Computers and Mathematics with Applications 60 (2010) 1940–1947

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The variational iteration method for solving Riesz fractional partial differential equations

A. Elsaid

Mathematics & Engineering Physics Department, Faculty of Engineering, Mansoura University, PO 35516, Mansoura, Egypt

ARTICLE INFO

Article history:
Received 7 August 2009
Received in revised form 20 July 2010
Accepted 20 July 2010

Keywords:
Fractional partial differential equations
Riesz
Variational iteration method

ABSTRACT

In this paper, the variational iteration method is applied to obtain the solution for space fractional partial differential equations where the space fractional derivative is in the Riesz sense. On the basis of the properties and definition of the fractional derivative, the iterative technique is carried out in a straightforward manner without the need for transforms or numerical approximations. Examples demonstrate that the series solution obtained shows agreement with the exact solutions of the problems solved.

1. Introduction

The variational iteration method (VIM) was proposed by the Chinese mathematician He [1–4] as a modification of a general Lagrange multiplier method [5]. It has been shown that this procedure is a powerful tool for solving various kinds of problems. VIM has been used to solve fractional differential equations with great success [6]. Following that, VIM was applied to more fractional differential equations, showing the effectiveness and accuracy of the method [7–11].

Fractional differential equations have attracted much attention recently due to the exact description of nonlinear phenomena that they afford. Here we consider fractional partial differential equations of the form

$$L_t u(x,t) = R_x^{\alpha} u(x,t), \quad t > 0, \tag{1}$$

where L_t is an integer order time derivative, and R_x^{α} is the Riesz space fractional derivative of order α where parameter α is a real number restricted to

$$0 < \alpha < 2, \quad \alpha \neq 1.$$

The Riesz fractional derivative and its generalizations are used in equations that describe applications in random walk models and anomalous diffusion characterized by nonlinear dependence of the mean square displacement of a diffusing particle over time [12–17]. Numerical methods have been recently suggested for solving Riesz fractional problems [18,19]. However, iterative techniques have not been applied to such problems due to the difficulty of repeated evaluation of Riesz fractional derivative of the solution components.

In this work, VIM is used to obtain the solution of the linear problem for a general zeroth-component function $u_0 = f(x)$ that belongs to $L^1(-\infty, \infty)$. The article begins with some basic definitions of the fractional derivatives used. A lemma is proved, showing how to carry out iterative steps to obtain the series solution. Examples are presented to illustrate the approach suggested.

E-mail address: a_elsaid@mans.edu.eg.

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.07.027

2. The Riesz fractional derivative

The Riesz fractional derivative R_x^α is defined as [20]

$$R_x^\alpha u(x) = -\frac{[D_+^\alpha u(x) + D_-^\alpha u(x)]}{2\cos(\alpha\pi/2)}, \quad 0 < \alpha < 2,\ \alpha \neq 1 \tag{2}$$

where $D_\pm^\alpha u(x)$ are the Weyl fractional derivatives

$$D_\pm^\alpha u(x) = \begin{cases} \pm\dfrac{\mathrm{d}}{\mathrm{d}x} I_\pm^{1-\alpha} u(x), & 0 < \alpha < 1 \\ \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} I_\pm^{2-\alpha} u(x), & 1 < \alpha < 2, \end{cases} \tag{3}$$

and $I_\pm^\beta$ denote the Weyl fractional integrals of order $\beta > 0$, given by

$$\begin{aligned} I_+^\beta u(x) &= \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} (x-z)^{\beta-1} u(z)\mathrm{d}z \\ I_-^\beta u(x) &= \frac{1}{\Gamma(\beta)} \int_{x}^{\infty} (z-x)^{\beta-1} u(z)\mathrm{d}z. \end{aligned} \tag{4}$$

When $\alpha = 0$ the Weyl fractional derivative degenerates into the identity operator

$$D_\pm^0 u(x) = Iu(x) = u(x). \tag{5}$$

For continuity we get

$$D_\pm^1 u(x) = \pm\frac{\mathrm{d}}{\mathrm{d}x} u(x), \qquad D_\pm^2 u(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x). \tag{6}$$

Evidently, in the case $\alpha = 2$ it takes the form of the second-derivative operator

$$R_x^2 u(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x). \tag{7}$$

For the case $\alpha = 1$ we have

$$R_x^1 u(x) = \frac{\mathrm{d}}{\mathrm{d}x} Hu(x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(z)}{z-x} \mathrm{d}z, \tag{8}$$

where H is the Hilbert transform and the integral is understood in the Cauchy principal value sense.

