Journal of Computational and Applied Mathematics 230 (2009) 173–186

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The fuzzy core in games with fuzzy coalitions☆

Xiaohui Yu *, Qiang Zhang

School of Management & Economics, Beijing Institute of Technology, Beijing 100081, PR China

ARTICLE INFO

Article history:
Received 24 May 2008
Received in revised form 6 November 2008

Keywords:
Cooperative game
Fuzzy coalition
Core
Shapley value

ABSTRACT

In this paper, the fuzzy core of games with fuzzy coalition is proposed, which can be regarded as the generalization of crisp core. The fuzzy core is based on the assumption that the total worth of a fuzzy coalition will be allocated to the players whose participation rate is larger than zero. The nonempty condition of the fuzzy core is given based on the fuzzy convexity. Three kinds of special fuzzy cores in games with fuzzy coalition are studied, and the explicit fuzzy core represented by the crisp core is also given. Because the fuzzy Shapley value had been proposed as a kind of solution for the fuzzy games, the relationship between fuzzy core and the fuzzy Shapley function is also shown. Surprisingly, the relationship between fuzzy core and the fuzzy Shapley value does coincide, as in the classical case.

1. Introduction

The general question raised by any cooperative game can be described as follows: how should the utility set available to all coalitions be used to determine an outcome from the set of feasible solutions? Most researchers who have investigated the solutions of cooperative games focus their attention on crisp coalition.

There are some situations in which some players do not fully take part in a coalition, but do to a certain extent. A coalition in which some players participate partially can be treated as a so-called fuzzy coalition introduced in [1,2]. Butnariu [3–5] defined a Shapley value and showed the explicit form of the Shapley function on a limited class of fuzzy games. Tsurumi et al. [6] defined new Shapley axioms and a new class of fuzzy games with Choquet integral form. This class of fuzzy games is both monotone nondecreasing and continuous with respect to players' participation. The core for fuzzy games is also studied in [7]. The lexicographical solution for fuzzy games is researched in [8].

The purpose of this paper is to study fuzzy cores for games with fuzzy coalitions. Note that the fuzzy core is different from core for fuzzy games defined by Tijs et al. [7]. The fuzzy core in this paper coincides with the fuzzy imputation defined for any fuzzy coalition. In other words, we consider how to allocate the total worth of a fuzzy coalition to the players whose participation rate is larger than zero.

The paper will be organized as follows: In Section 2, we review some definitions of the crisp cooperative game, such as the core, the Shapley value, and the imputation. Also, three kinds of games with fuzzy coalitions and their fuzzy Shapley values are reviewed. In Section 3, we define the fuzzy core of fuzzy games and study the nonempty condition of the fuzzy core. In Section 4, special attention is paid to three kinds of games with fuzzy coalitions and their relationship with fuzzy Shapley values. We also analyze the relationship between the fuzzy core and the corresponding core of crisp game.

☆ This work was supported by the National Natural Science Foundation of China (No. 70471063, 70771010), was funded by the National Innovation Base in Philosophy and Social Science, titled Management on National Defense Science and Technology and National Economy Mobilization, of the Second Phase of "985 Project" of China (No. 107008200400024), and was supported by Graduate student Scientific and Technological Innovation project of Beijing Institute of Technology (No: GB200818).

* Corresponding author. Tel.: +86 13146060603.
E-mail addresses: yuxiaohui@bit.edu.cn (X. Yu), qiangzhang@bit.edu.cn (Q. Zhang).

0377-0427/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2008.11.004

2. Preliminaries

2.1. Crisp cooperative game and its solutions

We consider cooperative games with the set of players $N = \{1, \dots, n\}$. A crisp coalition S is a subset of N, and the class of all crisp coalitions of S is denoted by $P(S)$. Then a crisp cooperative game is defined by (N, v), in which N is the set of players and the characteristic function $v : P(N) \to \mathbb{R}_+ = \{r \in \mathbb{R} | r \geq 0\}$ satisfies that $v(\varnothing) = 0$.

If S and T are disjoint crisp coalitions, it is clear that they can accomplish at least as much by joining forces as by remaining separate. Hence, we mainly discuss the supperadditive crisp cooperative games in this paper, i.e.

$$v(S \cup T) \geq v(S) + v(T), \quad \forall S, T \in P(N), \quad \text{s.t. } S \cap T = \varnothing,$$

and we denote by $G_o(N)$ all the supperadditve crisp cooperative games.

Also, the convexity and the imputation of crisp games are defined as follows.

Definition 2.1. A crisp cooperative game (N, v) is said to be convex when

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \quad \forall S, T \in P(N).$$

Definition 2.2. An imputation for a crisp cooperative game $v \in G_0(N)$ is a vector $x = (x_1, \dots, x_n)$ satisfying

(1) $\sum_{i \in N} x_i = v(N)$,
(2) $x_i \geq v(\{i\}), \forall i \in N$.

We shall use the notation $E(v)(N)$ for the set of all imputations of the crisp game $v \in G_0(N)$.

We have many methods to obtain imputations for crisp games, such as the core and Shapley value. The core of a game $v \in G_0(N)$ is the convex set

$$C(v)(N) = \left\{ x \in \mathbb{R}^n_+ \,\middle|\, \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in P(N) \right\}, \tag{1}$$

which is the set of all undominated imputations for a game $v \in G_0(N)$. Also, the Shapley value $Sh_i(v)$ of player i with respect to a game $v \in G_0(N)$ is a weighted average value of the marginal contribution $v(S) - v(S \setminus \{i\})$ of player i alone in all combinations, which is defined by

$$Sh_i(v) = \sum_{i \in S \in P(N)} \frac{(n-s)!(s-1)!}{n!} [v(S) - v(S \setminus \{i\})], \tag{2}$$

where n, s are the cardinality of N, S, i.e., $n = |N|, s = |S|$.

Eq. (2) is the unique expression that satisfies three axiomatic characterization of Shapley value [9].

Lemma 2.1. *Let $v \in G_0(N)$ be convex game. Then*

$$Sh(v) = (Sh_1(v), \dots, Sh_n(v)) \in C(v).$$

Note any subset T of N can be seen as the grand coalition relative to the $S \subseteq T$. Hence, we define the T-restricted game of $v \in G_0(N)$ as follows.

Definition 2.3. Let $v \in G_0(N)$,$T \in P(N)$ and $T \neq \varnothing$. Then the T-restricted game of v is a game (T, v^T) where $v^T(S) = v(S)$ for all $S \subseteq T$. The game (T, v^T) will also be denoted by (T, v).

Then the above definitions about crisp games are not only applicable to allocate $v(N)$ but also available to allocate T-restricted game of (N, v). In other words, we can also allocate the worth of every coalition$T \subseteq N$. In this paper, we denote the imputation $\{x\}_{i \in T}$ for game (T, v^T) by n dimensional vector $x = (x_1, \dots, x_n)$ with

$$x_i = 0 \quad \text{if } i \in N \setminus T.$$

In other word, we assume the payoff of ith player in game (T, v^T) is zero if i does not belong to T. For the convenience of depiction, we often denote the crisp core $C(v)(N)$ by $C(v)$ and $C(v^T)(T)$ by $C(v^T)$.

2.2. Game with fuzzy coalitions and fuzzy Shapley value

A fuzzy coalition U is a fuzzy subset of N, which is a vector $\boldsymbol{U} = \{U(1), \ldots, U(n)\}$ with coordinates $U(i)$ contained in the interval [0,1]. The number $U(i)$ describes the membership grade of i in U. For two fuzzy coalitions K and U, $K \subseteq U$means that$K(i) \leq U(i)$, $\forall i \in N$. The class of all fuzzy subsets of U is denoted by $F(U)$. For a fuzzy set U, the α-level set is defined as $[U]_\alpha = \{i \in N | U(i) \geq \alpha\}$ for any $\alpha \in [0, 1]$, and the support set is denoted by $\mathrm{Supp}(U) = \{i \in N | U(i) > 0\}$.

