^{3,2}_4(A_{mn}^{(2)}) := \text{span}\{A_{i,j} : (i, j) \in E\}. \]
It can also be considered as a subspace of the space
\[ \text{loc } S^{3,2}_4(A, \mathbb{R}^2) \]
of all bivariate \( C^{3,2} \) quartic spline functions on \( \mathbb{R}^2 \) with the grid partition \( \Lambda : x - i = 0, y - i = 0, x - y - i = 0, x + y - i = 0, i = \ldots, -1, 0, 1, \ldots \), which vanish identically outside some bounded sets containing \( D \). A spline \( s \) in \( S^{3,2}_4(A_{mn}^{(2)}) \) will be called locally supported relative to \( D \) if \( s \) is the restriction on \( D \) of some function in \( \text{loc } S^{3,2}_4(A, \mathbb{R}^2) \).

Let \( a, b, c, d \) be integers with \( b - a, d - c \geq 5 \) and \( G = [a, b] \times [c, d] \), and consider the subspace
\[ \text{loc } S^{3,2}_4(A, G) \]
of all functions in \( \text{loc } S^{3,2}_4(A, \mathbb{R}^2) \) whose supports lie in \( G \). Note that each basis element has a distinct support. In particular, the supports of the truncated powers are different half plane domains. It is clear that

**Lemma 2.1.** The space \( \text{loc } S^{3,2}_4(A, G) \) is of dimension \( (b - a - 4) \times (d - c - 4) \) and has a basis given by
\[ \mathcal{C} = \{A_{i,j} : i = a + 2, \ldots, b - 3; j = c + 2, \ldots, d - 3\}. \]

By Theorem 2.2 and Lemma 2.1, we have the following result:

**Theorem 2.3.** \( lS^{3,2}_4(A_{mn}^{(2)}) \) is the space of all functions in \( S^{3,2}_4(A_{mn}^{(2)}) \) which are locally supported relative to \( D \), and
\[ \dim lS^{3,2}_4(A_{mn}^{(2)}) = mn + 4m + 4n + 8. \]
Furthermore, each of the five collections \( \{A_{i,j} : (i, j) \in E_k\}, k = 1, 2, 3, 4, 5 \) is a basis of \( lS^{3,2}_4(A_{mn}^{(2)}) \).
It means that all locally supported functions in $S^{3,2}_4(A^{(2)}_{mn})$ can only span a proper subspace of $S^{3,2}_4(A^{(2)}_{mn})$.

2.3. The quasi-interpolation operators in $S^{3,2}_4(A^{(2)}_{mn})$

Let $D' = [0, 1] \times [0, 1]$, the grid lines of type-2 triangulation on $D'$ be

$$mx - i = 0, \quad ny - i = 0, \quad mx - ny - i = 0, \quad mx + ny - i = 0,$$

where $i = \ldots, -1, 0, 1, \ldots$. Then, the basis functions on $D'$ are

$$A'_{i,j} = A'_{i,j}(x, y) = A(mx - i, ny - j),$$

where $i, j = \ldots, -1, 0, 1, \ldots$. It is clear that the index set for which the functions $A'_{i,j}$ do not vanish identically on $D'$ is $E$ as well.

Define a variation diminishing operator $V_{mn} : C(\Omega) \rightarrow S^{3,2}_4(A^{(2)}_{mn})$:

$$V_{mn}(f) = \sum_{(i,j) \in E} f(x_i, y_j)A'_{i,j}, \quad (8)$$

where $\Omega$ denotes an open set containing $D'$, $(x_i, y_j) = ((2i + 1)/2m, (2j + 1)/2n)$ are the centers of the supports of $A'_{i,j}$, $(i, j) \in E$. Note that $V_{mn}$ is a linear operator, and by computing we have the following result:

