Journal of Computational and Applied Mathematics 28 (1989) 259–267
North-Holland

Estimates for Sard's best formulas for linear functionals on $C^s[a, b]$

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Received 2 June 1988

Abstract: Let $J: C^s[a, b] \to \mathbb{R}$ be a bounded linear functional on the space of s times continuously differentiable functions ($s \geqslant 0$), and let $Y = (y_{i,n})$ be a triangular matrix of nodes satisfying $a = y_{0,n} < y_{1,n} < \cdots < y_{n,n} = b$. Then one may approximate J by linear functionals Q_n of the form $Q_n[f] = \sum_{i=m_1}^{n-m_2} a_{i,n} f(y_{i,n})$ where $m_1, m_2 \in \{0, 1\}$. Among these, we consider the best formulas Q_n^B in the sense of Sard. For certain classes of nodes (which include, e.g., equidistant nodes, and the nodes of the Gauss quadrature formulas), and for arbitrary J, we give estimates for the weights $a_{i,n}^B$ of Q_n^B, for the corresponding Peano kernels, and for the approximation error, including the error for interpolation by natural splines. By choosing special J's, estimates for best formulas for numerical integration, interpolation and differentiation are obtained, and also exponential decay of the fundamental natural splines is proved.

Keywords: Best formulas in the sense of Sard, linear functionals, natural splines, spline interpolation.

1. Introduction

We are interested in approximating arbitrary bounded linear functionals $J \in (C^s[a, b])^*$ by ("quadrature") formulas Q_n. Any such J has a representation

$$J[f] = \sum_{i=0}^{s} J_i[f^{(i)}] \quad \text{where } J_i \in (C[a, b])^*.$$

Therefore, we may restrict to consider a single element of this sum, i.e., our problem is to approximate a functional of the form

$$J[f] = I[f^{(s)}] \quad \text{for } I \in (C[a, b])^* \quad \text{and} \quad f \in C^s[a, b]. \tag{1.1}$$

From now on, J (or I, respectively) always means a functional of the type (1.1). The formulas Q_n considered will be of the form

$$Q_n[f] = \sum_{i=m_1}^{n-m_2} a_{i,n} f(y_{i,n}), \tag{1.2}$$

where $a = y_{0,n} < y_{1,n} < \cdots < y_{n,n} = b$, and $m_1, m_2 \in \{0, 1\}$. The nodes $Y = (y_{i,n})$ are supposed to have the following property (Q).

Definition 1. If there exist constants $\delta = \delta(Y)$ and $\gamma = \gamma(Y) \geqslant 0$, such that

$$\frac{y_{j+1,n} - y_{j,n}}{y_{i+1,n} - y_{i,n}} \leqslant \delta(1 + |i-j|)^{\gamma} \quad \text{for all } i,\ j \text{ and } n,$$

then Y is said to have *property* (Q).

Remarks. (1) The nodes common in quadrature theory (e.g., equidistant nodes, with $\gamma = 0$, and the nodes of the Gauss formulas, with $\gamma = 1$) satisfy property (Q) (therefore the "Q").

(2) Let M_{Q} be the set of Y's having property (Q), and let M_{G} and M_{L} be the set of Y's having bounded global and local mesh ratio, respectively. Then $M_{\mathrm{G}} \subsetneqq M_{\mathrm{Q}} \subsetneqq M_{\mathrm{L}}$.

The question now is how to choose the weights $a_{i,n}$. One common strategy is to interpolate f by a polynomial p of degree $n - m_1 - m_2$, and then to define $Q_n[f] = J[p]$. But the convergence (or divergence) properties of the resulting formulas depend heavily on J and Y. Another possibility is to interpolate f by some spline function g, and to define $Q_n[f] = J[g]$. Here we will consider interpolation by natural polynomial splines, which is of interest because of its various optimality properties. E.g., $J[g]$, where g is the natural spline of degree $2r-1$, interpolating in $y_{m_1,n}, \ldots, y_{n-m_2,n}$, is identical with the best formula of order r in the sense of Sard ([16]; in the sequel, we use the approach of Sard). As will be seen, these formulas have a more regular behaviour than those obtained by polynomial interpolation.

