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Computation of sharp bounds on the distribution of a function of dependent risks

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ABSTRACT

We propose a new algorithm to compute numerically sharp lower and upper bounds on the distribution of a function of d dependent random variables having fixed marginal distributions. Compared to the existing literature, the bounds are widely applicable, more accurate and more easily obtained.

1. Introduction and preliminaries

Let $X_1, \dots, X_d$ be d real-valued random variables on some probability space $(\Omega, \mathfrak{A}, P)$. Given a measurable function $\psi : \mathbb{R}^d \to \mathbb{R}$, we calculate numerical bounds on $P(\psi(X_1, \dots, X_d) \geq s)$, where we assume that each X_j has known distribution $F_j(x) = P(X_j \leq x)$, $1 \leq j \leq d$, but the dependence structure of the vector $(X_1, \dots, X_d)'$ is unknown. Thus, for a fixed $s \in \mathbb{R}$, we look for

$$M_\psi(s) = \sup\left\{P(\psi(X_1, \dots, X_d) \geq s) : X_j \sim F_j,\ 1 \leq j \leq d\right\}, \tag{1.1a}$$

$$m_\psi(s) = \inf\left\{P(\psi(X_1, \dots, X_d) > s) : X_j \sim F_j,\ 1 \leq j \leq d\right\}. \tag{1.1b}$$

A simple compactness argument shows that the supremum in (1.1a) and the infimum in (1.1b) are attained (see [1]). From mass transportation theory, a dual representation is known for the problems in (1.1) (see [2]). This dual representation, however, is typically difficult to evaluate. For the case that $\psi = +$ is the sum operator, problems (1.1) are of particular interest in risk analysis (see [3]) and reliable dual bounds related to the dual representation were given in [4] for the case of homogeneous marginals. These were extended to the non homogeneous case and to overlapping marginals in [5,6]. While these bounds are well computable for any dimension d in the homogeneous case, the numerical evaluation in the non homogeneous case poses serious problems. For more details on the numerical calculation/approximation of $M_+(s)$ and $m_+(s)$, we refer the reader to [4] and references therein. Note that, in general, the sharp bound $M_\psi(s)$ is not attained when the structure of dependence of the vector $(X_1, \dots, X_d)'$ is comonotonic, that is when the risks are similarly ordered. Analogously,

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in the case $d = 2$, the sharp bound $m_\psi(s)$ might not be attained when the risks are countermonotonic, that is when the two risks are oppositely ordered.

In this paper, we propose a new method to approximate numerically the sharp bounds $m_\psi(s)$ and $M_\psi(s)$ for certain classes of functions ψ which include in particular the sum, min, max and product operators. This method is based on rearrangement arguments and on a simple rearrangement algorithm introduced in special cases in [7,1]. In comparison to the method of dual bounds, our method is easy and fast. It can handle reasonable dimensions d and, in particular, also the inhomogeneous case. A numerical evaluation and comparison is given in Section 4 of this paper. The fact that this algorithm is computationally less demanding should be relevant for practical applications. It is interesting to note that, in the homogeneous examples considered, the bounds calculated by our method provide numerical evidence for the sharpness of the analytical dual bounds in [4], an analytic proof of which is still open.

1.1. Assumptions on the function ψ

Given a vector $\boldsymbol{x} \in \mathbb{R}^d$, let $\boldsymbol{x}_{-j}$ be the vector in $\mathbb{R}^{d-1}$ obtained by deleting the j-th component of $\boldsymbol{x}$. Throughout the paper, we assume that the function $\psi : \mathbb{R}^d \to \mathbb{R}$ is measurable and coordinate-wise increasing, and there exist two measurable functions $\psi^{d-1} : \mathbb{R}^{d-1} \to \mathbb{R}$ and $\psi^2 : \mathbb{R}^2 \to \mathbb{R}$ such that ψ satisfies

$$\psi(x_1, \dots, x_d) = \psi^2(x_j, \psi^{d-1}(\boldsymbol{x}_{-j})), \quad 1 \le j \le d. \tag{1.2}$$

Relevant cases of such ψ are the sum ($\psi^2(x_1, x_2) = x_1 + x_2$), the product ($\psi^2(x_1, x_2) = x_1 x_2$, for $x_1, x_2 > 0$), the max ($\psi^2(x_1, x_2) = \max\{x_1, x_2\}$) and the min ($\psi^2(x_1, x_2) = \min\{x_1, x_2\}$) operators. Asymmetric functions do not satisfy the above requirements.

