Lyapunov sequences for exponential dichotomies

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Abstract

For a linear nonautonomous dynamics with discrete time, we study the relation between nonuniform exponential dichotomies and strict Lyapunov sequences. Given such a sequence, we obtain the stable and unstable subspaces from the intersection of the images and preimages of the cones defined by each element of the sequence. The main difficulty is to extract some information about the angles between the stable and unstable subspaces (or some appropriate notion in the case of Banach spaces) from the Lyapunov sequence. In particular, for a large class of nonuniform exponential dichotomies we give a complete characterization in terms of strict quadratic Lyapunov sequences, that is, strict Lyapunov sequences defined by quadratic forms. We also construct explicitly families of strict Lyapunov sequences for each nonuniform exponential dichotomy, in terms of Lyapunov norms.

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1. Introduction

The main theme of our paper is the relation between the notion of nonuniform exponential dichotomy and the notion of strict Lyapunov sequence. Our study is somewhat motivated by related results in the case of exponential contractions when there exists only contraction and not simultaneously contraction and expansion. However, the existence of stable and unstable
directions causes several additional complications, which also require new ideas. Due to the use of cones we need the compactness of the closed unit ball in the ambient space, and thus we only consider finite-dimensional spaces.

The notion of (uniform) exponential dichotomy, introduced by Perron in [18], plays a central role in dynamics, particularly in the study of stable and unstable invariant manifolds. The theory of exponential dichotomies and its applications are well developed. In particular, there exist large classes of linear differential equations possessing exponential dichotomies. We refer to the books [5,9,10,19] for details and further references. We particularly recommend [5] for historical comments. The interested reader may also consult the books [6,7,15].

On the other hand, the notion of exponential dichotomy is too stringent for the dynamics and it is of considerable interest to look for more general types of hyperbolic behavior. We consider the more general notion of nonuniform exponential dichotomy. We refer to [3] for a systematic study of some of its consequences, in particular in connection with the existence and smoothness of invariant manifolds, the Grobman–Hartman theorem, and the existence of center manifolds, among other topics. In comparison with the classical notion of (uniform) exponential dichotomy, the existence of a nonuniform exponential dichotomy is more typical, although not only because it is a weaker assumption. In fact, perhaps surprisingly, essentially any linear equation \( x' = A(t)x \) in a finite-dimensional space with global solutions and with at least one negative Lyapunov exponent, has a nonuniform exponential dichotomy (see [3] for details). Moreover, at least from the point of view of ergodic theory the nonuniform part of the dichotomy can be made arbitrarily small for almost every trajectory, although not necessarily zero. This is a simple consequence of Oseledets’ multiplicative ergodic theorem in [17] (see [1] for a detailed discussion). Furthermore, by work of Barreira and Schmeling in [2], for certain classes of measure-preserving transformations, the nonuniform part of the dichotomy cannot be made zero in a set of topological entropy and Hausdorff dimension equal respectively to the topological entropy and Hausdorff dimension on the whole space.

According to Coppel in [6], the connection between Lyapunov functions and (uniform) exponential dichotomies was first considered by Maǐzel’ in [14]. We refer to the book by Mitropolsky, Samoilenko and Kulik [16] for a detailed discussion in the case of continuous time of the relation between Lyapunov functions and uniform exponential dichotomies. The use of Lyapunov functions in the study of the stability of trajectories in the theories of differential equations and dynamical systems, both in the finite and in the infinite-dimensional settings, goes back to the seminal work of Lyapunov in his 1892 thesis (see [13] for the most recent edition). Among the first accounts of the theory are the books by LaSalle and Lefschetz [12], Hahn [8], and Bhatia and Szegő [4]. Unfortunately, there exists no general method to construct explicitly Lyapunov functions for a given dynamics. In the context of ergodic theory, there is a related powerful approach. It started essentially with the work of Wojtkowski in [20] pointing out that to establish the existence of positive Lyapunov exponents it is often sufficient to have an invariant family of cones.

Our main objective is to show how a nonuniform exponential dichotomy can be completely characterized in terms of strict Lyapunov sequences. In particular, we obtain a complete characterization using quadratic Lyapunov sequences (see Section 6). We emphasize that we always consider the general case of a nonautonomous linear dynamics, and of nonuniform exponential dichotomies. We also discuss plenty examples illustrating the main notions and the main difficulties.

The optimal characterization of nonuniform exponential dichotomies uses “natural” Lyapunov functions, obtained explicitly from what are usually called Lyapunov norms, and with respect
to which the exponential behavior becomes uniform. To the best of our knowledge these are used here for the first time in connection to the characterization of nonuniform exponential dichotomies in terms of Lyapunov functions. Our work can be partly seen as a development of somewhat related approaches in the books by Dalec’kii and Kre˘ın [7, Chapter 2] and Massera and Schäffer [15, Chapter 9], which go back to Lyapunov in the finite-dimensional setting, although they only consider the case of uniform exponential behavior. The changes that are needed to treat the general case of arbitrary nonuniform exponential behavior are nontrivial.

2. Lyapunov sequences

2.1. Preliminaries

Given a function \( V : \mathbb{R}^p \to \mathbb{R} \) we consider the cones
\[
C^u(V) = \{0\} \cup V^{-1}(0, +\infty) \quad \text{and} \quad C^s(V) = \{0\} \cup V^{-1}(-\infty, 0).
\]

Let \( (A_m)_{m \in \mathbb{Z}} \) be a sequence of invertible \( p \times p \) matrices. We say that a sequence \( (V_m)_{m \in \mathbb{Z}} \) of continuous functions \( V_m : \mathbb{R}^p \to \mathbb{R} \) is a Lyapunov sequence for \( (A_m)_{m \in \mathbb{Z}} \) if there exist \( r_u, r_s \in \mathbb{N} \) with \( r_u + r_s = p \) such that for each \( m \in \mathbb{Z} \):

1. \( r_u \) and \( r_s \) are respectively the maximal dimensions of the linear subspaces inside \( C^u(V_m) \) and \( C^s(V_m) \);
2. for every \( x \in \mathbb{R}^p \) we have
\[
V_{m+1}(A_m x) \geq V_m(x). \tag{1}
\]

It follows from (1) that
\[
A_mC^u(V_m) \subset C^u(V_{m+1}), \tag{2}
\]

and
\[
A_m^{-1}C^s(V_m) \subset C^s(V_{m-1}). \tag{3}
\]

The cocycle \( A(m, n) \) associated to the sequence \( (A_m)_{m \in \mathbb{Z}} \) is defined for each \( m, n \in \mathbb{Z} \) by
\[
A(m, n) = \begin{cases} 
A_{m-1} \cdots A_n & \text{if } m > n, \\
\text{Id} & \text{if } m = n, \\
A_m^{-1} \cdots A_{n-1}^{-1} & \text{if } m < n.
\end{cases} \tag{4}
\]

For each \( m, n \in \mathbb{Z} \) we consider the sets
\[
C^u_{n,m} = A(n, m)C^u(V_m), \tag{5}
\]

and
\[
C^s_{n,m} = A(n, m)C^s(V_m). \tag{6}
\]
Proposition 1. If \((V_m)_{m \in \mathbb{Z}}\) is a Lyapunov sequence for \((A_m)_{m \in \mathbb{Z}}\), then for each \(n \in \mathbb{Z}\) the intersections

\[ E^u_n = \bigcap_{m \in \mathbb{Z}} C^u_{n,m} \subset \overline{C^u(V_n)} \quad \text{and} \quad E^s_n = \bigcap_{m \in \mathbb{Z}} C^s_{n,m} \subset \overline{C^s(V_n)} \quad (7) \]

contain subspaces respectively of dimensions \(r_u\) and \(r_s\).