In Section 2, the main results are stated. In Section 3, estimates for the weights for some special functionals are given. The proofs are contained in Section 4.

2. The main results

Now let J (or I, respectively) and Y be given, and let $r > s \geqslant 0$, and $n \geqslant r + m_1 + m_2$. Let

$$R_n^s[f] = J[f] - Q_n^s[f] = I[f^{(s)}] - \sum_{i=m_1}^{n-m_2} a_{i,n}^s f(y_{i,n}),$$

for arbitrary $a_{i,n}^s$. The best formulas $Q_n^{Bs,r}$ in the sense of Sard, with weights $a_{i,n}^{Bs,r}$, are those who satisfy

$$\sup_{\|f^{(r)}\|_2 \leqslant 1} |R_n^{Bs,r}[f]| = \inf_{a_{i,n}^s} \sup_{\|f^{(r)}\|_2 \leqslant 1} |R_n^s[f]|, \tag{2.1}$$

where $f \in W_2^r$, and, more generally, $W_p^r = \{f;\, f^{(r-1)} \text{ abs. cont.}, f^{(r)} \in L_p[a, b]\}$, with the usual L_p-spaces and norms. Because of (2.1), it is sufficient to consider formulas which are exact for P_{r-1} (polynomials of degree $r-1$), and therefore admit a Peano representation

$$R_n^s[f] = \int_a^b f^{(r)}(x) K_{r,n}^s(x)\, \mathrm{d}x \quad \text{for } f \in W_1^r. \tag{2.2}$$

As a consequence of (2.1) and (2.2), the Peano kernel $K_{r,n}^{Bs,r}$ of $Q_n^{Bs,r}$ minimizes $\| K_{r,n}^s \|_2$ (see [13] for details about linear functionals, Peano's theorem, and best formulas; it should be noted here, that our use of "best" is not always consistent with that of Sard [12,13], where the best formulas may depend on the support of the Peano kernels).

To formulate our results, we need some information on the functional I. As is well known, I can be represented as a Riemann–Stieltjes integral,

$$I[f]=\int_a^b f(x)\,\mathrm{d}w(x)\quad \text{for } f\in C[a,b], \tag{2.3}$$

where w is a function of bounded variation. Here, we will assume that w is continuous from the right on the open interval (a, b). Let $V(w, c, d)$ denote the variation of w on $[c, d]\subset[a, b]$, and let

$$w_\mu=V(w, y_{\mu,n}, y_{\mu+1,n}),\quad \mu=0,\dots,n-1. \tag{2.4}$$

In terms of the w_μ, the weights and Peano kernels of Sard's best formulas can be bounded in the following way.

Theorem 2. *Let $I\in(C[a, b])^*$, let Y satisfy property* (Q), *and let $0\leqslant s<r$ and $n\geqslant r+m_1+m_2$. Then there exist constants $c_j=c_j(r, Y)$, $j=1, 2$, and $q=q(r, Y)\in(0, 1)$, such that the weights and Peano kernels of Sard's best formulas satisfy the following inequalities*:

(*a*) *for $j=m_1,\dots,n-m_2$,*

$$\left|a_{j,n}^{Bs,r}\right|\leqslant c_1(y_{j+1,n}-y_{j-1,n})^{-s}\sum_{\mu=0}^{n-1}w_\mu q^{|j-\mu|};$$

(*b*) *for $x\in[y_{j,n}, y_{j+1,n}]$, $j=0,\dots,n-1$,*

$$\left|K_{r,n}^{Bs,r}(x)\right|\leqslant c_2(y_{j+1,n}-y_{j,n})^{r-s-1}\sum_{\mu=0}^{n-1}w_\mu q^{|j-\mu|}.$$

(In (a), $y_{-1,n}$ and $y_{n+1,n}$ may be replaced by a and b, respectively.)

