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The Dirichlet problem for Hessian quotient equations in exterior domains

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ABSTRACT

In this paper, we obtain the uniqueness and existence of viscosity solutions with prescribed asymptotic behavior at infinity to Hessian quotient equations in exterior domains.

1. Introduction

In this paper, we consider the Dirichlet problem for the following Hessian quotient equations in exterior domains

$$S_{k,l}(D^2u) = \frac{S_k(D^2u)}{S_l(D^2u)} = 1 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \tag{1.1}$$

$$u = \varphi \quad \text{on } \partial\Omega. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, $0 \leqslant l < k \leqslant n$, D^2u denotes the Hessian of the function u, and $S_j(D^2u)$ is defined to be the jth elementary symmetric function of the eigenvalues $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of D^2u, i.e.,

$$S_j(D^2u) = \sigma_j\big(\lambda(D^2u)\big) = \sum_{1\leqslant i_1<\cdots<i_j\leqslant n} \lambda_{i_1}\cdots\lambda_{i_j}, \quad j = 1, 2, \dots, n.$$

When $l = 0$, we denote $S_0(D^2u) \equiv 1$.

Eq. (1.1) is an important class of fully nonlinear elliptic equations which is closely related to geometric problem. Some well-known equations can be regarded as its special cases. When $l = 0$, it is a k-Hessian equation. In particular, it is a Poisson equation if $k = 1$, while it is a Monge–Ampère equation if $k = n$. When $k = n = 3$, $l = 1$, i.e., $\det D^2u = \Delta u$, Eq. (1.1) arises from special Lagrangian geometry [8]: if u is a solution of (1.1), the graph of Du over $\mathbb{R}^3$ in $\mathbb{C}^3$ is a special Lagrangian submanifold in $\mathbb{C}^3$, i.e., its mean curvature vanishes everywhere and the complex structure on $\mathbb{C}^3$ sends the tangent space of the graph to the normal space at every point. Therefore Eq. (1.1) has drawn much attention, see [1,3,10,11].

The Dirichlet problem of Monge–Ampère equations in exterior domains in $\mathbb{R}^2$ was studied by Ferrer, Martínez and Milán in [6,7] using complex variable methods and in exterior domains in $\mathbb{R}^n$ with prescribed asymptotic behavior at infinity was investigated by Caffarelli and Li in [2] using Perron's method. Recently, the Dirichlet problem of Hessian equations has been studied by Dai and Bao in [5] using Perron's method. In this paper, we consider the existence of viscosity solutions to Hessian quotient equations using Perron's method.

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To work in the realm of elliptic equations, we have to restrict the class of functions. Let

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ j = 1, 2, \ldots, k\}.$$

A function $u \in C^2(\mathbb{R}^n \setminus \overline{\Omega})$ is called k-convex (uniformly k-convex) if $\lambda \in \overline{\Gamma_k}(\Gamma_k)$, where $\lambda = \lambda(D^2u) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the eigenvalues of the Hessian matrix D^2u.

From [3] and [11], we know that Eq. (1.1) is elliptic and

$$\big(S_{k,l}(D^2u)\big)^{\frac{1}{k-l}} = \left(\frac{S_k(D^2u)}{S_l(D^2u)}\right)^{\frac{1}{k-l}}$$

is a concave function of the second derivatives of u if u is uniformly k-convex. It is natural for the solutions of Eq. (1.1) to be considered in the class of uniformly k-convex functions.

An extensive study of viscosity solutions of second order partial differential equations can be found in [4] and [9].

For the reader's convenience, we recall the definition of viscosity solutions to Eq. (1.1). Let D be an open subset of $\mathbb{R}^n$, and $f \in C^0(D)$ be nonnegative. A function $u \in C^0(D)$ is called a viscosity subsolution to

$$S_{k,l}(D^2u) = f \quad \text{in } D, \tag{1.3}$$

if for any $y \in D$, $\xi \in C^2(D)$ satisfying

$$u(x) \leqslant \xi(x), \quad x \in D \quad \text{and} \quad u(y) = \xi(y),$$

we have

$$S_{k,l}\big(D^2\xi(y)\big) \geqslant f(y).$$

A function $u \in C^0(D)$ is called a viscosity supersolution to (1.3), if for any $y \in D$, any k-convex function $\xi \in C^2(D)$ satisfying

$$u(x) \geqslant \xi(x), \quad x \in D \quad \text{and} \quad u(y) = \xi(y),$$

we have

$$S_{k,l}\big(D^2\xi(y)\big) \leqslant f(y).$$

A function $u \in C^0(D)$ is called a viscosity solution to (1.3), if u is both a viscosity subsolution and a viscosity supersolution to (1.3).

