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AN EXTENDED DIRECT BRANCHING ALGORITHM FOR CHECKING EQUIVALENCE OF DETERMINISTIC PUSHDOWN AUTOMATA*

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Abstract. This paper extends the direct branching algorithm of [25] for checking equivalence of deterministic pushdown automata. It does so by providing a technique called 'halting' for dealing with nodes with unbounded degree in the comparison tree. This may occur when a skipping step may be applied infinitely many times to a certain node, as a result of infinite sequences of ε-moves.

This extension allows the algorithm to check equivalence of two deterministic pushdown automata when none of them is real-time, but in a certain condition that properly contains a case where one of them is real-time strict.

1. Introduction

While the equivalence problem for deterministic context-free languages still remains open at the present moment, various algorithms have been devised to give as wide as possible partial solutions to it [1–18, 20–32]. Among them, Tomita [25] has recently presented a new technique which can be used to check equivalence of two strict deterministic pushdown automata (dpda), only one of which is real-time. It is especially distinguished that it does not need to 'mix' the two languages in question. So, it is very direct and simple. Such an approach is also applicable to a pair of strict deterministic vs. LL(k) grammars, or two LL(k) grammars as well [26].

The aim of this paper is to extend the former result to give a more powerful algorithm that works for a pair of strict dpda's provided that the pair satisfies a certain properly weaker condition than that of real-timeness for one of them. It also inherits such a property that it does not need to mix the two languages in question, that is, it mainly deals with only equivalence equations each of whose left-hand sides consists of a reachable configuration of one dpda and each of whose right-hand sides that of the other.

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Our equivalence checking is carried out by expanding step by step a comparison tree by 'branching' or 'skipping' as in [25]. Then, the condition under which the algorithm works is that a skipping step is eventually applicable when it is to be applied. Such a condition (described in Definition 3.4) may be satisfied even if both dpda's have infinite sequences of ε-moves. But if this is the case, we may have an infinite number of 'skipping-ends', and consequently the comparison tree may grow unboundedly wide. Then, a new technique is required for terminating the expansion of the tree. Such a technique is here named 'halting'.

The basic steps of branching and skipping are almost the same as in [25] but some extensions. For the sake of completeness, however, we repeat the definitions and the arguments from [25] in Section 2, and in Sections 4.1 and 4.2. Section 3 introduces the so-called 'segmental property' for a pair of dpda's for which our algorithm works, and shows some basic properties which are directly relevant to our algorithm. A new step of halting is given in Section 4.3. Then the whole algorithm is presented in Section 4.4, followed by an example in Section 4.5. The exact proofs of termination and correctness are summarized in Section 5. It is assumed in Sections 3 through 5 that dpda's are strict and all of whose reachable configurations are live, and then Section 6 explains that with such assumptions generality is not lost.

The reader is advised that prior understanding of at least one of [25] and [26] may be very helpful to read this paper.

2. Definitions and notation

Our definitions and notation are almost as in [25].

Definition 2.1. A *deterministic pushdown automaton* (*dpda* for short) is denoted by

$$M=(Q, \Gamma, \Sigma, \delta, q_0, Z_0, F),$$

where Q is the finite set of states, Γ the finite set of stack symbols, Σ the finite set of input symbols, δ the finite set of transition rules as described below, $q_0 \in Q$ is the initial state, $Z_0 \in \Gamma$ the initial stack symbol, and $F \subseteq Q$ the set of final states. We denote an empty string in Γ^* or Σ^* by ε.

The set of *transition rules* δ is a set of rules of the form

$$(p, A) \xrightarrow{a} (q, \theta), \quad p, q \in Q, A \in \Gamma, a \in \Sigma \cup \{\varepsilon\}, \theta \in \Gamma^*,$$

that satisfies the following conditions:

(i) If $(p, A) \xrightarrow{a} (q, \theta)$ with $a \in \Sigma \cup \{\varepsilon\}$ is in δ, then δ contains no rule of the form $(p, A) \xrightarrow{a} (r, \gamma)$ for any $(r, \gamma) \neq (q, \theta)$.

(ii) If $(p, A) \xrightarrow{\varepsilon} (q, \theta)$ is in δ, then $\theta = \varepsilon$ and δ contains no rule of the form $(p, A) \xrightarrow{a} (r, \gamma)$ for any $a \in \Sigma$, r, γ. Such a rule as $(p, A) \xrightarrow{\varepsilon} (q, \varepsilon)$ is called an *ε-rule*.

A dpda is said to be *real-time* if it has no ε-rules.

A dpda accepting by empty stack is called a *strict* dpda. In case the previously given dpda M is strict, we may let $M = (Q, \Gamma, \Sigma, \delta, q_0, Z_0, \emptyset)$. The class of strict dpda's is denoted by D_0, and that of real-time strict dpda's by R_0.

Definition 2.2. A *configuration* (p, α) of the dpda M is an element of $Q \times \Gamma^*$, where the *leftmost* symbol of α is the *top* symbol on the stack. In particular, (q_0, Z_0) is called the *initial configuration.*

A configuration (p, α) is said to be in *reading mode* if $\alpha = A\alpha'' \in \Gamma^+$ and $(p, A) \xrightarrow{a} (q, \theta)$ is in δ for some $a \in \Sigma$ and $(q, \theta) \in Q \times \Gamma^*$, while it is said to be in *ε-mode* if $\alpha = A\alpha'' \in \Gamma^+$ and $(p, A) \xrightarrow{\varepsilon} (q, \varepsilon)$ is in δ for some $q \in Q$. A configuration $(p, A\alpha'')$ in reading mode is also said to have a *nondecreasing mode* if $(p, A) \xrightarrow{a} (q, \theta)$ is in δ for some $a \in \Sigma$ and $(q, \theta) \in Q \times \Gamma^+$.

The *height* of a configuration (p, α) is $|\alpha|$. Here, for a string α, $|\alpha|$ denotes the length of α.

Definition 2.3. The dpda M makes a *move* $(p, A\omega) \underset{M}{\xrightarrow{a}} (q, \theta\omega)$ from one configuration to another for any $\omega \in \Gamma^*$ if and only if δ contains a rule $(p, A) \xrightarrow{a} (q, \theta)$ with $a \in \Sigma \cup \{\varepsilon\}$.

A sequence of such moves through successive configurations as

$$(p_1, \alpha_1) \underset{M}{\xrightarrow{a_1}} (p_2, \alpha_2), (p_2, \alpha_2) \underset{M}{\xrightarrow{a_2}} (p_3, \alpha_3), \ldots, (p_m, \alpha_m) \underset{M}{\xrightarrow{a_m}} (p_{m+1}, \alpha_{m+1})$$

($\alpha_i \neq \varepsilon$ and $a_i \in \Sigma \cup \{\varepsilon\}$ for $1 \leq i \leq m$) is called a *derivation*, and is written as

$$(p_1, \alpha_1) \underset{M}{\overset{x}{\Rightarrow}}{}^{(m)} (p_{m+1}, \alpha_{m+1}), \quad x = a_1 a_2 \ldots a_m,$$

or simply

$$(p_1, \alpha_1) \underset{M}{\overset{x}{\Rightarrow}} (p_{m+1}, \alpha_{m+1}).$$

If, in the above derivation, there exists $\alpha'' \in \Gamma^*$ such that, for each $1 \leq i \leq m+1$, $\alpha_i = \alpha'_i\alpha''$ for some $\alpha'_i \in \Gamma^*$ where $\alpha'_i \neq \varepsilon$ for $1 \leq i \leq m$, then it may be written as

$$(p_1, \alpha'_1 | \alpha'') \underset{M}{\overset{x}{\Rightarrow}}{}^{(m)} (p_{m+1}, \alpha'_{m+1} | \alpha''),$$

or

$$(p_1, \alpha'_1 | \alpha'') \underset{M}{\overset{x}{\Rightarrow}} (p_{m+1}, \alpha'_{m+1} | \alpha'').$$

where '|' is a metasymbol not in Γ.

By convention, we let $(p, \alpha) \underset{M}{\overset{\varepsilon}{\Rightarrow}} (p, \alpha)$ for any $(p, \alpha) \in Q \times \Gamma^*$.

A derivation $(p, \alpha) \overset{x}{\underset{M}{\Rightarrow}} (q, \beta)$ is also written as $(p, \alpha) \overset{x}{\underset{M}{\Rightarrow\!\!>}} (q, \beta)$ if no such derivation as $(q, \beta) \overset{\varepsilon}{\underset{M}{\Rightarrow}} (r, \gamma)$ is possible for any $(r, \gamma) \neq (q, \beta)$. On the other hand, a derivation $(p, \alpha) \overset{x}{\underset{M}{\Rightarrow}} (r, \gamma)$ is also written as $(p, \alpha) \overset{x}{\underset{M}{\Rightarrow\!|}} (r, \gamma)$ if no such derivation as $(p, \alpha) \overset{x}{\underset{M}{\Rightarrow}} (q, \beta) \overset{\varepsilon}{\underset{M}{\Rightarrow}} (r, \gamma)$ is possible for any $(q, \beta) \neq (r, \gamma)$.

A configuration (p, α) is said to be *reachable* if $(q_0, Z_0) \overset{u}{\underset{M}{\Rightarrow}} (p, \alpha)$ for some $u \in \Sigma^*$.

Definition 2.4. Let (p, α) be a configuration of the dpda M.

In case acceptance by M is by final states, define

$$L(p, \alpha) = \{x \in \Sigma^* \mid (p, \alpha) \overset{x}{\underset{M}{\Rightarrow}} (q, \beta) \text{ for some } (q, \beta) \in F \times \Gamma^*\}.$$

In case acceptance by M is by empty stack, i.e., M is strict, define

$$L(p, \alpha) = \{x \in \Sigma^* \mid (p, \alpha) \overset{x}{\underset{M}{\Rightarrow}} (q, \varepsilon) \text{ for some } q \in Q\}.$$

In either case, the language accepted by M is $L(M) = L(q_0, Z_0)$.

A configuration (p, α) is said to be *live* if $L(p, \alpha) \neq \emptyset$.

Definition 2.5. In case the dpda M is strict, let (p, α) be a configuration of M, and $(p, \alpha) \overset{\varepsilon}{\underset{M}{\Rightarrow\!\!>}} (p', \alpha')$. Then if $\alpha' = A'\alpha'' \neq \varepsilon$ with $A' \in \Gamma$, define

$$\text{FIRST}(p, \alpha) = \text{FIRST}(p', A') = \{a \in \Sigma \mid (p', A') \overset{a}{\rightarrow} (q, \theta) \text{ is in } \delta$$
$$\text{for some } (q, \theta) \in Q \times \Gamma^*\}.$$

Otherwise (i.e., $\alpha' = \varepsilon$), define

$$\text{FIRST}(p, \alpha) = \text{FIRST}(p', \varepsilon) = \{\varepsilon\}.$$

Definition 2.6. For a configuration (p, α) of the dpda M, define

$$\text{EMP}(p, \alpha) = \{q \in Q \mid (p, \alpha) \overset{x}{\underset{M}{\Rightarrow}} (q, \varepsilon) \text{ for some } x \in \Sigma^*\}.$$

Definition 2.7. Let (p, α) be a configuration of a dpda M_i and $(\bar{p}, \beta)$ be a configuration of a dpda M_2 ($i = 1$ or 2). If $L(p, \alpha) = L(\bar{p}, \beta)$, then the two configurations are *equivalent*, and it is written as $(p, \alpha) \equiv (\bar{p}, \beta)$. Such a formula is named an *equivalence equation*.

If $L(M_1) = L(M_2)$, then the two dpda's are equivalent, and it is written as $M_1 \equiv M_2$. Otherwise, $M_1 \not\equiv M_2$.

Definition 2.8. For a string $\beta(\in \Gamma^*)$ and nonnegative integers $\mathscr{R}$ and h, define

$$^{(\mathscr{R})}\beta = \begin{cases} \beta' & \text{if } |\beta| > \mathscr{R} \text{ and } \beta = \beta'\beta'' \text{ with } |\beta'| = \mathscr{R}, \\ \beta & \text{if } |\beta| \leq \mathscr{R}, \end{cases}$$

and

$$\beta^{(h)} = \begin{cases} \beta'' & \text{if } |\beta| > h \text{ and } \beta = \beta'\beta'' \text{ with } |\beta''| = h, \\ \beta & \text{if } |\beta| \leq h. \end{cases}$$

3. Prior conditions and basic properties

We shall consider checking equivalence of a pair of strict dpda's $M_i = (Q_i, \Gamma_i, \Sigma, \delta_i, q_{0i}, Z_{0i}, \emptyset)(\in D_0)$, $i = 1, 2$, which is under a certain condition described later (in Definition 3.4). Furthermore, we shall assume, without loss of generality, that all reachable configurations are live. Throughout this paper, we are only concerned with configurations that are reachable, and hence live. For more general cases, see Section 6.

Proposition 3.1. *The necessary and sufficient condition for strict dpda's M_1 and M_2 to be equivalent is:*

If $(q_{01}, Z_{01}) \underset{M_1}{\overset{u}{\Longrightarrow}} (p, \alpha)$ *for some* $u \in \Sigma^*$ *and* $(p, \alpha) \in Q_1 \times \Gamma_1^*$, *then*
$(q_{02}, Z_{02}) \underset{M_2}{\overset{u}{\Longrightarrow}} (\bar{p}, \beta)$ *for some* $(\bar{p}, \beta) \in Q_2 \times \Gamma_2^*$; *and*
$\text{FIRST}(p, \alpha) = \text{FIRST}(\bar{p}, \beta)$.

Proof. This is a direct consequence of the assumption that all reachable configurations are live (see for the details [25, Proposition 3.1, pp. 194–195]). □

Definition 3.2. Let $(p, \alpha\beta\gamma) \in Q_2 \times \Gamma_2^*$.

