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Designs with the Symmetric Difference Property on 64 Points and Their Groups

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The automorphism groups of the symmetric 2-(64, 28, 12) designs with the *symmetric difference property* (SDP), as well as the groups of their derived and residual designs, are computed. The symmetric SDP designs all have transitive automorphism groups. In addition, they all admit transitive regular subgroups, or equivalently, (64, 28, 12) difference sets. These results are used for the enumeration of certain binary codes achieving the Grey–Rankin bound and point sets of elliptic or hyperbolic type in $PG(5, 2)$. © 1994 Academic Press, Inc.

1. INTRODUCTION

We assume familiarity with the basic facts and ideas from design and coding theory. Our notation follows that from [1, 3, 13, 14, 16]. We also use some notions from the theory of strongly regular graphs and regular two-graphs [3, 15].

A symmetric 2-design has the *symmetric difference property*, or is an *SDP design* [11], if the symmetric difference of any *three* blocks is either

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a block or the complement of a block. The parameters (v, k, λ) of a symmetric SDP design with $k < v/2$ are of the form

$$v = 2^{2m}, \qquad k = 2^{2m-1} - 2^{m-1}, \qquad \lambda = 2^{2m-2} - 2^{m-1}. \tag{1}$$

A non-symmetric 2-(v, k, λ) design with parameters of the form

$$v = 2^{2m-1} - 2^{m-1}, \qquad k = 2^{2m-2} - 2^{m-1}, \qquad \lambda = 2^{2m-2} - 2^{m-1} - 1, \tag{2}$$

or

$$v = 2^{2m-1} + 2^{m-1}, \qquad k = 2^{2m-2}, \qquad \lambda = 2^{2m-2} - 2^{m-1} \tag{3}$$

is said to have the symmetric difference property, or to be an SDP design, if the symmetric difference of any *two* blocks is either a block or the complement of a block.

A 2-design is *quasi-symmetric* with intersection numbers x, y $(x < y)$ if any two blocks intersect in either x or y points. Clearly, a non-symmetric SDP design is quasi-symmetric.

The parameters (2) correspond to that of a derived design of a symmetric SDP design with parameters (1), while the parameters (3) are the same as those of a residual design of a symmetric design (1). The derived and residual designs of a symmetric SDP design are quasi-symmetric SDP designs [10]. Quasi-symmetric SDP designs that are derived or residual designs of non-isomorphic symmetric SDP designs are also non-isomorphic [9]. It was recently proved [17] that every quasi-symmetric SDP design is indeed the derived or residual of a unique symmetric SDP design. Consequently, two quasi-symmetric SDP designs can be isomorphic only if they are derived or residual designs of isomorphic symmetric SDP designs. Furthermore, two derived or residual designs of a given symmetric SDP design D with respect to a pair of blocks B_1, B_2 are isomorphic if and only if B_1 and B_2 are in the same orbit under the automorphism group of D. This reduces the classification of the quasi-symmetric SDP designs up to isomorphism to the classification of the symmetric SDP designs and computing the orbits of their automorphism groups.,

The symmetric SDP designs were characterized by Dillon and Schatz [5] as designs formed by the minimum weight vectors in a binary code spanned by a first-order Reed–Muller code and the incidence vector of a bent function (or an elementary Abelian difference set in the affine space $AG(2m, 2)$). The quasi-symmetric SDP designs were characterized in [17] as designs formed by the minimum weight vectors in a code spanned by the simplex code and the incidence vector of a set of points in the projective space $PG(2m-1, 2)$ that intersects every hyperplane in one of two prescribed number of points. Previous examples of sets with such intersection property include elliptic or hyperbolic quadrics in $PG(2m-1, 2)$ [2].

It follows from the characterization of quasi-symmetric SDP designs [17] and some previous results from [9, 12] that the number of inequivalent point sets in $PG(2m-1, 2)$ with the same intersection property as an elliptic or hyperbolic quadric grows exponentially with m. Furthermore, two sets of elliptic or hyperbolic type are projectively equivalent if and only if they correspond to points from one and the same orbit under the automorphism group of a given quasi-symmetric SDP design.

There is a one-to-one correspondence between the codes spanned by the blocks of quasi-symmetric SDP designs and certain binary self-complementary codes that are optimal in the sense of the Grey–Rankin bound [9]. Thus the classification of the SDP designs for given m and the knowledge of their groups implies also the classification of the corresponding optimal codes.

There is a unique SDP design of type (1), (2), or (3) for $m=2$, and precisely four non-isomorphic symmetric SDP 2-(64, 28, 12) designs for $m=3$ [5]. In this paper we report the results of the computation of the automorphism groups of the four symmetric 2-(64, 28, 12) SDP designs, as well as the groups of their derived and residual designs (Section 5). All four symmetric SDP designs have transitive automorphism groups, only the symplectic one having a primitive (in fact, a doubly transitive) group. In addition, all four designs admit regular transitive subgroups; that is, they all come from (64, 28, 12) difference sets. We show in Theorem 4 that the only design that admits an Abelian difference set is the symplectic design, which admits non-Abelian difference sets as well. All four designs admit polarities with no absolute points and all four designs are in the switching class of the regular two-graph on 64 points defined by the Kronecker cube of $J_4 - 2I_4$.

As an application, the inequivalent point sets of elliptic and hyperbolic type in $PG(5, 2)$ are enumerated in Section 2. There are five projective classes of sets of elliptic type and seven classes of sets of hyperbolic type. The automorphism groups of the derived and residual designs enabled us also to compute the precise number of sets of every class.

Another application is the enumeration of the binary self-complementary (28, 7, 12) and (36, 7, 16) codes up to equivalence. There are four inequivalent codes of each type (Section 3).

Section 4 contains some open questions concerning SDP designs.

2. Point Sets in $PG(5, 2)$ with Two Intersection Numbers

An (n, k, h_1, h_2) *set* S in the projective space $PG(k-1, q)$ is a set of n points in $PG(k-1, q)$ with the property that every hyperplane meets S in either h_1 or h_2 points [2]. Two point sets with the same parameters are equivalent if they are in one orbit under the action of the projective group $PGL(k, q)$; that is, there is a projectivity in $PG(k-1, q)$ that transforms one of the sets into the other.

It has been proved in [17] that the point sets in $PG(2m-1, 2)$ with parameters

$$\begin{aligned} n &= 2^{2m-1} - 2^{m-1} - 1, \qquad k = 2m, \\ h_1 &= 2^{2m-2} - 2^{m-1} - 1, \qquad h_2 = 2^{2m-2} - 1 \end{aligned} \tag{4}$$

are in one-to-one correspondence with the points of quasi-symmetric SDP designs of type (2). Moreover, sets corresponding to non-isomorphic designs are inequivalent, and two sets belonging to a given design are equivalent if and only if they correspond to points that are in the same orbit under the automorphism group of the design.

Similarly, the point sets in $PG(2m-1, 2)$ with parameters

$$\begin{aligned} n &= 2^{2m-1} + 2^{m-1} - 1, \qquad k = 2m, \\ h_1 &= 2^{2m-2} + 2^{m-1} - 1, \qquad h_2 = 2^{2m-2} - 1 \end{aligned} \tag{5}$$

are in one-to-one correspondence with the points of quasi-symmetric SDP designs complementary to the designs of type (3). Two sets are inequivalent if they correspond to points of non-isomorphic designs, and two sets corresponding to a pair of points from the same design are equivalent if and only if these two points are in one orbit under the group of the design.