**Theorem 2.4.** For all $(x, y) \in D'$,

$$f \in \mathbb{P}_1 \cup \text{span}\{xy\}, \quad V_{mn}(f) = f,$$

$$f = x^2, \quad V_{mn}(f) = x^2 + \frac{5}{12m^2}, \quad f = y^2, \quad V_{mn}(f) = y^2 + \frac{5}{12n^2},$$

$$f = x^3, \quad V_{mn}(f) = x^3 + \frac{5}{4m^2}x, \quad f = y^3, \quad V_{mn}(f) = y^3 + \frac{5}{4n^2}y,$$

$$f = x^2y, \quad V_{mn}(f) = x^2y + \frac{5}{12m^2}y, \quad f = xy^2, \quad V_{mn}(f) = xy^2 + \frac{5}{12n^2}x,$$

$$f = x^4, \quad V_{mn}(f) = x^4 + \frac{5}{2m^2}x^2 + \frac{23}{48m^4}, \quad f = y^4, \quad V_{mn}(f) = y^4 + \frac{5}{2n^2}y^2 + \frac{23}{48n^4}.$$
In order to preserve identities for all polynomials in \( \mathbb{P}_2 \) and \( \mathbb{P}_3 \), we define another linear operator \( W_{mn} : C(\Omega) \to S^{3,2}_4(\Delta_{mn}) \):

\[
W_{mn}(f) = \sum_{(i,j) \in E} \lambda_{i,j}(f)A'_{i,j},
\]

where

\[
\lambda_{i,j}(f) = \frac{8}{3} f \left( \frac{2i + 1}{2m}, \frac{2j + 1}{2n} \right) - \frac{5}{12} \left[ f \left( \frac{i}{m}, \frac{j}{n} \right) + f \left( \frac{i+1}{m}, \frac{j}{n} \right) + f \left( \frac{i}{m}, \frac{j+1}{n} \right) + f \left( \frac{i+1}{m}, \frac{j+1}{n} \right) \right].
\]

Note that each linear functional \( \lambda_{i,j} \) depends on five function values of \( f \) at the grid points in the support of \( A'_{i,j} \). We have the following result:

**Theorem 2.5.** For all \((x, y) \in D'\),

\[
\begin{align*}
f & \in \mathbb{P}_3, \quad W_{mn}(f) = f, \\
f = x^4, \quad W_{mn}(f) = x^4 - \frac{2}{3m^4}, & \quad f = y^4, \quad W_{mn}(f) = y^4 - \frac{2}{3n^4}, \\
f = x^5, \quad W_{mn}(f) = \frac{5}{2m} x^4 - \frac{5}{3m^2} x^3 - \frac{19}{6m^4} x + \frac{5}{m} \sum_{i=1}^{m-1} \left( x - \frac{i}{m} \right)^4, & \\
f = y^5, \quad W_{mn}(f) = \frac{5}{2n} y^4 - \frac{5}{3n^2} y^3 - \frac{19}{6n^4} y + \frac{5}{n} \sum_{j=1}^{n-1} \left( y - \frac{j}{n} \right)^4, \\
f = x^6, \quad W_{mn}(f) = \frac{5}{2m^2} x^4 - \frac{23}{2m^4} x^2 - \frac{65}{24m^6} + \frac{30}{m} \sum_{i=1}^{m-1} \frac{i}{m} \left( x - \frac{i}{m} \right)^4, & \\
f = y^6, \quad W_{mn}(f) = \frac{5}{2n^2} y^4 - \frac{23}{2n^4} y^2 - \frac{65}{24n^6} + \frac{30}{n} \sum_{j=1}^{n-1} \frac{j}{n} \left( y - \frac{j}{n} \right)^4.
\end{align*}
\]

2.4. Some properties of the quasi-interpolation operators

We consider the uniform approximation to \( S^{3,2}_4(\Delta_{mn}) \) by the spline series \( \sum_{(i,j) \in E} \lambda_{i,j}A'_{i,j} \). The case of \( S^{1,2}_2(\Delta_{mn}) \) was discussed in paper [5]. The Euclid norm of the ordered pair \((x, y)\) is defined by

\[
|(x, y)| = (x^2 + y^2)^{1/2}.
\]

Let \( K \subset \mathbb{R}^2 \) be a compact set, and denote the continuous module of \( f \in C(K) \) by

\[
w_K(f; \delta) = \sup\{ |f(x, y) - f(u, v)| : (x, y), (u, v) \in K, |(x, y) - (u, v)| < \delta \}.
\]
and

\[ \delta_{mn} = \max \left( \frac{1}{m}, \frac{1}{n} \right), \]

\[ \delta'_{mn} = \frac{1}{2mn} \max(\sqrt{25m^2 + n^2}, \sqrt{m^2 + 25n^2}). \]