If $L(q, \beta) \cap \Sigma^+ = \emptyset$ for any $q \in \text{EMP}(p, \alpha)$, then $\beta \in \Gamma_2^*$ is said to be an *ε-segment* in $(p, \alpha\beta\gamma)$. In this case, $\beta = \varepsilon$, or else if we let $\beta = \beta'\beta''$, $\beta'' \in \Gamma_2^+$ and $q' \in \text{EMP}(p, \alpha\beta')$, then (q', β'') is in ε-mode (when (q', β'') is live).

Otherwise, i.e., $L(q, \beta) \cap \Sigma^+ \neq \emptyset$ for some $q \in \text{EMP}(p, \alpha)$, then $\beta \in \Gamma_2^+$ is said to be a *reading segment* in $(p, \alpha\beta\gamma)$. Moreover, a reading segment $\beta \in \Gamma_2^+$ in $(p, \alpha\beta\gamma)$ is said to be *canonical* if no substring $\beta_2 \in \Gamma_2^+$ of $\beta = \beta_1\beta_2\beta_3$ $(\beta_1, \beta_3 \in \Gamma_2^*)$ is an ε-segment in $(p, \alpha\beta\gamma)$, and both $\alpha^{(1)}$ and $^{(1)}\gamma$ are ε-segments in $(p, \alpha\beta\gamma)$. In this case, if we let $\beta = \beta'\beta''$, $\beta'' \in \Gamma_2^+$, then (q', β'') is in reading mode for some $q' \in \text{EMP}(p, \alpha\beta')$.

For $(\bar{p}, \beta) \in Q_2 \times \Gamma_2^*$, if

$$\beta = \lambda_0\beta_1\lambda_1\beta_2\lambda_2 \ldots \beta_l\lambda_l,$$

where $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_l \in \Gamma_2^*$ are ε-segments in $(\bar{p}, \beta)$ and $\beta_1, \beta_2, \ldots, \beta_l \in \Gamma_2^+$ are canonical reading segments in $(\bar{p}, \beta)$, then we write

$$\text{Reading-Seg}(\bar{p}, \beta) = \beta_1\beta_2 \ldots \beta_l.$$

That is, Reading-Seg$(\bar{p}, \beta)$ is the concatenation of the whole successive canonical reading segments in $(\bar{p}, \beta)$.

Remark 3.3. Given a certain configuration, a sequence of the whole successive canonical reading segments in it uniquely exists, and can be detected easily.

Definition 3.4. A pair of live configurations $(p, \alpha) \in Q_1 \times \Gamma_1^+$ and $(\bar{p}, \beta) \in Q_2 \times \Gamma_2^+$ is said to have the *segmental property* if $(p, \alpha) \equiv (\bar{p}, \beta)$ implies the following property:

There exists a constant $\mathscr{R} \geq 1$, depending on only M_1 and M_2, such that if

$$(p, A|\alpha'') \overset{x}{\underset{M_1}{\Longrightarrow}} (q, \varepsilon|\alpha'') \quad \text{where } \alpha = A\alpha'' \text{ with } A \in \Gamma_1,$$

for some $x \in \Sigma^*$ and $q \in Q_1$ (with $L(q, \alpha'') \neq \emptyset$), then

$$(\bar{p}, \beta'|\beta'') \overset{x}{\underset{M_2}{\Longrightarrow}} (\bar{q}_j, \gamma_j|\beta'')$$

for some factorization of $\beta = \beta'\beta''$ and $(\bar{q}_j, \gamma_j) \in Q_2 \times \Gamma_2^*$ such that

$$|\text{Reading-Seg}(\bar{p}, \beta')| \leq \mathscr{R}.$$

(That is, the corresponding derivation of M_2 from $(\bar{p}, \beta'\beta'')$ can be restricted to be independent of β'' which is initially below a certain finite upper segment β'. Note here that the total length of β' may not be bounded when M_2 has infinite sequences of ε-moves.)

A pair of dpda's M_1 and M_2 is said to have the *segmental property* if, in case $M_1 \equiv M_2$, every pair of live configurations $(p, \alpha) \in Q_1 \times \Gamma_1^+$ and $(\bar{p}, \beta) \in Q_2 \times \Gamma_2^+$ such that $(q_{01}, Z_{01}) \overset{u}{\underset{M_1}{\Longrightarrow}} (p, \alpha)$ and $(q_{02}, Z_{02}) \overset{u}{\underset{M_2}{\Longrightarrow}} (\bar{p}, \beta)$ for some $u \in \Sigma^*$ has the segmental property.

Remark 3.5. Both dpda's may have infinite sequences of ε-moves in Definition 3.4. In particular, however, if M_2 is real-time strict, then the above condition is necessarily satisfied with Reading-Seg$(\bar{p}, \beta') = \beta'$ (see [25, Proposition 3.2, pp. 195–196]). Hence, for a pair of strict dpda's, the segmental property is properly more general than real-timeness for one of them.

We are henceforth exclusively concerned with a pair of strict dpda's M_1 and M_2 which is assumed to have the segmental property. Now the emphasis in this paper is on the equivalence checking algorithm itself, and the idea of the segmental property is introduced merely to demonstrate the increased generality of our technique over previous ones. So, we do not go into the details of the property further except the following which are directly relevant to termination of our algorithm.

Lemma 3.6. *Let M_1 and M_2 be a pair of equivalent strict dpda's which has the segmental property, and*

$$(q_{01}, Z_{01}) \underset{M_1}{\overset{u}{\Longrightarrow}} (p, \alpha) \quad \text{and} \quad (q_{02}, Z_{02}) \underset{M_2}{\overset{u}{\Longrightarrow}} (\bar{p}, \beta)$$

for some $u \in \Sigma^*$ and live configurations $(p, \alpha) \in Q_1 \times \Gamma_1^+$, $(\bar{p}, \beta) \in Q_2 \times \Gamma_2^+$, hence

$$(p, \alpha) \equiv (\bar{p}, \beta).$$

Then

$$|\text{Reading-Seg}(\bar{p}, \beta)| \leq \mathcal{B}(\alpha),$$

where

$$\mathcal{B}(\alpha) = \mathcal{R}(1 + |Q_2|)^{|\alpha|-1}.$$

Here, for a set Q_2, $|Q_2|$ denotes the cardinality of Q_2.

Proof. The proof is by induction on $|\alpha|$.

The basis, $|\alpha| = 1$, is obvious by Definition 3.4, where $\mathcal{B}(\alpha) = \mathcal{R}$.

Next, we assume for some n (≥ 1) that the lemma is true for any α such that $|\alpha| \leq n$. Then let

$$(q_{01}, Z_{01}) \underset{M_1}{\overset{u_0}{\Longrightarrow}} (p_0, A\alpha) \quad \text{and} \quad (q_{02}, Z_{02}) \underset{M_2}{\overset{u_0}{\Longrightarrow}} (\bar{p}_0, \beta_0'\beta_0'')$$

for some $u_0 \in \Sigma^*$ and live configurations $(p_0, A\alpha) \in Q_1 \times \Gamma_1^{n+1}$, $(\bar{p}_0, \beta_0'\beta_0'') \in Q_2 \times \Gamma_2^*$, where $A \in \Gamma_1$ and $|\text{Reading-Seg}(\bar{p}_0, \beta_0')| = \mathcal{R}$ or else $\beta_0'' = \varepsilon$. Now if

$$(p_0, A) \underset{M_1}{\overset{v}{\Longrightarrow}} (p, \varepsilon) \quad \text{for some } v \in \Sigma^* \text{ and } p \in Q_1,$$

then

$$(\bar{p}_0, \beta_0'|\beta_0'') \underset{M_2}{\overset{v}{\Longrightarrow}} (\bar{p}, \beta'|\beta_0'') \quad \text{for some } (\bar{p}, \beta') \in Q_2 \times \Gamma_2^*,$$

since the pair of M_1 and M_2 has the segmental property. In addition,

$$(p, \alpha) \equiv (\bar{p}, \beta'\beta_0'') \quad \text{with } |\alpha| = n.$$

Hence,

$$|\text{Reading-Seg}(\bar{p}, \beta'\beta_0'')| \leq \mathcal{R}(1 + |Q_2|)^{|\alpha|-1}$$

by the induction hypothesis. Thus, for any $\bar{q} \in \text{EMP}(\bar{p}, \beta') \subseteq \text{EMP}(\bar{p}_0, \beta_0')$,

$$|\text{Reading-Seg}(\bar{q}, \beta_0'')| \leq |\text{Reading-Seg}(\bar{p}, \beta'\beta_0'')| \leq \mathcal{R}(1 + |Q_2|)^{|\alpha|-1}.$$

Therefore, if we let $\mathrm{EMP}(\bar{p}_0, \beta_0') = \{\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_l\}$ where $l \leqslant |Q_2|$, we have

$$|\text{Reading-Seg}(\bar{p}_0, \beta_0'\beta_0'')| \leqslant |\text{Reading-Seg}(\bar{p}_0, \beta_0')| + \sum_{i=1}^{l} |\text{Reading-Seg}(\bar{q}_i, \beta_0'')|$$
$$\leqslant \mathcal{R} + |Q_2|\mathcal{R}(1+|Q_2|)^{|\alpha|-1}$$
$$\leqslant \mathcal{R}(1+|Q_2|)^{|A\alpha|-1}.$$

So, the lemma has been induced. □

Recall that, for a pair of equivalent live configurations (p, α) of a strict dpda ($\in D_0$) and $(\bar{p}, \beta)$ of a real-time strict dpda ($\in R_0$), we have

$$|\text{Reading-Seg}(\bar{p}, \beta)| = |\beta| \leqslant k_1|\alpha|$$

for some constant k_1 which is defined by M_1 (see [25, Definition 3.1 and Lemma 3.1, pp. 196–197]).

Definition 3.7. (i) Let (p, β) and $(p, \gamma) \in Q_2 \times \Gamma_2^*$ be two live configurations such that

$$\beta = \lambda_0\beta_1\lambda_1\beta_2\lambda_2 \ldots \beta_l\lambda_l, \qquad \gamma = \mu_0\beta_1\mu_1\beta_2\mu_2 \ldots \beta_l\mu_l,$$

where

$$\text{Reading-Seg}(p, \beta) = \text{Reading-Seg}(p, \gamma) = \beta_1\beta_2 \ldots \beta_l$$

and

$$\mathrm{EMP}(p, \lambda_0) = \mathrm{EMP}(p, \mu_0).$$

Then we write

$$(p, \beta) \approx (p, \gamma)$$

in case, for every $q \in \mathrm{EMP}(p, \lambda_0\beta_1\lambda_1 \ldots \beta_i) = \mathrm{EMP}(p, \mu_0\beta_1\mu_1 \ldots \beta_i)$,

$$(q, \lambda_i) \underset{M_2}{\overset{\varepsilon}{\Longrightarrow}} (r, \varepsilon) \text{ for } r \in Q_2 \text{ if and only if } (q, \mu_i) \underset{M_2}{\overset{\varepsilon}{\Longrightarrow}} (r, \varepsilon) \text{ for } r \in Q_2,$$

for $i = 1, 2, \ldots, l$.

(ii) For two live configurations $(p, \beta), (p, \gamma) \in Q_2 \times \Gamma_2^*$ and an integer $h \geqslant 0$, let

$$\beta = \beta'\beta'' \quad \text{with } \beta'' = \beta^{(h)},$$

and

$$\gamma = \gamma'\gamma'' \quad \text{with } \gamma'' = \gamma^{(h)}.$$

Then we write

$$(p, \beta) \overset{h}{\approx} (p, \gamma)$$

if $(p, \beta') \approx (p, \gamma')$ and $\beta'' = \gamma''$.

Remark 3.8. (i) By definition, $\overset{h}{\approx}$ coincides with $\approx$ and $=$ in case $h=0$ and $h=\infty$, respectively. In addition, relation $\overset{h}{\approx}$ is reflexive, symmetric, and transitive, i.e., $\overset{h}{\approx}$ is an equivalence relation. Furthermore, $(p,\beta)\overset{h}{\approx}(p,\gamma)$ implies $(p,\beta)\equiv(p,\gamma)$.

(ii) If $(p,\beta)\overset{h}{\approx}(p,\gamma)$ as in Definition 3.7(ii), then $(p,\beta)\overset{h'}{\approx}(p,\gamma)$ for any $h'\leq h$, and more generally, $(p,\beta'\partial)\overset{h''}{\approx}(p,\gamma'\partial)$ for any $\partial\in\Gamma_2^*$ and $h''\leq|\partial|$.

(iii) If $(p,\beta)\overset{h}{\approx}(p,\gamma)$ as above, and $(p,\beta)\overset{x}{\underset{M_2}{\Longrightarrow}}|(q_1,\partial_1)$, $(p,\gamma)\overset{x}{\underset{M_2}{\Longrightarrow}}|(q_2,\partial_2)$ for some $x\in\Sigma^*$, $(q_1,\partial_1)\in Q_2\times\Gamma_2^*$, $(q_2,\partial_2)\in Q_2\times\Gamma_2^*$, then $q_1=q_2$ and $(q_1,\partial_1)\overset{h}{\approx}(q_2,\partial_2)$. Besides, if either $|\partial_1|\leq h$ or $|\partial_2|\leq h$ then $\partial_1=\partial_2$.

(iv) Given (p,β), (p,γ), and h, it is easy to check whether $(p,\beta)\overset{h}{\approx}(p,\gamma)$ or not. Summarizing the above all, if $(p,\beta)\overset{h}{\approx}(p,\gamma)$ then we may regard that (p,β) and (p,γ) are almost equal.