Proof. By (2), for each \(n \in \mathbb{Z}\) we have

\[ \cdots \supset C^u_{n,1} \supset C^u_{n,0} \supset C^u_{n,-1} \supset \cdots. \]

On the other hand, by (5) and condition 1 in the notion of Lyapunov sequence, each set \(C^u_{n,m}\) contains a subspace of dimension \(r_u\). By the compactness of the closed unit ball in \(\mathbb{R}^p\), the intersection \(E^u_n\) also contains a subspace of dimension \(r_u\).

For the set \(E^s_n\) we note that it follows from (3) that for each \(n \in \mathbb{Z}\),

\[ \cdots \supset C^s_{n,-1} \supset C^s_{n,0} \supset C^s_{n,1} \supset \cdots. \]

Using (6) and identical arguments to those for \(E^u_n\), we conclude that \(E^s_n\) contains a subspace of dimension \(r_s\). \(\square\)

We emphasize that without further assumptions in general the intersections \(E^u_n\) and \(E^s_n\) need not be subspaces.

2.2. Lyapunov sequences and exponential behavior

Now we introduce the notion of strict Lyapunov sequence. Let \((V_m)_{m \in \mathbb{Z}}\) be a Lyapunov sequence for \((A_m)_{m \in \mathbb{Z}}\), and assume that there exist \(C > 0\) and \(\delta \geq 0\) such that

\[ \|V_m(x)\| \leq Ce^{\delta|m|}\|x\| \quad (8) \]

for every \(m \in \mathbb{Z}\) and \(x \in \mathbb{R}^p\). We say that \((V_m)_{m \in \mathbb{Z}}\) is a strict Lyapunov sequence if there exists \(\gamma \in (0, 1)\) such that for every \(m \in \mathbb{Z}\) and \(x \in \mathbb{R}^p\) we have:

1. \(V_{m+1}(A_m x) - V_m(x) \geq \gamma \|V_m(x)\|\); \( (9) \)
2. \(\|V_m(x)\| \geq \|x\|\) whenever \(V_m(x), V_{m+1}(A_m x) \leq 0\) or \(V_m(x), V_{m-1}(A_{m-1}^{-1} x) \geq 0\). \( (10) \)

Condition 2 in the notion of strict Lyapunov sequence essentially means that sufficiently close to the sets \(E^u_m\) and \(E^s_m\) the function \(x \mapsto \|V_m(x)\|\) behaves as a norm, up to the multiplicative factor \(Ce^{\delta|m|}\). Indeed, by (8), if (10) holds then

\[ \|x\| \leq \|V_m(x)\| \leq Ce^{\delta|m|}\|x\|. \]

We note that in Theorem 1, under the existence of a strict Lyapunov sequence, the sets \(E^u_m\) and \(E^s_m\) are shown to be subspaces. But when this is not yet known, condition (10) plays a corresponding
role. A posteriori our notion of strict Lyapunov sequence is completely justified by Theorems 1, 2 and 3 below which give a characterization of nonuniform exponential dichotomies in terms of strict Lyapunov sequences.

More generally, we say that \((V_m)_{m\in\mathbb{Z}}\) is an eventually strict Lyapunov sequence if there exist \(\gamma \in (0, 1)\) and \(N \in \mathbb{N}\) such that for every \(m \in \mathbb{Z}\) and \(x \in \mathbb{R}^p\) we have:

1. \[ V_{m+N}(A(m + N, m)x) - V_m(x) \geq \gamma |V_m(x)|; \] (11)

2. \[ |V_m(x)| \geq \|x\| \text{ whenever } V_m(x), V_{m+N}(A(m + N, m)x) \leq 0 \]

or

\[ V_m(x), V_{m-N}(A(m - N, m)x) \geq 0. \]

We note that any strict Lyapunov sequence is eventually strict (with \(N = 1\)).

The following result describes the consequences of the existence of an eventually strict Lyapunov sequence (and thus also of the existence of a strict Lyapunov sequence).

**Theorem 1.** If there exists an eventually strict Lyapunov sequence \((V_m)_{m\in\mathbb{Z}}\) for \((A_m)_{m\in\mathbb{Z}}\) satisfying

\[ (1 + \gamma)/(1 - \gamma) > e^{N\delta}, \] (12)

then:

1. for each \(n \in \mathbb{Z}\) the sets \(E^u_n\) and \(E^s_n\) in (7) are linear subspaces respectively of dimensions \(r_u\) and \(r_s\), and
   \[ \mathbb{R}^p = E^u_n \oplus E^s_n; \] (13)

2. for each \(m, n \in \mathbb{Z}\) we have
   \[ A(m, n)E^s_n = E^s_m \text{ and } A(m, n)E^u_n = E^u_m; \] (14)

3. there exist constants
   \[ \tilde{a} < 0 < b, \quad \varepsilon \geq 0, \quad \text{and } D \geq 1 \] (15)

such that for every \(m, n \in \mathbb{Z}\) with \(m \geq n\) we have

\[ \|A(m, n)E^s_n\| \leq De^{\tilde{a}(m-n)+\varepsilon|n|}; \] (16)

and

\[ \|A(m, n)^{-1}E^u_m\| \leq De^{-b(m-n)+\varepsilon|m|}. \] (17)

**Proof.** We start with an auxiliary statement.
Lemma 1. For each $n \in \mathbb{Z}$ we have
\[
\limsup_{m \to +\infty} \frac{1}{m} \log \|A(m, n)x\| > \frac{1}{N} \log(1 - \gamma) \quad \text{for } x \in E^u_n \setminus \{0\},
\]  
and
\[
\limsup_{m \to +\infty} \frac{1}{m} \log \|A(m, n)x\| \leq \frac{1}{N} \log(1 - \gamma) \quad \text{for } x \in E^s_n \setminus \{0\}.
\]