Examples of point sets of type (4) or (5) are provided by the elliptic or hyperbolic quadric in $PG(2m-1, 2)$, respectively [2, 7, 8, 18]. Therefore, a set with parameters (4) is called a set of *elliptic type*, and a set with parameters (5) is a set of *hyperbolic type*.

Since the automorphism group of any quasi-symmetric SDP design is a subgroup of the projective group $PGL(2m, 2)$ [17], the classification of the point sets in $PG(2m-1, 2)$ of elliptic or hyperbolic type is thus reduced to the classification of the SDP designs (1), (2), (3) and computing their automorphism groups.

As was already shown in [10], the derived 2-(28, 12, 11) designs and the residual 2-(36, 16, 12) designs of the four symmetric SDP 2-(64, 28, 12) designs ($m = 3$) provide at least four non-isomorphic quasi-symmetric SDP 2-(28, 12, 11)- and 2-(36, 16, 12) designs. Now the transitivity of the automorphism groups of the four symmetric SDP 2-(64, 28, 12) designs (Section 5) implies that up to isomorphism, there are precisely four quasi-symmetric SDP 2-(28, 12, 11) designs, and precisely four quasi-symmetric SDP 2-(36, 16, 12) designs. Note that there are more (at least 7) quasi-symmetric 2-(36, 16, 12) designs (see [9]), but only four are SDP designs.

Three of the four SDP 2-(28, 12, 11) designs have point transitive automorphism groups, while one design has two point orbits. This implies that there are precisely five equivalence classes of $(27, 6, 11, 5)$ sets (of elliptic type) in $PG(5, 2)$, namely one class of elliptic quadrics and four further classes of sets that are not quadrics but have the same intersection properties as a quadric.

In the case of the residual 2-(36, 16, 12) SDP designs, there is only one design with point transitive group, and each of the remaining three designs has two point orbits. This implies that there are precisely seven equivalence classes of $(35, 6, 19, 15)$ sets of hyperbolic type in $PG(5, 2)$, one quadric and six non-quadrics.

Having the automorphism groups, one can also compute the exact number of distinct sets from a given equivalence class. The number of distinct sets that are projectively equivalent to a given set S is equal to

$$\frac{|PGL(6, 2)|}{|G_S|},$$

where G_S is the stabilizer of S in $PGL(6, 2)$. Note that the order of $PGL(6, 2)$ is

$$|PGL(6, 2)| = 2^{15}(2^6-1)(2^5-1)(2^4-1)(2^3-1)(2^2-1) = 2^{15}\cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31.$$

From the other side, G_S is also a point stabilizer in a corresponding design D. Thus

$$|G_S| = \frac{|\mathrm{Aut}\, D|}{l},$$

where $\mathrm{Aut}\, D$ is the automorphism group of D and l is the orbit length of the point of D corresponding to S. Therefore, the data from Section 5 for the groups of the derived 2-(28, 12, 11) designs imply the following

THEOREM 1. *There are*

$2^8 \cdot 7^2 \cdot 31$ *elliptic sets of type* 1 (*elliptic quadrics*),

$2^8 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 31$ *elliptic sets of type* 2,

$2^{10} \cdot 3^4 \cdot 7^2 \cdot 31$ *elliptic sets of type* 3a,

$2^{12} \cdot 3^3 \cdot 7^2 \cdot 31$ *elliptic sets of type* 3b,

$2^{12} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 31$ *elliptic sets of type* 4

in $PG(5, 2)$ *with parameters* $(27, 6, 11, 15)$.

Here the notation for the sets 1, 2, 3a, 3b, 4 is taken from [17], where explicit representatives of such sets are listed.

Similarly, using the groups of the residual 2-(36, 16, 12) designs imply

THEOREM 2. *There are*

$2^8 \cdot 3^2 \cdot 7 \cdot 31$ *elliptic sets of type* 1 (*hyperbolic quadrics*),
$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$ *hyperbolic sets of type* 2a,
$2^8 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 31$ *hyperbolic sets of type* 2b,
$2^{10} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 31$ *hyperbolic sets of type* 3a,
$2^{12} \cdot 3^3 \cdot 7^2 \cdot 31$ *hyperbolic sets of type* 3b,
$2^{12} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 31$ *hyperbolic sets of type* 4a,
$2^{13} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$ *hyperbolic sets of type* 4b

in $PG(5, 2)$ *with parameters* (35, 6, 19, 15).

Again, the notation 1, 2a, ..., is in accordance with [17].

A general formula for the number of quadrics of elliptic or hyperbolic type is available in [7; 8, Chap. 22.6].

3. THE SELF-COMPLEMENTARY (28, 7, 12) AND (36, 7, 16) CODES

The binary code spanned by the characteristic vectors of the blocks of a quasi-symmetric SDP design is a self-complementary (n, k, d) code with

$$n = 2^{2m-1} - 2^{m-1}, \qquad k = 2m+1, \qquad d = 2^{2m-2} - 2^{m-1} \tag{6}$$

for a design of type (2), and

$$n = 2^{2m-1} + 2^{m-1}, \qquad k = 2m+1, \qquad d = 2^{2m-2} \tag{7}$$

for a design of type (3) [10].

Any such code is optimal in the sense that it achieves the Gray–Rankin bound, and any self-complementary code with parameters (6) or (7) is the code of some quasi-symmetric SDP design [9]. Therefore, the characterization of the quasi-symmetric SDP designs as derived or residual designs of symmetric SDP designs [17] reduces the enumeration of such codes to the enumeration of the related SDP designs. In addition, since the blocks constitute the set of all minimum weight codewords, the automorphism groups of the code and the design coincide.

It was pointed out previously in [10] that the derived and residual designs of the four symmetric SDP 2-(64, 28, 12) designs produce (at least) four inequivalent self-complementary (28, 7, 12) codes and (at least) four self-complementary (36, 7, 16) codes in this way. Now the transitivity of the groups of the four symmetric 2-(64, 28, 12) designs implies that there are precisely four quasi-symmetric SDP designs corresponding to each of the parameter sets 2-(28, 12, 11) and 2-(36, 16, 12). Thus we have

THEOREM 3. *Up to equivalence, there exist precisely four binary self-complementary* (28, 7, 12) (*resp.* (36, 7, 16)) *codes. Their automorphism groups are of orders* 1451520, 10752, 1920, *and* 672, *respectively.*

An alternative enumeration of the self-complementary (28, 7, 16) codes is described in [6].

4. OPEN PROBLEMS

The computation of the groups of the SDP designs for $m = 3$ raises some questions concerning these designs as well as SDP designs in general.

Problem 1. Classify all regular subgroups (up to conjugacy) of the groups of the four symmetric SDP 2-(64, 28, 12) designs.

We have found several non-conjugate regular subgroups (see Section 5) but we do not know if these are all of them.

Some more general questions are:

QUESTION 2. *When does a symmetric SDP design admit a difference set?*

QUESTION 3. *Given m, are all symmetric SDP designs obtainable by switching from the regular graph defined by the Kronecker product* $(J_4 - 2I_4)^m$, *and if not, do SDP designs obtained in this way by switching have additional geometric or combinatorial properties that distinguish them?*

5. THE AUTOMORPHISM GROUPS

All four symmetric SDP 2-(64, 28, 12) designs can be obtained from the symplectic one, defined by the Kronecker (tensor) cube of $J_4 - 2I_4$, by switching its incidence matrix with respect to maximal 4-arcs [10]. An interesting question raised by Bill Kantor (private communication) whose answer we do not know how yet is what is the geometric nature of the 4-arcs that switch the symplectic design into the other three designs. Here we use a slightly different presentation which features the designs as ones that admit polarities with no absolute points, or equivalently, as strongly regular graphs with parameters $n = 64$, $k = 28$, $\lambda = \mu = 12$.