A function $u \in C^0(\overline{D})$ is called a viscosity subsolution (supersolution, solution) to (1.3) and $u = \varphi(x)$ on ∂D if u is a viscosity subsolution (supersolution, solution) to (1.3) and $u \leqslant (\geqslant, =)\, \varphi(x)$ on ∂D.

A function $u \in C^0(\mathbb{R}^n \setminus \overline{\Omega})$ is called k-convex if in the viscosity sense $\sigma_j(\lambda(D^2u)) \geqslant 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$, $j = 1, 2, \ldots, k$.

2. Preliminaries

From [9, Proposition 2.2], we know the supremum of a set of subsolutions is still a subsolution. Moreover, a comparison principle of viscosity solutions to Hessian quotient equations holds [4, Theorem 3.3]. Then we can state the following existence and uniqueness results [9, Proposition 2.3].

Lemma 2.1. *Let B be a ball in $\mathbb{R}^n$ and $f \in C^0(\overline{B})$ be nonnegative. Suppose $\underline{u}, \bar{u} \in C^0(\overline{B})$ are respectively viscosity subsolution and supersolution of*

$$S_{k,l}(D^2u) = f \quad \text{in } B, \tag{2.1}$$

and satisfy $\underline{u}|_{\partial B} = \bar{u}|_{\partial B} = \varphi \in C^0(\partial B)$, then there exists a unique k-convex function $u \in C^0(\overline{B})$ satisfying (2.1) and

$$u = \varphi \quad \text{on } \partial B.$$

Lemma 2.2. *Let B be a ball in $\mathbb{R}^n$ and $f \in C^0(\overline{B})$ be nonnegative. Suppose $\underline{u} \in C^0(\overline{B})$ satisfies in the viscosity sense $S_{k,l}(D^2\underline{u}) \geqslant f$ in B. Then the Dirichlet problem*

$$S_{k,l}(D^2u) = f \quad \text{in } B, \tag{2.2}$$

$$u = \underline{u} \quad \text{on } \partial B \tag{2.3}$$

has a unique k-convex viscosity solution $u \in C^0(\overline{B})$.

Proof. Clearly, $\underline{u}$ is a viscosity subsolution of (2.2), (2.3). From Lemma 2.1, we only need to prove (2.2), (2.3) has a viscosity supersolution $\bar{u} \in C^0(\overline{B})$ satisfying $\bar{u} = \underline{u}$ on ∂B.

Let $v \in C^2(B) \cap C^0(\overline{B})$ satisfy

$$\Delta v = 0 \quad \text{in } B,$$
$$v = \underline{u} \quad \text{on } \partial B.$$

We claim v is a viscosity supersolution of (2.2). Indeed, suppose v is not a viscosity supersolution of (2.2), then there exist $y \in B$ and some k-convex function $\xi \in C^2(B)$ such that

$$v(x) \geqslant \xi(x), \quad x \in B, \qquad v(y) = \xi(y), \tag{2.4}$$

but

$$S_{k,l}\big(D^2\xi(y)\big) > f(y).$$

By the k-convexity of ξ and the Newton–Maclaurin inequality

$$\left(\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}\right)^{\frac{1}{k-l}} \leqslant C\left(\frac{\sigma_r(\lambda)}{\sigma_s(\lambda)}\right)^{\frac{1}{r-s}}, \quad \lambda \in \overline{\Gamma_k}, \qquad C = C(n,k,l,r,s), \quad k \geqslant r,\ l \geqslant s,\ k-l \geqslant r-s,$$

we know

$$\begin{aligned}\Delta\xi(y) = S_1\big(D^2\xi(y)\big) &\geqslant \frac{1}{C}\big(S_{k,l}\big(D^2\xi(y)\big)\big)^{\frac{1}{k-l}} \\ &> \frac{1}{C}\big(f(y)\big)^{\frac{1}{k-l}} \geqslant 0.\end{aligned}$$