2. A combinatorial problem

In this section, we describe a combinatorial problem which will turn out to be strictly connected to (1.1). Let $X = (x_{i,j})$, $x_{i,j} \in \mathbb{R} \cup \{-\infty, +\infty\}$, $1 \le i \le n$, $1 \le j \le d$, be a $(n \times d)$-matrix. Let X_{-j} be the $(n \times (d-1))$-matrix obtained from X by deleting its j-th column $X_{(j)}$. Denote by $\boldsymbol{\psi}(X)$ (respectively, $\boldsymbol{\psi}_{-j}(X)$) the n-dimensional vectors obtained by applying the function ψ (resp., ψ^{d-1}), to each row of X (resp., X_{-j}). Formally,

$$\boldsymbol{\psi}(X) = \begin{pmatrix} \psi(x_{1,1}, \dots, x_{1,d}) \\ \vdots \\ \psi(x_{i,1}, \dots, x_{i,d}) \\ \vdots \\ \psi(x_{n,1}, \dots, x_{n,d}) \end{pmatrix}, \qquad \boldsymbol{\psi}_{-j}(X) = \begin{pmatrix} \psi^{d-1}(x_{1,1}, \dots, x_{1,j-1}, x_{1,j+1}, \dots, x_{1,d}) \\ \vdots \\ \psi^{d-1}(x_{i,1}, \dots, x_{i,j-1}, x_{i,j+1}, \dots, x_{i,d}) \\ \vdots \\ \psi^{d-1}(x_{n,1}, \dots, x_{n,j-1}, x_{n,j+1}, \dots, x_{n,d}) \end{pmatrix}.$$

Using (1.2), for $1 \le j \le d$, we have

$$\boldsymbol{\psi}(X)_i = \psi^2(x_{i,j}, \boldsymbol{\psi}_{-j}(X)_i), \quad 1 \le i \le n.$$

Let $\mathcal{P}(X)$ be the set of all $(n \times d)$-matrices obtained from X by rearranging the elements within a number of its columns in a different order, that is

$$\mathcal{P}(X) = \left\{ \tilde{X} = (\tilde{x}_{i,j}) : \tilde{x}_{i,j} = x_{\pi_j(i),j}, \pi_1, \dots, \pi_d \text{ are permutations of } \{1, \dots, n\} \right\}.$$

We study the problem of how to rearrange the columns of X such that the minimal component of $\boldsymbol{\psi}(X)$ is maximized. Using the notation introduced above, this problem can be written as

$$G_\psi(X) = \max_{\tilde{X} \in \mathcal{P}(X)} \min_{1 \le i \le n} \boldsymbol{\psi}(\tilde{X})_i. \tag{2.1}$$

Similarly, we consider the problem of how to rearrange the columns of X such that the maximal component of $\boldsymbol{\psi}(X)$ is minimized, that is

$$H_\psi(X) = \min_{\tilde{X} \in \mathcal{P}(X)} \max_{1 \le i \le n} \boldsymbol{\psi}(\tilde{X})_i. \tag{2.2}$$

Given two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, we denote by $\boldsymbol{a}_{[i]}$ the i-largest component of $\boldsymbol{a}$ ($\boldsymbol{a}_{[n]}$ is the minimal). We write $\boldsymbol{a} \perp \boldsymbol{b}$ to indicate that the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ are oppositely ordered, that is $(a_j - a_k)(b_j - b_k) \le 0$ for all $1 \le j, k \le n$. Let

$$\mathcal{O}_\psi(X) = \left\{ X^* \in \mathcal{P}(X) : X^*_{(j)} \perp \boldsymbol{\psi}_{-j}(X^*), 1 \le j \le d \right\}$$

be the set of those permutation matrices X^* such that $X^*_{(j)}$ is oppositely ordered to $\boldsymbol{\psi}_{-j}(X^*)$.

Theorem 2.1. *Assume that the function ψ is strictly increasing in each coordinate. Using the above notation, it is possible to rewrite the problems (2.1) and (2.2) as*

$$G_\psi(X) = \max_{X^* \in \mathcal{O}_\psi(X)} \boldsymbol{\psi}(\tilde{X})_{[n]} \quad \text{and} \quad H_\psi(X) = \min_{X^* \in \mathcal{O}_\psi(X)} \boldsymbol{\psi}(\tilde{X})_{[1]}.$$

Proof. Take $\tilde{X} \in \mathcal{P}(X) \setminus \mathcal{O}_\psi(X)$. Then, it is possible to find an index $j \in \{1, \dots, d\}$ such that $\boldsymbol{a} = \tilde{X}_{(j)}$ is not oppositely ordered to $\boldsymbol{b} = \boldsymbol{\psi}_{-j}(\tilde{X})$. Therefore, there exist two indexes $i_1, i_2 \in \{1, \dots, n\}$ such that $a_{i_1} < a_{i_2}$ and $b_{i_1} < b_{i_2}$. Since ψ^2 is strictly increasing in each coordinate, we have that

$$\psi^2(a_{i_1}, b_{i_1}) < \psi^2(a_{i_1}, b_{i_2}) < \psi^2(a_{i_2}, b_{i_2}) \quad \text{and} \quad \psi^2(a_{i_1}, b_{i_1}) < \psi^2(a_{i_2}, b_{i_1}) < \psi^2(a_{i_2}, b_{i_2}).$$