A cooperative game with fuzzy coalition is a pair (N, v) in which the function $v : F(N) \to \mathbb{R}_+$ is such that $v(\varnothing) = 0$.

In this paper, we adopt the usual definition of union and intersection of fuzzy subset given by the maximum and minimum operators, i.e.

$$(K \cup U)(i) = \max\{K(i), U(i)\}, \quad \forall i \in N,$$
$$(K \cap U)(i) = \min\{K(i), U(i)\}, \quad \forall i \in N.$$

Corresponding to the crisp cooperative games, we mainly discuss the supperaddtive games with fuzzy coalition in this paper, i.e.

$$v(K \cup U) \geq v(U) + v(K), \quad \forall U, K \in F(N), \quad \text{s.t. } U \cap K = \varnothing,$$

and we denote by $G_F(N)$ all the supperadditve games with fuzzy coalition.

Following [6], the extended convexity for games with fuzzy coalition is defined as follows.

Definition 2.4. A fuzzy game $v \in G_F(N)$ is said to be convex when

$$v(K \cup U) + v(K \cap U) \geq v(K) + v(U), \quad \forall U, K \in F(N).$$

Now we extend imputation to fuzzy imputation so that it will be available for games with fuzzy coalitions. Preparatory to its definition, let us define $S_U \in F(U)$ by

$$S_U(i) = \begin{cases} U(i), & \text{if } i \in S, \\ 0, & \text{otherwise,} \end{cases}$$

for any $U \in F(N)$ and $S \subseteq N$. We often write i_U instead of $\{i\}_U$, where $i \in N$.

Definition 2.5. A function $x : F(U) \to \mathbb{R}^n_+$ is said to be imputation for a fuzzy game $v \in G_F(N)$ in fuzzy coalition $U \in F(N)$ if

(1) $x_i(U) = 0, \forall i \notin \mathrm{Supp}(U)$,
(2) $\sum_{i \in N} x_i(U) = v(U)$,
(3) $x_i(U) \geq v(i_U), \forall i \in \mathrm{Supp}(U)$,

where $x(U) = (x_1(U), \ldots, x_n(U))$.

We shall use the notation $\widetilde{E}(v)(U)$ for the set of all imputations of the fuzzy game $v \in G_F(N)$ in fuzzy coalition $U \in F(N)$.

Note that the definition above is also applicable to crisp games by restricting the domain. Butnariu [4] and Tsurumi et al. [6] have proposed the imputation too, but the two kinds of definitions are different from Definition 2.5.

In general, it is difficult to identify a characteristic function of a game with fuzzy coalitions in practice. Hence, a fuzzy characteristic function is often constructed on the basis of the characteristic function of the original crisp game when a decision maker tries to incorporate fuzzy coalitions in a model. Extending the crisp game to the game with fuzzy coalition can be represented by a mapping from the characteristic function of the crisp game to that of the game fuzzy coalition, such as the Owen's extension [10], Butnariu's extension [4] and the Tsurumi et al.'s extension [6].

Let $v \in G_0(N)$,$U \in F(N)$, $Q(U) = \{U(i) | U(i) > 0, i \in N\}$, $q(U)$ be the cardinality of $Q(U)$, i.e. $q(U) = |Q(U)|$, and $r_m(U) = \{i | i \in N, U(i) = r_m\}$. The element in $Q(U)$ are written in the increasing order as $r_1 < \cdots < r_{q(U)}$, and let $r_0 = 0$. Then the Owen's extension $ov \in G_F(N)$, Butnariu's extension $bv \in G_F(N)$ and the Tsurumi et al.'s extension $tv \in G_F(N)$ are defined as follows.

$$ov(U) = \sum_{T \subseteq N} \left\{ \prod_{i \in T} U(i) \prod_{i \notin T} (1 - U(i)) \right\} \cdot v(T), \tag{3}$$

$$bv(U) = \sum_{m=1}^{q(U)} v(r_m(U)) \cdot r_m, \tag{4}$$

$$tv(U) = \sum_{m=1}^{q(U)} v([U]_{r_m}) \cdot (r_m - r_{m-1}). \tag{5}$$

Note that the Owen's extension is also called multilinear extension, the games defined by Butnariu in Eq. (4) are also named games with proportional value, and the games defined by Tsurumi et al. in Eq. (5) are also the games with choquet integral form. There is a one-to-one correspondence between a crisp game and a fuzzy game $ov, bv, tv \in G_F(N)$, respectively. We call the crisp game corresponding to fuzzy game $ov, bv, tv \in G_F(N)$ the associated crisp game.

In order to get the imputations for games with fuzzy coalitions, Butnariu and Tsurumi et al. have given the fuzzy Shapley value $f(bv)$ and $f(tv)$ as the solution for $bv \in G_F(N)$ and $tv \in G_F(N)$,

$$f_i(bv)(U) = \begin{cases} sh_i(v^{r_m(U)}) \cdot U(i), & \text{if } i \in r_m(U), r_m \in Q(U), \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

$$f_i(tv)(U) = \sum_{m=1}^{q(U)} sh_i(v^{[U]_{r_m}}) \cdot (r_m - r_{m-1}), \tag{7}$$

where $v^{r_m(U)}$ is $r_m(U)$-restricted game of v, $sh_i(v^{r_m(U)})$ is the crisp Shapley value for player $i \in N$ in game $v^{r_m(U)}$, and $v^{[U]_{r_m}}$ is $[U]_{r_m}$-restricted game game of v, and $sh_i(v^{[U]_{r_m}})$ is the crisp Shapley value for player $i \in N$ in game $v^{[U]_{r_m}}$.

3. The fuzzy core in games with fuzzy coalitions

In this section, we will give another solution for games with fuzzy coalitions, i.e., the fuzzy core. Firstly, we extend the core of crisp game as the imputations for game with fuzzy coalitions.

Definition 3.1. Let $U \in F(N)$. The fuzzy core of a game $v \in G_F(N)$ in fuzzy coalition U is the convex set $\widetilde{C}(v)(U)$, i.e.,

$$\widetilde{C}(v)(U) = \left\{ x \in \mathbb{R}^n_+ \,\middle|\, \sum_{i \in N} x_i = v(U), \sum_{i \in \mathrm{Supp}(S_U)} x_i \geq v(S_U) \text{ for each } S \in P(N) \right\}. \tag{8}$$

We define an excess of a fuzzy coalition as well as that of a crisp coalition. Let $U \in F(N)$, $S \in P(N)$ and $x = (x_1, \ldots, x_n)$ be an imputation for U. Then an excess of the fuzzy coalition S_U with respect to the payoff vector x is denoted by

$$\widetilde{e}(S, x) = v(S_U) - \sum_{i \in \mathrm{Supp}(S_U)} x_i. \tag{9}$$

Thus, the fuzzy core of a game $v \in G_F(N)$ in fuzzy coalition U can also been thought as the set of all imputation x satisfying that all the excess function are not positive, i.e.,

$$\widetilde{C}(v)(U) = \left\{ x \in \widetilde{E}(v)(U) | \widetilde{e}(S, x) \leq 0 \text{ for each } S \in P(N) \right\}. \tag{10}$$

As special cases of cooperative games with fuzzy coalitions, crisp cooperative games have the excess of $S \in P(N)$,

$$e(S, x) = v(S) - \sum_{i \in S} x_i, \tag{11}$$

where $x = (x_1, \ldots, x_n)$ is an imputation of $v \in G_0(N)$.