But from (2.4), we get

$$\lambda\big(D^2 v(y)\big) \geqslant \lambda\big(D^2\xi(y)\big).$$

Hence

$$\Delta\xi(y) \leqslant \Delta v(y) = 0.$$

This is a contradiction. Lemma 2.2 is proved. □

Lemma 2.3. *Let D be an open set in $\mathbb{R}^n$ and $f \in C^0(\mathbb{R}^n)$ be nonnegative. Assume k-convex functions $v \in C^0(\overline{D})$, $u \in C^0(\mathbb{R}^n)$ satisfy respectively*

$$S_{k,l}\big(D^2 v\big) \geqslant f(x), \quad x \in D,$$
$$S_{k,l}\big(D^2 u\big) \geqslant f(x), \quad x \in \mathbb{R}^n.$$

Moreover,

$$\begin{aligned}&u \leqslant v, \quad x \in \overline{D},\\ &u = v, \quad x \in \partial D.\end{aligned} \tag{2.5}$$

Set

$$w(x) = \begin{cases} v(x), & x \in D,\\ u(x), & x \in \mathbb{R}^n \setminus D.\end{cases}$$

Then $w \in C^0(\mathbb{R}^n)$ is a k-convex function and satisfies in the viscosity sense

$$S_{k,l}\big(D^2 w\big) \geqslant f(x), \quad x \in \mathbb{R}^n.$$

Proof. From the proof of Lemma 2 in [5], we know w is k-convex.

Let $y \in \mathbb{R}^n$, $\xi \in C^2(\mathbb{R}^n)$ satisfying $w(y) = \xi(y)$,

$$w(x) \leqslant \xi(x), \quad x \in \mathbb{R}^n. \tag{2.6}$$

If $y \in D$, we have

$$v(y) = w(y) = \xi(y), \qquad v(x) = w(x) \leqslant \xi(x), \quad x \in D.$$

Therefore

$$S_{k,l}\big(D^2\xi(y)\big) \geqslant f(y).$$

If $y \in \mathbb{R}^n \setminus D$, we have

$$u(y) = w(y) = \xi(y), \qquad u(x) = w(x) \leqslant \xi(x), \quad x \in \mathbb{R}^n \setminus D.$$

By (2.5), (2.6),

$$u(x) \leqslant \xi(x), \quad x \in \mathbb{R}^n.$$

Therefore

$$S_{k,l}\big(D^2\xi(y)\big) \geqslant f(y).$$

This completes the proof of Lemma 2.3. □

The following lemma can be found in [2].

Lemma 2.4. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$, $\partial\Omega \in C^2$, $\varphi \in C^2(\overline{\Omega})$. Then there exists a constant c only dependent on n, φ and Ω such that for any $\xi \in \partial\Omega$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying*

$$\big|\bar{x}(\xi)\big| \leqslant c, w_\xi < \varphi, \quad x \in \overline{\Omega} \setminus \{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}\big(\big|x - \bar{x}(\xi)\big|^2 - \big|\xi - \bar{x}(\xi)\big|^2\big), \quad x \in \mathbb{R}^n.$$

The main result of this paper is the following theorem.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded and strictly convex domain and $0 \in \Omega$, $\varphi \in C^2(\partial\Omega)$. Then there exists a constant c_0 such that for any $c > c_0$ there exists a unique k-convex function $u \in C^0(\overline{\mathbb{R}^n \setminus \Omega})$ satisfying (1.1), (1.2) in the viscosity sense and*

$$\limsup_{|x|\to\infty}\left(|x|^{k-l-2}\left|u(x) - \left[\frac{c_*}{2}|x|^2 + c\right]\right|\right) < \infty, \tag{2.7}$$

where $c_* = (C_n^l/C_n^k)^{\frac{1}{k-l}}$, $k - l \geqslant 3$.

If $l = 0$, Theorem 2.1 corresponds to Theorem 1 in [5]. Thus Theorem 2.1 generalizes Theorem 1 in [5]. And it seems interesting to study the entire solution problem for Hessian quotient equations and the exterior problem for other partial differential equations.