Then, we obtain

$$\min\left\{\psi^2(a_{i_1}, b_{i_1}), \psi^2(a_{i_2}, b_{i_2})\right\} = \psi^2(a_{i_1}, b_{i_1}) < \min\left\{\psi^2(a_{i_1}, b_{i_2}), \psi^2(a_{i_2}, b_{i_1})\right\}$$
$$\max\left\{\psi^2(a_{i_1}, b_{i_1}), \psi^2(a_{i_2}, b_{i_2})\right\} = \psi^2(a_{i_2}, b_{i_2}) > \max\left\{\psi^2(a_{i_1}, b_{i_2}), \psi^2(a_{i_2}, b_{i_1})\right\}.$$

Thus, if we rearrange $\boldsymbol{a}$ by switching the indexes i_1 and i_2, we obtain a new matrix X' for which $\boldsymbol{\psi}(X')$ has the maximal component decreased and the minimal component increased. Note that $\boldsymbol{\psi}(X')$ is different from $\boldsymbol{\psi}(\tilde{X})$ as, for some j, the j-th greatest component of $\boldsymbol{\psi}(X')$ is strictly smaller than the j-th greatest component of $\boldsymbol{\psi}(\tilde{X})$. Being the set $\mathcal{P}(X)$ finite, repeating the above procedure on the columns of $\tilde{X}$ we can pass from the matrix $\tilde{X}$ to a matrix $X^* \in \mathcal{O}_\psi(X)$ in a finite number of steps, obtaining that

$$\min_{1\le i\le n} \boldsymbol{\psi}(X^*)_i = \min_{1\le i\le n} \psi^2(x^*_{i,j}, \boldsymbol{\psi}_{-j}(X^*)_i) \ge \min_{1\le i\le n} \psi^2(\tilde{x}_{i,j}, \boldsymbol{\psi}_{-j}(\tilde{X})_i) = \min_{1\le i\le n} \boldsymbol{\psi}(\tilde{X})_i.$$
$$\max_{1\le i\le n} \boldsymbol{\psi}(X^*)_i = \max_{1\le i\le n} \psi^2(x^*_{i,j}, \boldsymbol{\psi}_{-j}(X^*)_i) \le \max_{1\le i\le n} \psi^2(\tilde{x}_{i,j}, \boldsymbol{\psi}_{-j}(\tilde{X})_i) = \max_{1\le i\le n} \boldsymbol{\psi}(\tilde{X})_i.$$

Consequently, we can restrict the domains of the problems in (2.1) and (2.2) to the set $\mathcal{O}_\psi(X)$. □

The proof of Theorem 2.1 indicates a simple algorithm to find elements in $\mathcal{O}_\psi(X)$ and, hence, possible solutions to (2.1) and (2.2). This algorithm is a more general version of the algorithm described in Section 3 in [7].

> **Rearrangement algorithm** to find elements in $\mathcal{O}_\psi(X)$. Start with any $\tilde{X} \in \mathcal{P}(X)$. Define $\tilde{X}_1$ by iteratively rearranging its j-th column $\tilde{X}_{(j)}$ such that $\tilde{X}_{(j)} \perp \boldsymbol{\psi}_{-j}(\tilde{X})$, for $1 \le j \le d$. Then, repeat using $\tilde{X}_1$ as the initial matrix until an element $X^* \in \mathcal{O}_\psi(X)$ is found.

The rearrangement algorithm can be used also when the function ψ is non-strictly increasing, provided that the set $\mathcal{O}_\psi(X)$ is nonempty. If this is the case, it is always possible to rearrange finitely many times the columns of a matrix $\tilde{X} \in \mathcal{P}(X) \setminus \mathcal{O}_\psi(X)$ in order to obtain a matrix $X^* \in \mathcal{O}_\psi(X)$ such that $\boldsymbol{\psi}(X^*)_{[n]} \ge \boldsymbol{\psi}(\tilde{X})_{[n]}$ and $\boldsymbol{\psi}(X^*)_{[1]} \le \boldsymbol{\psi}(\tilde{X})_{[1]}$. In Section 4.1, we illustrate the relevant cases that $\psi = \max$ and $\psi = \min$.

3. Numerical approximation

In the remainder, let $F_j^{-1}(\alpha) := \sup\left\{x \in \mathbb{R} : F_j(x) \le \alpha\right\}, \alpha \in [0, 1]$ be the generalized inverse of F_j, $1 \le j \le d$. For a subset $A \subset [0, 1]$, we denote by $F_j^{-1}|A$ the restriction of F_j^{-1} to A. We write $f_j \sim F_j^{-1}|A$ to indicate that the function $f_j : A \to \mathbb{R}$ is a rearrangement of $F_j^{-1}|A$. We refer to [1] for a basic introduction to the theory of rearrangements. The following representation of (1.1) is given in Theorem 2 in [1].

If U is a random variable uniformly distributed in $[0, 1]$, then

$$M_\psi(s) = \sup\left\{P(\psi(f_1(U), \dots, f_d(U)) \ge s) : f_j \sim F_j^{-1}, 1 \le j \le d\right\}, \tag{3.1a}$$
$$m_\psi(s) = \inf\left\{P(\psi(f_1(U), \dots, f_d(U)) > s) : f_j \sim F_j^{-1}, 1 \le j \le d\right\}. \tag{3.1b}$$

In order to establish a link between (3.1) and the rearrangement algorithm described in Section 2, we need the following theorem.