For a functional F, let

$$\|F\|_{j,p}=\sup_{\|f^{(j)}\|_p\leqslant 1}|F[f]|,$$

provided that the right-hand side makes sense; here $f\in W_p^j$ for $1\leqslant p<\infty$, and $f\in C^j[a, b]$ for $p=\infty$. Especially, we have

$$\sum_{\mu=0}^{n-1}w_\mu=V(w, a, b)=\|I\|_{0,\infty}.$$

Part (a) and (c) of the following theorem are direct consequences of Theorem 2, whereas the proof of part (b) is somewhat more complicated, but follows the same lines.

Theorem 3. *Let the assumptions of Theorem* 2 *hold. Then*

(a) $$\left\|Q_n^{Bs,r}\right\|_{0,\infty}=\sum_{i=m_1}^{n-m_2}\left|a_{i,n}^{Bs,r}\right|\leqslant c_1\sum_{\mu=0}^{n-1}(y_{\mu+1,n}-y_{\mu,n})^{-s}w_\mu\leqslant c_1\delta_n^{-s}\|I\|_{0,\infty},$$

where $c_1=c_1(r, Y)$, and $\delta_n=\min_{0\leqslant i\leqslant n-1}(y_{i+1,n}-y_{i,n})$;

(b) $$\left\|R_n^{Bs,r}\right\|_{s,\infty}\leqslant c_2\|I\|_{0,\infty},$$

where $c_2 = c_2(r, Y)$, *and*

$$\text{(c)}\quad \| R_n^{Bs,r} \|_{r,p} \leqslant c_3 \Delta_n^{r-s-1/p} \| I \|_{0,\infty},$$

where $1 \leqslant p \leqslant \infty$, $c_3 = c_3(r, Y, p)$, *and* $\Delta_n = \max_{0 \leqslant i \leqslant n-1}(y_{i+1,n} - y_{i,n})$.

From (c), we obtain the error estimate

$$| R_n^{Bs,r}[f] | \leqslant c_3 \Delta_n^{r-s-1/p} \| I \|_{0,\infty} \| f^{(r)} \|_p \quad \text{for } f \in W_p^r, \tag{2.5}$$

and from (b) and (c) (with $p = \infty$)

$$| R_n^{Bs,r}[f] | \leqslant c_4 \| I \|_{0,\infty} \omega_{r-s}(f^{(s)}, \Delta_n) \quad \text{for } f \in C^s[a, b], \tag{2.6}$$

where ω_j is the jth modulus of continuity (this is obtained by an application of the K-functional; see, e.g., [14]). Therefore, $Q_n^{Bs,r}[f]$ converges for any $f \in C^s[a, b]$, if $\Delta_n \to 0$. For $I[f] = f(u)$, (2.5) and (2.6) are estimates for the error of natural spline interpolation. Conversely, from estimates for the interpolation error there follow estimates for $R_n^{Bs,r}[f]$. E.g., it follows from [1, Theorem 5.9.1], that (2.5) holds for arbitrary Y, if $p = 2$. For further estimates for the error in natural spline interpolation, see [1,14,18].

3. Interpolation, differentiation, integration

In this section, we state, for some special functionals I, estimates for the weights $a_{i,n}^{Bs,r}$ of Sard's best formulas. This estimates follow more or less directly from Theorem 2(a); the proofs will be omitted. Of course, the estimates for $K_{r,n}^{Bs,r}(x)$ can be specialized in the same way; moreover, in some cases the order in Theorem 3(c) can be improved to Δ_n^{r-s} for all p (e.g., for example(b) below).

(a) Interpolation, differentiation. For $u \in (a, b]$ (similar for $u = a$), let

$$I[f] = f(u) = \int_a^b f(x)\, \mathrm{d}w(x),$$

where

$$w(x) = \begin{cases} 0 & \text{if } x \in [a, u), \\ 1 & \text{if } x \in [u, b]. \end{cases}$$