Let \( k \) be a positive integer, and denote

\[ \mathcal{D}^k f = \sum_{l=0}^{k} C_k^l q^l \partial_x^{k-l} \partial_y^l, \]

where \( l = 0, \ldots, k \). Denote

\[ \omega^k f = \max_{l=0, \ldots, k} w_{D'}(f, \delta_{mn}/2), \]

\[ \| \mathcal{D}^k f \| = \max_{l=0, \ldots, k} \sup_{(x,y) \in D'} |f_{x^{k-l}, y^l}(x, y)|. \]

Let the compact set \( K \) be the closure of the open set \( D' \) containing \( D' \). For sufficiently large numbers \( m \) and \( n \) (e.g., \( m, n \geq N_0 \)), the centers of the supports of all \( A'_{i,j} \) are located in the interior of \( K \). Let \( \| \cdot \|_{D'} \) be the supremum over \( D' \). We have the following result on \( V_{mn} \):

**Theorem 2.6.** Let \( f \in C(K) \), for all \( m, n \geq N_0 \),

\[ \| f - V_{mn}(f) \|_{D'} \leq \omega_K f, \]  

if \( f \in C^1(D') \), then

\[ \| f - V_{mn}(f) \|_{D'} \leq 2\delta_{mn} \omega^1 f, \]  

if \( f \in C^2(D') \), then

\[ \| f - V_{mn}(f) \|_{D'} \leq \delta_{mn}^2 \| \mathcal{D}^2 f \|. \]  

**Proof.** (1) Note that \( \delta'_{mn} \) is the semi-diameter of the support of \( A'_{i,j} \). By Theorem 2.4 it is easy to obtain (10).

(2) When \( f \in C^1(D') \), let \( T \) denote the triangular cell in \( A^{(2)}_{mn} \), such that

\[ \| f - V_{mn}(f) \|_{D'} = \| f - V_{mn}(f) \|_T. \]

Let \((x_0, y_0)\) be the point \((i/m, (j+1/2)/n)\) or \((((i+1/2)/m, j/n)\) \( \in T \), then

\[ \forall (x, y) \in T, \quad |x - x_0|, \quad |y - y_0|, \quad |(x, y) - (x_0, y_0)| \leq \delta_{mn}/2. \]

Denote

\[ p_1(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0), \]
then, by the Taylor expansion
\[ f(x, y) = p_1(x, y) + (f_x(x, y) - f_x(x_0, y_0))(x - x_0) + (f_y(x, y) - f_y(x_0, y_0))(y - y_0), \]
where \((u, v) = t(x, y) + (1 - t)(x_0, y_0), 0 \leq t \leq 1.\]

By Theorem 2.4 and \(\|V_{mn}\| = 1\), we have
\[ \|f - V_{mn}(f)\|_T \leq \|f - p_1\|_T + \|V_{mn}(f - p_1)\|_T \leq 2\|f - p_1\|_T \leq 2(\delta_{mn}/2+\delta_{mn}/2 + \delta_{mn}/2) \leq 2\delta_{mn} \omega f. \]

(3) When \(f \in C^2(D')\), by the Taylor expansion
\[ f(x, y) = p_1(x, y) + \frac{1}{2} \left( (x - x_0) \frac{\partial}{\partial x} + (y - y_0) \frac{\partial}{\partial y} \right)^2 f(u, v), \]
then
\[ \|f - V_{mn}(f)\|_T \leq 2\|f - p_1\|_T \leq 2 \times \frac{1}{2} (\delta_{mn}/2 + \delta_{mn}/2)^2 \|D^2 f\| = \delta_{mn}^2 \|D^2 f\|. \quad \Box \]

For the approximation of the operator \(W_{mn}\), we can make some improvements.