The Kronecker product $(J_4 - 2I_4) \times (J_4 - 2I_4) \times (J_4 - 2I_4)$ gives a symmetric Hadamard matrix H of order 64, such that $H + I$ is a regular two-graph. In its switching class it has (64, 28, 12, 12) strongly regular graphs, four of which correspond to the SDP 2-(64, 28, 12) designs. They are obtained by switching $H + I$ with respect to the following four vertex sets:

1.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
2.	1	2	3	4	5	6	8	9	10	11	12	13	15	16	20	25
	33	44	49	50	51	52	53	55	56	57	58	59	60	61	62	64
3.	1	2	3	4	5	7	8	9	10	11	12	13	14	16	20	25
	33	34	37	43	44	48	49	51	52	54	56	57	58	60	61	63
4.	1	2	3	4	5	6	11	12	13	16	17	20	33	34	35	36
	37	40	41	44	46	47	49	50	51	52	55	56	57	58	61	64

We call these designs D_1, D_2, D_3, and D_4 in accordance with the bent functions f_1, f_2, f_3, f_4 from [10]. The full automorphism groups of these designs were found by computer. All four designs have transitive automorphism groups, only the symplectic one, $\mathrm{Aut}(D_1)$, is doubly transitive and hence primitive. Using CAYLEY interactively, we were able to find regular subgroups for all of the designs. Thus all four SDP designs can be defined by (64, 28, 12) difference sets. Previously, only the group of the symplectic design was known to contain an elementary Abelian regular subgroup. We have not attempted to find all regular subgroups up to conjugacy (see Problem 1 in the previous section). Our Theorem 4, below, shows that $\mathrm{Aut}(D_2)$, $\mathrm{Aut}(D_3)$, and $\mathrm{Aut}(D_4)$ do not contain regular Abelian subgroups and hence that the corresponding designs do not admit Abelian difference sets (compare with [4]).

To describe the structure of the groups we use the following notation: $G \sim 2^{a+(b+c+\cdots)}: H$ means that G has a normal subgroup Q of order $2^{a+b+c+\cdots}$ and complement $H \cong G/Q$. Further, $Z(Q)$ has order 2^a, and $Q/Z(Q)$ is elementary Abelian of order $2^{b+c+\cdots}$, which, as a $GF(2)H \cong GF(2)G/Q$-module, has irreducible composition factors of order $2^a, 2^b, \ldots$. If H has more than one non-isomorphic $GF(2)$-modules of a given dimension then we distinguish them by subscripts.

THEOREM 4. *Only the automorphism group of design D_1 has an Abelian regular subgroup.*

Proof. It is known that the group of D_1 contains a normal elementary Abelian regular subgroup [11].

Using CAYLEY we find that $G = \mathrm{Aut}(D_2) \sim Q: GL(3, 2)$, where $Q \sim 2^{3+(3_1+3+3_1)}$ is a normal special subgroup of G of order 2^{12} with $|Z(Q)| = 2^3$ on which $GL(3,2) \cong G/Q$ induces an irreducible $GF(2)\,GL(3,2)$-module. We define a subgroup $Q_1 = Z(Q)\,\mathrm{Stab}_Q(1)$ of Q. By CAYLEY, Q_1 is normal in G and $|Q_1| = 2^9$. Using CAYLEY again we find that Q/Q_1 is a non-central chief factor for G and that Q/Q_1 is, as a G/Q-module, dual to $Z(Q)$. Now suppose that R is an Abelian regular subgroup of G. Since every element of R is fixed-point free, we obviously must have $R \cap Q_1 \leqslant Z(Q)$. Suppose that R is not a subgroup of Q. Then, as R is Abelian, $|RQ/Q| \leqslant 2^2$ and $|R \cap Q| \geqslant 2^4$. If $Z(Q) \leqslant R$, then $Z(Q)$ is centralized by $\langle R^G \rangle Q$, which is impossible. So $|R \cap Z(Q)| \leqslant 2^2$ and similarly $|(Q \cap R) Q_1/Q_1| \leqslant 2^2$. Thus we deduce that all the above inequalities are in fact equalities. But then

$$|C_{Q/Q_1}(RQ/Q)| = |C_{Z(Q)}(RQ/Q)|,$$

which implies that $Q/Q_1 \cong Z(Q)$ as $GF(2)G/Q$-modules, which is not the case. Therefore, $R \leqslant Q$, $R \geqslant Z(Q)$, and $Q = RQ_1$. Using CAYLEY we find that all of the $2^{12} - 2^9$ elements of Q/Q_1 are fixed-point free. So to show that there are no Abelian regular subgroups we have to show that there is no Abelian group R_1 of order 2^6 containing $Z(Q)$ with $Q = R_1 Q_1$. We reduce the size of the problem by noting that G/Q is two-transitive on Q/Q_1 and so we can assume that $R_1 = \langle Z(Q), x_1, y_1, z \rangle$, where x_1 and y_1 are in fixed distinct cosets xQ_1 and yQ_1 of Q_1 in Q and z is chosen from $C_Q(\langle x_1, y_1 \rangle) \backslash \langle Q_1, x, y \rangle$. Further, of the 2^9 elements of xQ_1 and yQ_1 we only need to consider those elements which fall into different $Z(Q)$ cosets. Thus we finish with at the very most 64^2 possibilities to check that $C_Q(\langle x_1, y_1 \rangle) \backslash \langle Q_1, x, y \rangle$ is the empty set. This is done using CAYLEY.

Using CAYLEY we find that $G = \mathrm{Aut}(D_3) \sim Q{:}\,\mathrm{Sym}(5)$, where Q is a normal special subgroup of G or order $2^{10} = 2^{2+(4+4)}$ with center of order 4 on which G acts non-trivially with centralizer G'. We also find that $Q/Z(Q)$ is, as a G/Q-module, a direct sum of two isomorphic modules which are, on restriction to $(G/Q)'$, both natural $SL(2, 4)$-modules (to see this we consider the fixed points on $Q/Z(Q)$ of an element of order 3; $Q/Z(Q)$ has only one). We also observe that G' is not transitive. Thus if R is an Abelian regular subgroup of G, $R \nleqslant G'$. Hence $2^2 \geqslant |RQ/Q| \geqslant 2$, $|R \cap Z(Q)| = 2$, and $2^4 \geqslant |(R \cap Q)\, Z(Q)/Z(Q)| \geqslant 2^3$. Because $R \nleqslant G'$ and $|C_{Q/Z(Q)}(RQ/Q)| \geqslant 2^3$, the structure of the module $Q/Z(Q)$ implies that $|RQ/Q| = 2$, $|(R \cap Q)\, Z(Q)/Z(Q)| = 2^4$ and $R = \langle x, C_Q(x) \rangle$, where $x \in Q \backslash Q$. We finally use CAYLEY to check that there are no such regular subgroups.

Finally, from CAYLEY we find that $G = \mathrm{Aut}(D_3) \sim Q{:}\, GL(3, 2)$, where Q is an elementary Abelian group of order 2^8 on which C/Q acts irreducibly (as is seen by noting that, for $S \in \mathrm{Syl}_2(G)$, $C_Q(S)$ has order 2 and closes under the action of G to Q; it's the Steinberg module over $GF(2)$ for $GL(3, 2)$). Further, we find that Q is not transitive. Hence, if R is an Abelian regular subgroup of G, $R \nleqslant G$ and, from the structure of the Steinberg representation of $GL(3, 2)$, if $|RQ/Q| = 4$, then $|C_Q(R)| \leqslant 2^2$ and, if $|RQ/Q| = 2$, $|C_Q(R)| = 2^4$; thus $|R| \leqslant 2^5$, which is nonsense.