Then $J[f] = I[f^{(s)}] = f^{(s)}(u)$, i.e., we consider interpolation ($s = 0$) or differentiaton ($1 \leqslant s \leqslant r - 1$), respectively. Let $m = m_n(u)$ be chosen such that $y_{m,n} < u \leqslant y_{m+1,n}$. Theorem 2(a) yields

$$| a_{i,n}^{Bs,r} | \leqslant c_1 (y_{i+1,n} - y_{i-1,n})^{-s} q^{|i-m|}, \quad i = m_1, \ldots, n - m_2, \tag{3.1}$$

i.e., the weights are of order $(y_{i+1,n} - y_{i-1,n})^{-s}$ near u, and decay exponentially away from u. Let $l_{i,n} \in S_{2r-1}^{\text{nat}}(y_{m_1,n}, \ldots, y_{n-m_2,n})$ be the natural spline of degree $2r-1$ satisfying $l_{i,n}(y_{j,n}) = \delta_{i,j}$ (Kronecker's symbol). Then

$$l_{i,n}^{(s)}(u) = Q_n^{Bs,r}[l_{i,n}] = a_{i,n}^{Bs,r} \quad \text{for } m_1 \leqslant i \leqslant n - m_2 \quad \text{and} \quad 0 \leqslant s < r.$$

By (3.1), the fundamental spline $l_{i,n}$ and its first $r-1$ derivatives decay exponentially away from $y_{i,n}$. For further results on the exponential decay of fundamental splines in the cubic case, see [4].

(b) Integration. Let w be absolutely continuous, and let $w' \in L_\infty[a, b]$, i.e.,

$$J[f] = I[f^{(s)}] = \int_a^b f^{(s)}(x) w'(x)\, dx.$$

Then

$$|a_{i,n}^{Bs,r}| \leqslant c_2 \| w' \|_\infty (y_{i+1,n} - y_{i-1,n})^{1-s}. \tag{3.2}$$

If, moreover, $w' \in C[a, b]$, and $w'(u) = 0$ for some $u \in [a, b]$, then it can be shown that

$$|a_{i,n}^{Bs,r}| \leqslant c_3 (y_{i+1,n} - y_{i-1,n})^{1-s} \omega(w', \max(|y_{i+1,n} - u|, |y_{i-1,n} - u|)), \tag{3.3}$$

where ω is the modulus of continuity of w'. E.g., for $s = 0$, the weights are of order $O(y_{i+1,n} - y_{i-1,n})$ in general, but of order $o(y_{i+1,n} - y_{i-1,n})$ near to a zero of w'.

(c) Integration. Again let w be absolutely continuous, but $w'(x) = (x-a)^\alpha (b-x)^\beta \tilde{w}(x)$, where $\tilde{w} \in L_\infty[a, b]$, and $\alpha, \beta \in (-1, 0)$. Then

$$|a_{i,n}^{Bs,r}| \leqslant c_4 \| \tilde{w} \|_\infty (y_{i+1,n} - y_{i-1,n})^{1-s} (y_{i+1,n} - a)^\alpha (b - y_{i-1,n})^\beta. \tag{3.4}$$

E.g., let $y_{i,n} = -\cos i\pi/n$, $a = -1$, $b = 1$, $\tilde{w} \equiv 1$, $m_1 = m_2 = 1$, and $\alpha = \beta$. Then, from (3.4), one obtains

$$|a_{i,n}^{Bs,r}| \leqslant c_5 \sin\frac{\pi}{n} \left(\sin \frac{i\pi}{n} \right)^{1+2\alpha} \quad \text{for } i = 1, \ldots, n-1.$$

(For Sard's best quadrature formulas, see also [2, pp.251–256; 7,8,10,11,15,17].)