**Theorem 2.7.** Let \(f \in C(K)\), for all \(m, n \geq N_0\), if \(f \in C^2(D')\), then
\[ \|f - W_{mn}(f)\|_{D'} \leq \frac{8}{3} \delta_{mn}^2 \omega^2 f, \]
if \(f \in C^3(D')\), then
\[ \|f - W_{mn}(f)\|_{D'} \leq \frac{8}{3} \delta_{mn}^3 \omega^3 f, \]
if \(f \in C^4(D')\), then
\[ \|f - W_{mn}(f)\|_{D'} \leq \frac{8}{3} \delta_{mn}^4 \|D^4 f\|. \]

**Proof.** (1) When \(f \in C^2(D')\), by the Taylor expansion
\[ f(x, y) = p_2(x, y) + \frac{1}{2} \left( (f_{xx}(x_0, y_0)) - (f_{xx}(x, y)) (x - x_0)^2 + 2(f_{xy}(x, y) - f_{xy}(x_0, y_0)) (x - x_0)(y - y_0) \right), \]
where
\[ p_2(x, y) = p_1(x, y) + \frac{1}{2} \{ f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 \}. \]

By Theorem 2.5 and \(\|W_{mn}\| \leq 13/3\), we have
\[ \|f - W_{mn}(f)\|_T \leq \frac{16}{3} \|f - p_2\|_T \leq \frac{16}{3} \times \frac{1}{2} \times 4 \omega^2 f \times \left( \frac{\delta_{mn}}{2} \right)^2 = \frac{8}{3} \delta_{mn}^2 \omega^2 f. \]

(2) When \(f \in C^3(D')\), by the Taylor expansion
\[ f(x, y) = p_3(x, y) + \frac{1}{6} \left( (f_{xxx}(x_0, y_0)) - (f_{xxx}(x, y)) (x - x_0)^3 + 3(f_{xxy}(x, y) - f_{xxy}(x_0, y_0)) (x - x_0)^2 (y - y_0) + 3(f_{xyy}(x, y) - f_{xyy}(x_0, y_0)) (x - x_0)(y - y_0)^2 \right), \]
where
\[ p_3(x, y) = p_1(x, y) + \frac{1}{6} \{ f_{xxx}(x_0, y_0)(x - x_0)^3 + 3f_{xxy}(x_0, y_0)(x - x_0)^2(y - y_0) + 3f_{xyy}(x_0, y_0)(x - x_0)(y - y_0)^2 \}. \]
where
\[ p_3(x, y) = p_2(x, y) + \frac{1}{6} \{ f_{xxx}(x_0, y_0)(x - x_0)^3 + 3f_{xxy}(x_0, y_0)(x - x_0)^2(y - y_0) \\
+ 3f_{xyy}(x_0, y_0)(x - x_0)(y - y_0)^2 + f_{yyy}(x_0, y_0)(y - y_0)^3 \} . \]

Then
\[ \| f - W_{mn}(f) \|_T \leq \frac{16}{3} \| f - p_3 \|_T \leq \frac{16}{3} \times \frac{1}{6} \times 8\omega^3 f \times \left( \frac{\delta_{mn}}{2} \right)^3 = \frac{8}{9} \delta_{mn}^3 \omega^3 f. \]

(3) When \( f \in C^4(D') \), by the Taylor expansion
\[ f(x, y) = p_3(x, y) + \frac{1}{24} \left( \frac{\partial}{\partial x} (x - x_0) + (y - y_0) \frac{\partial}{\partial y} \right)^4 f(u, v). \]

Then
\[ \| f - W_{mn}(f) \|_T \leq \frac{16}{3} \| f - p_3 \|_T \leq \frac{16}{3} \times \frac{1}{24} (\delta_{mn}/2 + \delta_{mn}/2)^4 \| D^4 f \| = \frac{2}{9} \delta_{mn}^4 \| D^4 f \|. \]

\section{The bivariate spline space \( S_{2,3}^{2,3}(\Lambda_{mn}) \)}

\subsection{The dimension and basis of \( S_{2,3}^{2,3}(\Lambda_{mn}) \)}

Similar to the discussion on \( S_{3,2}^{3,2}(\Lambda_{mn}) \), we can obtain some corresponding results on \( S_{2,3}^{2,3}(\Lambda_{mn}) \) as follows:

A spline \( s \in S_{2,3}^{2,3}(\Lambda_{mn}) \) is a piecewise polynomial of degree 4 with the following two continuous conditions:

(a) \( s \) is \( C^2 \) continuous on the rectangle grid segments: \( x - i = 0, y - i = 0 \);
(b) \( s \) is \( C^3 \) continuous on the diagonal grid segments: \( x - y - i = 0, x + y - i = 0 \), where \( i = \ldots, -1, 0, 1, \ldots \).