Below we list base blocks, group order, group structure, blocks of imprimitivity, group generators, generators of regular subgroups, and the order of the normalizer of the non-Abelian regular subgroup together with an element of largest order in the normalizer that fixes the base block for the four 2-(64, 28, 12) designs $D_1, ..., D_4$, as well as similar data for their derived 2-(28, 12, 11) designs and residual 2-(36, 16, 12) designs.

Design D_1

Base block:
2 3 4 5 9 13 18 19 20 21 25 29 34 35 36 37 41 45 49 54 55 56 58 59 60 62 63 64.

Group order: $92897280 = 2^{15} \cdot 3^4 \cdot 5 \cdot 7$.

Group structure: $\mathrm{Aut}(D_1) \sim 2^6\colon \mathrm{Sp}(6, 2)$. Primitive, doubly transitive. The stabilizer of a point is Sp(6, 2).

Generators:

1. (33, 49) (34, 50) (35, 51) (36, 52) (37, 53) (38, 54) (39, 55) (40, 56) (41, 57) (42, 58) (43, 59) (44, 60) (45, 61) (46, 62) (47, 63) (48, 64)
2. (17, 33) (18, 34) (19, 35) (20, 36) (21, 37) (22, 38) (23, 39) (24, 40) (25, 41) (26, 42) (27, 43) (28, 44) (29, 45) (30, 46) (31, 47) (32, 48)
3. (9, 13) (10, 14) (11, 15) (12, 16) (25, 29) (26, 30) (27, 31) (28, 32) (41, 45) (42, 46) (43, 47) (44, 48) (57, 61) (58, 62) (59, 63) (60, 64)
4. (9, 25) (10, 26) (11, 27) (12, 28) (13, 29) (14,30) (15, 31) (16, 32) (33, 53) (34, 54) (35, 55) (36, 56) (37, 49) (38, 50) (39, 51) (40, 52) (41, 45) (42, 46) (43, 47) (44, 48) (57, 61) (58, 62) (59, 63) (60, 64)
5. (5, 9) (6, 10) (7, 11) (8, 12) (21, 25) (22, 26) (23, 27) (24, 28) (37, 41) (38, 42) (39, 43) (40, 44) (53, 57) (54, 58) (55, 59) (56, 60)
6. (3, 4) (7, 8) (11, 12) (15, 16) (19, 20) (23, 24) (27, 28) (31, 32) (35, 36) (39, 40) (43, 44) (47, 48) (51, 52) (55, 56) (59, 60) (63, 64)
7. (3, 7) (4, 8) (9, 14) (10, 13) (11, 12) (15, 16) (19, 23) (20, 24) (25, 30) (26, 29) (27, 28) (31, 32) (35, 39) (36, 40) (41, 46) (42, 45) (43, 44) (47, 48) (51, 55) (52, 56) (57, 62) (58, 61) (59, 60) (63, 64)
8. (2, 3) (6, 7) (10, 11) (14, 15) (18, 19) (22, 23) (26, 27) (30, 31) (34, 35) (38, 39) (42, 43) (46, 47) (50, 51) (54, 55) (58, 59) (62, 63)
9. (1, 2) (5, 6) (9, 10) (13, 14) (17, 18) (21, 22) (25, 26) (29, 30) (33, 34) (37, 38) (41, 42) (45, 46) (49, 50) (53, 54) (57, 58) (61, 62)

Generators of an elementary Abelian regular subgroup:

1. (1, 9) (2, 10) (3, 11) (4, 12) (5, 13) (6, 14) (7, 15) (8, 16) (17, 25) (18, 26) (19, 27) (20, 28) (21, 29) (22, 30) (23, 31) (24, 32) (33, 41) (34, 42) (35, 43) (36, 44) (37, 45) (38, 46) (39, 47) (40, 48) (49, 57) (50, 58) (51, 59) (52, 60) (53, 61) (54, 62) (55, 63) (56, 64)
2. (1, 3) (2, 4) (5, 7) (6, 8) (9, 11) (10, 12) (13, 15) (14, 16) (17, 19) (18, 20) (21, 23) (22, 24) (25, 27) (26, 28) (29, 31) (30, 32) (33, 35) (34, 36) (37, 39) (38, 40) (41, 43) (42, 44) (45, 47) (46, 48) (49, 51) (50, 52) (53, 55) (54, 56) (57, 59) (58, 60) (61, 63) (62, 64)

3. (1, 4) (2, 3) (5, 8) (6, 7) (9, 12) (10, 11) (13, 16) (14, 15) (17, 20) (18, 19) (21, 24) (22, 23) (25, 28) (26, 27) (29, 32) (30, 31) (33, 36) (34, 35) (37, 40) (38, 39) (41, 44) (42, 43) (45, 48) (46, 47) (49, 52) (50, 51) (53, 56) (54, 55) (57, 60) (58, 59) (61, 64) (62, 63)
4. (1, 5) (2, 6) (3, 7) (4, 8) (9, 13) (10, 14) (11, 15) (12, 16) (17, 21) (18, 22) (19, 23) (20, 24) (25, 29) (26, 30) (27, 31) (28, 32) (33, 37) (34, 38) (35, 39) (36, 40) (41, 45) (42, 46) (43, 47) (44, 48) (49, 53) (50, 54) (51, 55) (52, 56) (57, 61) (58, 62) (59, 63) (60, 64)
5. (1, 17) (2, 18) (3, 19) (4, 20) (5, 21) (6, 22) (7, 23) (8, 24) (9, 25) (10, 26) (11, 27) (12, 28) (13, 29) (14, 30) (15, 31) (16, 32) (33, 49) (34, 50) (35, 51) (36, 52) (37, 53) (38, 54) (39, 55) (40, 56) (41, 57) (42, 58) (43, 59) (44, 60) (45, 61) (46, 62) (47, 63) (48, 64)
6. (1, 33) (2, 34) (3, 35) (4, 36) (5, 37) (6, 38) (7, 39) (8, 40) (9, 41) (10, 42) (11, 43) (12, 44) (13, 45) (14, 46) (15, 47) (16, 48) (17, 49) (18, 50) (19, 51) (20, 52) (21, 53) (22, 54) (23, 55) (24, 56) (25, 57) (26, 58) (27, 59) (28, 60) (29, 61) (30, 62) (31, 63) (32, 64)

Generators of a non-Abelian regular subgroup, R:

1. (1, 9) (2, 10) (3, 11) (4, 12) (5, 13) (6, 14) (7, 15) (8, 16) (17, 25) (18, 26) (19, 27) (20, 28) (21, 29) (22, 30) (23, 31) (24, 32) (33, 41) (34, 42) (35, 43) (36, 44) (37, 45) (38, 46) (39, 47) (40, 48) (49, 57) (50, 58) (51, 59) (52, 60) (53, 61) (54, 62) (55, 63) (56, 64)
2. (1, 3) (2, 4) (5, 7) (6, 8) (9, 11) (10, 12) (13, 15) (14, 16) (17, 19) (18, 20) (21, 23) (22, 24) (25, 27) (26, 28) (29, 31) (30, 32) (33, 35) (34, 36) (37, 39) (38, 40) (41, 43) (42, 44) (45, 47) (46, 48) (49, 51) (50, 52) (53, 55) (54, 56) (57, 59) (58, 60) (61, 63) (62, 64)
3. (1, 5) (2, 6) (3, 7) (4, 8) (9, 13) (10, 14) (11, 15) (12, 16) (17, 21) (18, 22) (19, 23) (20, 24) (25, 29) (26, 30) (27, 31) (28, 32) (33, 37) (34, 38) (35, 39) (36, 40) (41, 45) (42, 46) (43, 47) (44, 48) (49, 53) (50, 54) (51, 55) (52, 56) (57, 61) (58, 62) (59, 63) (60, 64)
4. (1, 51, 18, 36) (2, 52, 17, 35) (3, 50, 20, 33) (4, 49, 19, 34) (5, 55, 22, 40) (6, 56, 21, 39) (7, 54, 24, 37) (8, 53, 23, 38) (9, 59, 26, 44) (10, 60, 25, 43) (11, 58, 28, 41) (12, 57, 27, 42) (13, 63, 30, 48) (14, 64, 29, 47) (15, 62, 32, 45) (16, 61, 31, 46)

Order of $N_G(R)$: $49152 = 2^{14} \cdot 3$.