Lemma 3.9. *Given a positive integer $\mathscr{B}$, consider a set $S[\mathscr{B}]$ of configurations defined by*

$$S[\mathscr{B}]=\{(\bar{p},\beta')\in Q_2\times\Gamma_2^*\mid(q_{02},Z_{02})\overset{u}{\underset{M_2}{\Longrightarrow}}(\bar{p},\beta'\beta'')\ \textit{for some}\ u\in\Sigma^*,\beta''\in\Gamma_2^*,$$

$$\textit{with}\ (\bar{p},\beta')\overset{\varepsilon}{\underset{M_2}{\Longrightarrow}}(\bar{p},\beta')\ \textit{and}\ 0\leq|\text{Reading-Seg}(\bar{p},\beta')|\leq\mathscr{B}\},$$

and partition it into the equivalence classes under the relation $\approx$. Then the number of the equivalence classes into which $S[\mathscr{B}]$ is partitioned is at most

$$|Q_2|^{|Q_2|\mathscr{B}+1}(|\Gamma_2|+1)^{\mathscr{B}}.$$

Proof. We have

$$|\{\bar{p}\in Q_2\mid(\bar{p},\beta')\in S[\mathscr{B}]\}|\leq|Q_2|$$

and

$$|\{\text{Reading-Seg}(\bar{p},\beta')\in\Gamma_2^*\mid(\bar{p},\beta')\in S[\mathscr{B}]\}|\leq(|\Gamma_2|+1)^{\mathscr{B}}.$$

Furthermore, for each $(\bar{p},\beta')=(\bar{p},\beta_1\lambda_1\beta_2\lambda_2\ldots\beta_i\lambda_i\ldots\beta_l\lambda_l)\in S[\mathscr{B}]$ and i, where Reading-Seg$(\bar{p},\beta')=\beta_1\beta_2\ldots\beta_i\ldots\beta_l$ and $1\leq i\leq l\leq\mathscr{B}$, the number of the equivalence classes into which $\{(\bar{p},\beta_1\lambda_1\beta_2\lambda_2\ldots\beta_i\lambda_i'\ldots\beta_l\lambda_l)\in S[\mathscr{B}]\mid\lambda_i'\in\Gamma_2^*$ is an ε-segment in $(\bar{p},\beta_1\lambda_1\beta_2\lambda_2\ldots\beta_i\lambda_i'\ldots\beta_l\lambda_l)\}$ is partitioned under $\approx$ is at most

$$|Q_2|^{|Q_2|}.$$

Therefore, the objective number of the equivalence classes is at most

$$|Q_2|(|\Gamma_2|+1)^{\mathscr{B}}(|Q_2|^{|Q_2|})^{\mathscr{B}}=|Q_2|^{|Q_2|\mathscr{B}+1}(|\Gamma_2|+1)^{\mathscr{B}}.\quad\square$$

Proposition 3.10. *Let M_1 and M_2 be a pair of equivalent strict dpda's which has the segmental property, and*

$$(q_{01}, Z_{01}) \underset{M_1}{\overset{u}{\Longrightarrow}} (p, \alpha) \quad \text{and} \quad (q_{02}, Z_{02}) \underset{M_2}{\overset{u}{\Longrightarrow}} (\bar{p}, \beta)$$

for some $u \in \Sigma^*$ and live configurations $(p, \alpha) \in Q_1 \times \Gamma_1^+$, $(\bar{p}, \beta) \in Q_2 \times \Gamma_2^+$, where $\alpha = A\alpha''$ with $A \in \Gamma_1$. Furthermore, let

$$(p, A|\alpha'') \underset{M_1}{\overset{x_0}{\Longrightarrow}} (q, \varepsilon|\alpha'')$$

for some $x_0 \in \Sigma^*$ and $q \in Q_1$, and

$$\beta = \beta'\beta'' \quad \text{with } \beta^{(h)} = \beta'',$$

for some nonnegative integer $h < |\beta|$.

Now consider the set $S[(p, \alpha) \equiv (\bar{p}, \beta), q; h]$ of configurations defined by

$$S[(p, \alpha) \equiv (\bar{p}, \beta), q; h]$$

$$= \{(\bar{q}_i, \gamma_i\beta'') \in Q_2 \times \Gamma_2^* \mid (p, A) \underset{M_1}{\overset{x}{\Longrightarrow}} (q, \varepsilon) \text{ and}$$

$$(\bar{p}, \beta') \underset{M_2}{\overset{x}{\Longrightarrow}}\!| \; (\bar{q}_i, \gamma_i) \text{ for some } x \in \Sigma^*\},$$

and partition it into the equivalence classes under the relation $\overset{h}{\approx}$. Then the number of the equivalence classes into which $S[(p, \alpha) \equiv (\bar{p}, \beta), q; h]$ is partitioned is at most

$$|Q_2|^{|Q_2|\mathscr{B}(\alpha'')+2}(|\Gamma_2|+1)^{\mathscr{B}(\alpha'')},$$

where $\mathscr{B}(\alpha'')$ is defined in Lemma 3.6.

Proof. Let

$$S'[(p, \alpha) \equiv (\bar{p}, \beta), q; h]$$

$$= \{(\bar{q}_i', \gamma_i'\beta'') \in Q_2 \times \Gamma_2^* \mid (\bar{q}_i, \gamma_i) \underset{M_2}{\overset{\varepsilon}{\Longrightarrow}} (\bar{q}_i', \gamma_i'),$$

$$(\bar{q}_i, \gamma_i) \in S[(p, \alpha) \equiv (\bar{p}, \beta), q; h]\}.$$

Then each $(\bar{q}_i', \gamma_i'\beta'') \in S'[(p, \alpha) \equiv (\bar{p}, \beta), q; h]$ is such that $(q, \alpha'') \equiv (\bar{q}_i', \gamma_i'\beta'')$, and hence

$$|\text{Reading-Seg}(\bar{q}_i', \gamma_i')| \leq |\text{Reading-Seg}(\bar{q}_i', \gamma_i'\beta'')| \leq \mathscr{B}(\alpha'')$$

by Lemma 3.6. So, the number of the equivalence classes into which $S'[(p, \alpha) \equiv (\bar{p}, \beta), q; h]$ is partitioned under $\overset{h}{\approx}$ is at most $|Q_2|^{|Q_2|\mathscr{B}(\alpha'')+1}(|\Gamma_2|+1)^{\mathscr{B}(\alpha'')}$ by Lemma 3.9.

Moreover, $|\{\bar{q}_j \in Q_2 | (\bar{q}_j, \gamma_j\beta'') \in S[(p, \alpha) \equiv (\bar{p}, \beta), q; h]\}| \leq |Q_2|$.
So, combining these gives the final bound. □

In case M_2 is real-time in the above proposition, $S[(p, \alpha) \equiv (\bar{p}, \beta), q; h]$ is finite (cf. [25, Lemma 4.4, p. 205]). But it is not necessary so in our case, nevertheless we can resort to the above property instead for finite termination of the algorithm. It should be noted here that our algorithm needs to know neither $\mathcal{R}$ nor $\mathcal{B}(\alpha'')$ in advance.

4. The equivalence checking algorithm

The equivalence checking is carried out by developing step by step a so-called comparison tree.

At the initial stage, the comparison tree contains only the root labeled $(q_{01}, Z_{01}) \equiv (q_{02}, Z_{02})$ which is said to be in *unchecked* status. In each step the algorithm considers a node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ such that, for some $u \in \Sigma^*$,

$$(q_{01}, Z_{01}) \overset{u}{\underset{M_1}{\Longrightarrow}} (p, \alpha) \quad \text{and} \quad (q_{02}, Z_{02}) \overset{u}{\underset{M_2}{\Longrightarrow}} (\bar{p}, \beta),$$

and tries to prove or disprove this equivalence. In case $\alpha = \beta = \varepsilon$, or another internal node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ has already appeared elsewhere in the tree, then we turn the node to be in *checked* status. Otherwise, except a special case where the node is in *halting* status defined in Section 4.3, we expand it by branching or skipping almost as in [25].

4.1. Branching

Lemma 4.1. *If $(p, \alpha) \equiv (\bar{p}, \beta)$ holds, then the following conditions* (i) *and* (ii) *hold*:

(i) *In case neither (p, α) nor $(\bar{p}, \beta)$ is in ε-mode,*

$$\text{FIRST}(p, \alpha) = \text{FIRST}(\bar{p}, \beta).$$

(ii) (a) *In case both (p, α) and $(\bar{p}, \beta)$ are in reading mode, for each $a_i \in \text{FIRST}(p, \alpha) = \{a_1, a_2, \ldots, a_l\} \subseteq \Sigma$, let*

$$(p, \alpha) \overset{a_i}{\underset{M_1}{\longrightarrow}} (p_i, \alpha_i) \quad \textit{and} \quad (\bar{p}, \beta) \overset{a_i}{\underset{M_2}{\longrightarrow}} (\bar{p}_i, \beta_i).$$

(b) *In case (p, α) is in ε-mode, let $a_1 = \varepsilon$, $l = 1$,*

$$(p, \alpha) \overset{\varepsilon}{\underset{M_1}{\longrightarrow}} (p_1, \alpha_1) \quad \textit{and} \quad (\bar{p}, \beta) = (\bar{p}_1, \beta_1).$$

(c) *In case (p, α) is not in ε-mode and $(\bar{p}, \beta)$ is in ε-mode, let $a_1 = \varepsilon$, $l = 1$,*

$$(p, \alpha) = (p_1, \alpha_1) \quad \textit{and} \quad (\bar{p}, \beta) \overset{\varepsilon}{\underset{M_2}{\Longrightarrow}} (\bar{p}_1, \beta_1).$$

Then, in every case,

$$(p_i, \alpha_i) \equiv (\bar{p}_i, \beta_i), \quad i = 1, 2, \ldots, l.$$

Proof. Part (i) follows from Proposition 3.1, and part (ii) is obvious. □

Checking whether condition (i) holds or not is named branch checking to the node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ in question. When it is verified to hold, the checking is said to be successful. Then, we expand the above node by adding to it l sons labeled $(p_i, \alpha_i) \equiv (\bar{p}_i, \beta_i)$ in *unchecked* status, $i = 1, 2, \ldots, l$. In addition, let the new edges connecting it and these sons be labeled a_i, $i = 1, 2, \ldots, l$, in the same order as above. The step of developing the comparison tree in this way is named *branching* to the node in question. We may apply branching not only to a leaf but also to an internal node (in *skipping* status defined in Section 4.2) in the developing comparison tree. In the latter case, old and new sons may coexist. In either case, the node to which branching has been applied is called the *branching* node, and is in *checked* status.

If condition (i) does not hold, then we conclude that "$M_1 \not\equiv M_2$".

4.2. Skipping

In order to prevent the comparison tree from growing larger and larger infinitely by successive application of branching steps, certain nodes are expanded by other steps of skipping.

Now let the comparison tree which has just been constructed up to a certain stage be denoted by $T(M_1 : M_2)$. Here, it is assumed that the father–son relations in the tree may have been realized not only by branching but also by skipping which will be described hereafter. No other step is ever used to realize any father–son relation in the comparison tree.

Definition 4.2. (i) If $(p, \alpha\gamma) \equiv (\bar{p}, \bar{\alpha}\bar{\gamma})$ and $(q, \beta\gamma) \equiv (\bar{q}, \bar{\beta}\bar{\gamma})$ are labels of two nodes in $T(M_1 : M_2)$ which are connected by an edge labeled $x \in \Sigma^*$ such that

$$(p, \alpha \mid \gamma) \overset{x}{\underset{M_1}{\Longrightarrow}} (q, \beta \mid \gamma) \quad \text{and} \quad (\bar{p}, \bar{\alpha} \mid \bar{\gamma}) \overset{x}{\underset{M_2}{\Longrightarrow}} (\bar{q}, \bar{\beta} \mid \bar{\gamma}),$$

then we write

$$\langle (p, \alpha \mid \gamma) \equiv (\bar{p}, \bar{\alpha} \mid \bar{\gamma}) \rangle \xrightarrow[T(M_1 : M_2)]{x} \langle (q, \beta \mid \gamma) \equiv (\bar{q}, \bar{\beta} \mid \bar{\gamma}) \rangle.$$

(ii) A sequence of such father–son relations as

$$\langle (p_i, \alpha_i \mid \gamma_1) \equiv (\bar{p}_i, \bar{\alpha}_i \mid \bar{\gamma}_1) \rangle \xrightarrow[T(M_1 : M_2)]{x_i} \langle (p_{i+1}, \alpha_{i+1} \mid \gamma_1) \equiv (\bar{p}_{i+1}, \bar{\alpha}_{i+1} \mid \bar{\gamma}_1) \rangle,$$

for $i = 1, 2, \ldots, m$, is named a *derivation path*, and is written as

$$\langle\!\langle (p_1, \alpha_1 | \gamma_1) \equiv (\bar{p}_1, \bar{\alpha}_1 | \bar{\gamma}_1) \rangle\!\rangle \xrightarrow[T(M_1:M_2)]{y} \langle\!\langle (p_{m+1}, \alpha_{m+1} | \gamma_1)$$

$$\equiv (\bar{p}_{m+1}, \bar{\alpha}_{m+1} | \bar{\gamma}_1) \rangle\!\rangle,$$

where $y = x_1 x_2 \ldots x_m$, or simply

$$\langle\!\langle (p_1, \alpha_1 \gamma_1) \equiv (\bar{p}_1, \bar{\alpha}_1 \bar{\gamma}_1) \rangle\!\rangle \xrightarrow[T(M_1:M_2)]{y} \langle\!\langle (p_{m+1}, \alpha_{m+1} \gamma_1) \equiv (\bar{p}_{m+1}, \bar{\alpha}_{m+1} \bar{\gamma}_1) \rangle\!\rangle.$$

In particular, if $(\bar{p}_1, \bar{\alpha}_1) \xRightarrow[M_2]{y}\!\!| \; (\bar{p}_{m+1}, \bar{\alpha}_{m+1})$ (see Definition 2.3), then the above $\xRightarrow[T(M_1:M_2)]{y}$ may be replaced by $\xRightarrow[T(M_1:M_2)]{}\!\!|$.