Hence, the core of a game $v \in G_o(N)$ can also be represented by

$$C(v)(N) = \{ x \in E(v)(N) | e(S, x) \leq 0 \text{ for each } S \in P(N) \}. \tag{12}$$

It is not hard to see that the fuzzy core for fuzzy game $v \in G_F(N)$ in Eq. (10) is generalized form of core for crisp game $v \in G_0(N)$ in Eq. (12). The fuzzy core may be an empty set just as the core for the crisp games. Therefore, it is necessary to find the condition that the fuzzy core is nonempty.

Lemma 3.1. *A game $v \in G_F(N)$ is convex, then for all $i \in N$ and any $U \in F(N)$,*

$$v(S_U \cup i_U) - v(S_U) \leq v(T_U \cup i_U) - v(T_U) \quad \textit{for all } S \subseteq T \subseteq N \setminus \{i\}.$$

Proof. If $S \subseteq T \subseteq N \setminus \{i\}$, then $(S \cup i)_U \cap T_U = S_U$ and $(S \cup i)_U \cup T_U = T_U \cup i_U$. Due to the convexity of $v \in G_F(N)$, we have $v(T_U \cup i_U) + v(S_U) \geq v(S \cup i_U) + v(T_U)$. □

Theorem 3.1. *Let fuzzy game $v \in G_F(N)$ and any fuzzy coalition $U \in F(N)$. If v is convex, then $\widetilde{C}(v)(U) \neq \varnothing$.*

Proof. Let the set of players be $N = \{1, \ldots, n\}$, and π be a permutation of N. Then we denote by

$$P_i^\pi = \{j \in N | \pi(j) < \pi(i)\}$$

the set of members of N which precede i with respect to the order π. Also, we define $x_i^\pi \in \mathbb{R}_+$ by

$$x_i^\pi = v([P_i^\pi \cup \{i\}]_U) - v([P_i^\pi]_U), \quad \forall i \in N. \tag{13}$$

Next, we will prove that$x^\pi = (x_1^\pi, \ldots, x_n^\pi) \in \widetilde{C}(v)(U)$.

Summing up the equalities (13), we obtain $\sum_{i \in N} x_i^\pi = v(N_U) - v(\varnothing_U) = v(U)$.

Then we have to show $\sum_{i \in \mathrm{Supp}(S_U)} x_i^\pi \geq v(S_U)$ for any $S \in P(N)$.

Let $i_1, \ldots, i$ be chosen such that $S = \{i_1, \ldots, i_s\}$ and $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_s)$, where $s = |S|$. Hence, $\{i_1, \ldots, i_{j-1}\} \subseteq P_{i_j}^\pi$ for every $j = 1, \ldots, s$. Thus by Lemma 3.1,

$$v([P_{i_j}^\pi \cup \{i_j\}]_U) - v([P_{i_j}^\pi]_U) \geq v(\{i_1, \ldots, i_j\}_U) - v(\{i_1, \ldots, i_{j-1}\}_U), \tag{14}$$

for $j = 1, \ldots, s$. Summing up the inequalities (14), we get that $\sum_{i \in \mathrm{Supp}(S_U)} x_i^\pi \geq v(S_U)$.

The proof is completed. □

Example 3.1. Let $N = \{1, 2\}$, v be a characteristic function on N as follows,
$v(\varnothing) = 0$, $v(\{1\}) = 1$, $v(\{2\}) = 2$, $v(\{1, 2\}) = 4$. Then Owen 's multilinear extension is

$$ov(U) = U(1) \cdot (1 - U(2)) + 2U(2) \cdot (1 - U(1)) + 4U(1) \cdot U(2),$$

equivalently,

$$ov(U) = U(1) + 2U(2) + U(1) \cdot U(2).$$

Obviously, this game $ov \in G_F(N)$ is convex, so the fuzzy core of the game $ov \in G_F(N)$ is

$$\widetilde{C}_o(v)(U) = \left\{y \in \mathbb{R}_+^2 | y_1 + y_2 = U(1) + 2U(2) + U(1) \cdot U(2), y_1 \geq U(1), y_2 \geq 2U(2)\right\}.$$

(1) If $U(1) = U(2) = 1$, then
$$\widetilde{C}_o(v)(U) = C(v) = \left\{y \in \mathbb{R}_+^2 | y_1 + y_2 = 4, y_1 \geq 1, y_2 \geq 2\right\}.$$
(2) If $U(1) = 0.2$ and $U(2) = 0.3$, then
$$\widetilde{C}_o(v)(U) = \left\{y \in \mathbb{R}_+^2 | y_1 + y_2 = 0.86, y_1 \geq 0.2, y_2 \geq 0.6\right\}.$$

Although the fuzzy core has been defined above, it is not easy to get the fuzzy core by the expression of $\widetilde{C}(v)(U)$. However, it is not hard to find the core $C(v)$ for the associated crisp game $v \in G_0(N)$ because there are many methods can be used. Hence, it forces us to build the relationship between the fuzzy core and crisp core from the advantage that several solution concepts in crisp games can be used without modification. In the next section, we will take research on the fuzzy cores for the game $ov, bv, tv \in G_F(N)$, respectively.

4. The relation between the fuzzy core and the fuzzy Shapley value

In this section the fuzzy cores for three main kinds of games with fuzzy coalitions $ov, bv, tv \in G_F(N)$ are given. Also, the relationship between fuzzy core and fuzzy Shapley value will be studied.

4.1. The fuzzy core for the fuzzy games $ov \in G_F(N)$ defined by Owen

Let $ov \in G_F(N)$. The excess $\widetilde{e}_o(S, x)$ of the fuzzy coalition S_U with respect to the payoff vector x in $ov \in G_F(N)$ is

$$\widetilde{e}_o(S, x) = ov(S_U) - \sum_{i \in \mathrm{Supp}(S_U)} x_i = \sum_{T \subseteq N} \left\{\prod_{i \in T} S_U(i) \cdot \prod_{i \notin T} (1 - S_U(i))\right\} \cdot v(T) - \sum_{i \in \mathrm{Supp}(S_U)} x_i$$

then the fuzzy core of $ov \in G_F(N)$ is represented as

$$\widetilde{C}_o(v)(U) = \left\{x \in \mathbb{R}_+^n \left| \sum_{i \in N} x_i = \sum_{T \subseteq N} \left\{\prod_{i \in T} U(i) \cdot \prod_{i \notin T} (1 - U(i))\right\} \cdot v(T), \widetilde{e}_o(S, x) \leq 0 \text{ for } \forall S \in P(N)\right.\right\}.$$

Lemma 4.1. *Let $ov \in G_F(N)$, $U \in F(N)$. Then for any $S \subseteq N$,*

$$ov(S_U) = \sum_{T\subseteq N}\left\{\prod_{i\in T} S_U(i)\cdot\prod_{i\notin T}(1-S_U(i))\right\}\cdot v(T) = \sum_{T\subseteq \mathrm{Supp}(S_U)}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\in \mathrm{Supp}(S_U)\setminus T}(1-U(i))\right\}\cdot v(T).$$

Proof. Let $S, T \subseteq N$. If $i \in T \setminus \mathrm{Supp}(S_U)$, then $S_U(i) = 0$; if $i \in \mathrm{Supp}(S_U)$, then $S_U(i) = U(i)$; if $i \notin \mathrm{Supp}(S_U)$, then $S_U(i) = 0$. Hence, we get

$$\begin{aligned}&\sum_{T\in P(N)\setminus P(\mathrm{Supp}(S_U))}\left\{\prod_{i\in T} S_U(i)\cdot\prod_{i\notin T}(1-S_U(i))\right\}\cdot v(T)\\ &\quad=\sum_{T\in P(N)\setminus P(\mathrm{Supp}(S_U))}\left\{\prod_{i\in T\cap\mathrm{Supp}(S_U)} S_U(i)\cdot\prod_{i\in T\setminus\mathrm{Supp}(S_U)} S_U(i)\cdot\prod_{i\notin T}(1-S_U(i))\right\}\cdot v(T)=0.\end{aligned}$$