An element of largest order in $N_G(R)$ which fixes the base block:

(1, 10, 14, 17, 26, 30) (2, 9, 13, 18, 25, 29) (3, 4) (5, 21) (6, 22) (7, 31, 12, 8, 32, 11) (15, 28, 24, 16, 27, 23) (19, 20) (33, 61, 53, 50, 46, 38) (34, 62, 54, 49, 45, 37) (35, 55, 60) (36, 56, 59) (39, 44, 51) (40, 43, 52) (41, 58) (42, 57)

Design D_2

Base block:
2 3 4 5 7 9 13 14 18 19 21 29 33 34 35 36 37 41 44 45 49 55 56 58 59 60 62 64.

Group order: $688128 = 2^{15} \cdot 3 \cdot 7$.

Group structure: $\mathrm{Aut}(D_2) \sim 2^{3+(3_1+3+3_1)}: GL(3,2)$. The stabilizer of a point is $2^{(3_1+3)}: GL(3,2)$.

Blocks of imprimitivity:
[1, 12, 22, 31, 39, 46, 52, 57], [2, 11, 21, 32, 40, 45, 51, 58], [3, 10, 24, 29, 37, 48, 50, 59], [4, 9, 23, 30, 38, 47, 49, 60], [5, 16, 18, 27, 35, 42, 56, 61], [6, 15, 17, 28, 36, 41, 55, 62], [7, 14, 20, 25, 33, 44, 54, 63], [8, 13, 19, 26, 34, 43, 53, 64]

Generators:

1. (5, 61) (6, 62) (7, 63) (8, 64) (13, 53) (14, 54) (15, 55) (16, 56) (17, 41) (18, 42) (19, 43) (20, 44) (25, 33) (26, 34) (27, 35) (28, 36)
2. (5, 64) (6, 63) (7, 62) (8, 61) (13, 56) (14, 55) (15, 54) (16, 53) (17, 20) (18, 19) (21, 45) (22, 46) (23, 47) (24, 48) (25, 28) (26, 27) (29, 37) (30, 38) (31, 39) (32, 40) (33, 36) (34, 35) (41, 44) (42, 43)
3. (5, 7) (6, 8) (9, 49) (10, 50) (11, 51) (12, 52) (13, 55) (14, 56) (15, 53) (16, 54) (17, 43) (18, 44) (19, 41) (20, 42) (21, 45) (22, 46) (23, 47) (24, 48) (25, 27) (26, 28) (33, 35) (34, 36) (61, 63) (62, 64)
4. (3, 4) (5, 18) (6, 17) (7, 19) (8, 20) (9, 29) (10, 30) (11, 32) (12, 31) (13, 14) (23, 24) (25, 26) (33, 34) (37, 49) (38, 50) (39, 52) (40, 51) (41, 62) (42, 61) (43, 63) (44, 64) (47, 48) (53, 54) (59, 60)
5. (3, 13) (4, 14) (7, 9) (8, 10) (19, 29) (20, 30) (23, 25) (24, 26) (33, 60) (34, 59) (35, 56) (36, 55) (37, 64) (38, 63) (39, 52) (40, 51) (41, 62) (42, 61) (43, 50) (44, 49) (45, 58) (46, 57) (47, 54) (48, 53)
6. (2, 3) (5, 13) (6, 15) (7, 14) (8, 16) (10, 11) (18, 19) (21, 29) (22, 31) (23, 30) (24, 32) (26, 27) (34, 35) (37, 45) (38, 47) (39, 46) (40, 48) (42, 43) (50, 51) (53, 61) (54, 63) (55, 62) (56, 64) (58, 59)
7. (1, 2) (3, 4) (5, 17) (6, 18) (7, 19) (8, 20) (9, 10) (11, 12) (13, 25) (14, 26) (15, 27) (16, 28) (21, 22) (23, 24) (29, 30) (31, 32) (33, 53) (34, 54) (35, 55) (36, 56) (37, 38) (39, 40) (41, 61) (42, 62) (43, 63) (44, 64) (45, 46) (47, 48) (49, 50) (51, 52) (57, 58) (59, 60)

Generators of a non-Abelian regular subgroup, R:

1. (1, 18) (2, 36) (3, 7) (4, 53) (5, 22) (6, 40) (8, 49) (9, 64) (10, 14) (11, 41) (12, 27) (13, 60) (15, 45) (16, 31) (17, 51) (19, 38) (20, 24) (21, 55) (23, 34) (25, 29) (26, 47) (28, 58) (30, 43) (32, 62) (33, 37) (35, 52) (39, 56) (42, 57) (44, 48) (46, 61) (50, 54) (59, 63)

2. (1, 45) (2, 46) (3, 4) (5, 17) (6, 18) (7, 64) (8, 63) (9, 10) (11, 39) (12, 40) (13, 54) (14, 53) (15, 27) (16, 28) (19, 44) (20, 43) (21, 57) (22, 58) (23, 24) (25, 34) (26, 33) (29, 30) (31, 51) (32, 52) (35, 55) (36, 56) (37, 38) (41, 61) (42, 62) (47, 48) (49, 50) (59, 60)
3. (1, 63) (2, 53) (3, 18) (4, 28) (5, 24) (6, 30) (7, 57) (8, 51) (9, 17) (10, 27) (11, 64) (12, 54) (13, 58) (14, 52) (15, 23) (16, 29) (19, 40) (20, 46) (21, 34) (22, 44) (25, 39) (26, 45) (31, 33) (32, 43) (35, 50) (36, 60) (37, 56) (38, 62) (41, 49) (42, 59) (47, 55) (48, 61)

Order of $N_G(R)$: $4096 = 2^{12}$.

An element of largest order in $N_G(R)$ which fixes the base block:
(1, 9, 46, 38) (2, 50, 45, 29) (3, 32, 48, 51) (4, 39, 47, 12) (5, 26, 42, 53) (6, 33, 41, 14) (7, 15, 44, 36) (8, 56, 43, 27) (10, 21, 37, 58) (11, 59, 40, 24) (13, 61, 34, 18) (16, 19, 35, 64) (17, 54, 62, 25) (20, 28, 63, 55) (22, 30, 57, 49) (23, 52, 60, 31)

Design D_3

Base block:
2 3 4 5 6 9 13 15 18 19 21 29 33 35 36 41 43 44 45 48 49 50 53 54 56 58 60 63.

Group order: $122880 = 2^{13} \cdot 3 \cdot 5$.

Group structure: $\mathrm{Aut}(D_3) \sim 2^{2+(4+4)}$: Sym(5). The stabilizer of a point is $2^{(1+4)}$: Alt(5).