By formula (1), and \( d_{2,3}^{2,3}(4) = d_{4}^{3,2}(4) \), the dimension of the bivariate quartic spline space \( S_{2,3}^{2,3}(\Lambda_{mn}) \) is
\[ \dim S_{2,3}^{2,3}(\Lambda_{mn}) = \left( \begin{array}{c} 4 + 2 \\ 2 \end{array} \right) + 3[(m - 1) + (n - 1)] + 2(m + n - 1) + (m - 1)(n - 1)d_{4}^{2,3}(4) \\
= mn + 4m + 4n + 8. \] (16)

By using the smoothing cofactor-conformality method, we obtain another spline denoted by \( B(x, y) \) whose support and Bézier representation are shown in Fig. 2, where the center of the support is at \((0, 0)\). The Bézier coefficients (B-net) which are not shown are either zero or can be easily obtained by symmetry, and in order to avoid fractions, we have multiplied the coefficients by 192. The local support of \( B(x, y) \) is minimal.

Denote
\[ B_{i,j} = B_{i,j}(x, y) = B(x - i, y - j), \]
where \( i, j = \ldots, -1, 0, 1, \ldots \). It is clear that the index set for which the functions \( B_{i,j} \) do not vanish identically on \( D \) is

\[
F = \{(i, j) = (x, \beta) : -1 \leq x \leq m + 1, -1 \leq \beta \leq n + 1\}.
\]

By checking the sums of the appropriate Bézier coefficients, we have the following identities:

**Theorem 3.1.** For all \( (x, y) \in D \),

\[
\sum_{(i,j) \in F} (-1)^{i+j} B_{i,j}(x, y) = 0, \quad (17)
\]

\[
\sum_{(i,j) \in F} B_{i,j}(x, y) = 1. \quad (18)
\]

Because the number of \( B_{i,j} \) (equals to the cardinality of \( F \)), namely

\[
mn + 3m + 3n + 9
\]

is less than the dimension of \( S^2_4(D^{(2)}_{mn}) \), \( \{B_{i,j} : (i, j) \in F\} \) cannot be a spanning set of all of \( S^2_4(D^{(2)}_{mn}) \).

We add \( m + n \) truncated powers \((x - r)^{4_+}, (y - s)^{4_+}, r = 0, \ldots, m - 1; s = 0, \ldots, n - 1\) to construct the basis of \( S^2_4(D^{(2)}_{mn}) \). We have the following result:

**Theorem 3.2.** Let \((x_0, \beta_0) \in F\) be arbitrarily chosen. Then the collection

\[
\mathcal{B} = \{B_{i,j}, (x - r)^{4_+}, (y - s)^{4_+} : (i, j) \in F \setminus \{(x_0, \beta_0)\}, r = 0, \ldots, m - 1; s = 0, \ldots, n - 1\}
\]

is a basis of \( S^2_4(D^{(2)}_{mn}) \).
Similarly, we can prove that there are no other locally supported splines that can be used in place of some of the basis elements \((x - r)^4_+, (x - s)^4_+\) in the basis \(\mathcal{B}\) of \(S_4^2, 3(A_{mn}^{(2)})\). All locally supported functions in \(S_4^2, 3(A_{mn}^{(2)})\) can only span a proper subspace of \(S_4^2, 3(A_{mn}^{(2)})\).

### 3.2. The quasi-interpolation operators in \(S_4^2, 3(A_{mn}^{(2)})\)

Denote the basis functions on \(D' = [0, 1] \times [0, 1]\) by

\[
B_{i,j}' = B_{i,j}(x, y) = B(mx - i, ny - j),
\]

where \(i, j = \ldots, -1, 0, 1, \ldots\). It is clear that the index set for which the functions \(B_{i,j}'\) do not vanish identically on \(D'\) is \(F\) as well.