Consequently, the following holds

$$\begin{aligned}ov(S_U) &= \sum_{T\subseteq N}\left\{\prod_{i\in T} S_U(i)\cdot\prod_{i\notin T}(1-S_U(i))\right\}\cdot v(T)\\ &= \sum_{T\in P(\mathrm{Supp}(S_U))}\left\{\prod_{i\in T} S_U(i)\cdot\prod_{i\notin T}(1-S_U(i))\right\}\cdot v(T) + \sum_{T\in P(N)\setminus P(\mathrm{Supp}(S_U))}\left\{\prod_{i\in T} S_U(i)\cdot\prod_{i\notin T}(1-S_U(i))\right\}\cdot v(T)\\ &= \sum_{T\in P(\mathrm{Supp}(S_U))}\left\{\prod_{i\in T} S_U(i)\cdot\prod_{i\notin \mathrm{Supp}(S_U)}(1-S_U(i))\cdot\prod_{i\in\mathrm{Supp}(S_U)\setminus T}(1-S_U(i))\right\}\cdot v(T)\\ &= \sum_{T\subseteq\mathrm{Supp}(S_U)}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\in\mathrm{Supp}(S_U)\setminus T}(1-U(i))\right\}\cdot v(T).\quad\square\end{aligned}$$

Proposition 4.1. *Let $v \in G_0(N)$ be the associated crisp game of $ov \in G_F(N)$. If all the T-restricted games of $v \in G_0(N)$ are convex, then $\widetilde{C}_o(v)(U) \neq \varnothing$ and*

$$\widetilde{C}_o(v)(U) = \left\{y \,\middle|\, y = \sum_{T\subseteq N}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x^T\right\}, \forall x^T = (x_1^T, x_2^T, \ldots, x_n^T)\in C(v^T), \forall T\subseteq N\right\},$$

where $U \in F(N)$.

Proof. Let $C(v^T) \neq \varnothing$ for $\forall T \subseteq N$. Given any $U \in F(N)$ and any $x^T \in C(v^T)$, let

$$y \triangleq \sum_{T\subseteq N}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x^T\right\}.$$

Firstly, we shall show $y \in \widetilde{C}_o(v)(U)$.
Due to $x^T = (x_1^T, x_2^T, \ldots, x_n^T) \in C(v^T)$, we get

$$\begin{aligned}\sum_{j\in N} y_j &= \sum_{j\in N}\sum_{T\subseteq N}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x_j^T\right\} = \sum_{T\subseteq N}\sum_{j\in N}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x_j^T\right\}\\ &= \sum_{T\subseteq N}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\sum_{j\in N} x_j^T\right\} = \sum_{T\subseteq N}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot v(T)\right\} = ov(U).\end{aligned}$$

The following always holds,

$$\begin{aligned}\sum_{j\in\mathrm{Supp}(S_U)} y_j &= \sum_{j\in\mathrm{Supp}(S_U)}\sum_{T\subseteq N}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x_j^T\right\}\\ &= \sum_{j\in\mathrm{Supp}(S_U)}\left\{\sum_{T\subseteq N: T\cap\mathrm{Supp}(S_U)=\varnothing}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x_j^T\right\}\right.\end{aligned}$$

$$
\begin{aligned}
&+\sum_{T\subseteq N:T\cap \mathrm{Supp}(S_U)\neq\varnothing}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x_j^T\right\}\Bigg\}\\
&=\sum_{j\in \mathrm{Supp}(S_U)}\ \sum_{T\subseteq N:T\cap \mathrm{Supp}(S_U)\neq\varnothing}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\right\}\cdot x_j^T\\
&=\sum_{T\subseteq \mathrm{Supp}(S_U)}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\sum_{j\in \mathrm{Supp}(S_U)}x_j^T\right\}\\
&\quad+\sum_{T\subseteq N:T\not\subseteq \mathrm{Supp}(S_U),T\cap \mathrm{Supp}(S_U)\neq\varnothing}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\sum_{j\in \mathrm{Supp}(S_U)}x_j^T\right\}\\
&=\sum_{T\subseteq \mathrm{Supp}(S_U)}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot v(T)\right\}\\
&\quad+\sum_{T\subseteq N:T\not\subseteq \mathrm{Supp}(S_U),T\cap \mathrm{Supp}(S_U)\neq\varnothing}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\sum_{j\in \mathrm{Supp}(S_U)}x_j^T\right\}.
\end{aligned}
$$

Given any $T\not\subseteq \mathrm{Supp}(S_U)$, let $T_1=T\cap \mathrm{Supp}(S_U)$ and $T_2=T\setminus T_1$. Then $T=T_1\cup T_2$, $T_1\subseteq \mathrm{Supp}(S_U)$, $T_2\subseteq N\setminus \mathrm{Supp}(S_U)$ and $T_1\cap T_2=\varnothing$. Thus,

$$
\begin{aligned}
&\sum_{T\subseteq N:T\not\subseteq \mathrm{Supp}(S_U),T\cap \mathrm{Supp}(S_U)\neq\varnothing}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\sum_{j\in \mathrm{Supp}(S_U)}x_j^T\right\}\\
&\geq\sum_{T_1\subseteq \mathrm{Supp}(S_U)}\ \sum_{T_2\subseteq N\setminus \mathrm{Supp}(S_U):T_2\neq\varnothing}\left\{\prod_{i\in T_1}U(i)\cdot\prod_{i\in \mathrm{Supp}(S_U)\setminus T_1}(1-U(i))\cdot\prod_{i\in T_2}U(i)\cdot\prod_{i\in N\setminus(T_2\cup \mathrm{Supp}(S_U))}(1-U(i))\cdot v(T_1)\right\}\\
&=\sum_{T_1\subseteq \mathrm{Supp}(S_U)}\left\{\prod_{i\in T_1}U(i)\cdot\prod_{i\in \mathrm{Supp}(S_U)\setminus T_1}(1-U(i))\cdot v(T_1)\sum_{T_2\subseteq N\setminus \mathrm{Supp}(S_U):T_2\neq\varnothing}\left\{\prod_{i\in T_2}U(i)\cdot\prod_{i\in N\setminus(T_2\cup \mathrm{Supp}(S_U))}(1-U(i))\right\}\right\}.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
\sum_{j\in \mathrm{Supp}(S_U)}y_j&=\sum_{T\subseteq \mathrm{Supp}(S_U)}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot v(T)\right\}\\
&\quad+\sum_{T\subseteq N:T\not\subseteq \mathrm{Supp}(S_U),T\cap \mathrm{Supp}(S_U)\neq\varnothing}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\sum_{j\in \mathrm{Supp}(S_U)}x_j^T\right\}\\
&\geq\sum_{T_1\subseteq \mathrm{Supp}(S_U)}\left\{\prod_{i\in T_1}U(i)\cdot\prod_{i\in \mathrm{Supp}(S_U)\setminus T_1}(1-U(i))\cdot v(T_1)\right.\\
&\quad\times\left.\left(\prod_{i\in N\setminus \mathrm{Supp}(S_U)}(1-U(i))+\sum_{T_2\subseteq N\setminus \mathrm{Supp}(S_U):T_2\neq\varnothing}\ \prod_{i\in T_2}U(i)\cdot\prod_{i\in N\setminus(T_2\cup \mathrm{Supp}(S_U))}(1-U(i))\right)\right\}\\
&=\sum_{T_1\subseteq \mathrm{Supp}(S_U)}\left\{\prod_{i\in T_1}U(i)\cdot\prod_{i\in \mathrm{Supp}(S_U)\setminus T_1}(1-U(i))\cdot v(T_1)\right\},
\end{aligned}
$$

where in the last line we have used that

$$
\begin{aligned}
&\prod_{i\in N\setminus \mathrm{Supp}(S_U)}(1-U(i))+\sum_{T_2\subseteq N\setminus \mathrm{Supp}(S_U):T_2\neq\varnothing}\ \prod_{i\in T_2}U(i)\cdot\prod_{i\in N\setminus(T_2\cup \mathrm{Supp}(S_U))}(1-U(i))\\
&=\sum_{T_2\subseteq N\setminus \mathrm{Supp}(S_U)}\ \prod_{i\in T_2}U(i)\cdot\prod_{i\in N\setminus(T_2\cup \mathrm{Supp}(S_U))}(1-U(i))=1.
\end{aligned}
$$

By Lemma 4.1, we can get the conclusion that

$$
\sum_{j\in \mathrm{Supp}(S_U)}y_j\geq\sum_{T_1\subseteq \mathrm{Supp}(S_U)}\left\{\prod_{i\in T_1}U(i)\cdot\prod_{i\in \mathrm{Supp}(S_U)\setminus T_1}(1-U(i))\cdot v(T_1)\right\}=ov(S_U).
$$

Hence, for $\forall U \subseteq F(N)$, $\widetilde{C}_o(v)(U) \neq \varnothing$.