Proof. It follows from condition 2 in the notion of eventually strict Lyapunov sequence that the inclusions in (7) can be replaced by
\[
E^u_n \subset C^u(V_n) \quad \text{and} \quad E^s_n \subset C^s(V_n).
\]  
Indeed, if $x \in E^u_n \setminus \{0\}$, then by (7) we have $V_m(A(m, n)x) \geq 0$ for every $m \in \mathbb{Z}$. By condition 2 in the notion of eventually strict Lyapunov sequence we obtain $V_n(x) \geq \|x\| > 0$. This establishes the first inclusion in (20). A similar argument establishes the second one. By (20), the function $V_n$ is positive in $E^u_n \setminus \{0\}$ and negative in $E^s_n \setminus \{0\}$. In particular, we can set
\[
\kappa^u_{m,n} = \inf \left\{ \frac{V_m(A(m, n)x)}{V_n(x)} : x \in E^u_n \setminus \{0\} \right\}
\]  
and
\[
\kappa^s_{m,n} = \sup \left\{ \frac{|V_m(A(m, n)x)|}{|V_n(x)|} : x \in E^s_n \setminus \{0\} \right\}
\]
for each $m, n \in \mathbb{Z}$. Clearly, $\kappa^u_{m,n} > 0$ and $\kappa^s_{m,n} > 0$. Furthermore, since $A(l, n)A(n, m) = A(l, m)$ we have
\[
A(l, n)E^s_n = \bigcap_{m \in \mathbb{Z}} A(l, n)A(n, m)\overline{C^s(V_m)} = E^s_l
\]
for every $l, n \in \mathbb{Z}$. Using (23) we obtain
\[
\kappa^s_{m,n} = \sup \left\{ \frac{|V_m(A(m, n)x)|}{|V_l(A(l, n)x)|} : x \in E^s_n \setminus \{0\} \right\}
\]  
\[
\leq \sup \left\{ \frac{|V_m(A(m, l)y)|}{|V_l(y)|} : y \in A(l, n)E^s_n \setminus \{0\} \right\} \kappa^s_{l,n} = \kappa^s_{m,l} \kappa^s_{l,n}
\]  
for every $m, l, n \in \mathbb{Z}$. In particular, if $m \geq n$ then
\[
\kappa^s_{m,n} \leq \kappa^s_{m,n+rN} \prod_{j=0}^{r-1} \kappa^s_{n+(j+1)N,n+jN},
\]
where $r = \lfloor (m - n)/N \rfloor$ (here $\lfloor \cdot \rfloor$ denotes the integer part).
On the other hand, by (11), for each \( x \in E^s_m \setminus \{0\} \) we have

\[
V_{m+N}(A(m + N, m)x) \geq V_m(x) + \gamma |V_m(x)|,
\]

and thus,

\[
\frac{|V_{m+N}(A(m + N, m)x)|}{|V_m(x)|} \leq 1 - \gamma \in (0, 1).
\]

This implies that for each \( j \),

\[
\kappa^s_{n+(j+1)N,n+jN} \leq 1 - \gamma.
\]

Moreover, by (1) we have

\[
\frac{|V_{m+1}(A_mx)|}{|V_m(x)|} \leq 1 \quad \text{for every } x \in E^s_m \setminus \{0\}.
\]

By (24) this implies that \( \kappa^s_{m,n+rN} \leq 1 \). Therefore, it follows from (25) that

\[
\kappa^s_{m,n} \leq (1 - \gamma)^{(m-n)/N} - 1. \quad (26)
\]

Furthermore, by (23), for each \( x \in E^s_n \) we have \( A(m, n)x \in E^s_m \) for every \( m \in \mathbb{Z} \). Hence,

\[
V_m(A(m, n)x), V_{m+N}(A(m + N, n)x) \leq 0
\]

for every \( x \in E^s_n \), and it follows from condition 2 in the notion of eventually strict Lyapunov sequence that \( |V_m(A(m, n)x)| \geq \|A(m, n)x\| \). Therefore,

\[
\|A(m, n)x\| \leq |V_m(A(m, n)x)| \leq \kappa^s_{m,n} |V_n(x)|, \quad (27)
\]

and by (26) we obtain

\[
\limsup_{m \to +\infty} \frac{1}{m} \log \|A(m, n)x\| \leq \limsup_{m \to +\infty} \frac{\log \kappa^s_{m,n}}{m} \leq \frac{1}{N} \log(1 - \gamma) < 0.
\]

This establishes (19).

Now we consider the subspaces \( E^u_n \). Similarly, we have

\[
A(l, n)E^u_n = E^u_l \quad \text{for every } l, n \in \mathbb{Z}. \quad (28)
\]

Using (28) we can easily show that

\[
\kappa^u_{m,n} \geq \kappa^u_{m,n+rN} \prod_{j=0}^{r-1} \kappa^u_{n+(j+1)N,n+jN}, \quad (29)
\]

with \( r = \lfloor (m - n)/N \rfloor \). By (11) and (28), for each \( x \in E^u_n \setminus \{0\} \) we have

\[
\frac{V_{m+N}(A(m + N, m)x)}{V_m(x)} \geq 1 + \gamma > 1.
\]
Therefore,
\[ \kappa_{n+(j+1)N,n+jN}^u \geq 1 + \gamma. \]

Moreover, by (1) we have
\[ V_{m+1}(A_m x)/V_m(x) \geq 1 \quad \text{for every } x \in E^u_m \setminus \{0\}. \]

By (29) this implies that \( \kappa_{m,n+rN}^u \geq 1 \). Therefore, it follows from (29) that
\[ \kappa_{m,n}^u \geq (1 + \gamma)^{(m-n)/N-1}. \tag{30} \]

Furthermore, by (8), for each \( x \in E^u_n \) we have
\[ \|A(m,n)x\| \geq \frac{1}{C} e^{-\delta|m|} V_m(A(m,n)x) \geq \frac{1}{C} e^{-\delta|m|} \kappa_{m,n}^u V_n(x). \]

It follows from (30) and (12) that
\[ \limsup_{m \to +\infty} \frac{1}{m} \log \|A(m,n)x\| \geq -\delta + \frac{1}{N} \log(1 + \gamma) > \frac{1}{N} \log(1 - \gamma). \]

This establishes (18). \( \square \)

We proceed with the proof of Theorem 1.

**Lemma 2.** For each \( n \in \mathbb{Z} \) the sets \( E^u_n \) and \( E^s_n \) are linear subspaces respectively of dimensions \( r_u \) and \( r_s \).

**Proof.** Let \( D^u_n \subset E^u_n \) be any \( r_u \)-dimensional subspace, and let \( D^s_n \subset E^s_n \) be any \( r_s \)-dimensional subspace. Their existence is guaranteed by Proposition 1. By (20), we have \( E^u_n \cap E^s_n = \{0\} \), and hence \( D^u_n \cap D^s_n = \{0\} \). This implies that
\[ \mathbb{R}^p = D^u_n \oplus D^s_n. \tag{31} \]