Definition 4.3. (cf. [25, Definition 4.2, pp. 206–207]). For a derivation $(\bar{p}, \beta) \xRightarrow[M_2]{x} (\bar{q}_j, \gamma)$, if $(\bar{p}, \beta) \xRightarrow[M_2]{x}{}^{(n)} (\bar{q}_j, \gamma)$ with $n \geq 1$, then define

$$|(\bar{p}, \beta) \xRightarrow[M_2]{x} (\bar{q}_j, \gamma)|$$

$$= \min\{|\tilde{\beta}| \mid (\bar{p}, \beta) \xRightarrow[M_2]{x'} (\tilde{p}, \tilde{\beta}) \xRightarrow[M_2]{x''}{}^{(n'')} (\bar{q}_j, \gamma),$$

$$x = x'x'', x'' \in \Sigma^*, (\tilde{p}, \tilde{\beta}) \in Q_2 \times \Gamma_2^+, 1 \leq n'' \leq n\} - 1.$$

Define also

$$|(\bar{p}, \beta) \xRightarrow[M_2]{\varepsilon} (\bar{p}, \beta)| = |\beta| - 1$$

for any $(\bar{p}, \beta) \in Q_2 \times \Gamma_2^+$.

Then consider a node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ in $T(M_1 : M_2)$ where (p, α) has a nondecreasing mode (see Definition 2.2) and $(\bar{p}, \beta)$ is in reading mode, and rewrite the node label as

$$(p, A\alpha'') \equiv (\bar{p}, \beta'\beta''), \quad \text{where } \alpha = A\alpha'' \text{ with } A \in \Gamma_1 \tag{4.1}$$

for some factorization $\beta = \beta'\beta''$ with $\beta' \in \Gamma_2^+$. Furthermore, suppose that $T(M_1 : M_2)$ contains another *branching* node labeled

$$(p, A\omega_1) \equiv (\bar{p}, \underline{\beta}'\omega_2) \tag{4.2}$$

for some $\omega_1 \in \Gamma_1^*$ and $\beta'\omega_2 \in \Gamma_2^*$ such that $(\bar{p}, \beta') \approx (\bar{p}, \underline{\beta}')$. Here, let such $(p, A\omega_1) \equiv (\bar{p}, \underline{\beta}'\omega_2)$ be with the longest possible Reading-Seg$(\bar{p}, \underline{\beta}) \in \Gamma_2^+$.

Definition 4.4. (i) *Applicability of skipping*: Skipping to the node labeled $(p, A\alpha'') \equiv (\bar{p}, \beta'\beta'')$ (equation (4.1)) in question with respect to the *branching* node labeled $(p, A\omega_1) \equiv (\bar{p}, \underline{\beta}'\omega_2)$ (equation (4.2)) is said to be applicable in $T(M_1 : M_2)$ if every derivation path of the form

$$\langle (p, A \mid \omega_1) \equiv (\bar{p}, \underline{\beta}' \omega_2 \mid \varepsilon) \rangle \xRightarrow[T(M_1:M_2)]{x}\!\!| \ \langle (q, \varepsilon \mid \omega_1) \equiv (\bar{q}_j, \underline{\gamma} \mid \varepsilon) \rangle,$$

for any $x \in \Sigma^*$, $(\bar{q}_j, \gamma) \in Q_2 \times \Gamma_2^*$, can be rewritten as

$$\langle (p, A \mid \omega_1) \equiv (\bar{p}, \underline{\beta}' \mid \omega_2) \rangle \xRightarrow[T(M_1:M_2)]{x}\!\!| \ \langle (q, \varepsilon \mid \omega_1) \equiv (\bar{q}_j, \underline{\gamma}_j \mid \omega_2) \rangle, \tag{4.3}$$

with $\underline{\gamma} = \gamma_j \omega_2$.

(ii) *A skipping-end*: A skipping-end from the node labeled $(p, A\alpha'') \equiv (\bar{p}, \beta'\beta'')$ (equation (4.1)) in question with respect to the ***branching*** node labeled $(p, A\omega_1) \equiv (\bar{p}, \underline{\beta}'\omega_2)$ (equation (4.2)) in $T(M_1 : M_2)$ is defined to be a node labeled by each equivalence equation in

$$\{(q, \alpha'') \equiv (\bar{q}_j, \gamma_j\beta'') \mid \langle (p, A \mid \omega_1) \equiv (\bar{p}, \underline{\beta}' \mid \omega_2) \rangle \xRightarrow[T(M_1:M_2)]{x}\!\!|$$

$\langle (q, \varepsilon \mid \omega_1) \equiv (\bar{q}_j, \underline{\gamma}_j \mid \omega_2) \rangle$, where the node labeled $(q, \omega_1) \equiv$

$(\bar{q}_j, \underline{\gamma}_j\omega_2)$ is not in *halting* status, $(\bar{p}, \beta') \xRightarrow[M_2]{x}\!| \ (\bar{q}_j, \gamma_j)$,

$x \in \Sigma^*$, $q \in \mathrm{EMP}(p, A)\}$.

(iii) *An edge-label*: For a skipping-end labeled $(q, \alpha'') \equiv (\bar{q}_j, \gamma_j\beta'')$ from the node labeled $(p, A\alpha'') \equiv (\bar{p}, \beta'\beta'')$ (equation (4.1)) in question, an edge-label between them in $T(M_1 : M_2)$ is defined to be a shortest input string x_0 such that

$$\langle (p, A \mid \omega_1) \equiv (\bar{p}, \underline{\beta}' \mid \omega_2) \rangle \xRightarrow[T(M_1:M_2)]{x_0} \langle (q, \varepsilon \mid \omega_1) \equiv (\bar{q}_j, \underline{\gamma}_j \mid \omega_2) \rangle$$

with the following property: For any $x \in \Sigma^*$ such that

$$\langle (p, A \mid \omega_1) \equiv (\bar{p}, \underline{\beta}' \mid \omega_2) \rangle \xRightarrow[T(M_1:M_2)]{x}\!\!| \ \langle (q, \varepsilon \mid \omega_1) \equiv (\bar{q}_j, \underline{\gamma}_j \mid \omega_2) \rangle,$$

it holds that

$$|(\bar{p}, \underline{\beta}') \xRightarrow[M_2]{x_0} (\bar{q}_j, \underline{\gamma}_j)| \leq |(\bar{p}, \underline{\beta}') \xRightarrow[M_2]{x} (\bar{q}_j, \underline{\gamma}_j)|.$$

Hence,

$$|(\bar{p}, \beta') \xRightarrow[M_2]{x_0} (\bar{q}_j, \gamma_j)| \leq |(\bar{p}, \beta') \xRightarrow[M_2]{x} (\bar{q}_j, \gamma_j)|.$$

Note, in Definition 4.4(ii), the restriction that the node labeled $(q, \omega_1) \equiv (\bar{q}_j, \underline{\gamma}_j\omega_2)$ is not in *halting* status. The reader, however, may suppose for a while that no node is in *halting* status until *halting* status is introduced for the first time in Section 4.3.

When skipping is applicable to the node in question with respect to some *branching* node in $T(M_1 : M_2)$, we expand it by adding all its skipping-ends in *unchecked* status to it as its sons, with the new edges connecting them labeled by edge-labels defined above. The step of developing the comparison tree in this way is named *skipping* to the node in question. We introduce here *skipping* status in which a node turns to be when skipping has been applied to it.

Remark 4.5. If all the branch checkings along the derivation path (4.3) have been successful, then so are those along x starting from the node labeled (4.1) except the end (see [25, Lemma 4.3, pp. 203–204]). Hence, the latter intermediate checkings can be skipped as described above.

Nodes in *skipping* status should be visited over and over again. When skipping to a node in *skipping* status has turned to be not applicable at some later stage, then a branching step is applied to it. When a skipping status is applied again to a node in *skipping* status to which skipping keeps applicable, if additional skipping-ends are found then they are added as new sons. Besides, the edges between the *skipping* node and its sons are relabeled, if necessary, so that the latest labeling should satisfy the conditions of Definition 4.4(iii).

A skipping step applied to some node reduces the height of the left-hand side of its equivalence equation by one to have its skipping-ends as its sons. Therefore, in case $M_1 \equiv M_2$, the height of the left-hand side of any node label in the comparison tree is bounded as in [25] (see Lemma 5.2). Unlike [25], however, successive application of conventional skipping steps (without taking *halting* status into consideration) may yield an infinite number of skipping-ends with unboundedly high right-hand sides in case M_2 has ε-moves (see for an example Remark 4.7). This is the reason why *halting* status shall be introduced in the next section.

4.3. *Halting*

This section is significant only if the dpda M_2 has ε-moves.

Consider a *skipping* node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ with $\alpha = A\alpha''$, $A \in \Gamma_1$, which has a skipping-end labeled

$$(q, \alpha'') \equiv (\bar{q}_j, \gamma_0) \tag{4.4}$$

such that

$$\langle (p, \alpha) \equiv (\bar{p}, \beta) \rangle \xrightarrow[T(M_1 : M_2)]{x_0} \langle (q, \alpha'') \equiv (\bar{q}_j, \gamma_0) \rangle, \tag{4.5}$$

for some $x_0 \in \Sigma^+$. Now suppose that it has been applied skipping steps sufficiently many times, and that another brother skipping-end labeled

$$(q, \alpha'') \equiv (\bar{q}_j, \gamma), \tag{4.6}$$

such that

$$\langle (p, \alpha) \equiv (\bar{p}, \beta) \rangle \xrightarrow[T(M_1 : M_2)]{x} \langle (q, \alpha'') \equiv (\bar{q}_j, \gamma) \rangle \tag{4.7}$$

for some $x \in \Sigma^+$, $\gamma \neq \gamma_0$, has just been contained in the comparison tree. In addition, assume that they satisfy

$$h = |(\bar{p}, \beta) \xRightarrow[M_2]{x_0} (\bar{q}_j, \gamma_0)| \leq |(\bar{p}, \beta) \xRightarrow[M_2]{x} (\bar{q}_j, \gamma)| \tag{4.8}$$

and

$$(\bar{q}_j, \gamma_0) \overset{h}{\approx} (\bar{q}_j, \gamma) \tag{4.9}$$

(see Definition 3.7). Here, both equivalence equations (4.4) and (4.6) should be checked to hold for $(p, \alpha) \equiv (\bar{p}, \beta)$ to hold. However, once (4.9) has been verified to hold, we know that $(\bar{q}_j, \gamma_0) \equiv (\bar{q}_j, \gamma)$ holds. Hence, it suffices to check only the former (4.4). Then the latter skipping-end labeled (4.6) is turned to be in newly introduced *halting* status, and it will not be expanded any more. Nodes which are not in *halting* status are said to be *nonhalting*.

Definition 4.6. A skipping-end labeled (4.6) with (4.7) is said to satisfy the *halting condition* if it can find a *nonhalting* brother skipping-end labeled (4.4) with (4.5) such that (4.8) and (4.9) hold.

Just after a skipping step to a node in question, we check whether its *unchecked* or *halting* skipping-ends satisfy the halting condition or not. Then we turn only each skipping-end that satisfies the halting condition to be in *halting* status. Such a step of turning appropriate skipping-ends to *halting* just after skipping to the node in question is named *halting* to them. We should recall here that the edge-labels between the *skipping* node and its skipping-ends may vary as the tree grows. Hence, the halting condition to a skipping-end in *halting* status should be rechecked as well in a halting step as described above. If a *halting* skipping-end has turned not to satisfy the halting condition, then its status is turned from *halting* back to *unchecked* in the halting step.

We are on the assumption that the given pair of dpda's M_1 and M_2 has the segmental property. So, in case $M_1 \equiv M_2$, the number of *nonhalting* skipping-ends is bounded owing to Proposition 3.10 (see Lemma 5.5).

Note that the brotherhood of the *halting* node labeled $(q, \alpha'') \equiv (\bar{q}_j, \gamma)$ and the *nonhalting* node labeled $(q, \alpha'') \equiv (\bar{q}_j, \gamma_0)$ is significant to have (4.8) and (4.9). To show the significance of (4.8) and (4.9), assume that the comparison tree contains a *branching* node labeled $(p_0, A_0\alpha'') \equiv (\bar{p}_0, \underline{\beta}'_0\beta'')$ with respect to which skipping to another node labeled $(p_0, A_0\alpha''_0) \equiv (\bar{p}_0, \beta'_0\beta''_0)$ is applicable, and that

$$\langle (p_0, A_0 | \alpha'') \equiv (\bar{p}_0, \underline{\beta}'_0 | \beta'') \rangle \xRightarrow[T(M_1 : M_2)]{t} \langle (p, A | \alpha'') \equiv (\bar{p}, \beta' | \beta'') \rangle$$

for some $v \in \Sigma^*$, where $(\bar{p}_0, \beta_0') \approx (\bar{p}_0, \underline{\beta}_0')$, $\alpha = A\alpha''$ with $A \in \Gamma_1$, $\beta = \beta'\beta''$ with $\beta' \in \Gamma_2^+$, preceding (4.5) and (4.7). So

$$\langle (p_0, A_0 | \alpha'') \equiv (\bar{p}_0, \underline{\beta}_0' | \beta'') \rangle \xRightarrow[T(M_1 : M_2)]{vx_0} | \langle (q, \varepsilon | \alpha'') \equiv (\bar{q}_j, \underline{\gamma}_{j0} | \beta'') \rangle,$$

where $\gamma_0 = \underline{\gamma}_{j0}\beta''$,

and

$$\langle (p_0, A_0 | \alpha'') \equiv (\bar{p}_0, \underline{\beta}_0' | \beta'') \rangle \xRightarrow[T(M_1 : M_2)]{vx} | \langle (q, \varepsilon | \alpha'') \equiv (\bar{q}_j, \underline{\gamma}_j | \beta'') \rangle,$$

where $\gamma = \underline{\gamma}_j\beta''$.