Next, we show that any $z \in \widetilde{C}_o(v)(U)$ can be written by

$$z = \sum_{T\subseteq N}\left\{\prod_{i\in T} U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x^T\right\}. \tag{15}$$

Let $\underline{x}_j^T = \min\left\{x_j^T|\forall x^T \in C(v^T)\right\}$ and $\overline{x}_j^T = \max\left\{x_j^T|\forall x^T \in C(v^T)\right\}$ for any $T \subseteq N$. Obviously,

$$\underline{x}_j^T = \begin{cases} v(\{j\}), & \text{if } j \in T,\\ 0, & \text{otherwise,}\end{cases} \qquad \overline{x}_j^T = \begin{cases} v(T) - v(T\setminus\{j\}), & \text{if } j \in T,\\ 0, & \text{otherwise.}\end{cases}$$

If there exists $S \subseteq N$ such that z_j can not be written by Eq. (15) for any $j \in S$, then there are only two cases for z_j,

(i) $z_j < \sum_{T\subseteq N}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\underline{x}_j^T$;
(ii) $z_j > \sum_{T\subseteq N}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\overline{x}_j^T$.

Case (i): Because $z \in \widetilde{C}_o(v)(U)$, it follows that

$$\begin{aligned} z_j &< \sum_{T\subseteq N}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\underline{x}_j^T\\ &= \sum_{T\subseteq N: j\in T}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\underline{x}_j^T + \sum_{T\subseteq N: j\notin T}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\underline{x}_j^T\\ &= \sum_{T\subseteq N: j\in T}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot v(\{j\})\\ &= U(j)\cdot v(\{j\})\cdot\sum_{T\subseteq N\setminus\{j\}}\prod_{i\in T}U(i)\cdot\prod_{i\in N\setminus(T\cup\{j\})}(1-U(i))\\ &= U(j)\cdot v(\{j\}).\end{aligned}$$

By Lemma 4.1, we have

$$ov(j_U) = \sum_{T\subseteq \mathrm{Supp}(j_U)}\left\{\prod_{i\in T}U(i)\cdot\prod_{i\in \mathrm{Supp}(j_U)\setminus T}(1-U(i))\right\}\cdot v(T) = \prod_{i\in\{j\}} j_U(i)\cdot\prod_{i\notin\{j\}}(1-j_U(i))\cdot v(\{j\}) = U(j)\cdot v(\{j\}).$$

Then

$$z_j < ov(j_U),$$

which contradicts with $z \in \widetilde{C}_o(v)(U)$.

Case (ii): If $z_j > \sum_{T\subseteq N}\prod_{i\in T}U(i)\prod_{i\notin T}(1-U(i))\overline{x}_j^T$ for any $j \in S$, then there must exist nonempty $S' \subseteq N$ such that $z_j < \sum_{T\subseteq N}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\underline{x}_j^T$ for any $j \in S'$. In fact, if $S' = \varnothing$, then any $j \in N\setminus S$ can be represented by $x^T = (x_1^T, x_2^T, \ldots, x_n^T) \in C(v^T)$, i.e.,

$$\begin{aligned}\sum_{j\in N} z_j &> \sum_{j\in S}\sum_{T\subseteq N}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\overline{x}_j^T + \sum_{j\in N\setminus S}\sum_{T\subseteq N}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x_j^T\\ &\geq \sum_{j\in N}\sum_{T\subseteq N}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot x_j^T = ov(U),\end{aligned}$$

which contradicts with $z \in \widetilde{C}_o(v)(U)$. Thus, $S' \neq \varnothing$. By the proof of Case (i), we know that $z_j < \sum_{T\subseteq N}\prod_{i\in T}U(i)\cdot\prod_{i\notin T}(1-U(i))\cdot\underline{x}_j^T$ can not hold true, either.

Hence, we get the conclusion that any $z \in \widetilde{C}_o(v)(U)$ can be written by

$$z = \sum_{T\subseteq N}\left\{\prod_{i\in T}U(i)\prod_{i\notin T}(1-U(i))\cdot x^T\right\}.$$

The proof is completed. □

Example 4.1 (*Cf.* Example 3.1). Let $N = \{1, 2\}$, v be a characteristic function on N as defined in Example 3.1, i.e.,

$$v(\varnothing) = 0, \qquad v(\{1\}) = 1, \qquad v(\{2\}) = 2, \qquad v(\{1,2\}) = 4.$$

It is not hard to get the core for the crisp game as follows,

$$x^{\{1\}} \in C(v^{\{1\}}) = \{(1,0)\}, \qquad x^{\{2\}} \in C(v^{\{2\}}) = \{(0,2)\},$$
$$x^{\{1,2\}} \in C(v) = \left\{x^{\{1,2\}} \in \mathbb{R}^2_+ | x_1^{\{1,2\}} + x_2^{\{1,2\}} = 4, x_1^{\{1,2\}} \geq 1, x_2^{\{1,2\}} \geq 2\right\}.$$

Letting

$$y = U(1) \cdot (1 - U(2))x^{\{1\}} + U(2) \cdot (1 - U(1))x^{\{2\}} + U(1) \cdot U(2)x^{\{1,2\}},$$

i.e.,

$$y = (U(1) + U(1) \cdot U(2)(x_1^{\{1,2\}} - 1), 2U(2) + U(1) \cdot U(2)(x_2^{\{1,2\}} - 2)).$$

Hence, the fuzzy core this game $ov \in G_F(N)$ is

$$\widetilde{C}_o(v)(U) = \left\{y \in \mathbb{R}^2_+ | y_1 + y_2 = U(1) + 2U(2) + U(1) \cdot U(2), y_1 \geq U(1), y_2 \geq 2U(2)\right\}.$$

Obviously, the fuzzy core coincides with the result in Example 3.1.