Define a variation diminishing operator \(V_{mn}' : C(\Omega) \rightarrow S_4^2, 3(A_{mn}^{(2)})\):

\[
V_{mn}'(f) = \sum_{(i,j) \in F} f(x_i, y_j)B_{i,j}',
\]

where \(\Omega\) denotes an open set containing \(D'\), \((x_i, y_j) = (i/m, j/n)\) are the centers of the supports of \(B_{i,j}'\), \((i, j) \in F\). Note that \(V_{mn}'\) is a linear operator, and by computing we have the following result:

**Theorem 3.3.** For all \((x, y) \in D'\),

\[
f \in \mathbb{P}_1 \cup \text{span}\{xy\}, \quad V_{mn}'(f) = f,
\]

\[
f = x^2, \quad V_{mn}'(f) = x^2 + \frac{1}{3m^2}, \quad f = y^2, \quad V_{mn}'(f) = y^2 + \frac{1}{3n^2},
\]

\[
f = x^3, \quad V_{mn}'(f) = x^3 + \frac{1}{m^2}x, \quad f = y^3, \quad V_{mn}'(f) = y^3 + \frac{1}{n^2}y,
\]

\[
f = x^2y, \quad V_{mn}'(f) = x^2y + \frac{1}{3m^2}y, \quad f = xy^2, \quad V_{mn}'(f) = xy^2 + \frac{1}{3n^2}x,
\]

\[
f = x^3y, \quad V_{mn}'(f) = x^3y + \frac{1}{m^2}xy, \quad f = xy^3, \quad V_{mn}'(f) = xy^3 + \frac{1}{n^2}xy,
\]

\[
f = x^4, \quad V_{mn}'(f) = \frac{2}{m}x^3 + \frac{1}{m^2}x^2 + \frac{1}{3m^4} + \frac{4}{m} \sum_{i=1}^{m-1} (x - \frac{i}{m})^3,
\]

\[
f = y^4, \quad V_{mn}'(f) = \frac{2}{n}y^3 + \frac{1}{n^2}y^2 + \frac{1}{3n^4} + \frac{4}{n} \sum_{j=1}^{n-1} (y - \frac{j}{n})^3,
\]

\[
f = x^2y^2, \quad V_{mn}'(f) = \frac{m^2}{6n^2}x^4 + \frac{n^2}{6m^2}y^4 - \frac{m}{3n^2}x^3 - \frac{n}{3m^2}y^3 + \frac{1}{2n^2}x^2 + \frac{1}{2m^2}y^2 + \frac{1}{12m^2n^2} - \frac{2m}{3n^2} \sum_{i=1}^{m-1} (x - \frac{i}{m})^3 - \frac{2n}{3m^2} \sum_{j=1}^{n-1} (y - \frac{j}{n})^3.
\]
In order to preserve identities for all polynomials in \( \mathbb{P}_2 \) and \( \mathbb{P}_3 \), we define another linear operator \( W_{mn}' : C(\Omega) \to S_4^{2,3}(A_{mn}) \):

\[
W_{mn}'(f) = \sum_{(i,j) \in E} \lambda_{i,j}'(f) B_{i,j}',
\]

where \( \lambda_{i,j}'(f) = \frac{2}{3} f\left( \frac{i}{m}, \frac{j}{n} \right) - \frac{1}{3} [f(\frac{2i-1}{2m}, \frac{2j-1}{2n}) + f(\frac{2i+1}{2m}, \frac{2j-1}{2n}) + f(\frac{2i-1}{2m}, \frac{2j+1}{2n}) + f(\frac{2i+1}{2m}, \frac{2j+1}{2n})]. \)