Theorem 3.1. *If the function ψ is increasing in each coordinate, then, for all $s \in \mathbb{R}$, we have that*

$$M_\psi(s) = 1 - \inf\left\{\alpha : \text{ there exist } f_j^\alpha \sim F_j^{-1}|[\alpha, 1], 1 \le j \le d \text{ s.t. } \psi(f_1^\alpha, \dots, f_d^\alpha) \ge s\right\}, \tag{3.2a}$$
$$m_\psi(s) = 1 - \sup\left\{\alpha : \text{ there exist } f_j^\alpha \sim F_j^{-1}|[0, \alpha], 1 \le j \le d \text{ s.t. } \psi(f_1^\alpha, \dots, f_d^\alpha) \le s\right\}. \tag{3.2b}$$

Proof. First, we prove (3.2a). If there exist such f_j^α's, we can easily extend them to rearrangements of $F_j^{-1}|[0,\,1]$ and, by (3.1), $M_\psi(s) \geq 1-\alpha$, hence (3.2a) holds with $\geq$. For the $\leq$ inequality, we use a similar argument as in Proposition 3(c) in [2]. Let $f_j^* \sim F_j^{-1}$ be solutions of (3.1a) and define the set

$$A = \{u \in [0,\,1] : \psi(f_1^*(u), \ldots, f_d^*(u)) \geq s\}.$$

Then, the Lebesgue measure of A is $\lambda(A) = M_\psi(s)$. With $\alpha = 1 - M_\psi(s)$, there exists a λ-preserving transformation $\phi : [0,\,1] \to [0,\,1]$ such that $A = \phi([\alpha,\,1])$. Therefore, we can assume w.l.o.g. that $A = [\alpha,\,1]$. Moreover, there exist $\phi_j : [0,\,1] \to [0,\,1]$, $\phi_j \sim F_j^{-1}$, $1 \leq j \leq d$, such that $f_j^\alpha = \phi_j|[\alpha,\,1] \sim F_j^{-1}|[\alpha,\,1]$ and $f_j^\alpha(u) \geq f_j^*(u)$, $u \in [\alpha,\,1]$. Define, for example,

$$A_j^* = \{u \in [\alpha,\,1] : f_j^*(u) \geq F_j^{-1}(\alpha)\},$$

and $f_j^\alpha|[\alpha,\,1] = f_j^* 1_{\{A_j^*\}} + F_j^{-1} 1_{\{[\alpha,1]\setminus A_j^*\}}$, $1 \leq j \leq d$. For the functions ϕ_j, we can use an extension of $f_j^\alpha|[\alpha,\,1]$ to $[0,\,1]$ such that $f_j^\alpha \sim F_j^{-1}$. This implies, by monotonicity of ψ, and since $A = [\alpha,\,1]$, that, for $u \in [\alpha,\,1]$, we have

$$\psi(f_1^\alpha(u), \ldots, f_d^\alpha(u)) \geq \psi(f_1^*(u), \ldots, f_d^*(u)) \geq s.$$

The proof for (3.2b) is analogous, considering that $m_\psi(s) = 1 - \sup\{P(\psi(X_1, \ldots, X_d) \leq s) : X_j \sim F_j,\ 1 \leq j \leq d\}$. □

The rearrangement algorithm can be applied to find solutions of (3.2) when the marginal distributions F_j are (rational) discrete. Assume that each $F_j^{-1}|[\alpha,\,1]$ takes only the n real values $\boldsymbol{x}_j^\alpha = \{x_{i,j}^\alpha,\, 1 \leq i \leq n\}$, for $1 \leq j \leq d$. We may assume, by using repetitions, that all the $x_{i,j}^\alpha$'s have the same probability $(1-\alpha)/n$. The rearrangements f_j^α of $F_j^{-1}|[\alpha,\,1]$ are then replaced by the rearrangements of the components of $\boldsymbol{x}_j^\alpha$. For instance, if the components of each $\boldsymbol{x}_j^\alpha$ are arranged in increasing order, the columns of the matrix $X^\alpha = (x_{i,j}^\alpha)$ represent the increasing rearrangement $(F_1^{-1}|[\alpha,\,1], \ldots, F_d^{-1}|[\alpha,\,1])$. Since, in the following, we only consider order induced rearrangements like increasing or decreasing order rearrangements, the transition to rearrangements of the components of the discrete vectors is justified.

