Logarithm of the scale factor as a generalised coordinate in a Lagrangian for dark matter and dark energy

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A B S T R A C T

A Lagrangian for the k-essence field is set up with canonical kinetic terms and incorporating the scaling relation of [R.J. Scherrer, Phys. Rev. Lett. 93 (2004) 011301]. There are two degrees of freedom, viz., $q(t) = \ln a(t)$ ($a(t)$ is the scale factor) and the scalar field ϕ, and an interaction term involving ϕ and $q(t)$. The Euler–Lagrange equations are solved for q and ϕ. Using these solutions quantities of cosmological interest are determined. The energy density ρ has a constant component which we identify as dark energy and a component behaving as a^{-3} which we call dark matter. The pressure p is *negative* for time $t \to \infty$ and the sound velocity $c_s^2 = \frac{\partial p}{\partial \rho} \ll 1$. When dark energy dominates, the deceleration parameter $Q \to -1$ while in the matter dominated era $Q \sim 1/2$. The equation of state parameter $w = p/\rho$ is shown to be consistent with $w = p/\rho \sim -1$ for dark energy domination and during the matter dominated era we have $w \sim 0$. Bounds for the parameters of the theory are estimated from observational data.

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1. Introduction

The universe consists of roughly 25 percent dark matter, 70 percent dark energy, about 4 percent free hydrogen and helium with the remaining one percent consisting of stars, dust, neutrinos and heavy elements. In [1] it was shown that it is possible to unify the dark matter and dark energy components into a single scalar field model with the scalar field ϕ having a non-canonical kinetic term. These scalar fields are known as k-essence fields. The idea of k-essence first came in models of inflation [2,3]. Subsequently k-essence fields were shown to lead to models of dark energy also [4–7]. The general form of the Lagrangian for these k-essence models is assumed to be a function $F(X)$ of the derivatives of the field (i.e. $X = \nabla_\mu \phi \nabla^\mu \phi$) and do not depend explicitly on ϕ to start with. In [1] the evolution of ϕ for an arbitrary functional form for the Lagrangian has been given in terms of an exact analytical solution. The solution was in the form of a general scaling relation between the function F of the derivatives of the scalar field and the scale factor $a(t)$ of the Robertson–Walker metric (a similar expression was first derived in [8]). To obtain this result the scalar field potential $V(\phi)$ was assumed to be a constant. In [1] specific forms (motivated from string theory [2,3,9,10]) for the Lagrangian (or pressure) p and $F(X)$ were assumed to show that self-consistent models can be built which account for both the dark matter and

dark energy components. Reviews on dark matter and dark energy can be found in Refs. [11–13]. Literature on k-essence models are in Refs. [14–25].

The motivation of the present work stems from the question whether the standard Lagrangian formalism can be used to understand the origins of dark matter and dark energy after preserving the scaling relation in [1]. By standard formalism we mean that the kinetic terms corresponding to fields in the Lagrangian should be canonical. Given the fact that the constituents of dark matter and dark energy are unknown to start with, it is extremely difficult to write down some sort of Lagrangian. Yet there exists schemes (like the one described above) where a Lagrangian can be written down although the kinetic terms are non-canonical. The problem with such Lagrangians are that one cannot use the well established methods of the Lagrangian formalism. Moreover, if ϕ is a quantum field then studying such fields outside the gambit of the Lagrangian formalism is problematic.

The basic results of this work can be summarised as follows. Using the zero–zero component of Einstein's field equations and incorporating the scaling relation of [1], an expression for the Lagrangian for the k-essence field is obtained. This Lagrangian has a non-canonical kinetic term. We now convert this Lagrangian into one with canonical kinetic terms after a redefinition of the variables. There are two degrees of freedom, viz. $q(t) = \ln a(t)$ and ϕ. Note that $\dot{q}(t)$ is nothing but the Hubble parameter. The resulting Lagrangian is in standard form of canonical kinetic terms corresponding to $q(t)$ and a complicated interaction term involving the scalar field ϕ and $q(t)$. We solve the Euler–Lagrange equations

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for q and ϕ. The solutions give realistic cosmological scenarios in the context of dark matter and dark energy. The energy density ρ has a dark energy component and a dark matter component. The pressure p is *negative* for time $t \to \infty$ and the sound velocity $c_s^2 = \frac{\partial p}{\partial \rho} \ll 1$. When dark energy dominates, i.e. $t \to \infty$, the deceleration parameter $Q \to -1$ while in the matter dominated era $Q \sim 1/2$. When the dark energy dominates, the equation of state parameter $w = p/\rho$ is shown to be consistent with $w = p/\rho \sim -1$ and during the matter dominated era we have $w \sim 0$. Bounds can be estimated for the constants of integration in the theory from observational data.