Now we assume that \( E^s_n \setminus D^s_n \neq \emptyset \) and we proceed by contradiction. Take \( x \in E^s_n \setminus D^s_n \), and write \( x = y + z \) with \( y \in D^s_n \) and \( z \in D^u_n \). If \( z \neq 0 \), then by (18) and (19) we have
\[
\limsup_{m \to +\infty} \frac{1}{m} \log \|A(m,n)x\| \geq \max \left\{ \limsup_{m \to +\infty} \frac{1}{m} \log \|A(m,n)y\|, \limsup_{m \to +\infty} \frac{1}{m} \log \|A(m,n)z\| \right\} \\
= \limsup_{m \to +\infty} \frac{1}{m} \log \|A(m,n)z\| > \frac{1}{N} \log(1 - \gamma),
\]

which contradicts (19). Therefore \( z = 0 \), and \( x = y \in D^s_n \). But by hypothesis we also have \( x \in E^s_n \setminus D^s_n \). This contradiction shows that \( E^s_n \setminus D^s_n = \emptyset \), and hence \( E^s_n = D^s_n \) for each \( n \in \mathbb{Z} \). We show in a similar manner that \( E^u_n = D^u_n \) for each \( n \in \mathbb{Z} \). \( \square \)
Lemma 2 and (31) establish property 1. Property 2 follows from (23) and (28). Now we establish property 3. By (8), (26) and (27), we have
\[ \| A(m, n)x \| \leq \kappa_{m, n}^s|V_n(x)| \leq C(1 - \gamma)^{(m-n)/N-1}e^{\delta|m|}\|x\| \] (32)
for every \( m \geq n \) and \( x \in E_n^s \). On the other hand, it follows from (8) and (30) that
\[ \| A(m, n)x \| \geq \frac{1}{C}e^{-\delta|m|}k_{m, n+N}^uV_{n+N}(A(n + N, n)x) \geq \frac{1}{C}(1 + \gamma)^{(m-n-N)/N-1}e^{-\delta|m|}\|x\| \] (33)
for every \( m \geq n \) and \( x \in E_n^u \), and hence,
\[ \| A(m, n)^{-1}x \| \leq C(1 + \gamma)^2(1 + \gamma)^{-(m-n)/N}e^{\delta|m|}\|x\| \] (34)
for every \( m \geq n \) and \( x \in E_m^u \). By (32) and (34), we conclude that the sequence \( (A_m)_{m \in \mathbb{Z}} \) satisfies (16) and (17) with
\[ \tilde{a} = \frac{1}{N} \log(1 - \gamma) < 0, \quad b = \frac{1}{N} \log(1 + \gamma) > 0, \quad \varepsilon = \delta, \]
and
\[ D = C \max \left\{ \frac{1}{1 - \gamma}, (1 + \gamma)^2 \right\}. \]
This completes the proof of the theorem. \( \square \)

3. Nonuniform exponential dichotomies

3.1. Preliminaries

We denote by \( B(X) \) the space of bounded linear operators in a Banach space \( X \). We say that a sequence \( (A_m)_{m \in \mathbb{Z}} \) of invertible operators in \( B(X) \) admits a nonuniform exponential dichotomy if there exist projections \( P_m \in B(X) \), \( m \in \mathbb{Z} \) such that
\[ P_mA(m, n) = A(m, n)P_n, \quad m, n \in \mathbb{Z}, \] (35)
and constants as in (15) such that for every \( m, n \in \mathbb{Z} \) with \( m \geq n \) we have
\[ \| A(m, n)P_n \| \leq De^{\tilde{a}(m-n)+\varepsilon|m|}, \quad \| A(m, n)^{-1}Q_m \| \leq De^{-b(m-n)+\varepsilon|m|}, \] (36)
where \( Q_m = \text{Id} - P_m \) for each \( m \in \mathbb{Z} \). When \( (A_m)_{m \in \mathbb{Z}} \) admits a nonuniform exponential dichotomy, for each \( m \in \mathbb{Z} \) we define the stable and unstable subspaces by
\[ F_m^s = P_m(X) \quad \text{and} \quad F_m^u = Q_m(X). \] (37)
We also say that \((A_m)_{m \in \mathbb{Z}}\) admits a uniform exponential dichotomy if it admits a nonuniform exponential dichotomy with \(\varepsilon = 0\).

We give an explicit example of a sequence \((A_m)_{m \in \mathbb{Z}}\) admitting a nonuniform exponential dichotomy.

**Example 1.** Given \(\omega < 0\) and \(\varepsilon \geq 0\), we consider the matrices

\[
A_m = \begin{pmatrix}
e^{\omega + \varepsilon \left[ (-1)^m m - 1/2 \right]} & 0 \\
0 & e^{-\omega + \varepsilon \left[ (-1)^{m+1} m - 1/2 \right]}
\end{pmatrix}, \quad m \in \mathbb{Z}.
\]

We also consider the projections \(P_m\) and \(Q_m\) given by

\[
P_m(x, y) = (x, 0) \quad \text{and} \quad Q_m(x, y) = (0, y).
\]

Clearly, for every \(m \geq n\) we have

\[
A(m, n) = \begin{pmatrix}
e^{(\omega - \varepsilon/2)(m-n) + \varepsilon \sum_{k=n}^{m-1} (-1)^k} & 0 \\
0 & e^{-(\omega + \varepsilon/2)(m-n) + \varepsilon \sum_{k=n}^{m-1} (-1)^{k+1}}
\end{pmatrix},
\]

and (35) holds. We note that

\[
\sum_{k=1}^{l} (-1)^k k = (-1)^l \left\lfloor \frac{(l + 1)}{2} \right\rfloor
\]

for each \(l \in \mathbb{N}\), where \(\lfloor \cdot \rfloor\) denotes the integer part. Indeed, if \(l\) is even then

\[
\sum_{k=1}^{l} (-1)^k k = -\sum_{j=1}^{l/2} (2j - 1) + \sum_{j=1}^{l/2} 2j = \frac{l}{2} = (-1)^l \left\lfloor \frac{(l + 1)}{2} \right\rfloor,
\]

and if \(l\) is odd then

\[
\sum_{k=1}^{l} (-1)^k k = -\sum_{k=1}^{l-1} (-1)^k k - l = \frac{l-1}{2} - l = (-1)^l \left\lfloor \frac{(l + 1)}{2} \right\rfloor.
\]

Moreover, for each \(l \in \mathbb{Z}^-\) we have

\[
\sum_{k=l}^{-1} (-1)^k k = \sum_{j=1}^{-l} (-1)^{-j} (-j) = -\sum_{j=1}^{l} (-1)^j j = (-1)^{|l|+1} \left\lfloor \frac{(|l| + 1)}{2} \right\rfloor.
\]

We claim that for every \(m, n \in \mathbb{Z}\) with \(m \geq n\) we have

\[
\sum_{k=n}^{m-1} (-1)^k k \leq \frac{|m| + |n| + 2}{2}.
\]

This follows from (38) when \(m, n \in \mathbb{N}\). If \(m \in \mathbb{N}\) and \(n \in \mathbb{Z}^-\), then
Finally, if \( m, n \in \mathbb{Z}^- \) with \( m \geq n \), then

\[
\sum_{k=n}^{m-1} (-1)^k k = \sum_{k=n}^{-1} (-1)^k k - \sum_{k=m}^{-1} (-1)^k k
\]

\[
= (-1)^{|n|+1} \left( \left( |n| + 1 \right)/2 \right) + (-1)^{|m|-1} \left( |m| + 1 \right)/2
\]

\[
\leq (|m| + |n| + 2)/2.
\]