We let here

$$(\bar{p}_0, \beta_0') \xRightarrow[M_2]{vx_0} | (\bar{q}_j, \gamma_{j0}) \quad \text{and} \quad (\bar{p}_0, \beta_0') \xRightarrow[M_2]{vx} | (\bar{q}_j, \gamma_j).$$

Then for $(p_0, A_0\alpha_0'') \equiv (\bar{p}_0, \beta_0'\beta_0'')$ to hold, not only $(q, \alpha_0'') \equiv (\bar{q}_j, \gamma_{j0}\beta_0'')$ but also $(q, \alpha_0'') \equiv (\bar{q}_j, \gamma_j\beta_0'')$ should be checked to hold. Here, the node labeled $(q, \alpha_0'') \equiv (\bar{q}_j, \gamma_{j0}\beta_0'')$ is yielded as a skipping-end from the node labeled $(p_0, A_0\alpha_0'') \equiv (\bar{p}_0, \beta_0'\beta_0'')$ so that

$$\langle (p_0, A_0\alpha_0'') \equiv (\bar{p}_0, \beta_0'\beta_0'') \rangle \xrightarrow[T(M_1 : M_2)]{w_0} \langle (q, \alpha_0'') \equiv (\bar{q}_j, \gamma_{j0}\beta_0'') \rangle$$

for some $w_0 \in L(p_0, A_0)$ such that

$$h_0 = |(\bar{p}_0, \beta_0'\beta_0'') \xRightarrow[M_2]{w_0} (\bar{q}_j, \gamma_{j0}\beta_0'')| \leq |(\bar{p}_0, \beta_0'\beta_0'') \xRightarrow[M_2]{vx_0} (\bar{q}_j, \gamma_{j0}\beta_0'')|$$

$$(\leq |(\bar{p}_0, \beta_0'\beta_0'') \xRightarrow[M_2]{vx} (\bar{q}_j, \gamma_j\beta_0'')|),$$

and it may be expanded hereafter. On the contrary, the node labeled $(q, \alpha_0'') \equiv (\bar{q}_j, \gamma_j\beta_0'')$ is not so, since the node labeled $(q, \alpha'') \equiv (\bar{q}_j, \underline{\gamma}_j\beta'')$ is in *halting* status. However, (4.9), i.e., $(\bar{q}_j, \underline{\gamma}_{j0}\beta'') \overset{h}{\approx} (\bar{q}_j, \underline{\gamma}_j\beta'')$, necessarily implies $(\bar{q}_j, \gamma_{j0}\beta_0'') \equiv (\bar{q}_j, \gamma_j\beta_0'')$ from Remark 3.8(i), (ii), and hence $(q, \alpha_0'') \equiv (\bar{q}_j, \gamma_j\beta_0'')$ if only $(q, \alpha_0'') \equiv (\bar{q}_j, \gamma_{j0}\beta_0'')$ is checked to hold. Therefore, avoidance of the node labeled $(q, \alpha_0'') \equiv (\bar{q}_j, \gamma_j\beta_0'')$ is justified under (4.8) and (4.9) (see also the proof of Claim $\mathbb{E}_n$, case (B)(iii′) in Section 5.2).

4.4. *The algorithm*

Continue the process so far described as far as possible so long as no branch checking failure is encountered. If we reach a stage where the comparison tree hitherto having been constructed is subject to no more change, then we conclude that "$M_1 \equiv M_2$". On the way, the next node to be visited is chosen as the 'smallest' of the *unchecked* and *skipping* nodes, where the *size* of a node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ is the pair $(\text{Max}\{|\alpha|, |\beta|\}, \text{Min}\{|\alpha|, |\beta|\})$, under lexicographic ordering.

The exact algorithm follows below.

Algorithm

[INITIALIZATION]
Let the comparison tree consist of only a root labeled $(q_{01}, Z_{01}) \equiv (q_{02}, Z_{02})$.

while the comparison tree contains an *unchecked* or a *skipping* node
  do let P be the smallest such node, and suppose it is labeled $(p, \alpha) \equiv (\bar{p}, \beta)$
[NO EXPANSION]
    if $\alpha = \beta = \varepsilon$ or $(p, \alpha) \equiv (\bar{p}, \beta)$ appears as the label of another internal node
      then turn P to *checked*
[SKIPPING, RE-SKIPPING]
      else if (p, α) has a nondecreasing mode, $(\bar{p}, \beta)$ is in reading mode, and
          skipping is applicable to P with respect to some *branching* node
          then apply the skipping to P
[HALTING]   followed by halting to its skipping-ends
            if the skipping has caused a change other than
              generation of only *halting* skipping-ends
              then turn P to *skipping*;
[BACK]   turn all *s-checked* nodes to *skipping*;
              else [no essential change has occurred]
                turn P to *s-checked*
            fi
[BRANCHING]
          else if branch checking is successful for P
            then apply the branching to P;
                turn P to *checked*;
[BACK]   turn all *s-checked* nodes to *skipping*
            else [branch checking fails]
                conclude that "$M_1 \not\equiv M_2$"; **halt**
            fi
        fi
    fi
  od
Conclude that "$M_1 \equiv M_2$"; **halt**

4.5. An example

Let us apply the above algorithm to the following pair of strict dpda's: $M_1 = (\{q_{01}, p, q\}, \{Z_{01}, A, B, C, D\}, \{a, b, c, d, e\}, \delta_1, q_{01}, Z_{01}, \emptyset)$ and $M_2 = (\{q_{02}, r, s\},$

$\{Z_{02}, E, F, G, H\}$, $\{a, b, c, d, e\}$, δ_2, q_{02}, Z_{02}, $\emptyset$), where

$$
\begin{array}{ll}
\delta_1 & \delta_2 \\
(q_{01}, Z_{01}) \xrightarrow{a} (p, ACD), & (q_{02}, Z_{02}) \xrightarrow{a} (r, EH), \\
(p, A) \xrightarrow{a} (p, AB), & (r, E) \xrightarrow{a} (r, EF), \\
(p, A) \xrightarrow{b} (p, B), & (r, E) \xrightarrow{b} (r, F), \\
(p, B) \xrightarrow{b} (p, \varepsilon), & (r, F) \xrightarrow{b} (r, \varepsilon), \\
(p, B) \xrightarrow{c} (q, \varepsilon), & (r, F) \xrightarrow{c} (r, G), \\
(p, C) \xrightarrow{c} (q, C), & (r, H) \xrightarrow{c} (r, GH), \\
(q, B) \xrightarrow{\varepsilon} (q, \varepsilon), & (r, G) \xrightarrow{d} (s, \varepsilon), \\
(q, C) \xrightarrow{d} (q, \varepsilon), & (s, F) \xrightarrow{\varepsilon} (s, \varepsilon), \\
(q, D) \xrightarrow{d} (q, CD), & (s, H) \xrightarrow{d} (r, GFH), \\
(q, D) \xrightarrow{e} (q, \varepsilon), & (s, H) \xrightarrow{e} (s, \varepsilon).
\end{array}
$$

Successive application of branching steps yields an intermediate tree containing early nodes numbered ①—⑪ in Fig. 1. When ⑫ $(p, A \cdot BCD) \equiv (r, E \cdot FH)$ is visited first, skipping is applied with respect to ② $(p, A \cdot CD) \equiv (r, E \cdot H)$ to yield *nonhalting* skipping-ends $(p, BCD) \equiv (r, FH)$ and $(q, BCD) \equiv (r, G \cdot FH)$. Furthermore, when the same node ⑯ $(p, A \cdot BCD) \equiv (r, E \cdot FH)$ is visited and applied skipping again, a new skipping-end $(q, BCD) \equiv (r, GF \cdot FH)$ is yielded to have the tree in Fig. 1. Then it is confirmed to satisfy the halting condition, since it can find a *nonhalting* brother skipping-end ⑭ $(q, BCD) \equiv (r, G \cdot FH)$ such that

$$2 = |(r, EFH) \underset{M_2}{\overset{bc}{\Longrightarrow}} (r, GFH)| \leq |(r, EFH) \underset{M_2}{\overset{abc\varepsilon}{\Longrightarrow}} (r, GFFH)|$$

and

$$(r, G \cdot FH) \overset{2}{\approx} (r, GF \cdot FH).$$

Here, G is a common canonical reading segment in (r, G) and (r, GF), and $\mathrm{EMP}(r, G) = \{s\}$. Moreover, ε and F are ε-segments in $(r, G\varepsilon)$ and (r, GF),

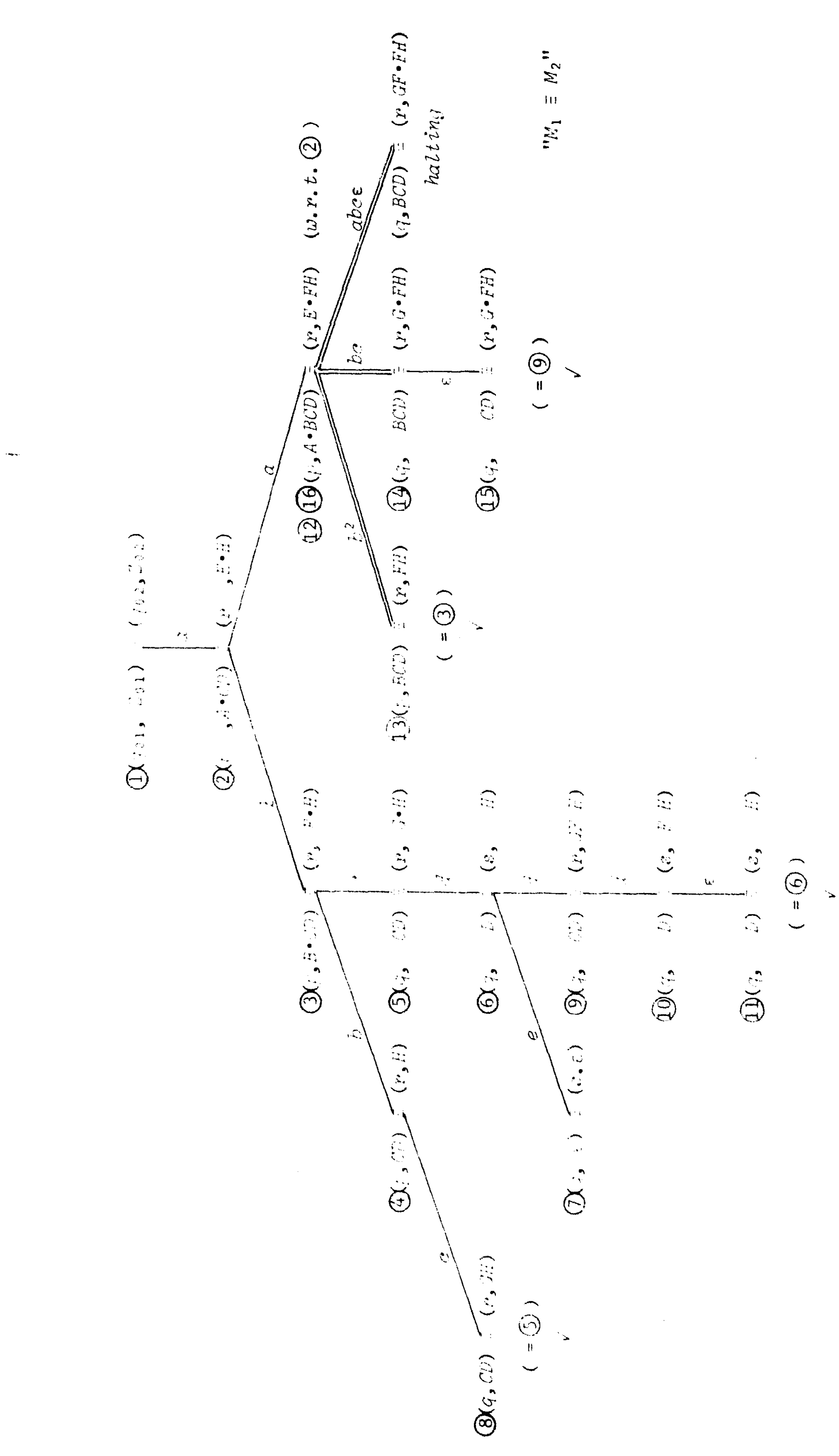

Fig. 1. Example.

respectively, and $(s,\varepsilon)\overset{\varepsilon}{\underset{M_2}{\Longrightarrow}}(s,\varepsilon)$, $(s,F)\overset{\varepsilon}{\underset{M_2}{\longrightarrow}}(s,\varepsilon)$. Hence, $(r,G)\approx(r,GF)$. Consequently, skipping-end $(q,BCD)\equiv(r,GF\cdot FH)$ is turned to *halting* and node (16) $(p,A\cdot BCD)\equiv(r,E\cdot FH)$ is turned from *skipping* to s-*checked*. Then algorithm halts with the correct conclusion that "$M_1\equiv M_2$". ($L(M_1)=L(M_2)=\{a^lb(c\cup b^mc)d^{2n-1}e \mid l\geq m\geq 1, n\geq 1\}$.)

Remark 4.7. If '*halting*' would not be taken into consideration, then skipping steps should be repeatedly applied to $(p,A\cdot BCD)\equiv(r,E\cdot FH)$ to yield an infinite number of skipping-ends $(q,BCD)\equiv(r,GF^iFH)$, $i=0,1,2,\ldots$, since

$$(p,A\,|\,BCD)\overset{a^ibcc^i}{\underset{M_1}{\Longrightarrow}}(q,\varepsilon\,|\,BCD)\quad\text{and}\quad(r,E\,|\,FH)\overset{a^ibc}{\underset{M_2}{\Longrightarrow}}(r,GF^i\,|\,FH)$$

for $i=0,1,2,\ldots$.

5. Termination and correctness of the algorithm

5.1. The case where M_1 and M_2 are equivalent

In what follows, the pair of dpda's M_1 and M_2 is assumed to have the segmental property (see Definition 3.4).