Therefore, all the possible rearrangements $(f_1^\alpha, \ldots, f_d^\alpha)$ in (3.2a) can be represented by a rearrangement of the columns of X^α, that is, using the notation introduced in Section 2, by a matrix $\tilde{X}^\alpha \in \mathcal{P}(X^\alpha)$. Thus, the condition $\psi(f_1^\alpha, \ldots, f_d^\alpha) \geq s$ becomes $\boldsymbol{\psi}(\tilde{X}^\alpha)_{[n]} \geq s$. In conclusion, we can rewrite (3.2a) as

$$\begin{aligned} M_\psi(s) &= 1 - \inf\left\{\alpha :\ \text{there exist } \tilde{X}^\alpha \in \mathcal{P}(X^\alpha) \text{ s.t. } \boldsymbol{\psi}(\tilde{X}^\alpha)_{[n]} \geq s\right\} \\ &= 1 - \inf\left\{\alpha : G_\psi(X^\alpha) \geq s\right\}, \end{aligned} \tag{3.3}$$

where X^α is the matrix having as columns the increasing rearrangements of the points of the domains $F_j^{-1}([\alpha,\,1])$, $1 \leq j \leq d$. Analogously, we can rewrite (3.2b) as

$$\begin{aligned} m_\psi(s) &= 1 - \sup\left\{\alpha :\ \text{there exist } \tilde{X}_\alpha \in \mathcal{P}(X_\alpha) \text{ s.t. } \boldsymbol{\psi}(\tilde{X}_\alpha)_{[1]} \leq s\right\} \\ &= 1 - \sup\left\{\alpha : H_\psi(X_\alpha) \leq s\right\}, \end{aligned} \tag{3.4}$$

where X_α is the matrix having as columns the increasing rearrangements of the points of the domains $F_j^{-1}([0,\,\alpha])$, $1 \leq j \leq d$.

The representations (3.3) and (3.4) hold only when the F_j's are discrete, yet they are useful also in the case of arbitrary marginals. Indeed, we can always find two discrete distribution functions which approximate any F_j from below and from above. For instance, we define the discrete distributions $\underline{F}_j$ and $\overline{F}_j$ as

$$\underline{F}_j(x) = \frac{1}{n}\sum_{r=0}^{n-1} 1_{[q_r,+\infty)}(x) \quad \text{and} \quad \overline{F}_j(x) = \frac{1}{n}\sum_{r=1}^{n} 1_{[q_r,+\infty)}(x), \tag{3.5}$$

where the jump points q_r are defined by $q_r := F_j^{-1}(r/n)$, $0 \leq r \leq n$. Since $\underline{F}_j \geq F_j \geq \overline{F}_j$ and ψ is increasing, it follows that, for every real s,

$$\begin{aligned} &\underline{M}_\psi(s) \leq M_\psi(s) \leq \overline{M}_\psi(s), \\ &\underline{m}_\psi(s) \leq m_\psi(s) \leq \overline{m}_\psi(s) \end{aligned}$$

where $\underline{m}_\psi(s)$ (respectively $\overline{m}_\psi(s)$) is the analogous of (1.1b) when $F_j = \underline{F}_j$ (resp. $F_j = \overline{F}_j$). Analogously, $\underline{M}_\psi(s)$ (resp. $\overline{M}_\psi(s)$) is the analogous of (1.1a) when $F_j = \underline{F}_j$ (resp. $F_j = \overline{F}_j$).

Note that we can always choose a different number of points n_j in the support of the discrete distributions $\underline{F}_j$ and $\overline{F}_j$ so that, for $1 \leq j \leq d$, the increasing rearrangement of the supports $\underline{F}_j^{-1}([\alpha,\,1])$ and $\overline{F}_j^{-1}([\alpha,\,1])$ all have the same number n

of components. Using (3.3), it is possible to write

$$\underline{M}_\psi(s) = 1 - \inf\{\alpha : G_\psi(\underline{X}^\alpha) \geq s\},$$
$$\overline{M}_\psi(s) = 1 - \inf\{\alpha : G_\psi(\overline{X}^\alpha) \geq s\},$$

where $\underline{X}^\alpha$, respectively $\overline{X}^\alpha$, are the $(n \times d)$-matrix having as columns the increasing rearrangements of the supports $\underline{F}_j^{-1}([\alpha, 1])$, respectively $\overline{F}_j^{-1}([\alpha, 1])$, for $1 \leq j \leq d$.

Similarly,

$$\underline{m}_\psi(s) = 1 - \sup\{\alpha : H_\psi(\underline{X}_\alpha) \leq s\},$$
$$\overline{m}_\psi(s) = 1 - \sup\{\alpha : H_\psi(\overline{X}_\alpha) \leq s\},$$

where $\underline{X}_\alpha$, respectively $\overline{X}_\alpha$, are the $(n \times d)$-matrix having as columns the increasing rearrangements of the supports $\underline{F}_j^{-1}([0, \alpha])$, respectively $\overline{F}_j^{-1}([0, \alpha])$, for $1 \leq j \leq d$.