Blocks of imprimitivity:
[1, 4, 6, 7, 9, 12, 14, 15, 17, 20, 22, 23, 25, 28, 30, 31, 33, 36, 38, 39, 41, 44, 46, 47, 49, 52, 54, 55, 57, 60, 62, 3], [2, 3, 5, 8, 10, 11, 13, 16, 18, 19, 21, 24, 26, 27, 29, 32, 34, 35, 37, 40, 42, 43, 45, 48, 50, 51, 53, 56, 58, 59, 61, 64]

Generators:
1. (5, 50) (6, 49) (7, 52) (8, 51) (13, 58) (14, 57) (15, 60) (16, 59) (17, 46) (18, 45) (19, 48) (20, 47) (21, 29) (22, 30) (23, 31) (24, 32) (25, 38) (26, 37) (27, 40) (28, 39) (33, 41) (34, 42) (35, 43) (36, 44)
2. (5, 51) (6, 52) (7, 49) (8, 50) (13, 59) (14, 60) (15, 57) (16, 58) (17, 20) (18, 19) (21, 34) (22, 33) (23, 36) (24, 35) (25, 28) (26, 27) (29, 42) (30, 41) (31, 44) (32, 43) (37, 40) (38, 39) (45, 48) (46, 47)
3. (3, 5) (4, 6) (9, 15) (10, 16) (19, 21) (20, 22) (25, 31) (26, 32) (33, 55) (34, 56) (35, 51) (36, 52) (37, 53) (38, 54) (39, 49) (40, 50) (41, 57) (42, 58) (43, 61) (44, 62) (45, 59) (46, 60) (47, 63) (48, 64)
4. (2, 3) (5, 58) (6, 60) (7, 57) (8, 59) (10, 11) (13, 50) (14, 52) (15, 49) (16, 51) (17, 46) (18, 48) (19, 45) (20, 47) (22, 23) (25, 38) (26, 40) (27, 37) (28, 39) (30, 31) (33, 36) (41, 44) (53, 56) (61, 64)

5. (1, 2, 52, 40) (3, 49, 37, 4) (5, 33, 35, 20) (6, 19, 55, 53) (7, 18, 54, 56) (8, 36, 34, 17) (9, 10, 60, 48) (11, 57, 45, 12) (13, 41, 43, 28) (14, 27, 63, 61) (15, 26, 62, 64) (16, 44, 42, 25) (21, 39, 51, 22) (23, 24, 38, 50) (29, 47, 59, 30) (31, 32, 46, 58)

Generators of a non-Abelian regular subgroup, R:

1. (1, 34, 49, 59, 52, 64, 9, 48) (2, 36, 56, 7, 32, 28, 35, 54) (3, 39, 13, 23, 24, 22, 19, 47) (4, 37, 12, 43, 60, 50, 57, 53) (5, 25, 40, 15, 18, 44, 58, 55) (6, 27, 33, 51, 62, 16, 20, 45) (8, 30, 29, 31, 26, 38, 10, 46) (11, 41, 61, 14, 21, 17, 42, 63)
2. (1, 5, 57, 27) (2, 30, 21, 47) (3, 54, 29, 14) (4, 45, 49, 58) (6, 34, 55, 53) (7, 10, 63, 24) (8, 17, 19, 36) (9, 40, 60, 51) (11, 23, 32, 38) (12, 16, 52, 18) (13, 28, 26, 41) (15, 43, 62, 64) (20, 59, 44, 37) (22, 56, 46, 42) (25, 50, 33, 48) (31, 61, 39, 35)
3. (1, 7, 57, 63) (2, 50, 21, 48) (3, 40, 29, 51) (4, 17, 49, 36) (5, 26, 27, 13) (6, 47, 55, 30) (8, 16, 19, 18) (9, 28, 60, 41) (10, 45, 24, 58) (11, 59, 32, 37) (12, 14, 52, 54) (15, 38, 62, 23) (20, 39, 44, 31) (22, 25, 46, 33) (34, 35, 53, 61) (42, 64, 56, 43)

Order $N_G(R)$: $256 = 2^8$.

An element of largest order in $N_G(R)$ which fixes the base block:
(1, 12) (4, 9) (6, 15) (7, 14) (17, 28) (20, 25) (22, 31) (23, 30) (33, 44) (36, 41) (38, 47) (39, 46) (49, 60) (52, 57) (54, 63) (55, 62)

Design D_4

Base block:
2 3 4 5 7 8 10 13 14 15 17 18 19 21 25 29 33 40 44 45 46 47 49 53 55 56 58 64.

Group order: $43008 = 2^{11} \cdot 3 \cdot 7$.

Group structure: $\mathrm{Aut}(D_4) \sim 2^8{:}\, GL(3, 2)$. The stabilizer of a point is $2^{(3+2)}{:}\, \mathrm{Frob}(21)$.

Blocks of imprimitivity:
[1, 4, 13, 16, 49, 52, 61, 64], [2, 3, 14, 15, 50, 51, 62, 63], [5, 8, 9, 12, 53, 56, 57, 60], [6, 7, 10, 11, 54, 55, 58, 59], [17, 20, 29, 32, 33, 36, 45, 48], [18, 19, 30, 31, 34, 35, 46, 47], [21, 24, 25, 28, 37, 40, 41, 44], [22, 23, 26, 27, 38, 39, 42, 43]

Generators:

1. (5, 9) (6, 10) (7, 11) (8, 12) (17, 36) (18, 35) (19, 34) (20, 33) (21, 44) (22, 43) (23, 42) (24, 41) (25, 40) (26, 39) (27, 38) (28, 37) (29, 48) (30, 47) (31, 46) (32, 45) (53, 57) (54, 58) (55, 59) (56, 60)

2. (5, 57) (6, 58) (7, 59) (8, 60) (9, 53) (10, 54) (11, 55) (12, 56) (17, 32) (18, 31) (19, 30) (20, 29) (21, 40) (22, 39) (23, 38) (24, 37) (25, 44) (26, 43) (27, 42) (28, 41) (33, 48) (34, 47) (35, 46) (36, 45)
3. (5, 22, 19) (6, 21, 20) (7, 24, 17) (8, 23, 18) (9, 39, 47) (10, 40, 48) (11, 37, 45) (12, 38, 46) (13, 52, 61) (14, 51, 62) (15, 50, 63) (16, 49, 64) (25, 33, 58) (26, 34, 57) (27, 35, 60) (28, 36, 59) (29, 54, 44) (30, 53, 43) (31, 56, 42) (32, 55, 41)
4. (2, 3) (5, 56) (6, 54) (7, 55) (8, 53) (9, 60) (10, 58) (11, 59) (12, 57) (14, 15) (18, 19) (21, 40) (22, 38) (23, 39) (24, 37) (25, 44) (26, 42) (27, 43) (28, 41) (30, 31) (34, 35) (46, 47) (50, 51) (62, 63)
5. (2, 14) (3, 15) (5, 56) (6, 59) (7, 58) (8, 53) (9, 60) (10, 55) (11, 54) (12, 57) (17, 29), (20, 32) (21, 44) (22, 39) (23, 38) (24, 41) (25, 40) (26, 43) (27, 42) (28, 37) (33, 45) (36, 48) (50, 62) (51, 63)
6. (2, 17, 18) (3, 29, 31) (4, 13, 16) (5, 26, 24) (6, 10, 7) (8, 22, 25) (9, 23, 21) (12, 27, 28) (14, 32, 19) (15, 20, 30) (33, 47, 62) (34, 63, 45) (35, 51, 36) (37, 56, 43) (38, 40, 60) (39, 44, 53) (41, 57, 42) (46, 50, 48) (49, 64, 61) (54, 55, 59)
7. (1, 2) (3, 4) (5, 25) (6, 26) (7, 27) (8, 28) (9, 21) (10, 22) (11, 23) (12, 24) (13, 14) (15, 16) (17, 18) (19, 20) (29, 30) (31, 32) (33, 47) (34, 48) (35, 45) (36, 46) (37, 56) (38, 55) (39, 54) (40, 53) (41, 60) (42, 59) (43, 58) (44, 57) (49, 63) (50, 64) (51, 61) (52, 62)