Lemma 5.1. *A branching node labeled $(p,\alpha)\equiv(\bar p,\beta)$ is said to be nondecreasing-reading if (p,α) has a nondecreasing mode and $(\bar p,\beta)$ is in reading mode. Then in case the given dpda's M_1 and M_2 are equivalent, the number of nondecreasing-reading branching nodes in the comparison tree is less than*

$$\mathcal{S}=|Q_1||\Gamma_1||Q_2|^{|Q_2|\mathcal{R}+1}(|\Gamma_2|+1)^{\mathcal{R}}$$

(see Definition 3.4 *for $\mathcal{R}$).*

Proof. Suppose for the sake of contradiction that the comparison tree could contain nondecreasing-reading *branching* nodes labeled $(p_i,A_i\alpha_i'')\equiv(\bar p_i,\beta_i'\beta_i'')$, $i=1,2,\ldots,m$, with $m\geq\mathcal{S}$, where $(p_i,A_i)\in Q_1\times\Gamma_1$, and $|\text{Reading-Seg}(\bar p_i,\beta_i')|=\mathcal{R}$ or else $1\leq|\text{Reading-Seg}(\bar p_i,\beta_i')|<\mathcal{R}$ and $\beta_i''=\varepsilon$. Here, we can show as in the proof of Lemma 3.9 that the number of the equivalence classes into which $\{(\bar p_i,\beta_i')\in Q_2\times\Gamma_2^+ \mid 1\leq i\leq m\}$ is partitioned under $\approx$ is at most

$$|Q_2|^{|Q_2|\mathcal{R}+1}\{(|\Gamma_2|+1)^{\mathcal{R}}-1\}\leq|Q_2|^{|Q_2|\mathcal{R}+1}(|\Gamma_2|+1)^{\mathcal{R}}-1.$$

Then, there could exist a pair of nondecreasing-reading *branching* node labeled $(p_k,A_k\alpha_k'')\equiv(\bar p_k,\beta_k'\beta_k'')$ and $(p_l,A_l\alpha_l'')\equiv(\bar p_l,\beta_l'\beta_l'')$ such that $(p_k,A_k)=(p_l,A_l)$ and $(\bar p_k,\beta_k')\approx(\bar p_l,\beta_l')$. So, one of these nodes should have been applied skipping. This is a contradiction. □

Lemma 5.2. *Let*

$$\mathscr{P}_1 = \mathrm{Max}\{|\theta| \,|\, (p, A) \xrightarrow{a} (q, \theta) \text{ is in } \delta_1\}, \qquad \mathscr{S}' = \mathrm{Max}\{(\mathscr{P}_1 - 1)\mathscr{S}, 1\}.$$

Then in case the given dpda's M_1 and M_2 are equivalent, every node label $(p, \alpha) \equiv (\bar{p}, \beta)$ in the comparison tree satisfies

$$|\alpha| \leq \mathscr{S}'.$$

Proof. The case where $\mathscr{P}_1 \leq 1$ is trivial, therefore, we shall consider only the case where $\mathscr{P}_1 > 1$.

Now let a node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ be with the maximum $|\alpha|$ in the comparison tree $T(M_1 : M_2)$, where

$$(\langle (q_{01}, Z_{01}) \equiv (q_{02}, Z_{02}) \rangle =) \quad \langle (p_1, \alpha_1) \equiv (\bar{p}_1, \beta_1) \rangle \xrightarrow[T(M_1 : M_2)]{u_1}$$

$$\langle (p_2, \alpha_2) \equiv (\bar{p}_2, \beta_2) \rangle \xrightarrow[T(M_1 : M_2)]{u_2} \cdots \xrightarrow[T(M_1 : M_2)]{u_{n-1}} \langle (p_n, \alpha_n) \equiv (\bar{p}_n, \beta_n) \rangle \xrightarrow[T(M_1 : M_2)]{u_n}$$

$$\langle (p_{n+1}, \alpha_{n+1}) \equiv (\bar{p}_{n+1}, \beta_{n+1}) \rangle \quad (= \langle (p, \alpha) \equiv (\bar{p}, \beta) \rangle)$$

for some $u_i \in \Sigma^*$, $(p_i, \alpha_i) \in Q_1 \times \Gamma_1^*$, $(\bar{p}_i, \beta_i) \in Q_2 \times \Gamma_2^*$, $1 \leq i \leq n$.

Then let $i_1 = 1$, and pick out as many indices i_j's, $j \geq 2$, as possible from $\{2, 3, \ldots, n\}$ such that

(1) $|\alpha_{i_{j-1}}| + 1 \leq |\alpha_{i_j}|$,

(2) $|\alpha_{i_j}| < |\alpha_{i'}|$ for any i', $i_j < i' \leq n+1$,

(3) $(\bar{p}_{i_j}, \beta_{i_j})$ is not in ε-mode.

Here, every internal node labeled $(p_{i_j}, \alpha_{i_j}) \equiv (\bar{p}_{i_j}, \beta_{i_j})(1 \leq i_j \leq n)$ as above is a *branching* node, since if it had been applied skipping then $|\alpha_{i_j+1}| = |\alpha_{i_j}| - 1$, violating condition (2). Also for the same reason, every (p_{i_j}, α_{i_j}) has a nondecreasing mode. Then, every $(\bar{p}_{i_j}, \beta_{i_j})$ is in reading mode, since $L(\bar{p}_{i_j}, \beta_{i_j}) = L(p_{i_j}, \alpha_{i_j}) \neq \{\varepsilon\}$. That is, every internal node labeled $(p_{i_j}, \alpha_{i_j}) \equiv (\bar{p}_{i_j}, \beta_{i_j})$ picked out above is a nondecreasing-reading *branching* node. Therefore, the number of these nodes is less than or equal to $\mathscr{S} - 1$ by Lemma 5.1. So,

$$|\alpha| = |\alpha_{n+1}| \leq (\mathscr{P}_1 - 1)(\mathscr{S} - 1) + (\mathscr{P}_1 - 1) = (\mathscr{P}_1 - 1)\mathscr{S}.$$

Hence, the result. □

(Note a misprint in [26, p. 147, line −1]: "$n>$" should read "$n\geq$".)

Definition 5.3 ([19, p. 52]). The *height of a tree* is defined to be the maximum number of edges along a path from the root of the tree to a leaf of the tree.

Lemma 5.4. *In case the given dpda's M_1 and M_2 are equivalent, the height of the comparison tree is at most $2(\mathscr{P}_1\mathscr{S} - \mathscr{P}_1 + 1)$.*

Proof. Let the comparison tree $T(M_1:M_2)$ have a path from the root labeled $(q_{01}, Z_{01}) \equiv (q_{02}, Z_{02})$ to a leaf labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ such that

$$(\langle (q_{01}, Z_{01}) \equiv (q_{02}, Z_{02}) \rangle =) \quad \langle (p_1, \alpha_1) \equiv (\bar{p}_1, \beta_1) \rangle \xrightarrow[T(M_1:M_2)]{u_1}$$

$$\langle (p_2, \alpha_2) \equiv (\bar{p}_2, \beta_2) \rangle \xrightarrow[T(M_1:M_2)]{u_2} \cdots \xrightarrow[T(M_1:M_2)]{u_{n-1}} \langle (p_n, \alpha_n) \equiv (\bar{p}_n, \beta_n) \rangle \xrightarrow[T(M_1:M_2)]{u_n}$$

$$\langle (p_{n+1}, \alpha_{n+1}) \equiv (\bar{p}_{n+1}, \beta_{n+1}) \rangle \quad (= \langle (p, \alpha) \equiv (\bar{p}, \beta) \rangle).$$

Here, these edges

$$\langle (p_i, \alpha_i) \equiv (\bar{p}_i, \beta_i) \rangle \xrightarrow[T(M_1:M_2)]{u_i} \langle (p_{i+1}, \alpha_{i+1}) \equiv (\bar{p}_{i+1}, \beta_{i+1}) \rangle,$$

$i = 1, 2, \ldots, n$, are classified into the following three types:

(a) $|\alpha_{i+1}| \geq |\alpha_i|$, and $(\bar{p}_i, \beta_i)$ is in reading mode. (An edge realized by branching of type (a) in Lemma 4.1(ii), where (p_i, α_i) has a nondecreasing mode. So a node labeled $(p_i, \alpha_i) \equiv (\bar{p}_i, \beta_i)$ is a nondecreasing-reading *branching* node in Lemma 5.1.)

(b) $|\alpha_{i+1}| = |\alpha_i| - 1$ (an edge realized by branching of type (a) or (b), or by skipping),

(c) $(p_{i+1}, \alpha_{i+1}) = (p_i, \alpha_i)$, and $(\bar{p}_i, \beta_i)$ is in ε-mode (an edge realized by branching of type (c)).

Then, for $t = a, b, c$, let

$$N_t = \{ i \mid \langle (p_i, \alpha_i) \equiv (\bar{p}_i, \beta_i) \rangle \xrightarrow[T(M_1:M_2)]{u_i} \langle (p_{i+1}, \alpha_{i+1}) \equiv (\bar{p}_{i+1}, \beta_{i+1}) \rangle \text{ is an edge of type } (t) \text{ along the above derivation path} \}.$$

So,

$$|N_a| + |N_b| + |N_c| = n.$$

Let us first consider the case where $\mathscr{P}_1 \geq 1$. Now Lemma 5.1 shows

$$|N_a| \leq \mathscr{S} - 1$$

Therefore,

$$|N_b| \leq 1 + \sum_{i \in N_a} (|\alpha_{i+1}| - |\alpha_i|) \leq 1 + (\mathscr{P}_1 - 1)(\mathscr{S} - 1).$$

Moreover,

$$|N_c| \leq |N_a| + |N_b|.$$

Hence,

$$n \leq 2(|N_a| + |N_b|) \leq 2(\mathscr{P}_1 \mathscr{S} - \mathscr{P}_1 + 1).$$

In case $\mathscr{P}_1 = 0$, we have $|N_a| = 0$ and $|N_b| = 1$, hence $n \leq 2$.

Thus, the number of edges along a path from the root to any leaf of the comparison tree is at most $2(\mathscr{P}_1\mathscr{S}-\mathscr{P}_1+1)$. This concludes the result. □

Lemma 5.5. *Let*

$$\mathscr{B}_1=\mathrm{Max}\{\mathscr{B}(\alpha'')\,|\,(q,\alpha'')\ \textit{is a reachable configuration of}\ M_1\ \textit{such that}\ |\alpha''|<\mathscr{S}'\},\quad \textit{see Lemma 3.6 for}\ \mathscr{B}(\alpha'').$$

Then in case the given dpda's M_1 and M_2 are equivalent, the number of nonhalting skipping-ends from any skipping node in the comparison tree is at most

$$\mathscr{R}|Q_1||Q_2|^{|Q_2|\mathscr{B}_1+2}(|\Gamma_2|+1)^{\mathscr{B}_1}.$$

Proof. Let a node labeled $(p,\alpha)\equiv(\bar{p},\beta)$ be a *skipping* node in the comparison tree $T(M_1:M_2)$, where $\alpha=A\alpha''$ with $A\in\Gamma_1$ and $|\alpha''|<\mathscr{S}'$ by Lemma 5.2, $\beta=\beta'\beta''$ with $\beta^{(h)}=\beta''$ for some nonnegative integer $h<|\beta|$, and let $q\in\mathrm{EMP}(p,A)$. Then, from Proposition 3.10, the number of the equivalence classes into which

$$S[(p,\alpha)\equiv(\bar{p},\beta),q;h]$$
$$=\{(\bar{q}_i,\gamma_i\beta'')\in Q_2\times\Gamma_2^*\,|\,(p,A)\underset{M_1}{\overset{x}{\Longrightarrow}}(q,\varepsilon)\ \text{and}$$
$$(\bar{p},\beta')\underset{M_2}{\overset{x}{\Longrightarrow}}|\,(\bar{q}_i,\gamma_i)\ \text{for some}\ x\in\Sigma^*\}$$

is partitioned under $\overset{h}{\approx}$ is at most

$$|Q_2|^{|Q_2|\mathscr{B}_1+2}(|\Gamma_2|+1)^{\mathscr{B}_1}$$

(cf. (4.9) in the halting condition). Moreover, the pair of (p,α) and $(\bar{p},\beta)$ has the segmental property. So the number of *nonhalting* skipping-ends whose labels are of the form $(q,\alpha'')\equiv(\bar{q}_i,\gamma_i\beta'')$ for given $q\in\mathrm{EMP}(p,A)$ is at most $\mathscr{R}|Q_2|^{|Q_2|\mathscr{B}_1+2}(|\Gamma_2|+1)^{\mathscr{B}_1}$ (cf. (4.8)). Here $|\mathrm{EMP}(p,A)|\le|Q_1|$, so we have the final bound as described above. □

Now, these lemmas are combined to have the following.

Theorem 5.6. *For the pair of equivalent dpda's M_1 and M_2 ($\in D_0$) which has the segmental property, the algorithm halts in a finite number of steps with the correct conclusion that "$M_1\equiv M_2$".*

Proof. From Lemmas 5.4 and 5.5, the height and the 'width' of the comparison tree with only *nonhalting* nodes are bounded. Though *halting* skipping-ends may be yielded from this finite part of the tree, too, no *halting* skipping-ends contribute to expand the comparison tree further. Hence, the development of the tree terminates in a finite number of steps. In addition, every node label $(p, \alpha) \equiv (\bar{p}, \beta)$ in the tree is such that $(q_{01}, Z_{01}) \overset{u}{\underset{M_1}{\Longrightarrow}} (p, \alpha)$ and $(q_{02}, Z_{02}) \overset{u}{\underset{M_2}{\Longrightarrow}} (\bar{p}, \beta)$ for some $u \in \Sigma^*$. Hence, by Proposition 3.1, no branch checking failure ever occurs. Therefore, the conclusion is "$M_1 \equiv M_2$". □

5.2. *The case where M_1 and M_2 are inequivalent*

Theorem 5.7. *In case the given dpda's M_1 and M_2 ($\in D_0$) are inequivalent, the algorithm halts in a finite number of steps with the correct conclusion that "$M_1 \not\equiv M_2$".*

Proof. Suppose for the sake of contradiction that no branch checking failure would ever occur. Here, let $T(M_1 : M_2)$ denote the comparison tree which has been developed as far as possible. Then it should follow that we have the following Claim $\mathbb{E}_n$ for any positive integer n.