2. The Lagrangian and the solutions of the Euler–Lagrange equations

The Lagrangian for the k-essence field is taken as

$$L = -V(\phi)F(X), \tag{1}$$

$$X = \nabla_\mu \phi \nabla^\mu \phi. \tag{2}$$

The pressure p is taken to be given by (1) and the energy density given by

$$\rho = V(\phi)\big[F(X) - 2XF_X\big] \tag{3}$$

with $F_X \equiv dF/dX$. The equation of state parameter

$$w = \frac{p}{\rho} = \frac{F}{2XF_X - F} \tag{4}$$

and we take the standard expression for the sound velocity as

$$c_s^2 = \frac{\partial p}{\partial \rho}. \tag{5}$$

For a flat Robertson–Walker metric the equation for the k-essence field is

$$(F_X + 2XF_{XX})\ddot{\phi} + 3HF_X\dot{\phi} + (2XF_X - F)\frac{V_\phi}{V} = 0, \tag{6}$$

$V_\phi \equiv dV/d\phi$, $H = \dot{a}(t)/a(t)$ is the Hubble parameter. In [1], $V(\phi)$ was a constant so that the third term in (6) was absent and the following scaling law was obtained:

$$XF_X^2 = Ca^{-6}, \tag{7}$$

where a is the scale factor and C a constant. We assume that $V(\phi)$ is not a constant but V_ϕ/V can be made sufficiently small so that the third term in (6) is still negligible and thus (7) still holds (we will show explicitly that this is possible in our approach, refer to the discussion after Eq. (38)). Using (7) and the zero–zero component of Einstein's field equations an expression for the Lagrangian for the k-essence field is obtained as described below. We take the Robertson–Walker metric:

$$ds^2 = c^2 dt^2 - a^2(t)$$
$$\times \left[\frac{dr^2}{(1 - kr^2)} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right], \tag{8}$$

where $k = 0, 1$ or -1 is the curvature constant. The zero–zero component of Einstein's equation reads:

$$R_{00} - \frac{1}{2}g_{00}R = -kT_{00}. \tag{9}$$

This gives with the metric (8)

$$\frac{k}{a^2} + H^2 = \frac{8\pi G}{3}\rho. \tag{10}$$

Using (1)–(3), (9) and (10), and for $k = 0$, we arrive at

$$XF_X = \frac{1}{2X}\left[F - \left(\frac{3}{8\pi GV(\phi)} \right)H^2 \right]. \tag{11}$$

Using (7) to eliminate F_X gives

$$F(X) = 2\sqrt{C}\sqrt{X}a^{-3} + 3\frac{H^2}{8\pi GV(\phi)}. \tag{12}$$

So the expression for the Lagrangian is obtained as

$$L = -2\sqrt{C}\sqrt{X}a^{-3}V(\phi) - \left(\frac{3}{8\pi G} \right)H^2$$
$$= -2\sqrt{C}\sqrt{\dot{\phi}^2 - (\nabla\phi)^2}a^{-3}V(\phi) - 3\frac{H^2}{8\pi G}, \tag{13}$$

where $X = \nabla_\mu \phi \nabla^\mu \phi$. Homogeneity and isotropy of spacetime imply $\phi(t, \mathbf{x}) = \phi(\mathbf{t})$. Then (13) becomes

$$L = -c_1\dot{q}^2 - c_2V(\phi)\dot{\phi}e^{-3q}, \tag{14}$$

where $q(t) = \ln a(t)$, $c_1 = 3(8\pi G)^{-1}$, $c_2 = 2\sqrt{C}$. Thus the non-standard Lagrangian in (1) has now been cast into a standard form. The new Lagrangian has two generalised coordinates $q(t)$ and $\phi(t)$. q has a standard kinetic term while ϕ does not have a kinetic part. There is a complicated polynomial interaction between q and ϕ and ϕ occurs purely through this interaction term. The two Euler–Lagrange equations corresponding to $q(t)$ and $\phi(t)$ are respectively:

$$\frac{d}{dt}\big(2c_1\dot{q}(t)\big) = -3c_2V(\phi)\dot{\phi}e^{-3q(t)}, \tag{15}$$

$$\dot{\phi} = -\frac{3V(\phi)}{\frac{\partial V}{\partial \phi}}\dot{q}. \tag{16}$$