At this point, the algorithm described at the end of Section 2 can be used to find numerical ranges for the sharp bounds M_ψ and m_ψ. Define

$$\mathcal{G}_\psi(X) = \left\{\boldsymbol{\psi}(X^*)_{[n]} : X^* \in \mathcal{O}_\psi(X)\right\} \quad \text{and} \quad \mathcal{H}_\psi(X) = \left\{\boldsymbol{\psi}(X^*)_{[1]} : X^* \in \mathcal{O}_\psi(X)\right\},$$

the set of possible values for the max in (2.1) and, respectively, the min in (2.2). First, we illustrate how to obtain a range on $M_\psi(s)$. Start selecting randomly a matrix $\tilde{X}^\alpha \in \mathcal{P}(\underline{X}^\alpha)$. Define $\tilde{X}_1^\alpha$ by rearranging the j-th column $\tilde{X}_{(j)}^\alpha$ of $\tilde{X}^\alpha$ such that $\tilde{X}_{(j)}^\alpha \perp \boldsymbol{\psi}_{-j}(\tilde{X}^\alpha)$, for all $j = 1, \ldots, d$. Then, repeat using $\tilde{X}_1^\alpha$ as the initial matrix, until an element $\underline{g}(\alpha) \in \mathcal{G}_\psi(\underline{X}^\alpha)$ is found. Denote by

$$\underline{\alpha}(s) = \inf\{\alpha \in [0, 1] : \underline{g}(\alpha) \geq s\}.$$

$\underline{\alpha}(s)$ can be computed numerically in several ways, as for example by iteratively bisecting the interval $[0, 1]$ and checking the condition $\underline{g}(\alpha) \geq s$. From (2.1), we have that $G_\psi(\underline{X}^\alpha) \geq \underline{g}(\alpha)$. Therefore, it follows that $\inf\{\alpha : G_\psi(\underline{X}^\alpha) \geq s\} \leq \underline{\alpha}(s)$ and, finally,

$$M_\psi(s) \geq \underline{M}_\psi(s) \geq 1 - \underline{\alpha}(s). \tag{3.6}$$

In order to find an upper bound on M_ψ, we proceed analogously by finding an element $\overline{g}(\alpha) \in \mathcal{G}_\psi(\overline{X}^\alpha)$. Denote by

$$\overline{\alpha}(s) = \inf\{\alpha \in [0, 1] : \overline{g}(\alpha) \geq s\}.$$

If $\overline{g}(\alpha)$ is optimal, that is $\overline{g}(\alpha) = G_\psi(\overline{X}^\alpha)$, we obtain

$$M_\psi(s) \leq \overline{M}_\psi(s) = 1 - \overline{\alpha}(s). \tag{3.7}$$

Note that, while (3.6) is always satisfied, (3.6) may fail to hold if $\overline{g}(\alpha)$ is not optimal. However, if $\overline{g}(\alpha)$ is a good approximation for $G_\psi(\overline{X}^\alpha)$, $\overline{\alpha}(s)$ represents a good approximation for $\overline{M}_\psi(s)$. In conclusion, combining (3.6) and (3.7), we obtain

$$1 - \underline{\alpha}(s) \leq M_\psi(s) \simeq 1 - \overline{\alpha}(s). \tag{3.8}$$

In order to find a range for the sharp bound $m_\psi(s)$, we proceed analogously. Applying the algorithm to some matrices $\tilde{X}_\alpha \in \mathcal{P}(\underline{X}_\alpha)$ and $\tilde{X}_\alpha \in \mathcal{P}(\overline{X}_\alpha)$, we find elements $\underline{h}(\alpha) \in \mathcal{H}_\psi(\underline{X}_\alpha)$ and $\overline{h}(\alpha) \in \mathcal{H}_\psi(\overline{X}_\alpha)$. Defining

$$\underline{\beta}(s) = \sup\{\alpha \in [0, 1] : \underline{h}(\alpha) \leq s\},$$
$$\overline{\beta}(s) = \sup\{\alpha \in [0, 1] : \overline{h}(\alpha) \leq s\},$$

it follows that

$$1 - \underline{\beta}(s) \simeq m_\psi(s) \leq 1 - \overline{\beta}(s). \tag{3.9}$$

For a fixed function ψ and marginals F_j, $1 \leq j \leq d$, the accuracy of the approximations given in (3.8) and in (3.9) can be increased by choosing:

- a larger value of n, so that the approximation to F_j given by the discrete distributions $\underline{F}_j$ and $\overline{F}_j$ is more accurate;
- a number of different random starting matrices in order to find different elements in the sets $\mathcal{G}_\psi(X)$ and $\mathcal{H}_\psi(X)$.

Having mainly applications to quantitative risk management in mind, in the following we will always compute the ranges (3.8) and (3.9) for continuous marginal distributions F_j. In these cases, we always find that any element in $\mathcal{G}_\psi(X)$ and $\mathcal{H}_\psi(X)$ yields a very good approximation of the real solutions $G_\psi(X)$ and $H_\psi(X)$, and the algorithm works very well with a single starting point and a high value for n.