Using (39), for every \( m \geq n \) we obtain

\[
\left\| \mathcal{A}(m, n) P_n \right\| = e^{(\omega - \varepsilon/2)(m-n) + \varepsilon \sum_{k=n}^{m-1} (-1)^k k}
\]

\[
\leq e^{(\omega - \varepsilon/2)(m-n) + \varepsilon (|m|+|n|+2)/2}
\]

\[
\leq e^{(\omega - \varepsilon/2)(m-n) + \varepsilon |m-n|/2 + \varepsilon |n| + \varepsilon}
\]

\[
= e^{\varepsilon} e^{\omega (m-n) + \varepsilon |n|},
\]

and

\[
\left\| \mathcal{A}(m, n)^{-1} Q_m \right\| = e^{(\omega + \varepsilon/2)(m-n) + \varepsilon \sum_{k=n}^{m-1} (-1)^k k}
\]

\[
\leq e^{(\omega + \varepsilon/2)(m-n) + \varepsilon (|m|+|n|+2)/2}
\]

\[
\leq e^{(\omega + \varepsilon/2)(m-n) + \varepsilon |m| + \varepsilon |m-n|/2 + \varepsilon}
\]

\[
= e^{\varepsilon} e^{\omega (m-n) + \varepsilon |m|}.
\]

Therefore, \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential dichotomy with

\[
\alpha = \omega, \quad \beta = -\omega - \varepsilon, \quad \text{and} \quad D = e^{\varepsilon}
\]

provided that \( \varepsilon \) is sufficiently small so that \( \omega + \varepsilon < 0 \).

The following statement is of particular interest in the case of infinite-dimensional spaces. It shows that the norms \( \|P_m\| \) and \( \|Q_m\| \) are uniformly proportional to the inverse of what can be interpreted as an “angle” between the subspaces \( F^s_m \) and \( F^u_m \) (see (37) for the definition). For each \( m \in \mathbb{Z} \) we set

\[
\alpha_m = \inf \{ \|x - y\| : x \in F^s_m, \ y \in F^u_m, \ \|x\| = \|y\| = 1 \}.
\]
Proposition 2. For each \( m \in \mathbb{Z} \) we have

\[
\frac{1}{\|P_m\|} \leq \alpha_m \leq \frac{2}{\|P_m\|} \quad \text{and} \quad \frac{1}{\|Q_m\|} \leq \alpha_m \leq \frac{2}{\|Q_m\|}.
\]  

(41)

Proof. Given \( x \in F^s_m \) and \( y \in F^u_m \) with \( \|x\| = \|y\| = 1 \) we have \( P_m(x - y) = x \), and hence,

\[
1 = \|P_m(x - y)\| \leq \|P_m\| \cdot \|x - y\|.
\]

This shows that \( \alpha_m \geq 1/\|P_m\| \). On the other hand, for each \( x \in F^s_m \setminus \{0\} \) and \( y \in F^u_m \setminus \{0\} \) we have

\[
\frac{\|x\|}{\|y\|} = \frac{\|x - y\|}{\|y\|} = \frac{\|x\| + y(\|y\| - \|x\|)}{\|x\| \cdot \|y\|} \leq 2\|x - y\|/\|x\|.
\]

Note that \( P_m(x - y) = x \). Given \( \varepsilon > 0 \) we can choose \( x \in F^s_m \setminus \{0\} \) and \( y \in F^u_m \setminus \{0\} \) such that for \( z = x - y \) we have

\[
\frac{\|z\|}{\|P_mz\|} \leq \frac{1}{\|P_m\|} + \varepsilon.
\]

Therefore,

\[
\frac{\|x\|}{\|y\|} \leq 2\|z\|/\|P_mz\| \leq \frac{2}{\|P_m\|} + 2\varepsilon.
\]

Since \( \varepsilon \) is arbitrary we obtain the lower bounds in (41).

When \( X \) is a Hilbert space (and thus in particular when \( X \) is finite-dimensional), one can easily show that for each \( m \in \mathbb{Z} \) we have

\[
\|P_m\| = \|Q_m\| = \frac{1}{\alpha_m} = \frac{1}{2\sin(\beta_m/2)},
\]

(42)

where \( \beta_m = \angle(F^s_m, F^u_m) \).

3.2. Lyapunov sequences and exponential dichotomies

We note that the requirement of the existence of a nonuniform exponential dichotomy is stronger than what is proven in Theorem 1. Indeed, in that theorem we never obtain bounds involving projections \( P_m \) and \( Q_m \) but instead only their images, that is, the subspaces \( E^s_m \) and \( E^u_m \) (for more details see the discussion after Theorem 2). This motivates the following criterion for the existence of nonuniform exponential dichotomies in finite-dimensional spaces.

Theorem 2. For \( X = \mathbb{R}^p \), if there exists an eventually strict Lyapunov sequence \((V_m)_{m \in \mathbb{Z}}\) for \((A_m)_{m \in \mathbb{Z}}\) satisfying (12), and there exist constants \( c, \mu > 0 \) such that the subspaces \( E^u_m \) and \( E^s_m \) in (7) satisfy

\[
\angle(E^u_m, E^s_m) \geq ce^{-\mu|m|}, \quad m \in \mathbb{Z},
\]

(43)
then \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential dichotomy, with

\[
F^u_m = E^u_m \quad \text{and} \quad F^s_m = E^s_m \quad \text{for every } m \in \mathbb{Z}.
\]  

**Proof.** It follows from (13) that for each \(m \in \mathbb{Z}\) there exist projections

\[P_m : \mathbb{R}^p \to E^s_m \quad \text{and} \quad Q_m : \mathbb{R}^p \to E^u_m\]

with \(P_m + Q_m = \text{Id}\). Note that

\[
\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1 \quad \text{for } x \in (0, \pi/2].
\]  

(45)

By (42) (since \(X = \mathbb{R}^p\) is a Hilbert space) and (43) we obtain

\[
\|P_m\| = \|Q_m\| = \frac{1}{2 \sin(\beta_m/2)} \leq \frac{\pi}{2\beta_m} \leq \frac{\pi}{2c} e^{\mu|m|}.
\]  

(46)

Now observe that

\[
\|A(m, n) P_n\| \leq \|A(m, n)\| E^s_n \cdot \|P_n\|,
\]  

(47)

and

\[
\|A(m, n)^{-1} Q_m\| \leq \|A(m, n)^{-1}\| E^u_m \cdot \|Q_m\|.
\]  

(48)

Hence, it follows from (16) and (17) in Theorem 1 and (46) that

\[
\|A(m, n) P_n\| \leq \frac{D\pi}{2c} e^{a(m-n)+(\varepsilon+\mu)|n|},
\]

and

\[
\|A(m, n)^{-1} Q_m\| \leq \frac{D\pi}{2c} e^{-b(m-n)+(\varepsilon+\mu)|n|}.
\]

Furthermore, if \(x \in E^s_n\) then \(P_n x = x\), and since \(A(m, n)x \in E^s_m\) (see (14)) we obtain

\[
P_m A(m, n)x = A(m, n)x = A(m, n)P_n x.
\]