**Theorem 3.4.** For all \((x, y) \in D'\),

\[
f \in \mathbb{P}_3 \cup \text{span}\{x^3y, xy^3\}, \quad W_{mn}'(f) = f,
\]

\[
f = x^4, \quad W_{mn}'(f) = \frac{2}{m} x^3 - \frac{1}{m^2} x^2 - \frac{5}{12m^4} + \frac{4}{m} \sum_{i=1}^{m-1} \left( x - \frac{i}{m} \right)^3,
\]

\[
f = y^4, \quad W_{mn}'(f) = \frac{2}{n} y^3 - \frac{1}{n^2} y^2 - \frac{5}{12n^4} + \frac{4}{n} \sum_{j=1}^{n-1} \left( y - \frac{j}{n} \right)^3,
\]

\[
f = x^2y^2, \quad W_{mn}'(f) = \frac{m^2}{6n^2} x^4 + x^2 y^2 + \frac{n^2}{6m^2} y^4 - \frac{m}{3m^2} x^3 - \frac{n}{3n^2} y^3 + \frac{1}{6n^2} x^2 + \frac{1}{6m^2} y^2
\]

\[
- \frac{2}{9m^2 n^2} - \frac{2m}{3n^2} \sum_{i=1}^{m-1} \left( x - \frac{i}{m} \right)^3 - \frac{2n}{3m^2} \sum_{j=1}^{n-1} \left( y - \frac{j}{n} \right)^3,
\]

\[
f = x^5, \quad W_{mn}'(f) = \frac{5}{3m^2} x^3 - \frac{11}{4m^4} x + \frac{20}{m} \sum_{i=1}^{m-1} \frac{i}{m} \left( x - \frac{i}{m} \right)^3,
\]

\[
f = y^5, \quad W_{mn}'(f) = \frac{5}{3n^2} y^3 - \frac{11}{4n^4} y + \frac{20}{n} \sum_{j=1}^{n-1} \frac{j}{n} \left( y - \frac{j}{n} \right)^3,
\]

\[
f = x^6, \quad W_{mn}'(f) = -\frac{21}{4m^4} x^2 - \frac{85}{48m^6} + \frac{60}{m} \sum_{i=1}^{m-1} \frac{i^2}{m^2} \left( x - \frac{i}{m} \right)^3,
\]

\[
f = y^6, \quad W_{mn}'(f) = -\frac{21}{4n^4} y^2 - \frac{85}{48n^6} + \frac{60}{n} \sum_{j=1}^{n-1} \frac{j^2}{n^2} \left( y - \frac{j}{n} \right)^3.
\]

We consider the uniform approximation to \( S_4^{2,3}(A_{mn}) \) by the spline series \( \sum_{(i,j) \in E} \lambda_{i,j} B_{i,j}' \). Denote

\[
\delta_{mn}' = \frac{1}{mn} \max \left( \sqrt{4m^2 + n^2}, \sqrt{m^2 + 4n^2} \right).
\]

We have the following result on \( V_{mn}' \):
Theorem 3.5. Let \( f \in C(K) \), for all \( m, n \geq N_0 \),
\[
\| f - V'_{mn}(f) \|_{D'} \leq w_K(f, \delta''_{mn}), \tag{21}
\]
if \( f \in C^1(D') \), then
\[
\| f - V'_{mn}(f) \|_{D'} \leq 2\delta_{mn} \omega^1 f, \tag{22}
\]
if \( f \in C^2(D') \), then
\[
\| f - V'_{mn}(f) \|_{D'} \leq \delta_{mn}^2 \| D^2 f \|. \tag{23}
\]
Note that \( \| W'_{mn} \| \leq 11/3 \), and we have the following result on \( W'_{mn} \):

Theorem 3.6. Let \( f \in C(K) \), for all \( m, n \geq N_0 \), if \( f \in C^2(D') \), then
\[
\| f - W'_{mn}(f) \|_{D'} \leq \frac{7}{3} \delta_{mn}^2 \omega^2 f, \tag{24}
\]
if \( f \in C^3(D') \), then
\[
\| f - W'_{mn}(f) \|_{D'} \leq \frac{7}{9} \delta_{mn}^3 \omega^3 f, \tag{25}
\]
if \( f \in C^4(D') \), then
\[
\| f - W'_{mn}(f) \|_{D'} \leq \frac{7}{36} \delta_{mn}^4 \| D^4 f \|. \tag{26}
\]

References