3. The variational iteration method (VIM)

Consider the partial differential equation

$$Lu(x,t) + Nu(x,t) = g(x,t), \tag{9}$$

with prescribed auxiliary conditions, where u is the unknown function, L and N are linear and nonlinear operators, respectively, and g is the source term. In the variational iteration method, a correction functional for Eq. (9) can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda[Lu_n(x,\xi) + N\widetilde{u}_n(x,\xi) - g(x,\xi)]\mathrm{d}\xi \tag{10}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\widetilde{u}_n$ is a restricted variation which means $\delta\widetilde{u}_n = 0$.

It is obvious now that the main steps of He's variational iteration method require first the determination of λ, the Lagrangian multiplier that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0, preferably chosen as the initial condition of the problem. Consequently, the solution is

$$u = \lim_{n\to\infty} u_n.$$

Applying the method to problem (1), the correction functional equation takes the form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda[L_\xi u_n(x,\xi) - R_x^\alpha \widetilde{u}_n(x,\xi)]\mathrm{d}\xi. \tag{11}$$

To carry out the procedure with relation (11), a repeated evaluation of the Riesz fractional derivative is needed. In the following lemma, we show how to overcome this obstacle.

Lemma 1. *Let f(x) be a function in $L^1(-\infty,\infty)$. Then for $\alpha \in (0,2)$, $\alpha \neq 1$, and a positive integer k, a k-times fractional Riesz derivative of $f(x)$ takes the form*

$$\left(R_x^\alpha\right)^k f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}(-)^k\omega^{k\alpha}f(v)\cos(\omega(x-v))\mathrm{d}\omega\mathrm{d}v.$$

Proof. Using the definition of Riesz and Weyl fractional derivatives (2) and (3), for the case $\alpha \in (0,1)$,

$$R_x^\alpha \sin(\omega x) = \frac{-C_1(\alpha)}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-\infty}^{x}\frac{\sin(\omega z)}{(x-z)^\alpha}\mathrm{d}z - \int_x^\infty \frac{\sin(\omega y)}{(y-x)^\alpha}\mathrm{d}y\right] \tag{12}$$

where $C_1(\alpha) = 1/(\cos(\alpha\pi/2)\Gamma(1-\alpha))$. By substitution and using trigonometric identities we have

$$R_x^\alpha \sin(\omega x) = -C_1(\alpha)\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \frac{-\cos(\omega x)\sin(\omega\tau)}{\tau^\alpha}\mathrm{d}\tau, \tag{13}$$

which yields

$$R_x^\alpha \sin(\omega x) = -\omega^\alpha \sin(\omega x). \tag{14}$$

For $\alpha \in (1,2)$,

$$R_x^\alpha \sin(\omega x) = \frac{-C_2(\alpha)}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\int_{-\infty}^{x}\frac{\sin(\omega z)}{(x-z)^{\alpha-1}}\mathrm{d}z + \int_x^\infty \frac{\sin(\omega y)}{(y-x)^{\alpha-1}}\mathrm{d}y\right] \tag{15}$$

where $C_2(\alpha) = 1/(\cos(\alpha\pi/2)\Gamma(2-\alpha))$. Then

$$R_x^\alpha \sin(\omega x) = -C_2(\alpha)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^\infty \frac{\sin(\omega x)\cos(\omega\tau)}{\tau^{\alpha-1}}\mathrm{d}\tau, \tag{16}$$

which yields

$$R_x^\alpha \sin(\omega x) = -\omega^\alpha \sin(\omega x). \tag{17}$$

A similar argument for $\cos(\omega x)$ shows that for the case $\alpha \in (0,1)$,

$$\begin{aligned} R_x^\alpha \cos(\omega x) &= -C_1(\alpha)\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \frac{\sin(\omega x)\sin(\omega\tau)}{\tau^\alpha}\mathrm{d}\tau, \\ &= -\omega^\alpha\cos(\omega x) \end{aligned} \tag{18}$$

and for the case $\alpha \in (1,2)$,

$$\begin{aligned} R_x^\alpha \cos(\omega x) &= -C_2(\alpha)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\int_0^\infty \frac{\cos(\omega x)\cos(\omega\tau)}{\tau^{\alpha-1}}\mathrm{d}\tau\right) \\ &= -\omega^\alpha\cos(\omega x). \end{aligned} \tag{19}$$