Moreover, if \(x \in E^u_n\) then \(A(m, n)x \in E^u_m\), and thus \(P_m A(m, n)x = 0 = A(m, n)P_n x\). This shows that (35) holds, and \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential dichotomy with stable and unstable subspaces as in (44). \(\square\)

By Theorem 1, if there exists an eventually strict Lyapunov sequence for \((A_m)_{m \in \mathbb{Z}}\) satisfying (12), then for each \(m \in \mathbb{Z}\) there exist subspaces \(E^s_m\) and \(E^u_m\) satisfying (13) and (14). Let \(P_n\) and \(Q_n\) be the projections obtained from the direct sum decomposition in (13). It follows easily from (14) that (35) holds. However, in general the sequence \((A_m)_{m \in \mathbb{Z}}\) in Theorem 1 need not admit a nonuniform exponential dichotomy. The reason is that the bounds in (16) and (17) are
only obtained respectively in the subspaces $E^s_n$ and $E^u_m$, and not in the whole space as in (36). More precisely, it follows from the first inequality in (36) that
\[ \| A(m, n) P_n x \| \leq D e^{a(m-n) + \varepsilon |n|} \| x \|, \quad x \in \mathbb{R}^p, \]  
(49)
while (16) gives
\[ \| A(m, n) P_n x \| \leq D e^{\bar{a}(m-n) + \varepsilon |n|} \| P_n x \|, \quad x \in \mathbb{R}^p. \]  
When $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential dichotomy it follows from (49) that $\| P_n \| \leq D e^{|n|}$. But otherwise, the norms $\| P_n \|$ may grow more than exponentially in $n$, in which case the sequence $(A_m)_{m \in \mathbb{Z}}$ does not admit a nonuniform exponential dichotomy. We give an explicit example.

**Example 2.** Consider the sequence of matrices $(A_m)_{m \in \mathbb{Z}}$ in Example 1. For each $m \in \mathbb{Z}$, take $\beta_m \in (0, \pi/2)$, and let
\[ R_m = \begin{pmatrix} 1 & \cos \beta_m \\ 0 & \sin \beta_m \end{pmatrix}. \]
Setting $B_m = R_{m+1} A_m R_m^{-1}$, by (4) we obtain
\[ \mathcal{B}(m, n) = R_m A(m, n) R_n^{-1}. \]  
(50)
For each $m \in \mathbb{Z}$, let $E^s_m$ and $E^u_m$ be respectively the one-dimensional subspaces generated by $(1, 0)$ and $(\cos \beta_m, \sin \beta_m)$. We have $\mathbb{R}^2 = E^s_m \oplus E^u_m$. Let also $P_m$ and $Q_m$ be the projections associated to this composition, with $P_m + Q_m = \text{Id}$. Since the entries of $R_m$ are bounded in $m$ we have
\[ D := \sup_{m \in \mathbb{Z}} \| R_m \| < \infty. \]  
(51)
Since the sequence $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential dichotomy (see Example 1), it follows from (50) and (51) that for every $m, n \in \mathbb{Z}$ with $m \geq n$ we have
\[ \| \mathcal{B}(m, n)| E^s_n \| = \| R_m A(m, n) | F^s_m \| \leq D C e^{ \bar{a}(m-n) + \varepsilon |n|}, \]
and
\[ \| \mathcal{B}(m, n)^{-1} | E^u_m \| = \| R_n A(m, n)^{-1} | F^u_m \| \leq D C e^{ -\underline{b}(m-n) + \varepsilon |m|},\]
with $\bar{a}$ and $\underline{b}$ as in (40). Moreover, we have $\beta_m = \angle (E^s_m, E^u_m)$ and it follows from (42) that
\[ \| P_m \| = \| Q_m \| = \frac{1}{2 \sin (\beta_m/2)}. \]
Therefore, the norms of the projections $P_m$ and $Q_m$ can be made arbitrarily large by making $\beta_m$ arbitrarily small. In particular, it is easy to choose the subspaces $E^s_m$ and $E^u_m$ so that condition (43) fails.
We note that condition (43) is not needed in the case of Lyapunov sequences obtained from quadratic forms. Namely, we show in Section 6 that at least in this case the strictness property implies condition (43), and thus also the existence of a nonuniform exponential dichotomy for a large class of sequences of matrices \((A_m)_{m \in \mathbb{Z}}\).

4. Construction of Lyapunov sequences

4.1. Lyapunov sequences for nonuniform exponential dichotomies

We show with an explicit construction that any sequence \((A_m)_{m \in \mathbb{Z}}\) admitting a nonuniform exponential dichotomy has a strict Lyapunov sequence. In fact, we obtain infinitely many strict Lyapunov sequences.

Theorem 3. For \(X = \mathbb{R}^p\), if a sequence \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential dichotomy, then it has a strict Lyapunov sequence. Moreover, if \(\varepsilon\) is sufficiently small, then (12) holds with \(N = 1\), i.e., \((1 + \gamma)/{(1 - \gamma)} > e^\delta\).

Proof. By hypothesis there exist projections \(P_n\) for each \(n \in \mathbb{Z}\), and constants \(a < b, \varepsilon \geq 0,\) and \(D \geq 1\) such that the cocycle \(A(m, n)\) satisfies (35) and (36). Choose \(\varrho > 0\) such that \(\varrho < \min\{-a, b\}\). Given \(r \in \mathbb{N}\), for each \(m \in \mathbb{Z}\) and \(x \in \mathbb{R}^p\) we set

\[
U_m(x) = -V_m^s(P_m x) + V_m^u(A_{m-1}^{-1}Q_m x),
\]

where

\[
V_m^s(x) = \sum_{k \geq m} \|A(k, m)x\|^{-r}(\varepsilon(a+\varrho)(k-m))e^{-r(a+\varrho)(k-m)}, x \in F^s_m,
\]

and

\[
V_m^u(x) = \sum_{k \leq m+1} \|A(m, k)^{-1}x\|^{-r}(\varepsilon(b-\varrho)(m+1-k))e^{r(b-\varrho)(m+1-k)}, x \in F^u_m.
\]

It is straightforward to verify that the two series converge, and that there exists a constant \(N > 0\) such that for every \(m \in \mathbb{Z}\) and \(x \in \mathbb{R}^p\) we have

\[
\|P_m x\| \leq V_m^s(P_m x)^{1/r} \leq N\varepsilon^{|m|}\|P_m x\|,
\]

and

\[
\|Q_m x\| \leq V_{m-1}^u(A_{m-1}^{-1}Q_m x)^{1/r} \leq N\varepsilon^{|m|}\|Q_m x\|.
\]

For each \(m \in \mathbb{Z}\) and \(x \in \mathbb{R}^p\) we set

\[
V_m(x) = \text{sign } U_m(x)[U_m(x)]^{1/r}.
\]