4. The proofs of Section 2

Let nodes outside $[a, b]$ be chosen such that property (Q) is satisfied for these nodes, too, and let

$$N_{j,r}(x) = (y_{j+r,n} - y_{j,n})[y_{j,n}, \ldots, y_{j+r,n}](\cdot - x)_+^{r-1}, \quad j = -r, \ldots, n,$$

be the B-splines of degree $r-1$ for these nodes. Given any formula of the form (1.2), which is exact for P_{r-1}, and with Peano kernel $K_{r,n}^s$, the Peano kernel $K_{r,n}^{Bs,r}$ of Sard's best formula is obtained by approximating $K_{r,n}^s$ by $N_{j,r}$, $j = m_1, \ldots, n - r - m_2$, in the norm of L_2. For $K_{r,n}^s$, we choose

$$K_{r,n}^s(x) = I[H_{r,n}^s(x, \cdot)], \tag{4.1}$$

where

$$H_{r,n}^s(x, t) = \frac{(-1)^{r-s}}{(r-1-s)!}(x-t)_+^{r-1-s} - \frac{(-1)^r}{(r-1)!} \sum_{j=-r}^{n} \psi_j^{(s)}(t)(\tau_j - t)_+^0 N_{j,r}(x),$$

$$\psi_j(t) = \prod_{l=1}^{r-1} (y_{j+l,n} - t),$$

and

$$\tau_j = \begin{cases} a- & \text{for } j < m_1, \\ y_{j+\frac{1}{2}r,n} & \text{for } j = m_1, \ldots, n-r-m_2, \\ b+ & \text{for } j > n-r-m_2, \end{cases}$$

(and $y_{j+\frac{1}{2}r,n}=\frac{1}{2}(y_{j+\frac{1}{2}(r-1),n}+y_{j+\frac{1}{2}(r+1),n})$, if r is odd). For (4.1) to be well-defined, we extend $I[f]=\int_a^b f(x)\,\mathrm{d}w(x)$ as a Lebesgue–Stieltjes integral. $(-1)^r H^0_{r,n}(x,\cdot)$ is the Peano kernel of the quasi-interpolant of de Boor and Fix [5]; moreover, $H^0_{r,n}(\cdot,t)$ is the Peano kernel of central polynomial interpolation (this may be seen from [5, Appendix]). The Peano kernel $K^0_{r,n}$ was already used in [9,11]. By Fubini's Theorem, we obtain, for $f\in W_1^r[a,b]$,

$$\int_a^b f^{(r)}(x)I\left[H^s_{r,n}(x,\cdot)\right]\mathrm{d}x=\int_a^b\int_a^b f^{(r)}(x)H^s_{r,n}(x,t)\,\mathrm{d}x\,\mathrm{d}w(t).$$

The inner integral on the right-hand side can be transformed by multiple partial integration and use of Marsden's identity, i.e.,

$$(x-t)^{r-1}=\sum_{j=i-r+1}^{i}\psi_j(t)N_{j,r}(x),$$

for $x\in[y_{i,n},y_{i+1,n}]$. After lengthy, but elementary calculations, which have to be omitted for reasons of space, one obtains the following lemma.

Lemma 4. *Let $K^{Bs,r}_{r,n}=K^s_{r,n}-\sum_{i=-r}^{n}\lambda^s_iN_{i,r}$, where $\lambda^s_i=0$ if $i\notin\{m_1,\ldots,n-r-m_2\}$. Then, for $f\in W_1^r$, and $0\leqslant s<r$, the following holds*:

$$\int_a^b f^{(r)}(x)K^{Bs,r}_{r,n}(x)\,\mathrm{d}x=I\left[f^{(s)}\right]-\sum_{j=m_1}^{n-m_2}a^{Bs,r}_{j,n}f(y_{j,n}),$$

where

$$a^{Bs,r}_{j,n}=I\left[\phi^s_j\right]+(-1)^r\sum_{i=j-r}^{j}\lambda^s_i\beta_{i,j},$$

$$\phi^s_j(t)=\frac{1}{(r-1)!}\sum_{i=j-r}^{j}\psi_i^{(s)}(t)(\tau_i-t)^0_+\beta_{i,j},$$

and

$$\beta_{i,j}=N^{(r-1)}_{i,r}(y_{j,n}+)-N^{(r-1)}_{i,r}(y_{j,n}-).$$

Before starting with the proof of Theorem 2, let us note that

$$\left|\int_{y_{\mu,n}}^{y_{\mu+1,n}}f(x)\,\mathrm{d}w(x)\right|\leqslant\int_{y_{\mu,n}}^{y_{\mu+1,n}}|f(x)|\,\mathrm{d}|w|(x)\leqslant\sup_{y_{\mu,n}\leqslant x\leqslant y_{\mu+1,n}}|f(x)|\,w_\mu,\qquad(4.2)$$

where the integrals are considered as Lebesgue–Stieltjes integrals over the intervals $(y_{\mu,n},y_{\mu+1,n}]$ in case $y_{\mu,n}>a$, and over $[y_{\mu,n},y_{\mu+1,n}]$, if $y_{\mu,n}=a$ (here it is important that w is right-continuous on (a,b)).