Now consider the Fourier integral representation of $f(x)$:

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^\infty f(v)\cos(\omega(x-v))\mathrm{d}\omega\mathrm{d}v. \tag{20}$$

Then, a k-times fractional Riesz derivative of $f(x)$ is given by

$$\left(R_x^\alpha\right)^k f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}(-)^k\omega^{k\alpha}f(v)\cos(\omega(x-v))\mathrm{d}\omega\mathrm{d}v. \quad \square \tag{21}$$

4. Examples

Example 1. Consider the fractional diffusion equation:

$$u_t(x,t) = R_x^\alpha u(x,t), \quad -\infty < x < \infty,\ t > 0, \tag{22}$$

with an initial condition

$$u(x,0) = f(x), \tag{23}$$

where f(x) is in $L^1(-\infty,\infty)$.

The correction functional equation for the problem is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - R_x^\alpha \widetilde{u}_n(x,\xi) \right] d\xi. \tag{24}$$

This yields the stationary conditions

$$\lambda'(\xi) = 0,$$
$$1 + \lambda(\xi) = 0.$$

This in turn gives

$$\lambda = -1.$$

Substituting this value of the Lagrangian multiplier into functional (24) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t R_x^\alpha \widetilde{u}_n(x,\xi) - \frac{\partial u_n(x,\xi)}{\partial \xi} d\xi. \tag{25}$$

The initial condition is used for the zeroth approximation in its Fourier integral representation. Eq. (25) yields the following successive approximations:

$$u_0(x,t) = f(x)$$
$$u_1(x,t) = (1/\pi) \int_{-\infty}^{\infty} \int_0^\infty (1 - t\omega^\alpha) f(v) \cos(\omega(x-v)) d\omega dv$$
$$u_2(x,t) = (1/\pi) \int_{-\infty}^{\infty} \int_0^\infty \left(1 - t\omega^\alpha + \frac{1}{2}\omega^{2\alpha} t^2\right) f(v) \cos(\omega(x-v)) d\omega dv$$
$$\vdots$$
$$u_n(x,t) = (1/\pi) \int_{-\infty}^{\infty} \int_0^\infty \sum_{k=0}^{n} \frac{(-t\omega^\alpha)^k}{k!} f(v) \cos(\omega(x-v)) d\omega dv.$$

Recall that $u = \lim_{k\to\infty} u_k$, which gives

$$u_\alpha(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^\infty e^{-\omega^\alpha t} f(v) \cos(\omega(x-v)) d\omega dv,$$

which is the exact solution of the problem. Fig. 1 shows at $t = 0.5$ the effect of changing α on the solution for $f(x) = \delta(x)$ where the problem describes discrete random walks [17]. It can be seen for Fig. 1 that as α increases the amplitude of the sinusoidal behavior in the solution decreases. In the figure, u_2 denotes the exact solution of the corresponding integer order problem.

Example 2. Consider the advection–dispersion problem [18]

$$u_t(x,t) = 0.25 R_x^\alpha u(x,t) + 0.25 R_x^\beta u(x,t), \quad 0 < x < \pi,\ t > 0,$$

subject to the conditions

$$u(x,0) = \sin(4x)$$
$$u(0,t) = u(\pi,t) = 0.$$

The correction functional equation takes the form

$$u_{n+1}(x,t) = u_n(x,t) + \lambda \int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - 0.25(R_x^\alpha \widetilde{u}_n(x,\xi) + R_x^\beta \widetilde{u}_n(x,\xi)) \right] d\xi. \tag{26}$$