3. Proof of Theorem 2.1

In this section, we prove Theorem 2.1. We divide the proof into six steps.

In the first step, we construct a viscosity subsolution w_a of (1.1).

Let $a > 0$. Set

$$w_a(x) = \min_{\partial\Omega}\varphi - \int_1^{\bar{r}}\big(s^{k-l} + a\big)^{\frac{1}{k-l}}\,ds + \int_1^{|\sqrt{c_*}x|}\big(s^{k-l} + a\big)^{\frac{1}{k-l}}\,ds, \quad x \in \mathbb{R}^n,$$

where $\bar{r} = 2\sqrt{c_*}\operatorname{diam}\Omega$. Then

$$D_{ij}w_a = \big(|y|^{k-l} + a\big)^{\frac{1}{k-l}-1}\left[\left(|y|^{k-l-1} + \frac{a}{|y|}\right)c_*\delta_{ij} - \frac{ac_*^2x_ix_j}{|y|^3}\right], \quad |x| > 0,$$

where $y = \sqrt{c_*}x$. By rotating the coordinates we may set $x = (r, 0, \ldots, 0)'$, $|y| = \sqrt{c_*}r$, therefore

$$D^2w_a = \big(R^{k-l} + a\big)^{\frac{1}{k-l}-1}\begin{pmatrix} R^{k-l-1}c_* & 0 & \cdots & 0 \\ 0 & (R^{k-l-1} + \frac{a}{R})c_* & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (R^{k-l-1} + \frac{a}{R})c_* \end{pmatrix},$$

where $R = |y|$. Consequently $\lambda(D^2 w_a) \in \Gamma_k$ for $0 < |x| < \infty$. Then

$$\begin{aligned} S_{k,l}\big(D^2 w_a\big) &= \frac{S_k(D^2 w_a)}{S_l(D^2 w_a)} \\ &= \frac{(R^{k-l}+a)^{\frac{k}{k-l}-k}\{C_{n-1}^k[(R^{k-l-1}+\frac{a}{R})c_*]^k + R^{k-l-1}c_* C_{n-1}^{k-1}[(R^{k-l-1}+\frac{a}{R})c_*]^{k-1}\}}{(R^{k-l}+a)^{\frac{l}{k-l}-l}\{C_{n-1}^l[(R^{k-l-1}+\frac{a}{R})c_*]^l + R^{k-l-1}c_* C_{n-1}^{l-1}[(R^{k-l-1}+\frac{a}{R})c_*]^{l-1}\}} \\ &= \big(R^{k-l}+a\big)c_*^{k-l}R^{l-k}\frac{C_n^k R^{k-l} + aC_{n-1}^k}{C_n^l R^{k-l} + aC_{n-1}^l} \\ &\geqslant \big(R^{k-l}+a\big)c_*^{k-l}R^{l-k}\frac{C_n^k R^{k-l}}{C_n^l R^{k-l} + aC_n^l} \\ &= c_*^{k-l}\frac{C_n^k}{C_n^l} = 1. \end{aligned} \tag{3.1}$$

Apparently,

$$w_a \leqslant \varphi, \quad x \in \partial\Omega. \tag{3.2}$$

Moreover,

$$\begin{aligned} w_a(x) &= \min_{\partial\Omega}\varphi - \int_1^{\bar r}\big(s^{k-l}+a\big)^{\frac{1}{k-l}}\,ds + \int_1^{|\sqrt{c_*}x|} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds + \int_1^{|\sqrt{c_*}x|} s\,ds \\ &= \frac{c_*}{2}|x|^2 - \int_1^{\bar r} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds - \frac{1}{2}\bar r^2 + \frac{1}{2} + \int_1^{\infty} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds - \frac{1}{2} \\ &\quad + \min_{\partial\Omega}\varphi - \int_{|\sqrt{c_*}x|}^{\infty} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds \\ &= \frac{c_*}{2}|x|^2 + \min_{\partial\Omega}\varphi + \int_{\bar r}^{\infty} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds - \frac{1}{2}\bar r^2 - \int_{|\sqrt{c_*}x|}^{\infty} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds. \end{aligned}$$