Generators of a non-Abelian regular subgroup, R:

1. (1, 17, 57, 35, 4, 33, 60, 19) (2, 25, 10, 42, 3, 41, 11, 26) (5, 30, 13, 45, 8, 46, 16, 29) (6, 22, 62, 40, 7, 38, 63, 24) (9, 34, 52, 36, 12, 18, 49, 20) (14, 37, 55, 39, 15, 21, 54, 23) (27, 50, 28, 58, 43, 51, 44, 59) (31, 61, 48, 56, 47, 64, 32, 53)
2. (1, 6) (2, 5) (3, 8) (4, 7) (9, 15) (10, 16) (11, 13) (12, 14) (17, 24) (18, 23) (19, 22) (20, 21) (25, 29) (26, 30) (27, 31) (28, 32) (33, 40) (34, 39) (35, 38) (36, 37) (41, 45) (42, 46) (43, 47) (44, 48) (49, 54) (50, 53) (51, 56) (52, 55) (57, 63) (58, 64) (59, 61) (60, 62)
3. (1, 8, 4, 5) (2, 6, 3, 7) (9, 64, 12, 61) (10, 62, 11, 63) (13, 57, 16, 60) (14, 59, 15, 58) (17, 30, 33, 46) (18, 32, 34, 48) (19, 29, 35, 45) (20, 31, 36, 47) (21, 27, 37, 43) (22, 25, 38, 41) (23, 28, 39, 44) (24, 26, 40, 42) (49, 56, 52, 53) (50, 54, 51, 55)
4. (1, 12, 4, 9) (2, 59, 3, 58) (5, 64, 8, 61) (6, 15, 7, 14) (10, 50, 11, 51) (13, 53, 16, 56) (17, 34, 33, 18) (19, 20, 35, 36) (21, 22, 37, 38) (23, 40, 39, 24) (25, 43, 41, 27) (26, 28, 42, 44) (29, 31, 45, 47) (30, 48, 46, 32) (49, 60, 52, 57) (54, 63, 55, 62)

Order of $N_G(R)$: $256 = 2^8$.

An element of largest order in $N_G(R)$ which fixes the base block:
(2, 3) (5, 8) (9, 12) (14, 15) (17, 45) (18, 47) (19, 46) (20, 48) (21, 44) (22, 42) (23, 43) (24, 41) (25, 40) (26, 38) (27, 39) (28, 37) (29, 33) (30, 35) (31, 34) (32, 36) (50, 51) (53, 56) (57, 60) (62, 63)

Derived 2-(28, 12, 11) *Design of* D_1

Group order: $1451520 = 2^9 \cdot 3^4 \cdot 5 \cdot 7$.

Group structure: Sp(6, 2).

Orbit lengths: points: 28 (two-transitive); blocks: 63 (transitive).

Base block: 2 3 8 9 14 15 21 22 24 25 27 28.

Generators:

1. (7, 13) (8, 14) (9, 15) (10, 16) (11, 17) (12, 18)
2. (5, 6) (11, 12) (17, 18) (23, 26) (24, 27) (25, 28)
3. (5, 11) (6, 12) (13, 20) (14, 21) (15, 22) (16, 19) (17, 18) (23, 26) (24, 27) (25, 28)
4. (4, 5) (10, 11) (16, 17) (20, 23) (21, 24) (22, 25)
5. (3, 4) (9, 10) (15, 16) (19, 22) (23, 27) (24, 26)
6. (2, 3) (8, 9) (14, 15) (21, 22) (24, 25) (27, 28)
7. (1, 2) (7, 8) (13, 14) (20, 21) (23, 24) (26, 27)

Derived 2-(28, 12, 11) *Design of* D_2

Group order: $10752 = 2^9 \cdot 3 \cdot 7$.

Group structure: $2^{(3_1+3)}$: $GL(3, 2)$.

Orbit lengths: points: 28 (transitive); blocks: $7 + 56$.

Blocks of imprimitivity:
[1, 11, 20, 24], [2, 12, 17, 25], [3, 6, 21, 26], [4, 9, 15, 23], [5, 8, 13, 19], [7, 10, 14, 28], [16, 18, 22, 27]

Base blocks:
2 3 5 10 15 16 19 22 23 25 26 28; 3 5 6 8 13 16 18 19 21 22 26 27.

Generators:

1. (4, 28) (5, 27) (7, 23) (8, 22) (9, 10) (11, 20) (12, 17) (13, 16) (14, 15) (18, 19)
2. (4, 27) (5, 28) (6, 21) (7, 8) (9, 18) (10, 19) (12, 17) (13, 14) (15, 16) (22, 23)
3. (3, 6) (5, 8) (13, 19) (16, 18) (21, 26) (22, 27)
4. (2, 3) (4, 9) (5, 10) (6, 12) (7, 8) (13, 14) (17, 21) (18, 27) (19, 28) (25, 26)
5. (2, 4, 10, 25, 9, 28) (3, 16, 5, 21, 27, 13) (6, 18, 8, 26, 22, 19) (7, 12, 15, 14, 17, 23) (11, 20, 24)

6. (1, 2) (4, 7) (5, 8) (9, 10) (11, 12) (14, 15) (17, 20) (22, 27) (23, 28) (24, 25)

Derived 2-(28, 12, 11) *Design of* D_3

Group order: $1920 = 2^7 \cdot 3 \cdot 5$.

Group structure: $2^{(1+4)}$: Alt(5).

Orbit lengths: points: $16 + 12$; blocks: $32 + 30 + 1$.

Base blocks:
2 3 8 10 14 15 17 18 20 25 27 28; 1 2 5 8 9 10 14 17 22 24 26 28; 3 5 6 8 13 15 16 18 21 24 27 28.

Generators:
1. (4, 22) (5, 21) (7, 26) (8, 27) (9, 19) (10, 20) (11, 12) (13, 16) (14, 17) (15, 18)
2. (3, 28) (4, 26) (6, 24) (7, 22) (9, 10) (11, 17) (12, 14) (13, 15) (16, 18) (19, 20)
3. (3, 6) (5, 8) (13, 18) (15, 16) (21, 27) (24, 28)
4. (2, 4) (3, 6) (5, 8) (10, 11) (13, 21) (14, 26) (15, 24) (16, 28) (17, 25) (18, 27) (19, 22) (20, 23)
5. (1, 2) (3, 6) (4, 26) (5, 21) (7, 22) (8, 27) (9, 20) (10, 19) (13, 16) (15, 18) (23, 25) (24, 28)

Derived 2-(28, 12, 11) *Design of* D_4

Group order: $672 = 2^5 \cdot 3 \cdot 7$.

Group structure: $2^{(3+2)}$: Frob(21).

Orbit lengths: points: 28 (transitive); blocks: $56 + 7$.

Blocks of imprimitivity:
[1, 2, 9, 10], [3, 8, 23, 28], [4, 6, 24, 26], [5, 7, 25, 27], [11, 16, 17, 20], [12, 13, 21, 22], [14, 15, 18, 19]

Base blocks:
2 3 5 6 10 13 18 19 22 25 26 28; 1 2 5 7 9 10 12 13 21 22 25 27.