Eqs. (15) and (16) may be looked upon as describing the evolution of the scale factor of the universe. Substituting (16) in (15) gives

$$\frac{d}{dt}(\dot{q}) = -8\pi G\sqrt{C}\frac{d}{dt}\big(e^{-3q}\big)\frac{V^2}{V_\phi}. \tag{17}$$

To solve this, let $V^2/V_\phi = -A_1$ where A_1 is a positive constant and $V_\phi = dV/d\phi$. This assumption means

$$V(\phi) = \frac{A_1}{\phi + A_2}, \tag{18}$$

where A_2 is a constant. We shall show that using the solutions to the equations of motion for q and ϕ, V_ϕ/V ($= -\frac{1}{\phi + A_2}$) can be made as small as we please so that the third term in (6) is ignorable and the scaling relation (7) can be made to remain valid. So (17) becomes

$$\frac{d}{dt}(\dot{q}) = 8\pi G\sqrt{C}A_1\frac{d}{dt}\big(e^{-3q}\big) \tag{19}$$

which gives after one integration

$$\dot{q} = 8\pi G\sqrt{C}A_1\big(e^{-3q}\big) + A_3. \tag{20}$$

Assumption here is that $H = \dot{q} = \dot{a}/a \neq 0$ for $q \to \infty$. Consequently $A_3 \neq 0$. Solving (20) gives

$$a_c(t) = \left[A_4 e^{3A_3 t} - \frac{8\pi G\sqrt{C}A_1}{A_3} \right]^{\frac{1}{3}}.$$

The subscript c in a_c means that this is a solution of the classical equations of motion. We choose $A_4 = 1$ and write $\alpha = 3A_3$ and $\beta = \frac{8\pi G\sqrt{C}A_1}{A_3} = \frac{24\pi G\sqrt{C}A_1}{\alpha}$. We have chosen one constant of integration in the solution for ϕ to be unity and α, β, A_1, A_2 to be all positive. Then the solutions for a, ϕ, H and $V(\phi)$ are

$$a_c(t) = \big[e^{\alpha t} - \beta\big]^{\frac{1}{3}}, \tag{21a}$$

$$\phi_c(t) = e^{\alpha t} - \beta - A_2, \tag{21b}$$

$$H_c = \frac{\dot{a}_c}{a_c} = \frac{\alpha e^{\alpha t}}{3(e^{\alpha t} - \beta)}, \tag{21c}$$

$$V_c(\phi) = V(\phi_c) = V(t) = \frac{A_1}{\phi_c + A_2} = \frac{A_1}{e^{\alpha t} - \beta}. \tag{21d}$$

We shall now calculate all cosmological quantities using these solutions only.

3. The energy density and the pressure

From (12), $F - 2XF_X = \frac{3H^2}{8\pi GV}$. Hence

$$\rho = V(F - 2XF_X) = \frac{3H^2}{8\pi G}, \tag{22}$$

$$p(\equiv L) = -VF = -\frac{3H^2}{8\pi G} - 2\sqrt{C}\sqrt{X}Va^{-3}. \tag{23}$$

Using the solutions (21) give

$$\rho_c = \frac{3H_c^2}{8\pi G}$$
$$= \frac{\alpha^2}{24\pi G} + \frac{\alpha^2\beta}{24\pi G}a_c^{-3} + \frac{\alpha^2\beta}{24\pi G}a_c^{-3}(1 - \beta e^{-\alpha t})^{-1}, \tag{24}$$

$$p_c = -\frac{3H_c^2}{8\pi G} - 2\sqrt{C}\dot{\phi}_c V_c a_c^{-3}$$
$$= -\rho_c + 2\sqrt{C}\alpha A_1(1 - \beta e^{-\alpha t})^{-1}a_c^{-3}. \tag{25}$$

ρ has a constant component, i.e. dark energy, a component varying as a^{-3}, i.e. dark matter; and a third component whose variation for large times is again like a^{-3}, i.e. dark matter. Substituting from (21a) into (25) and taking $t \to \infty$ the pressure p (i.e. the Lagrangian L) is surely *negative*. Thus for time scales very much larger than the matter domination era we have a negative pressure which may be the source for the observed acceleration of the universe.