4.2. The fuzzy core for the fuzzy games $bv \in G_F(N)$ by Butnariu

Let $bv \in G_F(N)$, $U \in F(N)$ and $Q(U) = \{U(i) | U(i) > 0, i \in N\}$, $q(U) = |Q(U)|$. The element in $Q(U)$ are written in the increasing order as $r_1 < \cdots < r_{q(U)}$.The excess $\widetilde{e}_b(S, x)$ of the fuzzy coalition S_U with respect to the payoff vector x in $bv \in G_F(N)$ is

$$\widetilde{e}_b(S, x) = bv(S_U) - \sum_{i \in \mathrm{Supp}(S_U)} x_i = \sum_{m=1}^{q(U)} v(S_{r_m(U)}) \cdot r_m - \sum_{i \in \mathrm{Supp}(S_U)} x_i,$$

then the fuzzy core of $bv \in G_F(N)$ is represented by

$$\widetilde{C}_b(v)(U) = \left\{x \in \mathbb{R}^n_+ \left| \sum_{i \in N} x_i = \sum_{m=1}^{q(U)} v(r_m(U)) \cdot r_m, \sum_{i \in \mathrm{Supp}(S_U)} x_i \geq \sum_{m=1}^{q(U)} v(S_{r_m(U)}) \cdot r_m \text{for } \forall S \in P(N) \right.\right\}.$$

Theorem 4.1. *Let $v \in G_0(N)$ be the associated crisp game of $bv \in G_F(N)$. Given any $U \in F(N)$, if $C(v^{r_m(U)}) \neq \varnothing$, $m = 1, 2, \ldots, q(U)$, then $\widetilde{C}_b(v)(U) \neq \varnothing$ and*

$$\widetilde{C}_b(v)(U) = \left\{ y \left| \begin{array}{l} y = \sum_{m=1}^{q(U)} r_m \cdot x^m = \left(\sum_{m=1}^{q(U)} r_m x_1^m, \sum_{m=1}^{q(U)} r_m x_2^m, \ldots, \sum_{m=1}^{q(U)} r_m x_n^m\right), \\ \forall x^m = (x_1^m, x_2^m, \ldots, x_n^m) \in C(v^{r_m(U)}), \quad m = 1, 2, \ldots, q(U) \end{array} \right.\right\} \tag{16}$$

where $r_m(U) = \{i | i \in N, U(i) = r_m\}$.

Proof. Let $C(v^{r_m(U)}) \neq \varnothing$ for $m = 1, 2, \ldots, q(U)$. Given any $U \in F(N)$ and any $x^m \in C(v^{r_m(U)})$, let $y \triangleq \sum_{m=1}^{q(U)} r_m \cdot x^m$. Firstly, we show $y \in \widetilde{C}_b(v)(U)$.

Due to $\sum_{i \in N} x_i^m = v(r_m(U))$, we have

$$\sum_{i \in N} y_i = \sum_{i \in N} \sum_{m=1}^{q(U)} r_m \cdot x_i^m = \sum_{m=1}^{q(U)} \left(r_m \cdot \sum_{i \in N} x_i^m\right) = \sum_{m=1}^{q(U)} (r_m \cdot v(r_m(U))) = bv(U).$$

If $r_1 < \cdots < r_{q(U)}$, then $\cup_{m=1}^{q(U)} r_m(U) = \mathrm{Supp}(U)$ and $r_i(U) \cap r_j(U) = \varnothing$ when $r_i \neq r_j$. In other words, $\{r_1(U), \ldots, r_{q(U)}(U)\}$ is a partition of $\mathrm{Supp}(U)$. Due to $x^m \in C(v^{r_m(U)})$, it follows that $x_i^m = 0$ for $\forall i \notin r_m(U)$. Hence, $\sum_{i \in \mathrm{Supp}(S_U)} x_i^m = \sum_{i \in S_{r_m(U)}} x_i^m$ holds. Further, we obtain

$$\sum_{i \in \mathrm{Supp}(S_U)} y_i = \sum_{i \in \mathrm{Supp}(S_U)} \sum_{m=1}^{q(U)} r_m \cdot x_i^m = \sum_{m=1}^{q(U)} \left(r_m \cdot \sum_{i \in \mathrm{Supp}(S_U)} x_i^m\right) = \sum_{m=1}^{q(U)} \left(r_m \cdot \sum_{i \in S_{r_m(U)}} x_i^m\right) \geq \sum_{m=1}^{q(U)} \left(r_m \cdot v(S_{r_m(U)})\right).$$

Hence, we obtain $y = \sum_{m=1}^{q(U)} r_m \cdot x^m \in \widetilde{C}_b(v)(U)$ and $\widetilde{C}_b(v)(U) \neq \varnothing$.

Given any $y \in \widetilde{C}_b(v)(U)$, let

$$x_i^m = \begin{cases} \dfrac{y_i}{r_m}, & \text{if } i \in r_m(U), \\ 0, & \text{otherwise,} \end{cases}$$

$m = 1, \dots, q(U)$. Next we will prove that $x^m = (x_1^m, x_2^m, \dots, x_n^m) \in C(v^{r_m(U)})$. By the definition of fuzzy core, we know

$$\sum_{i \in \mathrm{Supp}([r_m(U)]_U)} y_i \ge bv([r_m(U)]_U) = r_m \cdot v(r_m(U)) \quad \text{and}$$

$$\sum_{m=1}^{q(U)} \sum_{i \in \mathrm{Supp}([r_m(U)]_U)} y_i = \sum_{i=1}^{n} y_i = \sum_{m=1}^{q(U)} r_m \cdot v(r_m(U)).$$

Hence, we have that

$$\sum_{i \in \mathrm{Supp}([r_m(U)]_U)} y_i = r_m \cdot v(r_m(U)),$$

i.e.

$$\sum_{i \in \mathrm{Supp}([r_m(U)]_U)} \frac{y_i}{r_m} = \sum_{i \in r_m(U)} \frac{y_i}{r_m} = v(r_m(U)),$$

which means $\sum_{i \in r_m(U)} x_i^m = v(r_m(U))$.

On the other hand, $\sum_{i \in S_{r_m(U)}} y_i \ge r_m \cdot v(S_{r_m(U)})$ holds, which means

$$\sum_{i \in S_{r_m(U)}} x_i^m \ge v(S_{r_m(U)}).$$

So $x^m \in C(v^{r_m(U)})$. The proof is completed. □

It is apparent that $bv \in G_F(N)$ is convex if its associated crisp game $v \in G_o(N)$ is convex. Thus, if the crisp game $v \in G_o(N)$ is convex, then the fuzzy core of the game $bv \in G_F(N)$ defined by Eq. (4) is nonempty.

Theorem 4.2. *Let $v \in G_0(N)$ be convex game $bv \in G_F(N)$ be the game defined by Eq. (4). Then*

$$\{f_i(bv)(U)\}_{i \in N} \in \widetilde{C}_b(v)(U).$$

Proof. Because fuzzy Shapley is an imputation for game $bv \in G_F(N)$, it has been proof in Ref. [4] that $\sum_{i \in N} f_i(bv)(U) = bv(U)$. Thus, we have to prove that

$$\sum_{i \in \mathrm{Supp}(S_U)} f_i(bv)(U) \ge \sum_{m=1}^{q(U)} v(S_{r_m(U)}) \cdot r_m.$$

By Eq. (6),

$$\sum_{i \in \mathrm{Supp}(S_U)} f_i(bv)(U) = \sum_{m=1}^{q(U)} \sum_{i \in S_{r_m(U)}} f_i(bv)(U) = \sum_{m=1}^{q(U)} \sum_{i \in S_{r_m(U)}} sh_i(v^{r_m(U)}) \cdot r_m$$
$$\ge \sum_{m=1}^{q(U)} v^{r_m(U)}(S_{r_m(U)}) \cdot r_m = \sum_{m=1}^{q(U)} v(S_{r_m(U)}) \cdot r_m,$$

where in the last two lines we use the Lemma 2.1. The proof is completed. □

Example 4.2 (*Cf.* Example 3.1). Let $N = \{1, 2\}$, v be a characteristic function on N as defined in Example 3.1.