Generators:
1. (4, 24) (5, 25) (6, 26) (7, 27) (11, 20) (12, 21) (13, 22) (14, 15) (16, 17) (18, 19)
2. (3, 8) (4, 24) (5, 27) (6, 26) (7, 25) (11, 16) (12, 13) (14, 18) (15, 19) (17, 20) (21, 22) (23, 28)
3. (3, 23) (4, 6) (5, 25) (7, 27) (8, 28) (11, 17) (12, 13) (16, 20) (21, 22) (24, 26)

4. (2, 9, 10) (3, 12, 20, 8, 13, 17) (4, 14, 27, 24, 18, 5) (6, 15, 25, 26, 19, 7) (11, 23, 22, 16, 28, 21)
5. (1, 2) (4, 26) (5, 25) (6, 24) (7, 27) (9, 10) (12, 13) (14, 18) (15, 19) (21, 22)
6. (1, 3, 12, 7, 17, 15, 26, 9, 23, 13, 5, 16, 18, 6) (2, 8, 21, 25, 11, 14, 4, 10, 28, 22, 27, 20, 19, 24)

Residual 2-(36, 16, 12) *Design of* D_1

Group order: $1451520 = 2^9 \cdot 3^4 \cdot 5 \cdot 7$.

Group structure: Sp(6, 2).

Orbit lengths: points: 36 (two-transitive); blocks: 63 (transitive).

Base block: 1 2 5 8 11 12 15 18 21 22 25 28 31 34 35 36.

Generators:
1. (5, 18) (6, 19) (7, 20) (8, 15) (9, 16) (10, 17) (21, 34) (22, 31) (23, 32) (24, 33)
2. (11, 21) (12, 22) (13, 23) (14, 24) (15, 25) (16, 26) (17, 27) (18, 28) (19, 29) (20, 30)
3. (5, 8) (6, 9) (7, 10) (15, 18) (16, 19) (17, 20) (25, 28) (26, 29) (27, 30) (35, 36)
4. (4, 14) (7, 17) (8, 18) (9, 19) (21, 30) (22, 26) (23, 25) (24, 35) (27, 34) (28, 32) (29, 31) (33, 36)
5. (3, 4) (6, 7) (9, 10) (13, 14) (16, 17) (19, 20) (23, 24) (26, 27) (29, 30) (32, 33)
6. (2, 3) (5, 6) (8, 9) (12, 13) (15, 16) (18, 19) (22, 23) (25, 26) (28, 29) (31, 32)
7. (1, 2) (6, 10) (7, 9) (11, 12) (16, 20) (17, 19) (21, 22) (26, 30) (27, 29) (31, 34)

Residual 2-(36, 16, 12) *Design of* D_2

Group order: $10752 = 2^9 \cdot 3 \cdot 7$.

Group structure: $2^{(3_1+3)}$: $GL(3, 2)$.

Orbit lengths: points: $8 + 28$; blocks: $56 + 7$.

Base blocks:
1 2 4 9 11 14 15 18 21 24 26 29 32 33 34 35; 3 4 5 8 13 15 16 20 23 24 25 28 29 30 32 35.

Generators:
1. (4, 13) (5, 12) (6, 11) (7, 9) (8, 10) (14, 16) (18, 20) (21, 23) (24, 33) (25, 32) (26, 31) (27, 30) (28, 29) (35, 36)

2. (4, 13) (5, 10) (6, 11) (7, 9) (8, 12) (14, 20) (16, 18) (21, 35) (22, 34) (23, 36) (24, 27) (30, 33)
3. (3, 4) (5, 8) (10, 14) (12, 18) (13, 15) (16, 20) (22, 26) (23, 24) (25, 28) (29, 32) (30, 35) (31, 34)
4. (3, 21) (4, 36) (5, 20) (6, 19) (7, 17) (8, 16) (10, 29) (12, 32) (13, 33) (14, 28) (15, 27) (18, 25) (24, 35) (26, 34)
5. (2, 3, 5, 24, 21, 29, 33) (4, 10, 7, 15, 23, 35, 12) (6, 19, 26, 11, 31, 34, 22) (8, 27, 28, 14, 9, 32, 30) (13, 36, 17, 25, 20, 16, 18)
6. (1, 22) (4, 28) (6, 26) (8, 24) (10, 33) (11, 31) (13, 29) (14, 36) (16, 35) (19, 34)

Residual 2-(36, 16, 12) *Design of* D_3

Group order: $1920 = 2^7 \cdot 3 \cdot 5$.

Group structure: $2^{(1+4)}$: Alt(5).

Orbit lengths: points: $20 + 16$; blocks: $32 + 30 + 1$.

Base blocks:
1 4 7 9 11 14 15 18 21 22 23 26 27 32 34 35; 1 3 6 8 9 10 13 14 17 20 22 25 30 32 34 36; 3 4 5 8 13 15 16 20 21 25 26 29 33 34 36.

Generators:
1. (3, 5) (4, 8) (10, 14) (11, 19) (13, 16) (15, 20) (21, 25) (22, 26) (24, 27) (29, 36) (31, 35) (33, 34)
2. (3, 13) (4, 15) (5, 16) (8, 20) (21, 33) (22, 36) (23, 28) (24, 27) (25, 34) (26, 29) (30, 32) (31, 35)
3. (3, 22) (4, 21) (5, 26) (8, 25) (9, 17) (12, 18) (13, 36) (15, 33) (16, 29) (20, 34) (24, 27) (30, 32)
4. (2, 30) (4, 34) (5, 36) (7, 32) (9, 24) (10, 23) (11, 12) (13, 21) (14, 28) (15, 16) (17, 27) (18, 19) (20, 26) (22, 25)
5. (2, 27) (5, 21) (7, 24) (8, 22) (9, 19) (10, 31) (11, 17) (12, 30) (13, 15) (14, 35) (16, 36) (18, 32) (20, 33) (29, 34)
6. (1, 2, 28, 14, 30) (4, 21, 13, 34, 8) (5, 22, 33, 25, 36) (6, 7, 23, 10, 32) (9, 35, 24, 12, 11) (15, 16, 20, 29, 26) (17, 31, 27, 18, 19)

Residual 2-(36, 16, 12) *Design of* D_4

Group order: $672 = 2^5 \cdot 3 \cdot 7$.

Group structure: $2^{(3+2)}$: Frob(21).

Orbit lengths: points: $28 + 8$; blocks: $56 + 7$.

Base blocks:
1 2 3 8 11 14 17 20 21 23 24 27 30 31 34 35; 1 3 5 6 7 10 13 16 19 20 23 26 29 31 33 34.

Generators:

1. (3, 5) (7, 26) (8, 24) (9, 25) (10, 23) (11, 21) (12, 22) (13, 20) (14, 18) (15, 17) (16, 19) (27, 28) (31, 33) (35, 36)
2. (2, 4) (7, 16) (8, 9) (10, 13) (11, 12) (14, 15) (17, 18) (19, 26) (20, 23) (21, 22) (24, 25) (27, 36) (28, 35) (30, 32)
3. (2, 10, 17, 4, 20, 14) (3, 7, 27, 5, 19, 35) (6, 29, 34) (8, 9, 22, 25, 24, 11) (12, 21) (13, 18, 32, 23, 15, 30) (16, 28, 33, 26, 36, 31)
4. (1, 2, 10, 36, 14, 7, 31, 29, 4, 23, 28, 15, 26, 3) (8, 9, 25, 12, 24, 22, 11) (5, 34, 30, 20, 27, 18, 16, 33, 6, 32, 13, 35, 17, 19)

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