4. The equation of state

Expressing the second term on the right-hand side of (25) in terms of ρ_c gives the equation of state:

$$p_c = \rho_c - \frac{2\alpha}{\sqrt{24\pi G}}\rho_c^{\frac{1}{2}}. \tag{26}$$

The equation of state parameter

$$w = \frac{p_c}{\rho_c} = 1 - \frac{2\alpha}{\sqrt{24\pi G}\rho_c^{\frac{1}{2}}} = -1 + 2\beta e^{-\alpha t}. \tag{27}$$

Therefore when the dark energy dominates, i.e. at times $t \to \infty$, we have $w \approx -1$.

We show below that both α and β can be estimated to be positive in our scheme if we accept the current conjectures that the dark matter and dark energy densities were equal at a time one-tenth the present age of the universe ($\sim 10^{17}$ seconds) and that the dark energy density is roughly twice that of the dark matter density at present [1]. Moreover, from consistency arguments β will be shown to be greater than $1/2$. It will also be shown below that $\alpha t \sim$ very large number so that $e^{-\alpha t}$ is small and higher powers of $e^{-\alpha t}$ ignorable. From (24) we get

$$\rho_c = \frac{\alpha^2}{24\pi G} + \frac{\alpha^2\beta}{12\pi G}a_c^{-3}$$
$$+ \frac{\alpha^2}{24\pi G}a_c^{-3}\left[\beta^2 e^{-\alpha t} + \frac{\beta^3}{2!}e^{-2\alpha t} + \frac{\beta^4}{3!}e^{-3\alpha t} + \cdots\right]. \tag{28}$$

So for large times the third term on right-hand side of (28) is exponentially damped. This allows us to write the energy density as

$$\rho_c = \rho_{\text{DE}} + \rho_{\text{DM}} + \rho', \tag{29a}$$

where

$$\rho_{\text{DE}} = \frac{\alpha^2}{24\pi G}; \qquad \rho_{\text{DM}} = \frac{\alpha^2\beta}{12\pi G}a_c^{-3} \tag{29b}$$

and ρ' is the part that has negligible contribution at large times, viz.

$$\rho' = \frac{\alpha^2}{24\pi G}a_c^{-3}\left[\beta^2 e^{-\alpha t} + \frac{\beta^3}{2!}e^{-2\alpha t} + \frac{\beta^4}{3!}e^{-3\alpha t} + \cdots\right]. \tag{29c}$$

The age of the universe is $t_0 \sim 10^{17}$ seconds and at the present epoch indications are that the dark energy density is greater than the dark matter density. Let $\rho_{\text{DE}} \approx n\rho_{\text{DM}}$ where $n > 1$. Using (29b) this gives

$$t_0 \sim 10^{17} = A_0 \frac{1}{\alpha}\ln((2n+1)\beta), \tag{30a}$$

A_0 is a phenomenological parameter to be observationally determined. Now the dark matter and dark energy densities were equal roughly when the time $t_{eq} \sim 10^{16}$ s. Again using (29b) gives

$$t_{eq} \sim 10^{16} = A_{eq}\frac{1}{\alpha}\ln(3\beta), \tag{30b}$$

where A_{eq} is another parameter. Solving (30a) and (30b) gives

$$\beta \sim \left(\frac{(2n+1)^{A_0}}{3^{10A_{eq}}}\right)^{\frac{1}{10A_{eq}-A_0}},$$
$$\alpha \sim 10^{-16}\frac{A_0 A_{eq}}{10A_{eq} - A_0}\ln\left[\frac{(2n+1)}{3^{A_0}}\right]. \tag{31}$$

So for $n > 0$, β is always positive. For α to be positive we must have $\frac{2n+1}{3^{A_0}} > 1$. Current estimates are that $n \approx 2$ [1]. Then $A_0 \geqslant 1$. So positivity of α and β is ensured. We further clarify our estimate of β as follows. Consider the equation of state parameter (27). In the matter dominated era $w \sim 0$. Let the time scale be denoted by t_m. So

$$w = -1 + 2\beta e^{-\alpha t_m} \approx 0.$$

This gives

$$t_m \approx \frac{1}{\alpha}\ln(2\beta). \tag{32}$$

Since physical time scales cannot be negative, this means that $\ln(2\beta) > 0$, i.e. $2\beta > 1$ or $\beta > 1/2$. This is consistent with the similar requirement from (30b), viz., $\ln(3\beta) > 0$, i.e. $\beta > 1/3$.