Then Butnariu 's proportional extension is

$$bv(U) = \begin{cases} 4U(1), & \text{if } U(1) = U(2), \\ U(1) + 2U(2), & \text{otherwise.} \end{cases}$$

In Example 3.1, we have known that

$$C(v) = \left\{ x^{\{1,2\}} \in \mathbb{R}_+^2 | x_1^{\{1,2\}} + x_2^{\{1,2\}} = 4, x_1^{\{1,2\}} \ge 1, x_2^{\{1,2\}} \ge 2 \right\}.$$

(1) If $U(1) = U(2)$, then the fuzzy core of this game $bv \in G_F(N)$ is

$$\widetilde{C}_b(v)(U) = \left\{ (U(1) \cdot x_1^{\{1,2\}}, U(1) \cdot x_2^{\{1,2\}}) | x_1^{\{1,2\}} + x_2^{\{1,2\}} = 4, x_1^{\{1,2\}} \ge 1, x_2^{\{1,2\}} \ge 2 \right\}.$$

The fuzzy Shapley value is

$$f_1(bv)(U) = sh_1(v) \cdot U(1) = \left[\frac{(4-2)}{2} + \frac{1}{2}\right] \cdot U(1) = \frac{3 \cdot U(1)}{2},$$
$$f_2(bv)(U) = sh_2(v) \cdot U(1) = \left[\frac{(4-1)}{2} + \frac{2}{2}\right] \cdot U(1) = \frac{5 \cdot U(1)}{2}.$$

Obviously, $(\frac{3 \cdot U(1)}{2}, \frac{5 \cdot U(1)}{2}) \in \widetilde{C}_b(v)(U)$.

(2) If $U(1) \neq U(2)$, then

$$C(v^{\{1\}}) = \{(1,0)\}, \qquad C(v^{\{2\}}) = \{(0,2)\}, \qquad \widetilde{C}_b(v)(U) = \{(U(1), 2U(2))\}.$$

And the fuzzy Shapley value $(U(1), 2U(2)) \in \widetilde{C}_b(v)(U)$.

4.3. The fuzzy core for the fuzzy games $tv \in G_F(N)$ by Tsurumi

Let $tv \in G_F(N)$. The excess $\widetilde{e}_t(S, x)$ of the fuzzy coalition S_U with respect to the payoff vector x in $tv \in G_F(N)$ is

$$\widetilde{e}_t(S, x) = tv(S_U) - \sum_{i \in \mathrm{Supp}(S_U)} x_i = \sum_{m=1}^{q(U)} v(S_{[U]_{r_m}}) \cdot (r_m - r_{m-1}) - \sum_{i \in \mathrm{Supp}(S_U)} x_i,$$

then the fuzzy core of $tv \in G_F(N)$ is represented as

$$\widetilde{C}_t(v)(U) = \left\{ x \in \mathbb{R}^n_+ \,\middle|\, \sum_{i \in N} x_i = \sum_{m=1}^{q(U)} v([U]_{r_m}) \cdot (r_m - r_{m-1}), \sum_{i \in \mathrm{Supp}(S_U)} x_i \geq \sum_{m=1}^{q(U)} v(S_{[U]_{r_m}}) \cdot (r_m - r_{m-1}) \text{ for } \forall S \in P(N) \right\}.$$

Theorem 4.3. *Let $v \in G_0(N)$ be the associated crisp game of $tv \in G_F(N)$. Given any $U \in F(N)$ and $Q(U) = \{U(i) | U(i) > 0, i \in N\}$, $q(U) = |Q(U)|$. The element in $Q(U)$ are written in the increasing order as $r_1 < \cdots < r_{q(U)}$. If* Supp(U)*-restricted game of v is convex, then $\widetilde{C}_t(v)(U) \neq \varnothing$ and*

$$\widetilde{C}_t(v)(U) = \left\{ \begin{array}{l} y | y = \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot x^m = \left(\sum_{m=1}^{q(U)} (r_m - r_{m-1}) x_1^m, \ldots, \sum_{m=1}^{q(U)} (r_m - r_{m-1}) x_n^m \right), \\ \forall x^m = (x_1^m, x_2^m, \ldots, x_n^m) \in C(v^{[U]_{r_m}}), m = 1, 2, \ldots, q(U) \end{array} \right\}.$$

Proof. It is obvious that $C(v^{[U]_{r_m}}) \neq \varnothing$ for $m = 1, 2, \ldots, q(U)$. Given any $U \in F(N)$ and any $x^m \in C(v^{[U]_{r_m}})$, let $y \triangleq \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot x^m$. Firstly, we show $y \in \widetilde{C}_t(v)(U)$.

Because $x^m = (x_1^m, x_2^m, \ldots, x_n^m) \in C(v^{[U]_{r_m}})$, we have

$$\sum_{i \in N} y_i = \sum_{i \in N} \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot x_i^m = \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \sum_{i \in N} x_i^m = \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot v([U]_{r_m}) = tv(U).$$

Also, the following holds

$$\sum_{i \in \mathrm{Supp}(S_U)} y_i = \sum_{i \in \mathrm{Supp}(S_U)} \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot x_i^m = \sum_{m=1}^{q(U)} \left\{ (r_m - r_{m-1}) \cdot \sum_{i \in \mathrm{Supp}(S_U)} x_i^m \right\}$$
$$= \sum_{m=1}^{q(U)} \left\{ (r_m - r_{m-1}) \cdot \sum_{i \in S_{[U]_{r_m}}} x_i^m \right\} \geq \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot v(S_{[U]_{r_m}}).$$

Hence, $\widetilde{C}_t(v)(U) \neq \varnothing$.

Next, we will show that any $z \in \widetilde{C}_t(v)(U)$ can be denoted by

$$z = \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot x^m. \tag{17}$$

For any $r_m \in Q(U)$, let

$$\underline{x}_j^m \triangleq \min \left\{ x_j^m | x^m \in C(v^{[U]_{r_m}}) \right\}, \qquad \overline{x}_j^m \triangleq \max \left\{ x_j^m | x^m \in C(v^{[U]_{r_m}}) \right\}.$$

Also let

$$S \triangleq \left\{ j \in N \,\middle|\, z_j > \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \overline{x}_j^m \right\}, \qquad S^{/} \triangleq \left\{ j \in N \,\middle|\, z_j < \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \underline{x}_j^m \right\}.$$

If there exists $j \in N$ such that z_j can not be written as Eq. (17). Then there are only two cases for z_j,

(i) $z_j < \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \underline{x}_j^m$;
(ii) $z_j > \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \overline{x}_j^m$.

Case (i): Assume $z_j < \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \underline{x}_j^m$. Because $z \in \widetilde{C}_t(v)(U)$, we have

$$tv(S_U^{/}) \le \sum_{j \in \mathrm{Supp}(S_U^{/})} z_j$$

i.e.,

$$tv(S_U^{/}) < \sum_{j \in \mathrm{Supp}(S_U^{/})} \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \underline{x}_j^m.$$

Hence, the following holds:

$$\begin{aligned} 0 &< \sum_{j \in \mathrm{Supp}(S_U^{/})} \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \underline{x}_j^m - tv(S_U^{/}) \\ &= \sum_{m=1}^{q(U)} \left\{ (r_m - r_{m-1}) \cdot \sum_{j \in \mathrm{Supp}(S_U^{/})} \underline{x}_j^m \right\} - \sum_{m=1}^{q(U)} v(S^{/}_{[U]_{r_m}}) \cdot (r_m - r_{m-1}) \\ &= \sum_{m=1}^{q(U)} \left\{ (r_m - r_{m-1}) \cdot \left(\sum_{j \in \mathrm{Supp}(S_U^{/})} \underline{x}_j^m - v(S^{/}_{[U]_{r_m}}) \right) \right\}. \end{aligned}$$

Thus, there must exist $r_m \in Q(U)$ such that

$$\sum_{j \in \mathrm{Supp}(S_U^{/})} \underline{x}_j^m - v(S^{/}_{[U]_{r_m}}) = \sum_{j \in S^{/}_{[U]r_m}} \underline{x}_j^m - v(S^{/}_{[U]_{r_m}}) = \sum_{j \in S^{/}_{[U]r_m}} v(\{j\}) - v(S^{/}_{[U]_{r_m}}) > 0,$$

which contradicts with the supperadditivity of $tv \in G_F(N)$.