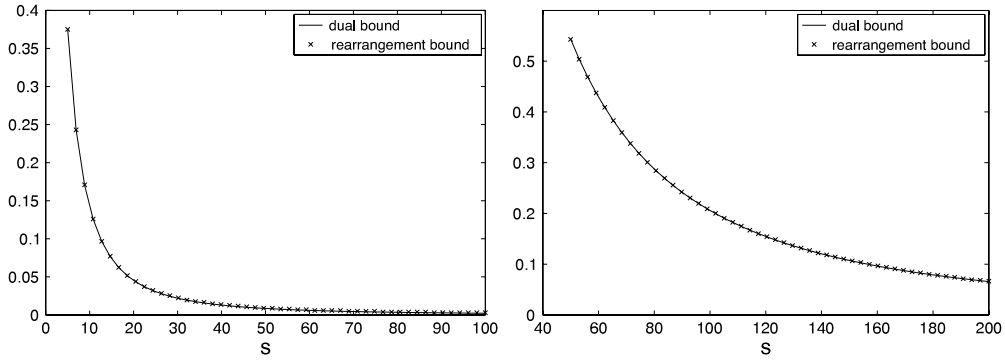

Fig. 1. Upper dual bound on $\mathbb{P}[X_1 + \cdots + X_d \geq s]$, calculated using Theorem 4.2 in [4], when the $X_j's$ are all Pareto(2)-distributed. We set $d = 3$ (left), and $d = 30$ (right). In both figures, the range for $M_+(s)$, calculated using (3.8), is provided at some threshold of interest.

Table 1
Range for $m_+(s)$ and $M_+(s)$ for the sum of three Pareto(2) random variables. Values for the upper dual bound on $M_+(s)$, as defined in [4], are also provided.

	$1-\underline{\beta}(s)$	$1-\overline{\beta}(s)$		$1-\underline{\alpha}(s)$	Dual bound	$1-\overline{\alpha}(s)$
$s = 0.5$	0.5101929	0.51025391	$s = 10$	0.1419678	0.142011834319527	0.1420288
$s = 1.0$	0.2500000	0.25006104	$s = 15$	0.0740356	0.074074074074074	0.0740967
$s = 1.5$	0.1599731	0.16003418	$s = 20$	0.0453491	0.045368620037807	0.0454102
$s = 2.0$	0.1110840	0.11114502	$s = 25$	0.0305786	0.030612244897959	0.0306397
$s = 2.5$	0.0816040	0.08166504	$s = 30$	0.0220337	0.022038567493113	0.0220947

Table 2
Range for $m_\times(s)$ and $M_\times(s)$ for the product of five Pareto(θ_j) random variables. We set $\boldsymbol{\theta} = (1.5, 1.8, 2.0, 2.2, 2.5)'$.

	$1-\underline{\beta}(s)$	$1-\overline{\beta}(s)$		$1-\underline{\alpha}(s)$	$1-\overline{\alpha}(s)$
$s = 0.001$	0.16113281	0.16210938	$s = 100$	0.2158203	0.2167969
$s = 0.002$	0.09852281	0.09863281	$s = 200$	0.1787109	0.1796875
$s = 0.003$	0.06347656	0.06445312	$s = 300$	0.1591797	0.1601562
$s = 0.004$	0.04101562	0.04199219	$s = 400$	0.1464844	0.1474609
$s = 0.005$	0.02441406	0.02539062	$s = 500$	0.1376953	0.1386719

4. Applications

In this section, we compute the ranges defined in (3.8) and in (3.9) for different functionals ψ and sets of marginals F_j, $1 \leq j \leq d$. In order to test the quality of the dual bound, Embrechts and Puccetti [4] calculate a numerical range for $M_+(s)$ via two linear problems and using a discretization of the F_j's identical to the one described in (3.5). Note that Embrechts and Puccetti [4] obtain bounds on $P(X_1 + \cdots + X_d < s)$ instead of $P(X_1 + \cdots + X_d \geq s)$.

Being only based on the iterative rearrangements of the columns of a matrix, an operation which can be performed efficiently with *R*, our algorithm turns out to be less demanding, in terms of computational time and memory, than the numerical procedure described in [4]. Indeed, both methods use discrete versions of the marginals with n points in their supports, and calculate $M_+(s)$ with an error that approximately decreases as $o(1/n)$. However, here we were able to use $n = 10^5$ as compared to $n = 180$ in [4]. In the case of $\psi = +$, using $n = 10^5$ gives $M_+(s)$ with an absolute error of about 10^{-4}.

In Fig. 1, we plot the dual bound functional introduced in [4] for the sum of d random variables being all Pareto(θ)-distributed, that is $P(X_j \leq x) = 1 - (1 + x)^{-\theta}$, $x > 0$. We set $\theta = 2$ and $d = 3$ (Fig. 1, left) and $d = 30$ (right). In the same figure, we provide the bounds obtained using (3.8), at some threshold of interest. Fig. 1 seems to indicate that the dual bound functional introduced in [4] is sharp.

In Table 1, we report the numerical ranges for $m_+(s)$ and $M_+(s)$, obtained using (3.8) and (3.9), under the same marginal assumptions, for $d = 3$. We used $n = 10^5$, and each figure is obtained within two minutes. In Table 2, we report the numerical range for $m_\times(s)$ and $M_\times(s)$, where $\psi = \times$ is the product operator. Here we use different marginal distributions, for $d = 5$ and $n = 10^5$.