5. The sound speed and the deceleration parameter

The (square of) sound speed is from Eq. (26)

$$c_s^2 = \frac{\partial p_c}{\partial \rho_c} = 1 - \frac{\alpha}{\sqrt{24\pi G}\rho_c^{\frac{1}{2}}}$$
$$= 1 - \frac{\alpha}{\sqrt{24\pi G}}\frac{\sqrt{24\pi G}}{\alpha}\left(\frac{e^{\alpha t} - \beta}{e^{\alpha t}}\right) = \beta e^{-\alpha t}. \tag{33}$$

So $c_s^2 \ll 1$ provided $e^{\alpha t} \gg \beta$, i.e. $e^{\alpha t} \gg \frac{24\pi G\sqrt{C}A_1}{\alpha}$. One way to achieved this is by choosing the constant $\sqrt{C} = \frac{\alpha c'}{24\pi GA_1}$ where $c' \gg 1$ so that $e^{\alpha t} \gg 1$. Note that α and t are always positive. So the sound speed $c_s^2 \ll 1$. Note that these results are consistent with our estimates obtained before.

Now consider the deceleration parameter defined by $Q = -a\ddot{a}/(\dot{a})^2$.

$$Q = -\frac{a_c\ddot{a}_c}{(\dot{a}_c)^2} = -1 + 3\beta e^{-\alpha t}. \tag{34a}$$

Therefore when dark energy dominates, i.e. $t \to \infty$ we have

$$Q_{t\to\infty} = -1. \tag{34b}$$

Let us now try to determine the behaviour of Q in the matter dominated era. During the matter dominated era $Q \sim 1/2$. Imposing this condition on Eq. (34a) gives

$$-1 + 3\beta e^{-\alpha t_m} \approx \frac{1}{2}, \quad \text{i.e. } t_m \approx \frac{1}{\alpha} \ln 2\beta. \tag{35}$$

6. Self consistency and fixing of parameters

Comparing (35) and (32) we see that we have obtained the same value for the time scale of the matter dominated era evaluated from two different view points viz. equation of state parameter $w \sim 0$ and deceleration parameter $Q \sim 1/2$. So our approach is internally self-consistent and we can write the following equations (A_0, A_{eq}, A_m are positive, phenomenological constants):

$$t_0 = A_0 \frac{1}{\alpha} \ln((2n+1)\beta) \sim 10^{17}, \tag{36a}$$

$$t_{eq} = A_{eq} \frac{1}{\alpha} \ln(3\beta) \sim 10^{16}, \tag{36b}$$

$$t_m = A_m \frac{1}{\alpha} \ln(2\beta). \tag{36c}$$

The thing to note here is that under current estimates [1], $n \approx 2$ and so $\ln[(2n+1)\beta] > \ln(3\beta) > \ln(2\beta)$, i.e. $t_0 > t_{eq} > t_m$ and this is what is physically expected. So the present theory is again shown to be self-consistent and this consistency will place bounds on the relative values of the parameters A_0, A_{eq}, A_m.

Let us now check whether the assumptions used in obtaining equation (28) were consistent with the other inputs. Note that for $t \to \infty$ only ρ_{DE} and ρ_{DM} survive. We now carry out the following order of magnitudes analysis. Consider the time set $t = (t_m, t_{eq}, t_0)$. Then (36) can be succinctly written as

$$\alpha t \sim \ln[(2n+1)\beta] \tag{37}$$

modulo respective constants (i.e. A_0, A_{eq}, A_m). Then $n = 1/2$ gives Eq. (36c), $n = 1$ gives (36b) and $n = n$ means (36a). So (37) implies

$$\beta e^{-\alpha t} = \frac{1}{(2n+1)} \equiv \left(\frac{1}{2}; \text{ or } \frac{1}{3}; \text{ or } \frac{1}{5}\right)$$

i.e. $n = \frac{1}{2}$ corresponds to the matter dominated era, $n = 1$ signifies the period when the dark matter and dark energy densities were equal and $n = 2$ characterises the present epoch when dark energy dominates. n is expected to grow larger and larger with time as the domination by dark energy increases. So Eq. (28) can also be written as