This yields the stationary conditions

$$\lambda'(\xi) = 0,$$
$$1 + \lambda(\xi) = 0,$$

which in turn gives

$$\lambda = -1.$$

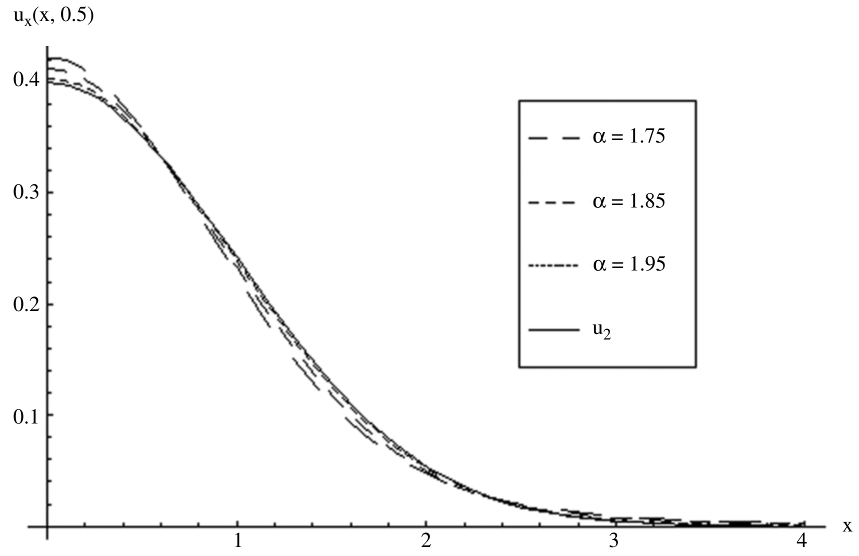

Fig. 1. $u(x, 0.5)$ with the change of α for $f(x) = \delta(x)$.

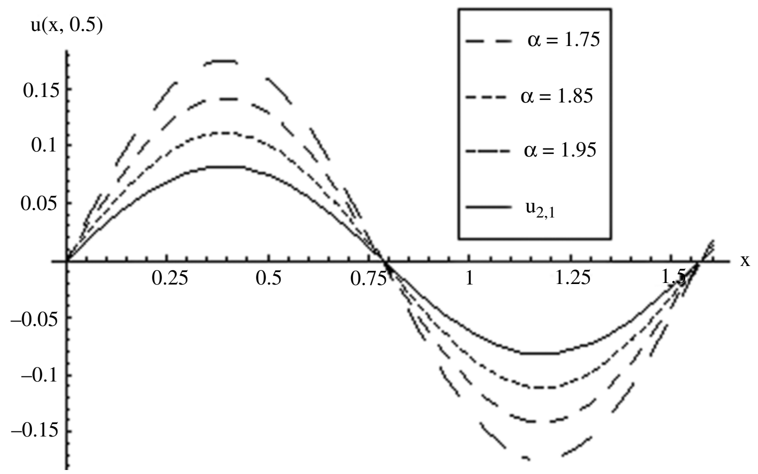

Fig. 2. $u(x, 0.5)$ with the change of α for $\beta = 0.7$.

The following successive approximations are obtained:

$$u_0(x, t) = \sin(4x)$$
$$u_1(x, t) = \sin(4x)(1 - 0.25(4^{\alpha} + 4^{\beta})t)$$
$$u_2(x, t) = \sin(4x)\left(1 - 0.25(4^{\alpha} + 4^{\beta})t + \frac{1}{2}(0.25)^2 t^2 (4^{\alpha} + 4^{\beta})^2\right)$$
$$\vdots$$
$$u_n(x, t) = \sin(4x)\sum_{k=0}^{n}\frac{1}{k!}(-0.25(4^{\alpha} + 4^{\beta})t)^k.$$

We have $u = \lim_{k\to\infty} u_k$, and as $\alpha \to 2$, $\beta \to 1$, from (8) and (7) we have

$$u(x, t) = \sin(4x)e^{-5t},$$

which is the exact solution of the corresponding integer order problem. Figs. 2 and 3 show at $t = 0.5$ the effect of changing α and β on the solution for $\beta = 0.7$ and $\alpha = 1.99$, respectively. In both figures, $u_{2,1}$ denotes the exact solution of the corresponding integer order problem.