Claim $\mathbb{E}_n$. *For every nonhalting node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ $(\alpha = \alpha_1\alpha_2 \in \Gamma_1^+, \beta \in \Gamma_2^*)$ in $T(M_1 : M_2)$, if*

$$(p, \alpha_1 | \alpha_2) \overset{w}{\underset{M_1}{\Longrightarrow}}{}^{(n)} (r, \varepsilon | \alpha_2)$$

for some $\alpha_1 \in \Gamma_1^+$, $w \in L(p, \alpha_1)$, and $r \in Q_1$, then the following hold:

(i) $(\bar{p}, \beta) \overset{w}{\underset{M_2}{\Longrightarrow}}| (\bar{r}_k, \partial)$ *for some $(\bar{r}_k, \partial) \in Q_2 \times \Gamma_2^*$.*

(ii) $\langle (p, \alpha_1\alpha_2) \equiv (\bar{p}, \beta) \rangle \overset{w_0}{\underset{T(M_1 : M_2)}{\Longrightarrow}}| \langle (r, \alpha_2) \equiv (\bar{r}_k, \partial_0) \rangle$

for some $w_0 \in L(p, \alpha_1)$ and a nonhalting node labeled $(r, \alpha_2) \equiv (\bar{r}_k, \partial_0)$ such that

(iii) $h = |(\bar{p}, \beta) \overset{w_0}{\underset{M_2}{\Longrightarrow}} (\bar{r}_k, \partial_0)| \leq |(\bar{p}, \beta) \overset{w}{\underset{M_2}{\Longrightarrow}} (\bar{r}_k, \partial)|$

and

$$(\bar{r}_k, \partial_0) \overset{h}{\approx} (\bar{r}_k, \partial).$$

Proof of Claim $\mathbb{E}_n$. The proof is by induction on n.

Basis. $n = 1$ $(\alpha_1 \in \Gamma_1)$. In case the internal node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ is a *branching* node, properties (i) and (ii) obviously hold with $w_0 = w \in \Sigma \cup \{\varepsilon\}$ and $\partial_0 = \partial$, since a branching step of type (a) or (b) in Lemma 4.1 has been successfully applied (after that of type (c)). Hence, property (iii) trivially holds. Consequently, in case the internal node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ has been applied skipping with respect to some *branching* node, the same properties as above also hold with $w \in \Sigma$, except that the node labeled $(r, \alpha_2) \equiv (\bar{r}_k, \partial)$ which is a skipping-end may possibly be *halting*. But, it can be so only if there exists a *nonhalting* brother skipping-end labeled $(r, \alpha_2) \equiv (\bar{r}_k, \partial_0)$ such that (ii) and (iii) hold. Hence, Claim $\mathbb{E}_1$ has been proved.

Induction step. Now we assume that $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_n$ are true for *some n* (≥ 1) and shall prove that $\mathbb{E}_{n+1}$ also holds.

Let a *nonhalting* internal node labeled $(p, \alpha) \equiv (\bar{p}, \beta)(\alpha = \alpha_1\alpha_2 \in \Gamma_1^+, \beta \in \Gamma_2^*)$ be in $T(M_1 : M_2)$ and

$$(p, \alpha_1 \mid \alpha_2) \overset{w}{\underset{M_1}{\Longrightarrow}}{}^{(n+1)} (r, \varepsilon \mid \alpha_2) \tag{5.1}$$

for some $\alpha_1 \in \Gamma_1^+$, $w \in L(p, \alpha_1)$, and $r \in Q_1$.

(A) In case the internal node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ is a *branching* node: Divide derivation (5.1) into

$$(p, \alpha_1 \mid \alpha_2) \overset{a}{\underset{M_1}{\longrightarrow}} (p', \alpha_1' \mid \alpha_2) \quad \text{and} \quad (p', \alpha_1' \mid \alpha_2) \overset{w'}{\underset{M_1}{\Longrightarrow}}{}^{(n)} (r, \varepsilon \mid \alpha_2), \tag{5.2}$$

where $w = aw'$ and $(p', \alpha_1'\alpha_2) \in Q_1 \times \Gamma_1^+$. Since the node in question has been applied a branching step of type (a) or (b) (after that of type (c)), we have

$$(\bar{p}, \beta) \overset{a}{\underset{M_2}{\Longrightarrow}} (\bar{p}', \bar{\beta}') \tag{5.3}$$

and

$$\langle (p, \alpha_1\alpha_2) \equiv (\bar{p}, \beta) \rangle \overset{a}{\underset{T(M_1 : M_2)}{\Longrightarrow}} \langle (p', \alpha_1'\alpha_2) \equiv (\bar{p}', \bar{\beta}') \rangle \tag{5.4}$$

for some $(\bar{p}', \bar{\beta}') \in Q_2 \times \Gamma_2^*$, where the node labeled $(p', \alpha_1'\alpha_2) \equiv (\bar{p}', \bar{\beta}')$ is *nonhalting*. Then the induction hypothesis $\mathbb{E}_n$ applies to this *nonhalting* node for the latter derivation of (5.2) to have the following:

$$(\bar{p}', \bar{\beta}') \overset{w'}{\underset{M_2}{\Longrightarrow}}| (\bar{r}_k, \partial) \quad \text{for some } (\bar{r}_k, \partial) \in Q_2 \times \Gamma_2^*,$$

and

$$\langle (p', \alpha_1'\alpha_2) \equiv (\bar{p}', \bar{\beta}') \rangle \overset{w_0'}{\underset{T(M_1 : M_2)}{\Longrightarrow}}| \langle (r, \alpha_2) \equiv (\bar{r}_k, \partial_0) \rangle$$

for some $w_0' \in L(p', \alpha_1')$ and a *nonhalting* node labeled $(r, \alpha_2) \equiv (\bar{r}_k, \partial_0)$ such that

$$h' = |(\bar{p}', \bar{\beta}') \overset{w_0'}{\underset{M_2}{\Longrightarrow}} (\bar{r}_k, \partial_0)| \leq |(\bar{p}', \bar{\beta}') \overset{w'}{\underset{M_2}{\Longrightarrow}} (\bar{r}_k, \partial)|$$

and

$$(\bar{r}_k, \partial_0) \overset{h'}{\approx} (\bar{r}_k, \partial). \tag{5.5}$$

So, these properties combined with the preceding (5.3) and (5.4) directly give the objective properties (i) through (iii) in case $h = |(\bar{p}, \beta) \overset{aw_0'}{\underset{M_2}{\Longrightarrow}} (\bar{r}_k, \partial_0)| = h'$. In the other case where $h < h'$, we can also derive $(\bar{r}_k, \partial_0) \overset{h}{\approx} (\bar{r}_k, \partial)$ among others from Remark 3.8(ii) applied to (5.5).

(B) In case the internal node labeled $(p, \alpha) \equiv (\bar{p}, \beta)$ $(\alpha = A\alpha'' = \alpha_1\alpha_2, A \in \Gamma_1, \beta = \beta'\beta'')$ has been applied skipping: There exists a *branching* node labeled $(p, A\omega_1) \equiv (\bar{p}, \underline{\beta}'\omega_2)$ such that $(\bar{p}, \underline{\beta}') \approx (\bar{p}, \beta')$ with respect to which skipping to the node in question has been applied, where Reading-Seg$(\bar{p}, \underline{\beta}') \in \Gamma_2^+$ is the longest possible one.

Now divide derivation (5.1) into

$$(p, A \mid \alpha'') \overset{x}{\underset{M_1}{\Longrightarrow}}{}^{(n')} (q, \varepsilon \mid \alpha'') \tag{5.6}$$

and

$$(q, \alpha_1'' \mid \alpha_2) \overset{y}{\underset{M_1}{\Longrightarrow}}{}^{(n'')} (r, \varepsilon \mid \alpha_2), \tag{5.7}$$

where $w = xy$, $\alpha_1 = A\alpha_1''$, $\alpha'' = \alpha_1''\alpha_2$ and $n' + n'' = n + 1$ (see Fig. 2).

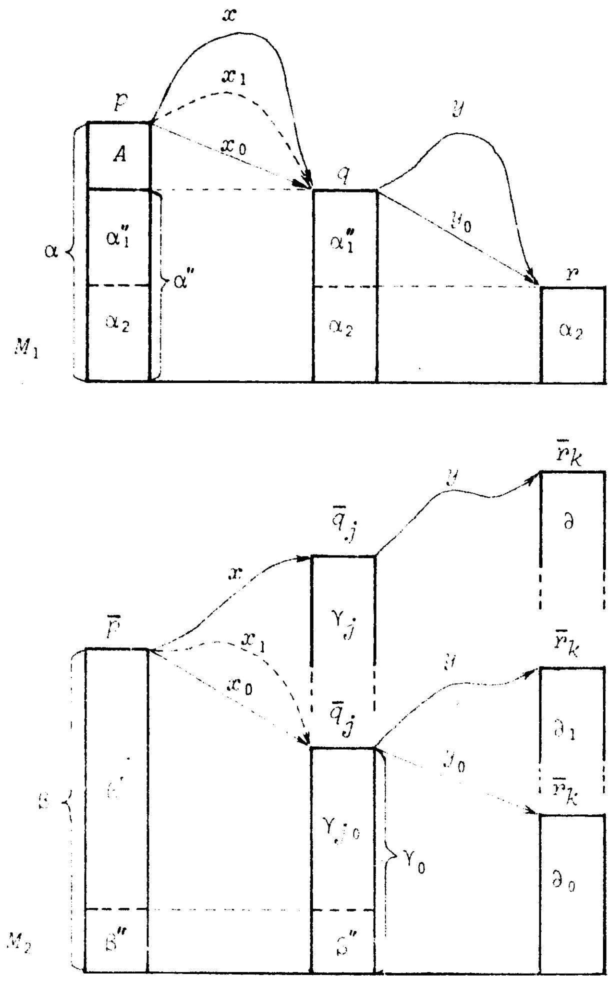

Fig. 2. Derivations in the proof of Claim $\mathbb{E}_n$.

The first half. The first derivation (5.6) implies

$$(p, A \mid \omega_1) \overset{x}{\underset{M_1}{\Longrightarrow}}{}^{(n')} (q, \varepsilon \mid \omega_1) \quad (1 \leq n' \leq n+1), \tag{5.8}$$

and the induction hypothesis $\mathbb{E}_{n'}$ (if $n' \leq n$) or the result in the preceding case (A) (if $n' = n+1$) applies to the *branching* node labeled $(p, A\omega_1) \equiv (\bar{p}, \underline{\beta}'\omega_2)$ for derivation (5.8) to have the following:

$$(\bar{p}, \underline{\beta}'\omega_2) \overset{x}{\underset{M_2}{\Longrightarrow|}} (\bar{q}_j, \underline{\gamma}) \quad \text{for some } (\bar{q}_j, \underline{\gamma}) \in Q_2 \times \Gamma_2^*,$$

and

$$\langle (p, A\omega_1) \equiv (\bar{p}, \underline{\beta}'\omega_2) \rangle \overset{x_1}{\underset{T(M_1:M_2)}{\Longrightarrow|}} \langle (q, \omega_1) \equiv (\bar{q}_j, \underline{\gamma}_0) \rangle \tag{5.9}$$

for some $x_1 \in L(p, A)$ and a *nonhalting* node labeled $(q, \omega_1) \equiv (\bar{q}_j, \underline{\gamma}_0)$ such that

$$h_1 = |(\bar{p}, \underline{\beta}'\omega_2) \overset{x_1}{\underset{M_2}{\Longrightarrow}} (\bar{q}_j, \underline{\gamma}_0)| \leq |(\bar{p}, \underline{\beta}'\omega_2) \overset{x}{\underset{M_2}{\Longrightarrow}} (\bar{q}_i, \underline{\gamma})| \tag{5.10}$$

and

$$(\bar{q}_j, \underline{\gamma}_0) \overset{h_1}{\approx} (\bar{q}_i, \underline{\gamma}). \tag{5.11}$$

Since skipping to the node labeled $(p, A\alpha'') \equiv (\bar{p}, \beta'\beta'')$ with respect to the *branching* node labeled $(p, A\omega_1) \equiv (\bar{p}, \underline{\beta}'\omega_2)$ has been checked to be applicable after the appearance of the above derivation path (5.9), it can be rewritten as

$$\langle (p, A \mid \omega_1) \equiv (\bar{p}, \underline{\beta}' \mid \omega_2) \rangle \overset{x_1}{\underset{T(M_1:M_2)}{\Longrightarrow|}} \langle (q, \varepsilon \mid \omega_1) \equiv (\bar{q}_j, \underline{\gamma}_{j0} \mid \omega_2) \rangle,$$

with $\underline{\gamma}_0 = \underline{\gamma}_{j0}\omega_2$. This implies $(\bar{p}, \underline{\beta}') \overset{x_1}{\underset{M_2}{\Longrightarrow|}} (\bar{q}_j, \underline{\gamma}_{j0})$, and hence $(\bar{p}, \underline{\beta}') \overset{x}{\underset{M_2}{\Longrightarrow|}} (\bar{q}_i, \underline{\gamma}_i)$, with $\underline{\gamma} = \underline{\gamma}_i\omega_2$, by (5.10). So,

$$(\bar{p}, \beta') \overset{x_1}{\underset{M_2}{\Longrightarrow|}} (\bar{q}_j, \gamma_{j0}) \quad \text{and} \quad (\bar{p}, \beta') \overset{x}{\underset{M_2}{\Longrightarrow|}} (\bar{q}_i, \gamma_i)$$

for some $\gamma_{j0}, \gamma_i \in \Gamma_2^*$, and

$$|(\bar{p}, \beta') \overset{x_1}{\underset{M_2}{\Longrightarrow}} (\bar{q}_j, \gamma_{j0})| \leq |(\bar{p}, \beta') \overset{x}{\underset{M_2}{\Longrightarrow}} (\bar{q}_i, \gamma_i)|. \tag{5.10'}$$

Therefore,

(i′) $(\bar{p}, \beta) \overset{x}{\underset{M_2}{\Longrightarrow|}} (\bar{q}_i, \gamma)$, where $\gamma = \gamma_i\beta''$.