$$\begin{aligned}
\rho_c &= \frac{\alpha^2}{24\pi G} + \frac{\alpha^2 \beta}{24\pi G} a_c^{-3} + \frac{\alpha^2 \beta}{24\pi G} a_c^{-3} \left(1 + \beta e^{-\alpha t} + \cdots\right) \\
&= \frac{\alpha^2}{24\pi G} + \frac{\alpha^2 \beta}{12\pi G} a_c^{-3} + \frac{\alpha^2 \beta}{24\pi G} a_c^{-3} \left(e^{\frac{1}{(2n+1)}} - 1\right) \\
&= \frac{\alpha^2}{24\pi G} + \frac{\alpha^2 \beta}{24\pi G} a_c^{-3} + \frac{\alpha^2 \beta}{24\pi G} a_c^{-3} e^{\frac{1}{(2n+1)}}.
\end{aligned} \tag{38}$$

As $n \to \infty$, $e^{\frac{1}{(2n+1)}} \to 1$ and $\rho_c \to \rho_{DE} + \rho_{DM}$ as before. Thus the assumptions leading to (28) are consistent with the other inputs.

There are five parameters of the theory to start with, viz., $\alpha, \beta, \sqrt{C}, A_1, A_2$, with $\alpha\beta = 24\pi G \sqrt{C} A_1$. So we are left with 4 independent parameters. We chose $\sqrt{C} = \frac{\alpha c'}{24\pi G A_1}$ while determining the sound speed and this means $\beta = c' \gg 1$ (discussion after Eq. (33)) leaving 3 independent parameters, viz., α, c' and A_2. Now A_0, A_{eq}, A_m are all positive numerical parameters and following the discussion after Eq. (31), $A_0 > 1$. In the present epoch $n \approx 2$, [1]. Hence $e^{\alpha t_0} \approx (5\beta)^{A_0}$. The scaling relation (7) is valid provided $\|\frac{V_\phi}{V}\| = \frac{1}{e^{\alpha t} - \beta} \approx 0$. This means $e^{\alpha t} - \beta = 5^{A_0} \beta^{A_0} - \beta$ is very large.

But $A_0 > 1$ and $\beta = c' \gg 1$. Therefore, $e^{\alpha t} - \beta \sim 5^{A_0} \beta^{A_0}$ is indeed large and (7) can be made to remain true by suitably choosing A_0, β to be very large. Note that bounds on α may be estimated consistently from Eqs. (36) and (21c) by using the current value of H along with the values of t_0, t_{eq}, t_m and suitably adjusting A_0, A_{eq}, A_m and c' (since $\beta = c'$). So effectively we have a theory where there are two free parameters viz., c', A_2, to be adjusted consistently taking into account observations as far as possible. The value of A_2 in $V(\phi)$ has to be consistently chosen with the other parameters. Beyond this nothing more can be said regarding this constant.

7. Discussion

There is considerable literature on k-essence [14–25] and references therein. The present work differs from all of these in that we have used a standard Lagrangian with canonical kinetic terms (obtained after a re-definition of variables) and used the solutions of the Euler–Lagrange equations to directly determine cosmologically relevant quantities. Realistic cosmological scenarios are obtained in the context of dark matter and dark energy. The basic results can be summarised thus: (a) $q(t)$ and ϕ can be looked upon as dynamical variables whose classical time evolution can be obtained by solving the classical Euler–Lagrange equations corresponding to a Lagrangian with canonical kinetic terms. Assuming $\frac{V^2}{dV/d\phi}$ to be a constant gives a potential of the form $V(\phi) = \frac{A_1}{\phi + A_2}$ where A_1, A_2 are constants. $\frac{dV/d\phi}{V}$ can be made negligible so that the scaling relation of Ref. [1] can be made to remain valid. (b) The energy density ρ has a dark energy component and a dark matter component. (c) The pressure p is *negative* for time $t \to \infty$ and the sound velocity $c_s^2 = \frac{\partial p}{\partial \rho} \ll 1$. (d) The deceleration parameter $Q \to -1$ for time $t \to \infty$ and $Q \sim \frac{1}{2}$ for the matter dominated era. (e) During the matter dominated era the equation of state parameter $w \approx 0$ and when the dark energy dominates (for times $t \to \infty$) we have $w \approx -1$.

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