Case (ii): Assume $z_j > \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \overline{x}_j^m$. Then $S^{/} \ne \varnothing$. In fact, if $S^{/} = \varnothing$, then any $j \in N \setminus S$,z_j can be represented by $x^m = (x_1^m, x_2^m, \ldots, x_n^m) \in C(v^{[U]_{r_m}})$, i.e.,

$$\sum_{j \in N} z_j > \sum_{j \in S} \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \overline{x}_j^m + \sum_{j \in N \setminus S} \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot x_j^m \ge \sum_{j \in N} \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot x_j^m = tv(U),$$

which contradicts with $z \in \widetilde{C}_t(v)(U)$. Thus, $S^{/} \ne \varnothing$. By the proof of Case (i), we know that $z_j < \sum_{m=1}^{q(U)} (r_m - r_{m-1}) \cdot \underline{x}_j^m$ can not hold true, either.

Hence, we get the conclusion that we show that any $z \in \widetilde{C}_t(v)(U)$ can be written by Eq. (17). The proof is completed. □

It is apparent that $tv \in G_F(N)$ is convex if its associated crisp game $v \in G_o(N)$ is convex. Thus, if the crisp game $v \in G_o(N)$ is convex, then the fuzzy core of the game $tv \in G_F(N)$ defined by Eq. (5) is nonempty.

Proposition 4.2. *Let $tv \in G_F(N)$, $U \in F(N)$ and $x \in \widetilde{C}_t(v)(U)$. If $K \subseteq U$, then x also satisfies*

$$\sum_{i \in N} x_i \ge tv(K). \tag{18}$$

Proof. Let $x \in \widetilde{C}_t(v)(U)$. If $K \subseteq U$, then by Eq. (5), we get that $tv(U) \geq tv(K)$.

Due to $x \in \widetilde{C}_t(v)(U)$, we can also have

$$\sum_{i \in N} x_i = tv(U) \geq tv(K).$$

The proof is completed. □

Theorem 4.4. *Let $v \in G_0(N)$ be convex game, $tv \in G_F(N)$ be the game defined by (5). Then*

$$\{f_i(tv)(U)\}_{i \in N} \in \widetilde{C}_t(v)(U).$$

Proof. Because fuzzy Shapley is an imputation for game $tv \in G_F(N)$, we have $\sum_{i \in N} f_i(tv)(U) = tv(U)$. Thus, we only need to show that

$$\sum_{i \in \mathrm{Supp}(S_U)} f_i(tv)(U) \geq \sum_{m=1}^{q(U)} v(S_{[U]_{r_m}}) \cdot (r_m - r_{m-1}).$$

By Eq. (7), we obtain

$$\sum_{i \in \mathrm{Supp}(S_U)} f_i(tv)(U) = \sum_{i \in \mathrm{Supp}(S_U)} \sum_{m=1}^{q(U)} sh_i(v^{[U]r_m}) \cdot (r_m - r_{m-1}) = \sum_{m=1}^{q(U)} \sum_{i \in S_{[U]_{r_m}}} sh_i(v^{[U]r_m}) \cdot (r_m - r_{m-1})$$
$$\geq \sum_{m=1}^{q(U)} v(S_{[U]_{r_m}}) \cdot (r_m - r_{m-1}) = tv(S_U).$$

The proof is completed. □

Example 4.3 (*Cf.* Example 3.1). Let $N = \{1, 2\}$, $U(1) = 0.2$ and $U(2) = 0.3$, v be a characteristic function on N as defined in Example 3.1.

Then Tsurumi et al.'s choquet integral extension is

$$tv(U) = 0.2 \cdot v(\{1, 2\}) + (0.3 - 0.2) \cdot v(\{2\}) = 1.$$

Because

$$C(v) = \left\{ x^{\{1,2\}} \in \mathbb{R}^2_+ | x_1^{\{1,2\}} + x_2^{\{1,2\}} = 4, x_1^{\{1,2\}} \geq 1, x_2^{\{1,2\}} \geq 2 \right\},$$

the fuzzy core of this game $tv \in G_F(N)$ is

$$\widetilde{C}_t(v)(U) = \left\{ (0.2x_1^{\{1,2\}} + 0.1x_1^{\{2\}}, 0.2x_2^{\{1,2\}} + 0.1x_2^{\{2\}}) | x_1^{\{1,2\}} + x_2^{\{1,2\}} = 4, x_1^{\{1,2\}} \geq 1, x_2^{\{1,2\}} \geq 2, x_1^{\{2\}} = 0, x_2^{\{2\}} = 2 \right\}$$

i.e.

$$\widetilde{C}_t(v)(U) = \left\{ (0.2x_1^{\{1,2\}}, 0.2x_2^{\{1,2\}} + 0.2) | x_1^{\{1,2\}} + x_2^{\{1,2\}} = 4, x_1^{\{1,2\}} \geq 1, x_2^{\{1,2\}} \geq 2 \right\}.$$

The fuzzy Shapley value is

$$f_1(tv)(U) = sh_1(v^{\{1,2\}}) \cdot 0.2 + sh_1(v^{\{2\}}) \cdot 0.1 = 0.2 \cdot \left[\frac{(4-2)}{2} + \frac{1}{2} \right] + 0 = 0.3,$$

$$f_2(tv)(U) = sh_2(v^{\{1,2\}}) \cdot 0.2 + sh_2(v^{\{2\}}) \cdot 0.1 = \left[\frac{(4-1)}{2} + \frac{2}{2} \right] \cdot 0.2 + 0.2 = 0.7.$$

Obviously, $(0.3, 0.7) \in \widetilde{C}_t(v)(U)$.

5. Conclusions

We have defined the fuzzy core of the games with fuzzy coalition. The nonempty condition of the fuzzy core has been given. As in the classical case of convex crisp games, games with fuzzy coalitions have a large core and the fuzzy Shapley value is contained in the fuzzy core when the fuzzy core is nonempty. Due to the three main kinds of games with fuzzy coalitions, we build the relationship between the core of the crisp game and the fuzzy core. This property will help us have a better understanding of the fuzzy core and avoid the complicated computation process.

However, we mainly study three kinds of fuzzy cores for games with fuzzy coalitions, and it will be interesting to find the fuzzy cores for other types of fuzzy games.

References

[1] J.P. Aubin, Coeur et valeur des jeux flous à paiements latéraux, Comptes Rendus Hebdomadaires des Séances de 1' Académie des Sciences 279-A (1974) 891–894.
[2] J.P. Aubin, Mathematical Methods of Game and Economic Theory (Revised Edition), North-Holland, Amsterdam, 1982.
[3] D. Butnariu, Fuzzy games: A description of the concept, Fuzzy Sets and Systems 1 (1978) 181–192.
[4] D. Butnariu, Stability and Shapley value for *n*-persons fuzzy game, Fuzzy Sets and Systems 4 (1980) 63–72.
[5] D. Butnariu, T. Kroupa, Shapley mapping and the cumulative value for *n*-person games with fuzz coalition, European Journal of Operational Research 186 (2008) 288–299.
[6] M. Tsurumi, T. Tanino, M. Inuiguchi, A Shapley function on a class of cooperative fuzzy games, European Journal of Operational Research 129 (2001) 596–618.
[7] S. Tijs, R. Branzei, S. Ishihara, S. Muto, On cores and stable sets for fuzzy games, Fuzzy Sets and Systems 146 (2004) 285–296.
[8] M. Sakawa, I. Nishizaki, A lexicographical solution concept in an *n*-person cooperative fuzzy game, Fuzzy Sets and Systems 61 (1994) 265–275.
[9] L.S. Shapley, A value for *n*-persons games, Annals of Mathematics Studies 28 (1953) 307–318.
[10] G. Owen, Multilinear extensions of games, Management Sciences 18 (1972) 64–79.