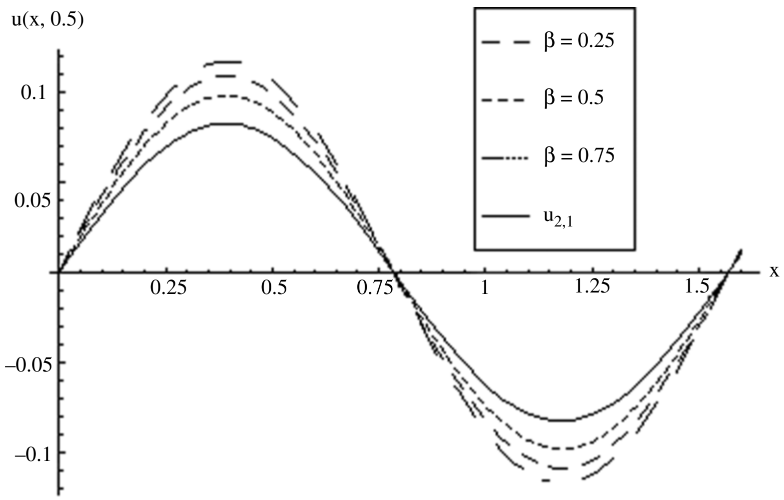

Fig. 3. $u(x, 0.5)$ with the change of β for $\alpha = 1.99$.

Example 3. Consider the space fractional wave equation

$$u_{tt}(x,t) = c^2 R_x^{\alpha} u(x,t), \quad c > 0,\ t > 0,$$

subject to the conditions

$$u(x,0) = \sin\left(\frac{\pi x}{l}\right), \qquad u_t(x,0) = 0$$

$$u(0,t) = 0, \qquad u_x(0,t) = \frac{\pi}{l}\cos\left(\frac{c\pi t}{l}\right).$$

This problem is a generalization of the diffusion equation in [21] obtained by replacing the integer second-order space derivative by the Riesz fractional derivative. The correction functional equation takes the form

$$u_{n+1}(x,t) = u_n(x,t) + \lambda \int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - c^2 R_x^{\alpha} \widetilde{u}_n(x,\xi)\right] d\xi. \tag{27}$$

This yields the stationary conditions

$$\lambda''(\xi) = 0, \qquad 1 - \lambda'(\xi)|_{\xi=t} = 0, \qquad \lambda(\xi)|_{\xi=t} = 0,$$

which in turn give

$$\lambda = \xi - t.$$

The correction functional equation now takes the form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left[\frac{\partial u_n(x,\xi)}{\partial \xi} - c^2 R_x^{\alpha} \widetilde{u}_n(x,\xi)\right] d\xi. \tag{28}$$

Taking the zeroth component as $u_0(x,t) = \sin\left(\frac{\pi x}{l}\right)$, the following successive approximations are obtained:

$$u_0(x,t) = \sin\left(\frac{\pi x}{l}\right)$$

$$u_1(x,t) = \sin\left(\frac{\pi x}{l}\right)\left(1 - \frac{1}{2!}\left(\frac{\pi}{l}\right)^{\alpha} c^2 t^2\right)$$

$$u_2(x,t) = \sin\left(\frac{\pi x}{l}\right)\left(1 - \frac{1}{2!}\left(\frac{\pi}{l}\right)^{\alpha} c^2 t^2 + \frac{1}{4!}\left(\frac{\pi}{l}\right)^{2\alpha} c^4 t^4\right)$$

$$\vdots$$

$$u_n(x,t) = \sin\left(\frac{\pi x}{l}\right)\sum_{k=0}^{n} \frac{(-1)^k}{2k!}\left(\frac{\pi}{l}\right)^{\alpha k} (ct)^{2k}.$$

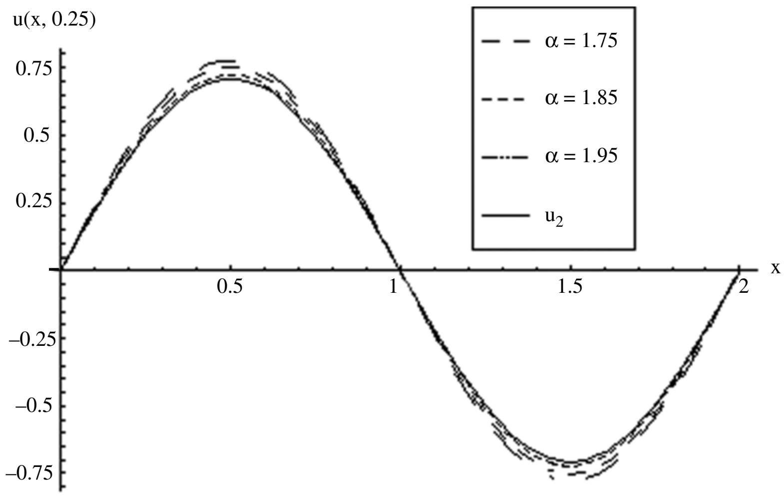

Fig. 4. $u(x, 0.25)$ with the change of α with $c = l = 1$.