Moreover, the above skipping to the node in question has yielded

(ii′) $\langle (p, A \mid \alpha'') \equiv (\bar{p}, \beta' \mid \beta'') \rangle \overset{x_1}{\underset{T(M_1:M_2)}{\longrightarrow}} \langle (q, \varepsilon \mid \alpha'') \equiv (\bar{q}_j, \gamma_{j0} \mid \beta'') \rangle$

for some $x_0 \in L(p, A)$, such that

$$\langle (p, A \mid \omega_1) \equiv (\bar{p}, \beta' \mid \omega_2) \rangle \xrightarrow[T(M_1:M_2)]{x_0} \langle (q, \varepsilon \mid \omega_1) \equiv (\bar{q}_j, \gamma_{j0} \mid \omega_2) \rangle$$

and

$$|(\bar{p}, \beta') \xRightarrow[M_2]{x_0} (\bar{q}_j, \gamma_{j0})| \leq |(\bar{p}, \beta') \xRightarrow[M_2]{x_1} (\bar{q}_j, \gamma_{j0})|. \tag{5.12}$$

Then, combining (5.12) and (5.10′) gives

$$\text{(iii')} \quad h_1' = |(\bar{p}, \beta) \xRightarrow[M_2]{x_0} (\bar{q}_j, \gamma_0)| \leq |(\bar{p}, \beta) \xRightarrow[M_2]{x} (\bar{q}_j, \gamma)|,$$

where $\gamma_0 = \gamma_{j0}\beta''$. In addition, Remark 3.8(ii) applied to (5.11), with (5.12), (iii′) and $(\bar{q}_j, \gamma_{j0}) \approx (\bar{q}_j, \gamma_j)$, gives

$$(\bar{q}_j, \gamma_0) \overset{h_1'}{\approx} (\bar{q}_j, \gamma). \tag{5.13}$$

Here we can assume, without loss of generality, that the skipping-end labeled $(q, \alpha'') \equiv (\bar{q}_j, \gamma_{j0}\beta'')$ is *nonhalting*. To show this, assume that it is a *halting* node. Then there exists another *nonhalting* brother skipping-end labeled $(q, \alpha'') \equiv (\bar{q}_j, \gamma_{j00}\beta'')$ such that, for some $x_{00} \in L(p, A)$,

$$\text{(ii'')} \quad \langle (p, A \mid \alpha'') \equiv (\bar{p}, \beta' \mid \beta'') \rangle \xrightarrow[T(M_1:M_2)]{x_{00}} \langle (q, \varepsilon \mid \alpha'') \equiv (\bar{q}_j, \gamma_{j00} \mid \beta'') \rangle,$$

$$h_{10}' = |(\bar{p}, \beta) \xRightarrow[M_2]{x_{00}} (\bar{q}_j, \gamma_{j00}\beta'')| \leq |(\bar{p}, \beta) \xRightarrow[M_2]{x_0} (\bar{q}_j, \gamma_{j0}\beta'')| \tag{5.14}$$

and

$$(\bar{q}_j, \gamma_{j00}\beta'') \overset{h_{10}'}{\approx} (\bar{q}_j, \gamma_{j0}\beta''). \tag{5.15}$$

Thus, combining (5.14) and (iii′) gives

$$\text{(iii'')} \quad |(\bar{p}, \beta) \xRightarrow[M_2]{x_{00}} (\bar{q}_j, \gamma_{j00}\beta'')| \leq |(\bar{p}, \beta) \xRightarrow[M_2]{x} (\bar{q}_j, \gamma)|.$$

In addition, (5.13) implies $(\bar{q}_j, \gamma_{j0}\beta'') \overset{h_{10}'}{\approx} (\bar{q}_j, \gamma)$, and hence

$$(\bar{q}_j, \gamma_{j00}\beta'') \overset{h_{10}'}{\approx} (\bar{q}_j, \gamma),$$

by (5.15). So, if we rename not only $(q, \alpha'') \equiv (\bar{q}_j, \gamma_{j00}\beta'')$ by $(q, \alpha'') \equiv (\bar{q}_j, \gamma_{j0}\beta'')$ but also x_{00} and h_{10}' by x_0 and h_1', respectively, then we have the desired result.

Now, if $n'=n+1$ and $x=w$ in (5.6) ($n''=0$, $y=\varepsilon$, $q=r$, and $\alpha_1''=\varepsilon$ in (5.7)), then (5.6) coincides with (5.1), and the proof is complete. So, we shall consider the other case in the following.

The second half. In case $\alpha_1''\in\Gamma_1^+$ and $n''>0$ ($n'\leqslant n$), we can apply induction hypothesis $\mathbb{E}_{n''}$ to the *nonhalting* node labeled $(q,\alpha'')\equiv(\bar{q}_j,\gamma_0)$ for derivation (5.7), since $n''\leqslant n$. Then,

$$(\bar{q}_j,\gamma_0)\overset{y}{\underset{M_2}{\Longrightarrow}}\!|\,(\bar{r}_k,\partial_1)\quad\text{for some }(\bar{r}_k,\partial_1)\in Q_2\times\Gamma_2^*,\tag{5.16}$$

and

$$\langle(q,\alpha_1''\alpha_2)\equiv(\bar{q}_j,\gamma_0)\rangle\overset{y_0}{\underset{T(M_1:M_2)}{\Longrightarrow}}\!|\,\langle(r,\alpha_2)\equiv(\bar{r}_k,\partial_0)\rangle\tag{5.17}$$

for some $y_0\in L(q,\alpha_1'')$ and a *nonhalting* node labeled $(r,\alpha_2)\equiv(\bar{r}_k,\partial_0)$ such that

$$h_2=\left|(\bar{q}_j,\gamma_0)\overset{y_0}{\underset{M_2}{\Longrightarrow}}(\bar{r}_k,\partial_0)\right|\leqslant\left|(\bar{q}_j,\gamma_0)\overset{y}{\underset{M_2}{\Longrightarrow}}(\bar{r}_k,\partial_1)\right|\tag{5.18}$$

and

$$(\bar{r}_k,\partial_0)\overset{h_2}{\approx}(\bar{r}_k,\partial_1).\tag{5.19}$$

The whole. Combining (i') and (5.16) by (5.13) gives

(i) $$(\bar{p},\beta)\overset{xy}{\underset{M_2}{\Longrightarrow}}\!|\,(\bar{r}_k,\partial)\quad\text{for some }(\bar{r}_k,\partial)\in Q_2\times\Gamma_2^*,$$

such that

$$(\bar{r}_k,\partial_1)\overset{h_1'}{\approx}(\bar{r}_k,\partial).\tag{5.20}$$

Furthermore, combining (ii') and (5.17) gives

(ii) $$\langle(p,\alpha_1\alpha_2)\equiv(\bar{p},\beta)\rangle\overset{x_0y_0}{\underset{T(M_1:M_2)}{\Longrightarrow}}\!|\,\langle(r,\alpha_2)\equiv(\bar{r}_k,\partial_0)\rangle,$$

where $x_0y_0\in L(p,\alpha_1)$ and the node labeled $(r,\alpha_2)\equiv(\bar{r}_k,\partial_0)$ is *nonhalting*. Now by (5.13), $|(\bar{q}_j,\gamma_0)\overset{y}{\underset{M_2}{\Longrightarrow}}(\bar{r}_k,\partial_1)|>h_1'$ if and only if $|(\bar{q}_j,\gamma)\overset{y}{\underset{M_2}{\Longrightarrow}}(\bar{r}_k,\partial)|>h_1'$, or else

$$\left|(\bar{q}_j,\gamma_0)\overset{y}{\underset{M_2}{\Longrightarrow}}(\bar{r}_k,\partial_1)\right|=\left|(\bar{q}_j,\gamma)\overset{y}{\underset{M_2}{\Longrightarrow}}(\bar{r}_k,\partial)\right|\leqslant h_1'$$

(see Remark 3.8(iii)). Therefore:

(a) If

$$h=\left|(\bar{p},\beta)\overset{x_0y_0}{\underset{M_2}{\Longrightarrow}}(\bar{r}_k,\partial_0)\right|=\left|(\bar{q}_j,\gamma_0)\overset{y_0}{\underset{M_2}{\Longrightarrow}}(\bar{r}_k,\partial_0)\right|=h_2\leqslant h_1',$$

then combining (iii′) and (5.18) easily derives

$$\text{(iii)}\quad |(\bar{p}, \beta) \overset{x_0 y_0}{\underset{M_2}{\Longrightarrow}} (\bar{r}_k, \partial_0)| \leq |(\bar{p}, \beta) \overset{xy}{\underset{M_2}{\Longrightarrow}} (\bar{r}_k, \partial)|.$$

In addition, if $h_2 \leq h_1'$ then (5.20) implies $(\bar{r}_k, \partial_1) \overset{h_2}{\approx} (\bar{r}_k, \partial)$. So, combining this and (5.19) gives

$$(\bar{r}_k, \partial_0) \overset{h_2}{\approx} (\bar{r}_k, \partial).$$

(b) If $h = h_1' < h_2$ then, by (5.18), we have $|(\bar{q}_j, \gamma_0) \overset{y}{\underset{M_2}{\Longrightarrow}} (\bar{r}_k, \partial_1)| > h_1'$, and hence $|(\bar{q}_j, \gamma) \overset{y}{\underset{M_2}{\Longrightarrow}} (\bar{r}_l, \partial)| > h_1'$. Thus, combining (iii′) and (5.18) also derives the same inequality as above. In addition, if $h_1' < h_2$, then (5.19) implies $(\bar{r}_k, \partial_0) \overset{h_1'}{\approx} (\bar{r}_k, \partial_1)$. So, combining this and (5.20) gives

$$(\bar{r}_k, \partial_0) \overset{h_1'}{\approx} (\bar{r}_k, \partial).$$

Therefore, in either case,

$$(\bar{r}_k, \partial_0) \overset{h}{\approx} (\bar{r}_k, \partial).$$

This concludes the proof of the whole part of case (B).

Thus, Claim $\mathbb{E}_n$ has been induced for *any* n. □

Proof of Theorem 5.7 (*continued*). Now if we apply this claim especially to the root labeled $(q_{01}, Z_{01}) \equiv (q_{02}, Z_{02})$, then we know that the condition of Proposition 3.1 holds, since we have encountered no branch checking failure (see [25, Lemma 5.3, Claim $\mathbb{E}_n$, property (ii), pp. 216–217]). Thus, $M_1 \equiv M_2$ should hold, contradicting the assumption that M_1 and M_2 are inequivalent. Therefore, branch checking failure does occur at some stage, concluding that "$M_1 \not\equiv M_2$". □

6. Concluding remarks

We have been so far concerned with only dpda's which accept by empty stack, i.e., which are strict. For dpda's M_1 and M_2 with either acceptance, we can transform them to strict dpda's M_1' and M_2' such that $M_1 \equiv M_2$ if and only if $M_1' \equiv M_2'$, as described below.

Let $M_i = (Q_i, \Gamma_i, \Sigma, \delta_i, q_{0i}, Z_{0i}, F_i)$, $i = 1, 2$, be a pair of dpda's accepting by either final states or empty stack which has the segmental property with a constant $\mathcal{R}$ defined in Definition 3.4.

Now introduce an endmarker $\# \notin \Sigma$, a new state $q_e \notin Q_i$, and a new stack symbol $Z \notin \Gamma_i$, and let

$$\delta_{0i} = \{(q_{0i}, Z_{0i}) \xrightarrow{a} (q, \theta) \text{ in } \delta_i \mid a \in \Sigma \cup \{\varepsilon\}, (q, \theta) \in Q_i \times \Gamma_i^*\},$$

$$\delta'_{0i} = \{(q_{0i}, Z_{0i}) \xrightarrow{a} (q, \theta Z) \mid (q_{0i}, Z_{0i}) \xrightarrow{a} (q, \theta) \text{ is in } \delta_{0i}\},$$

$$\delta_{fi} = \begin{cases} \{(q_f, A) \xrightarrow{\#} (q_e, \varepsilon) \mid q_f \in F_i, A \in \Gamma_i \cup \{Z\}\} & \\ \quad \cup \{(q_e, A) \xrightarrow{\varepsilon} (q_e, \varepsilon) \mid A \in \Gamma_i \cup \{Z\}\} & \text{if } M_i \text{ accepts by final states,} \\ \{(q, Z) \xrightarrow{\#} (q_e, \varepsilon) \mid q \in Q_i\} & \text{if } M_i \text{ accepts by empty stack,} \end{cases}$$

and

$$\delta'_i = (\delta_i - \delta_{0i}) \cup \delta'_{0i} \cup \delta_{fi}$$

for $i = 1, 2$. Then the strict dpda $M'_i = (Q_i \cup \{q_e\}, \Gamma_i \cup \{Z\}, \Sigma \cup \{\#\}, \delta'_i, q_{0i}, Z_{0i}, \emptyset)$ is such that $L(M'_i) = L(M_i)\#$, and the pair of M'_1 and M'_2 has also the segmental property with a constant $\mathscr{R} + 1$.

Hence, we can check a pair of dpda's for their equivalence so long as it has the segmental property, whether they accept by empty stack or by final states.

Secondly, we can easily generalize the algorithm so that it works even if some reachable configurations may not be live. Or, we can also convert an arbitrary strict dpda to an equivalent one all of whose reachable configurations are live, by, e.g., [2, Lemma 1.1, p. 259], preserving the segmental property. So, we lose no generality if we assume that all reachable configurations of M_i are live in case $L(M_i) \neq \emptyset$. Here, we can check whether $L(M_i) = \emptyset$ or not.

Summarizing, we have the following.

Corollary 6.1. *The equivalence of any pair of dpda's is decidable if it has the segmental property.*

Lastly, it is left open in this paper to find classes of dpda's with the segmental property holding for each pair. However, we may instead propose a class of dpda's in which every pair of live configurations has just the same segmental property as in Definition 3.4 except that M_1 and M_2 are identical. Then we conjecture here that the equivalence problem is solvable for dpda's in such a class.

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