Clearly, \((V_m)_{m \in \mathbb{Z}}\) satisfies condition 1 in the notion of Lyapunov sequence, with \(r_u = \dim F^u_m\) and \(r_s = \dim F^s_m\) (we note that by (35) the dimensions are independent of \(m\)). Since the strictness of the Lyapunov sequence is stronger than condition 2, we only show that \((V_m)_{m \in \mathbb{Z}}\) is strict.
Given $m \in \mathbb{Z}$ and $x \in \mathbb{R}^p$ we write $x = y + z$ with $y \in F^s_m$ and $z \in F^u_m$. Then

$$U_{m+1}(A_m x) - U_m(x) = -V^s_{m+1}(A_m y) + V^s_m(y) + V^u_m(z) - V^u_{m-1}(A^{-1}_{m-1}z).$$

(56)

Since $A_m y \in F^s_{m+1}$, we obtain

$$-V^s_{m+1}(A_m y) + V^s_m(y) = -\sum_{k \geq m+1} \|A(k, m + 1)A_m y\| r^{-r(\bar{a} + \varrho)(k-m-1)} + \sum_{k \geq m} \|A(k, m)y\| r^{-r(\bar{\alpha} + \varrho)(k-m)}$$

$$= -\sum_{k \geq m+1} \|A(k, m)y\| r^{-r(\bar{\alpha} + \varrho)(k-m-1)} + e^{-r(\bar{a} + \varrho)} \sum_{k \geq m} \|A(k, m)y\| r^{-r(\bar{\alpha} + \varrho)(k-m-1)}$$

$$= e^{r(\bar{\alpha} + \varrho)}\|y\| r + (1 - e^{r(\bar{\alpha} + \varrho)}) V^s_m(y)$$

$$\geq (1 - e^{r(\bar{\alpha} + \varrho)}) V^s_m(y).$$

(57)

Furthermore, since $A_m z \in F^u_{m+1}$ we have

$$V^u_m(z) - V^u_{m-1}(A^{-1}_{m-1}z)$$

$$= \sum_{k \leq m+1} \|A(m, k)^{-1}z\| r^r(\bar{b} - \varrho)(m+1-k)$$

$$- \sum_{k \leq m} \|A(m - 1, k)^{-1}A^{-1}_{m-1}z\| r^{r(\bar{b} - \varrho)(m-k)}$$

$$= \|A_m z\|^r + (e^{r(\bar{b} - \varrho)} - 1) \sum_{k \leq m} \|A(m - 1, k)^{-1}A^{-1}_{m-1}z\| r^{r(\bar{b} - \varrho)(m-k)}$$

$$\geq (e^{r(\bar{b} - \varrho)} - 1) V^u_{m-1}(A^{-1}_{m-1}z).$$

(58)

By (57) and (58), since

$$|U_m(x)| \leq V^s_m(y) + V^u_{m-1}(A^{-1}_{m-1}z),$$

setting

$$\eta = \min\left\{1 - e^{r(\bar{\alpha} + \varrho)}, e^{r(\bar{b} - \varrho)} - 1\right\}$$

it follows from (56) that

$$U_{m+1}(A_m x) - U_m(x) \geq \eta \left(V^s_m(y) + V^u_{m-1}(A^{-1}_{m-1}z)\right) \geq \eta |U_m(x)|.$$ 

(59)
Now we show that (9) holds. If $U_m(x) \geq 0$, then
\[ U_{m+1}(A_m x) \geq (1 + \eta) U_m(x), \]
and
\[ V_{m+1}(A_m x) = U_{m+1}(A_m x)^{1/r} \geq (1 + \eta)^{1/r} U_m(x)^{1/r} = (1 + \eta)^{1/r} V_m(x), \]
that is, (9) holds with $\gamma = (1 + \eta)^{1/r} - 1$. If $U_m(x) < 0$, then
\[ U_{m+1}(A_m x) \geq (1 - \eta) U_m(x) < 0. \]
We consider two subcases. If $U_m(x) < 0$ and $U_{m+1}(A_m x) \leq 0$, then
\[ 0 \leq -U_{m+1}(A_m x) \leq -(1 - \eta) U_m(x), \]
and
\[ V_{m+1}(A_m x) \geq (1 - \eta)^{1/r} V_m(x). \]
This shows that (9) holds with $\gamma = 1 - (1 - \eta)^{1/r}$. Finally, if $U_m(x) < 0$ and $U_{m+1}(A_m x) > 0$, then
\[ V_{m+1}(A_m x) - V_m(x) = V_{m+1}(A_m x) + |V_m(x)| \geq |V_m(x)|, \]
and (9) holds $\gamma = 1$. Therefore, (9) holds with
\[ \gamma < \min \{ 1, (1 + \eta)^{1/r} - 1, 1 - (1 - \eta)^{1/r} \}. \]

Now we establish condition 2 in the notion of strict Lyapunov sequence. By (53) and (54), we have
\[ V_m^S(y) \geq \|y\|^r \quad \text{and} \quad V_{m-1}^u(A_{m-1}^{-1} z) \geq \|z\|^r. \quad (60) \]
Thus, it follows from the first inequality in (59) that
\[ U_{m+1}(A_m x) - U_m(x) \geq \eta \left( \|y\|^r + \|z\|^r \right) \geq \eta \max \{ \|y\|^r, \|z\|^r \} \geq \eta \left( \frac{\|y\|^r + \|z\|^r}{2} \right)^r \geq \frac{\eta}{2^r} \|x\|^r. \quad (61) \]
If $V_m(x) \leq 0$ and $V_{m+1}(A_m x) \leq 0$, then $U_m(x) \leq 0$ and $U_{m+1}(A_m x) \leq 0$, which implies that
\[ |U_m(x)| \geq |U_m(x)| - |U_{m+1}(A_m x)| = U_{m+1}(A_m x) - U_m(x) \geq \frac{\eta}{2^r} \|x\|^r. \]
Therefore,

\[|V_m(x)| \geq \|x\|^{1/r}/2. \tag{62} \]

Now we consider the case when

\[V_m(x) \geq 0 \quad \text{and} \quad V_{m-1}(A_{m-1}^{-1}x) \geq 0. \tag{63} \]

By (56) we have

\[U_m(x) - U_{m-1}(A_{m-1}^{-1}x) = -V_m^s(y) + V_{m-1}^s(A_{m-1}^{-1}y) \]
\[+ V_{m-1}^u(A_{m-1}^{-1}z) - V_{m-2}^u(A_{m-2}^{-1}A_{m-1}^{-1}z). \tag{64} \]

Since \(y \in E_m^s\), proceeding as in (57) and using the first inequality in (60) we obtain

\[-V_m^s(y) + V_{m-1}^s(A_{m-1}^{-1}y) = -\sum_{k \geq m} \|A(k, m)y\|^r e^{-r(\bar{\alpha} + \varrho)(k-m)} \]
\[+ \sum_{k \geq m-1} \|A(k, m-1)A_{m-1}^{-1}y\|^r e^{-r(\bar{\alpha} + \varrho)(k-m+1)} \]
\[= -\sum_{k \geq m} \|A(k, m)y\|^r e^{-r(\bar{\alpha} + \varrho)(k-m)} \]
\[+ e^{-r(\bar{\alpha} + \varrho)} \sum_{k \geq m-1} \|A(k, m)y\|^r e^{-r(\bar{\alpha} + \varrho)(k-m)} \]
\[\geq \|A_{m-1}^{-1}y\|^r + (e^{-r(\bar{\alpha} + \varrho)} - 1)V_m^s(y) \]
\[\geq (e^{-r(\bar{\alpha} + \varrho)} - 1) \|y\|^r. \]