We have $u = \lim_{k\to\infty} u_k$, which gives

$$u_\alpha(x,t) = \sin\left(\frac{\pi x}{l}\right)\cos\left(\left(\frac{\pi}{l}\right)^{\alpha/2} ct\right),$$

whereas when $\alpha \to 2$, we have

$$u_2(x,t) = \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right),$$

which is the exact solution of the integer order problem. Fig. 4 show the effect of changing α on the solution at $t = 0.25$ with $c = l = 1$. In figure, u_2 denotes the exact solution of the corresponding integer order problem.

5. Conclusion

An iterative solution to the Riesz fractional partial differential equation is deduced. The scheme is based on the properties of the Riesz fractional derivative. The solution is obtained directly utilizing the variational iteration method without the need for any transforms, discretization of the operator, or numerical approximations. The series solutions obtained coincide with the exact solutions of the problems solved. The basic approach of this solution scheme can also be utilized with other iterative solution techniques.

References

[1] J.H. He, Non-perturbative methods for strongly nonlinear problems, Dissertation, de-Verlag im Internet GmbH, Berlin, 2006.
[2] J.H. He, Y.Q. Wan, Q. Guo, An iteration formulation for normalized diode characteristics, Int. J. Circuit Theory Appl. 32 (2004) 629–632.
[3] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2000) 115–123.
[4] J.H. He, Variational iteration method for delay differential equations, Commun. Nonlinear Sci. Numer. Simul. 2 (1997) 235–236.
[5] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in non-linear mathematical physics, in: Variational Methods in the Mechanics of Solids, Pergamon Press, New York, 1978, pp. 156–162.
[6] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167 (1998) 57–68.
[7] S. Das, Analytical solution of a fractional diffusion equation by variational, iteration method, Comput. Math. Appl. 57 (2009) 483–487.
[8] R. Yulita Molliq, M.S.M. Noorani, I. Hashim, Variational iteration method for fractional heat- and wave-like equations, Nonlinear Anal. RWA 10 (2009) 1854–1869.
[9] M. Inc, The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method, J. Math. Anal. Appl. 345 (2008) 476–484.
[10] S. Momani, Z. Odibat, Numerical approach to differential equations of fractional order, J. Comput. Appl. Math. 207 (2007) 96–110.
[11] G.E. Draganescu, Application of a variational iteration method to linear and nonlinear viscoelastic models with fractional derivatives, J. Math. Phys. 47 (8) (2006) doi:10.1063/1.2234273/082902.
[12] M. Ciesielski, J. Leszczynski, Numerical treatment of an initial–boundary value problem for fractional partial differential equations, Signal Process. 86 (2006) 2619–2631.
[13] J.R. Leith, Fractal scaling of fractional diffusion processes, Signal Process. 83 (2003) 2397–2409.
[14] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, P. Paradisi, Discrete random walk models for space–time fractional diffusion, Chem. Phys. 284 (2002) 521–541.
[15] H. Zhang, F. Liu, The fundamental solutions of the space, space–time Riesz fractional partial differential equations with periodic conditions, Numer. Math. J. Chin. Univ. (Engl. Ser.) 16 (2) (2007) 181–192.
[16] V.E. Tarasov, M. Zaslavsky, Fractional dynamics of systems with long-range interaction, Commun. Nonlinear Sci. Numer. Simul. 11 (2006) 885–898.
[17] R. Gorenflo, A. Vivoli, Fully discrete random walks for space–time fractional diffusion equations, Signal Process. 83 (2003) 2411–2420.

[18] Q. Yang, F. Liu, I. Turner, Numerical methods for fractional partial differential equations with Riesz space fractional derivatives, Appl. Math. Model. 34 (2010) 200–218.
[19] R. Lin, F. Liu, V. Anh, I. Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation, Appl. Math. Comput. 212 (2009) 435–445.
[20] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, New York, 1993.
[21] A.A. Hemeda, Variational iteration method for solving wave equation, Comput. Math. Appl. 56 (2008) 1948–1953.