Let

$$\mu(a) := \min_{\partial\Omega}\varphi + \int_{\bar r}^{\infty} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds - \frac{1}{2}\bar r^2.$$

Then $\mu(a)$ is increasing for a and

$$w_a(x) = \frac{c_*}{2}|x|^2 + \mu(a) - \int_{|\sqrt{c_*}x|}^{\infty} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds.$$

Therefore

$$\begin{aligned} w_a(x) - \frac{c_*}{2}|x|^2 &= \mu(a) - \int_{|\sqrt{c_*}x|}^{\infty} s\left[\left(1+\frac{a}{s^{k-l}}\right)^{\frac{1}{k-l}} - 1\right]ds \\ &< \mu(a), \quad x \in \mathbb{R}^n. \end{aligned} \tag{3.3}$$

In the second step, we define the Perron solution u_c of (1.1).

By the expression of $w_\xi(x)$ in Lemma 2.4, there exists a constant c_1 such that for any $\xi \in \partial\Omega$,

$$w_\xi(x) \leqslant \frac{c_*}{2}|x|^2 + c_1, \quad x \in \mathbb{R}^n \setminus \Omega, \qquad dist(x, \partial\Omega) \leqslant 1. \tag{3.4}$$

Fix $a_0 > 0$ such that $c_0 := \mu(a_0) \geqslant c_1$. For any $c > c_0$ and for $x \in \mathbb{R}^n \setminus \overline{\Omega}$, let $S_{c,x}$ denote the set of k-convex functions $w \in C^0(\mathbb{R}^n \setminus \Omega)$ satisfying in the viscosity sense,

$$S_{k,l}\big(D^2 w(y)\big) \geqslant 1, \quad y \in \mathbb{R}^n \setminus \overline{\Omega},$$
$$w \leqslant \varphi, \quad y \in \partial\Omega,$$

and for any $y \in \mathbb{R}^n \setminus \Omega$, $|y - x| \leqslant 2\operatorname{diam}\Omega$,

$$w(y) \leqslant \frac{c_*}{2}|y|^2 + c.$$

Then for all $\mu^{-1}(c_0) < a < \mu^{-1}(c)$, by (3.1), (3.2), (3.3), $w_a \in S_{c,x}$. Consequently $S_{c,x} \neq \emptyset$. Define

$$u_c(x) = \sup\big\{w(x):\ w \in S_{c,x}\big\}, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

In the third step, we prove u_c can be extended as a continuous function in $\mathbb{R}^n \setminus \Omega$ and $u_c = \varphi$ on $\partial\Omega$.

By (3.4), for $\bar{\xi} \in \partial\Omega$ and $x \in \mathbb{R}^n \setminus \overline{\Omega}$, x sufficiently close to $\bar{\xi}$, we have $w_{\bar{\xi}} \in S_{c,x}$. Consequently $u_c(x) \geqslant w_{\bar{\xi}}(x)$ for x sufficiently close to $\bar{\xi}$. And thus

$$\liminf_{x \to \bar{\xi}} u_c(x) \geqslant \liminf_{x \to \bar{\xi}} w_{\bar{\xi}}(x) = \varphi(\bar{\xi}).$$

On the other hand,

$$\limsup_{x \to \bar{\xi}} u_c(x) \leqslant \varphi(\bar{\xi}).$$

Indeed, if along a sequence $x_i \to \bar{\xi}$, $\lim_{i\to\infty} u_c(x_i) \geqslant \varphi(\bar{\xi}) + 3\delta$ for some $\delta > 0$. Then by the definition of u_c, there exists $w_i \in S_{c,x_i}$ such that $w_i(x_i) \geqslant \varphi(\bar{\xi}) + 2\delta$ for large i. But $w_i \in C^0(\mathbb{R}^n \setminus \Omega)$, then for any ξ close to $\bar{\xi}$, $w_i(\xi) \leqslant \varphi(\bar{\xi}) + \delta$. This is a contradiction.

In the fourth step, we prove u_c satisfies (1.1).

By the definition of u_c, u_c is a viscosity subsolution of (1.1). We only need to prove u_c is a viscosity supersolution of (1.1).