The results obtained for $n = 10^5$ and $d = 3$ in a two-minute time can be considered reasonably accurate. However, an important feature of our algorithm is that it can handle larger values of n and d without memory issues. Indeed, changing n and d means changing the dimensions of the matrices representing the rearrangements of the discrete marginals used. If extra-accuracy is required, with $n = 10^6$ one can obtain an estimate for $M_\psi(s)$ in about forty minutes. If one needs only two decimals for $M_\psi(s)$, using $n = 10^4$ provides an estimate in about 3 s. An analogous reasoning can be applied to an increase of the number of random variables d. With $n = 10^5$ we can handle up to $d = 30$ different marginals keeping the computational

Table 3
Range for $m_{\max}(s)$ and $M_{\max}(s)$ for the max of three Pareto(2) random variables. Values for the sharp bounds $m_{\max}(s)$ and $M_{\max}(s)$, as given in [8], are also provided.

	$1-\underline{\beta}(s)$	Sharp bound	$1-\overline{\beta}(s)$		$1-\underline{\alpha}(s)$	Sharp bound	$1-\overline{\alpha}(s)$
$s=1$	0.25000000	0.25000000	0.25000000	$s=1$	0.75000000	0.75000000	0.75000000
$s=2$	0.11035156	0.11111111	0.11132812	$s=2$	0.33300781	0.33333333	0.33398438
$s=3$	0.06250000	0.06250000	0.06250000	$s=3$	0.18750000	0.18750000	0.18750000
$s=4$	0.03906250	0.04000000	0.04003906	$s=4$	0.11914062	0.12000000	0.12011719
$s=5$	0.02734375	0.02777777	0.02832031	$s=5$	0.08300781	0.08333333	0.08398438

time under 40 min. If one needs to compute $M_\psi(s)$ and $m_\psi(s)$ at different thresholds s, the average computational time for a single estimate can be reduced by knowing the bounds calculated at a different threshold.

Though the dual bound given in [4] is mainly analytic and the rearrangement method in this paper is entirely numerical, it is useful to make a final comparison between the two. A dual upper bound on $M_\psi(s)$ has been given in [4] for homogeneous marginals ($F_j = F$, $1 \le j \le d$), and extended to general marginal settings in [5,6]. While the dual bound is stated for arbitrary marginals, its computational complexity increases with the number of different marginals used. It is easy to calculate the dual bound with an arbitrary number d of homogeneous random variables, while it is much more complicated to deal with a relatively small number $d \le 10$ of non homogeneous marginals. Moreover, the dual bound functional has been introduced only for the sum operator. The algorithm introduced in this paper can handle more general functionals ψ and inhomogeneous marginals. It approximates the sharp upper and lower bounds $M_\psi(s)$, $m_\psi(s)$ numerically while the dual bounds are constructed only as upper bounds for $M_\psi(s)$. However, the rearrangement method in this paper cannot be practically used to handle dimensions $d > 100$, where the computation of dual bounds is possible with homogeneous marginals. In the examples considered, the dual bounds in [4] for $M_\psi(s)$ seem to be sharp. It would be interesting to prove sharpness for certain classes of distributions.

4.1. Max and min operators

In the cases that $\psi = \max$ and $\psi = \min$ it is possible to write explicitly solutions for the problems $G_\psi(X)$ and $H_\psi(X)$ in (2.1). As an example, we consider the case that $\psi = \max$, for which sharp bounds have been given analytically in [8]. A solution to the problem $G_{\max}(X)$ is given by any matrix X^* in which each of the first n greatest components of X appear in a different row. The problem $H_{\max}(X)$ is trivial as the optimal value is given the greatest element of X. Combining the above solutions with the discretization procedure described in Section 3, one obtains, in the limit as n goes to infinity, the sharp bounds given in [8]. Analogous considerations hold for the case $\psi = \min$, for which sharp bounds have been given analytically in [9]. As remarked at the end of Section 2, the rearrangement algorithm can be successfully applied to case $\psi = \max$. In Table 3 we report the numerical ranges for $m_{\max}(s)$ and $M_{\max}(s)$, as well as the sharp bounds calculated in [8,9].

5. Conclusions and forthcoming research

In this paper, we introduce the rearrangement algorithm to calculate numerically the sharp bounds $M_\psi(s)$ and $m_\psi(s)$ on the distribution of a function of dependent random variables having fixed marginals. This algorithm is accurate, fast and can be used to handle random variables with inhomogeneous marginals, in moderately high dimensions. It provides evidence that the dual bounds in [4] are sharp for some classes of homogeneous distributions. Problems (1.1) have a wide range of application in quantitative risk management. For an overview of this kind of application, we refer the reader to [5,3].

In a forthcoming paper, we will describe how to use the rearrangement algorithm also in the case of overlapping marginals. Moreover, the authors propose to prove sharpness of the dual bounds in [4] in the case of the sum of risks, for some classes of homogeneous distributions. It is well known that the bound $M_\psi(s)$ and $m_\psi(s)$ can be improved if, for instance, the random variable $X_1, \ldots, X_d$ are known to be positively dependent. How to handle this extra information within the rearrangement algorithm will be the object of future research.

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