Proof of Theorem 2. (1) We are concerned with L_2-approximation of Peano kernels, the normal equations being

$$\int_a^b\left(K^s_{r,n}(x)-\sum_{i=m_1}^{n-r-m_2}\lambda^s_iN_{i,r}(x)\right)N_{j,r}(x)\,\mathrm{d}x=0,\quad j=m_1,\ldots,n-r-m_2.\qquad(4.3)$$

Let $\kappa_j = (y_{j+r,n} - y_{j,n})/r$ and $M_{i,j} = \int_a^b (\kappa_i \kappa_j)^{-\frac{1}{2}} N_{i,r}(x) N_{j,r}(x)\,\mathrm{d}x$, $i,\ j = m_1, \ldots, n-r-m_2$. Let $M_{i,j}^{-1}$ be the elements of the inverse matrix. By de Boor [3], there exist $c_1 = c_1(r)$ and $q_1 = q_1(r) \in (0, 1)$, such that (for arbitrary Y), $|M_{i,j}^{-1}| \leqslant c_1 q_1^{|i-j|}$. Inserting the definition of $K_{r,n}^s$ and using Fubini's theorem, (4.3) yields

$$\lambda_i^s = \int_a^b A_i(t)\,\mathrm{d}w(t),$$

where

$$A_i(t) = \int_a^b H_{r,n}^s(x, t) \sum_{j=m_1}^{n-r-m_2} M_{i,j}^{-1} (\kappa_i \kappa_j)^{-\frac{1}{2}} N_{j,r}(x)\,\mathrm{d}x.$$

Because of (4.2), we have

$$|\lambda_i^s| \leqslant \sum_{\mu=0}^{n-1} w_\mu \sup_{y_{\mu,n} \leqslant t \leqslant y_{\mu+1,n}} |A_i(t)|. \tag{4.4}$$

(2) From Marsden's identity, one obtains

$$H_{r,n}^s(x, t) = \frac{(-1)^r}{(r-1)!} \sum_{\nu=-r}^{n} \psi_\nu^{(s)}(t)\big((x-t)_+^0 - (\tau_\nu - t)_+^0\big) N_{\nu,r}(x).$$

Let $x \in [y_{\sigma,n}, y_{\sigma+1,n}]$. Then, since $\operatorname{supp} N_{\nu,r} = [y_{\nu,n} y_{\nu+r,n}]$, and $|N_{\nu,r}| \leqslant 1$,

$$|H_{r,n}^s(x, t)| \leqslant \frac{1}{(r-1)!} \sum_{\nu=\sigma-r+1}^{\sigma} |\psi_\nu^{(s)}(t)| \big|(x-t)_+^0 - (\tau_\nu - t)_+^0\big|.$$

For $t \notin [\tau_{\sigma-r+1}, \tau_\sigma]$, this gives $H_{r,n}^s(x, t) = 0$, and, for $t \in [\tau_{\sigma-r+1}, \tau_\sigma]$,

$$\begin{aligned} |H_{r,n}^s(x, t)| &\leqslant \frac{1}{(r-1)!} \sum_{\nu=\sigma-r+1}^{\sigma} (y_{\sigma+r,n} - y_{\sigma-r+1,n})^{r-1-s} \\ &\leqslant c_1 (y_{\sigma+1,n} - y_{\sigma,n})^{r-1-s}, \end{aligned} \tag{4.5}$$

where property (Q) was used.