For any $x \in \mathbb{R}^n \setminus \overline{\Omega}$, fix $0 < \varepsilon < 2\operatorname{diam}\Omega$ such that $B = B_\varepsilon(x) \subset \mathbb{R}^n \setminus \overline{\Omega}$. From Lemma 2.2, the Dirichlet problem

$$S_{k,l}\big(D^2\tilde{u}\big) = 1, \quad y \in B,$$
$$\tilde{u} = u_c, \quad y \in \partial B \tag{3.5}$$

has a unique k-convex viscosity solution $\tilde{u} \in C^0(\overline{B})$. By the comparison principle, $u_c \leqslant \tilde{u}$ in B. Define

$$\tilde{w}(y) = \begin{cases} \tilde{u}(y), & y \in B, \\ u_c(y), & y \in (\mathbb{R}^n \setminus \overline{\Omega}) \setminus B, \end{cases}$$

Then $\tilde{w} \in S_{c,x}$. Indeed, by the definition of u_c,

$$u_c(y) \leqslant \frac{c_*}{2}|y|^2 + c, \quad y \in \overline{B}.$$

Let

$$\tilde{v}(y) = \frac{c_*}{2}|y|^2 + c.$$

Then

$$S_{k,l}\big(D^2\tilde{u}\big) = 1 = S_{k,l}\big(D^2\tilde{v}\big), \quad y \in B,$$
$$\tilde{u} = u_c \leqslant \tilde{v}, \quad y \in \partial B.$$

From the comparison principle, for any $y \in B$, $\tilde{u} \leqslant \tilde{v}$, i.e. $\tilde{u}(y) \leqslant \frac{c_*}{2}|y|^2 + c$. By Lemma 2.3, $S_{k,l}(D^2\tilde{w}) \geqslant 1$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Therefore $\tilde{w} \in S_{c,x}$. And thus by the definition of u_c, $u_c \geqslant \tilde{w}$ in $\mathbb{R}^n \setminus \Omega$ and $u_c \geqslant \tilde{u}$ in B. Hence

$$u_c \equiv \tilde{u}, \quad y \in B. \tag{3.6}$$

But $\tilde{u}$ satisfies (3.5), we have in the viscosity sense,

$$S_{k,l}\big(D^2 u_c\big) = 1, \quad y \in B.$$

As a result, in the viscosity sense,

$$S_{k,l}\big(D^2u_c(x)\big)=1,\quad x\in\mathbb{R}^n\setminus\overline{\Omega}.$$

Because x is arbitrary, we know u_c is a viscosity supersolution of (1.1).

In the fifth step, we prove u_c satisfies (2.7).

By the definition of u_c,

$$u_c(x)\leqslant\frac{c_*}{2}|x|^2+c,\quad x\in\mathbb{R}^n\setminus\overline{\Omega}.$$

Then

$$u_c(x)-\frac{c_*}{2}|x|^2-c\leqslant 0\leqslant\frac{1}{|x|^{k-l-2}},\quad x\in\mathbb{R}^n\setminus\overline{\Omega}. \tag{3.7}$$

On the other hand, from (3.3), as $|x|\to\infty$,

$$w_a(x)=\frac{c_*}{2}|x|^2+\mu(a)-O\big(|x|^{2-k+l}\big).$$

Because $w_a\in S_{c,x}$, then as $|x|\to\infty$,

$$u_c(x)-\frac{c_*}{2}|x|^2-\mu(a)\geqslant -O\big(|x|^{2-k+l}\big).$$

Let $a\to\mu^{-1}(c)$, then

$$u_c(x)-\frac{c_*}{2}|x|^2-c\geqslant -O\big(|x|^{2-k+l}\big). \tag{3.8}$$

And thus from (3.7) and (3.8),

$$\limsup_{|x|\to\infty}\left(|x|^{k-l-2}\left|u_c(x)-\left[\frac{c_*}{2}|x|^2+c\right]\right|\right)<\infty.$$

In the sixth step, we prove the uniqueness.

Suppose u, v satisfy (1.1), (1.2) and (2.7). By the comparison principle of viscosity solutions to Hessian quotient equations, and $\lim_{|x|\to\infty}(u-v)=0$, we know $u\equiv v$ in $\mathbb{R}^n\setminus\Omega$. The proof of Theorem 2.1 is completed.

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