(3) Now let $t \in [y_{\mu,n}, y_{\mu+1,n}]$, $\sigma_1 = \max(0, \mu+1-r)$, $\sigma_2 = \min(n-1, \mu+r-1)$, and $j_1 = \max(m_1, \sigma - r + 1)$, $j_2 = \min(n - r - m_2, \sigma)$. From (4.5) and the restricted support of $H_{r,n}^s$ and $N_{j,r}$, one then obtains

$$\begin{aligned} |A_i(t)| &= \sum_{\sigma=0}^{n-1} \int_{y_{\sigma,n}}^{y_{\sigma+1,n}} |H_{r,n}^s(x, t)| \sum_{j=m_1}^{n-r-m_2} c_1 q_1^{|i-j|} (\kappa_i \kappa_j)^{-\frac{1}{2}} N_{j,r}(x)\,\mathrm{d}x \\ &\leqslant c_2 \sum_{\sigma=\sigma_1}^{\sigma_2} (y_{\sigma+1,n} - y_{\sigma,n})^{r-1-s} \sum_{j=j_1}^{j_2} q_1^{|i-j|} (\kappa_i \kappa_j)^{-\frac{1}{2}} (y_{\sigma+1,n} - y_{\sigma,n}). \end{aligned}$$

Multiple use of property (Q) now gives

$$\begin{aligned} |A_i(t)| &\leqslant c_3 (y_{i+1,n} - y_{i,n})^{r-s-1} \sum_{j=j_1}^{j_2} q_1^{|i-j|} \left(\frac{y_{\mu+1,n} - y_{\mu,n}}{y_{i+1,n} - y_{i,n}} \right)^{r-s-\frac{1}{2}} \\ &\leqslant c_4 (y_{i+1,n} - y_{i,n})^{r-s-1} q_1^{|i-\mu|} (1 + |i-\mu|)^{\gamma(r-s-\frac{1}{2})} \\ &\leqslant c_5 (y_{i+1,n} - y_{i,n})^{r-s-1} q_2^{|i-\mu|}, \end{aligned}$$

where $c_5 = c_5(r, Y)$, and $q_2 = q_2(r, Y)$, $q_1 \leqslant q_2 < 1$ (the decisive use of property (Q) is made in the last step of the above estimates for $A_i(t)$). In view of (4.4), we now obtain

$$|\lambda_i^s| \leqslant c_5(y_{i+1,n} - y_{i,n})^{r-s-1} \sum_{\mu=0}^{n-1} w_\mu q_2^{|i-j|}. \tag{4.6}$$

(For bounds on L_2-approximation by splines, see also [3,6].)

(4) The estimate for $K_{r,n}^{Bs,r}$ given in part (b) of the theorem, now follows from (4.2), (4.5), (4.6), property (Q) and

$$K_{r,n}^{Bs,r}(x) = \sum_{\mu=0}^{n-1} \int_{y_{\mu,n}}^{y_{\mu+1,n}} H_{r,n}^s(x, t)\, \mathrm{d}w(t) - \sum_{i=m_1}^{n-r-m_2} \lambda_i^s N_{i,r}(x).$$

(5) The estimate for the weights is derived from the representation given in Lemma 4. For $\beta_{i,j}$, one obtains from property (Q)

$$|\beta_{i,j}| = \left|N_{i,r}^{(r-1)}(y_{j,n}+) - N_{i,r}^{(r-1)}(y_{j,n}-)\right| = \left|\frac{(r-1)!(y_{i+r,n} - y_{i,n})}{\prod_{\nu=i,\nu\neq j}^{i+r} (y_{\nu,n} - y_{j,n})}\right|$$

$$\leqslant c_6(y_{j+1,n} - y_{j-1,n})^{1-r} \tag{4.7}$$

for $i = j-r, \ldots, j$. For ϕ_j^s, using Marsden's identity gives $\phi_j^s(t) = 0$ for $t \notin (\tau_{j-r}, \tau_j]$. By Lemma 4,

$$\left|a_{j,n}^{Bs,r}\right| \leqslant \sup_{\tau_{j-r} < t \leqslant \tau_j} |\phi_j^s(t)| V(w, \tau_{j-r}, \tau_j) + c_6 \sum_{i=j-r}^{j} |\lambda_i^s| (y_{j+1,n} - y_{j,n})^{1-r}.$$

From this and (4.6), (4.7) and again (Q), the estimate given in (a) follows. □

Proof of Theorem 3. (a) From Theorem 2(a) and (Q), we obtain

$$\sum_{j=m_1}^{n-m_2} \left|a_{j,n}^{Bs,r}\right| \leqslant c_1 \sum_{\mu=0}^{n-1} w_\mu (y_{\mu+1,n} - y_{\mu,n})^{-s} \sum_{j=m_1}^{n-m_2} q^{|j-\mu|} \left(\delta(1 + |j-\mu|)^\gamma\right)^s$$

$$\leqslant c_2 \sum_{\mu=0}^{n-1} w_\mu (y_{\mu+1,n} - y_{\mu,n})^{-s} \leqslant c_2 \delta_n^{-s} \sum_{\mu=0}^{n-1} w_\mu.$$

(b) Let $p_f(t) = \sum_{i=0}^{s-1} f^{(i)}(a)(t-a)^i/i!$. Then, by Taylor's Theorem,

$$R_n^{Bs,r}[f] = R_n^{Bs,r}[f - p_f] = J[f - p_f] - \sum_{j=m_1}^{n-m_2} a_{j,n}^{Bs,r} (f - p_f)(y_{j,n})$$

$$= I[f^{(s)}] - \int_a^b f^{(s)}(t) \sum_{j=m_1}^{n-m_2} a_{j,n}^{Bs,r} \frac{(y_{j,n} - t)_+^{s-1}}{(s-1)!}\, \mathrm{d}t,$$

and therefore

$$|R_n^{Bs,r}[f]| \leqslant \| f^{(s)} \|_\infty \left(\| I \|_{0,\infty} + \int_a^b \left| \sum_{j=m_1}^{n-m_2} a_{j,n}^{Bs,r} \frac{(y_{j,n} - t)_+^{s-1}}{(s-1)!} \right| \mathrm{d}t \right).$$

It is not sufficient here to use the estimate of Theorem 2(a) for $a_{j,n}^{Bs,r}$, but one has to start again with the representation of Lemma 4. The estimates given in the proof of Theorem 2 for λ_i^s etc. can be used. For reasons of space, we cannot give the details here.

(c) From Theorem 2(b) and the triangle inequality for l_n^p (discrete l^p-space), we obtain for $1 < p \leqslant \infty$ (the case $p = 1$ follows directly from Theorem 2(b)), with $1/p + 1/\tilde{p} = 1$,

$$\begin{aligned}
\| R_n^{Bs,r} \|_{r,p} = \| K_{r,n}^{Bs,r} \|_{\tilde{p}} &\leqslant \left(\sum_{j=0}^{n-1} \int_{y_{j,n}}^{y_{j+1,n}} \left(c_2 (y_{j+1,n} - y_{j,n})^{r-s-1} \sum_{\mu=0}^{n-1} w_\mu q^{|j-\mu|} \right)^{\tilde{p}} \mathrm{d}x \right)^{1/\tilde{p}} \\
&\leqslant c_2 \Delta_n^{r-s-1+1/\tilde{p}} \left(\sum_{j=0}^{n-1} \left(\sum_{\mu=0}^{n-1} w_\mu q^{|j-\mu|} \right)^{\tilde{p}} \right)^{1/\tilde{p}} \\
&\leqslant c_2 \Delta_n^{r-s-1+1/\tilde{p}} \sum_{\mu=0}^{n-1} w_\mu \left(\sum_{j=0}^{n-1} q^{|j-\mu|\tilde{p}} \right)^{1/\tilde{p}} \\
&\leqslant c_2 \Delta_n^{r-s-1/p} \sum_{\mu=0}^{n-1} w_\mu \left(2 \sum_{j=0}^{\infty} q^{j\tilde{p}} \right)^{1/\tilde{p}}. \qquad \square
\end{aligned}$$

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