Furthermore, by (58) we have

\[V_{m-1}^u(A_{m-1}^{-1}z) - V_{m-2}^u(A_{m-2}^{-1}A_{m-1}^{-1}z) \]
\[= \|z\|^r + (e^{r(\bar{\alpha} - \varrho)} - 1)V_{m-2}^u(A_{m-2}^{-1}A_{m-1}^{-1}z) \geq \|z\|^r. \]

Therefore, proceeding as in (61) and setting \(\bar{\eta} = \min\{e^{-r(\bar{\alpha} + \varrho)} - 1, 1\}\) it follows from (64) that

\[U_m(x) - U_{m-1}(A_{m-1}^{-1}x) \geq (e^{-r(\bar{\alpha} + \varrho)} - 1) \|y\|^r + \|z\|^r \]
\[\geq \bar{\eta}(\|y\|^r + \|z\|^r) \geq \frac{\bar{\eta}}{2r} \|x\|^r. \]

By (63), we have \(U_m(x) \geq 0\) and \(U_{m-1}(A_{m-1}^{-1}x) \geq 0\), which implies that

\[U_m(x) \geq U_m(x) - U_{m-1}(A_{m-1}^{-1}x) \geq \frac{\bar{\eta}}{2r} \|x\|^r. \]
Therefore,
\[
|V_m(x)| = V_m(x) \geq \|x\| \bar{\eta}^{1/r}/2.
\] (65)

By (62) and (65), a constant multiple \((V_m)m\in\mathbb{Z}\) of the sequence \((V_m)m\in\mathbb{Z}\) satisfies
\[
|V_m(x)| \geq \|x\|
\]
whenever (10) holds. We note that \((V_m)m\in\mathbb{Z}\) continues to satisfy (9), with the same constant \(\gamma\).

Finally, by (53) and (54) we obtain
\[
|U_m(x)| \leq V_m^s(y) + V_m^u(A^{-1}_mz)
\leq 2N'r e^{r|m|}(\|y\|^r + \|z\|^r)
\leq 2N'r e^{r|m|}(\|Pmx\|^r + \|Qmx\|^r)
\leq 4N'r D'e^{2r|m|}\|x\|^r,
\] (66)
and thus (8) holds. Therefore, \((V_m)m\in\mathbb{Z}\) is a strict Lyapunov sequence for \((A_m)m\in\mathbb{Z}\). For the last property it is sufficient to note that \(\delta = 2\epsilon\) (see (66)).

We note that the statement in Theorem 3 can be extended to any Banach space (with the same proof), although possibly with the dimensions \(r_u\) or \(r_s\) infinite.

**Example 3.** Consider the sequence of matrices \((A_m)m\in\mathbb{Z}\) in Example 1, and take \(r = 1\) and \(\varphi \in (0, -\omega)\). Using (52) and (55), for each \(m \in \mathbb{Z}\) and \((x, y) \in \mathbb{R}^2\) we set
\[
V_m(x, y) = -U_m^s(x) + U_m^u(y),
\]
where
\[
U_m^s(x) = \sum_{k \geq m} e^{(\omega - \epsilon/2)(k-m) + \epsilon \sum_{j=m}^{k-1} (-1)^j j |x| e^{-(\omega + \varphi)(k-m)}
\]
\[
= \sum_{k \geq m} e^{-(\varphi + \epsilon/2)(k-m) + \epsilon \sum_{j=m}^{k-1} (-1)^j j |x|}
\]
and
\[
U_m^u(y) = \sum_{k \leq m} e^{(\omega + \epsilon/2)(m-k) + \epsilon \sum_{j=k}^{m-1} (-1)^j j |y| e^{-(\omega + \varphi)(m-k)}
\]
\[
= \sum_{k \leq m} e^{-(\varphi + \epsilon/2)(m-k) + \epsilon \sum_{j=k}^{m-1} (-1)^j j |y|}
\]

It follows from the proof of Theorem 3 that \((V_m)m\in\mathbb{Z}\) is a strict Lyapunov sequence for \((A_m)m\in\mathbb{Z}\).

**4.2. The case of uniform exponential dichotomies**

The following result is a combination of appropriate versions of Theorems 1, 2, and 3 in the case of uniform exponential dichotomies.
Theorem 4. When $X = \mathbb{R}^p$ the following properties are equivalent:

1. $(A_m)_{m \in \mathbb{Z}}$ admits a uniform exponential dichotomy;
2. there exists a strict Lyapunov sequence for $(A_m)_{m \in \mathbb{Z}}$ with $\delta = 0$, and the subspaces $E_m^u$ and $E_m^s$ in (7) satisfy $\inf_{m \in \mathbb{Z}} \angle(E_m^u, E_m^s) > 0$.

Proof. We first assume that $(A_m)_{m \in \mathbb{Z}}$ admits a strict Lyapunov sequence $(V_m)_{m \in \mathbb{Z}}$ as in property 2. Proceeding as in (32) we obtain

$$\|A(m,n)P_n\| \leq C(1 - \gamma)^{m-n}$$

for every $m \geq n$. Moreover, proceeding as in (33) yields

$$\|A(m,n)^{-1}E_m^u\| \leq C(1 + \gamma)^{-(m-n)}$$

for every $m \geq n$. Finally, setting $\mu = 0$ in (43), it follows from (46) that

$$\|Pn\| = \|Qn\| \leq \pi/(2c)$$

for every $n \in \mathbb{Z}$, where $c = \inf_{m \in \mathbb{Z}} \angle(E_m^u, E_m^s)$. Combined with (47) and (48) this shows that the sequence $(A_m)_{m \in \mathbb{Z}}$ admits a uniform exponential dichotomy.

Now we assume that the sequence $(A_m)_{m \in \mathbb{Z}}$ admits a uniform exponential dichotomy. Let $(V_m)_{m \in \mathbb{Z}}$ be the strict Lyapunov sequence given by (55), for some $r \in \mathbb{N}$. It follows from (66) with $\epsilon = 0$ that we can take $\delta = 0$. Furthermore, by (36) with $\epsilon = 0$ and $m = n$ we obtain

$$\|P_n\| = \|Q_n\| \leq D \quad \text{for every } m \in \mathbb{Z}.$$ 

Finally, it follows from (42) and (45) that

$$\angle(F_m^u, F_m^s) \geq 2 \sin \frac{\angle(F_m^u, F_m^s)}{2} = \frac{1}{\|P_m\|} \geq \frac{1}{D}.$$ 

This completes the proof of the theorem. □

5. Strong nonuniform exponential dichotomies

We consider in this section a strong version of nonuniform exponential dichotomy. For simplicity of the exposition we consider a finite-dimensional setting from the beginning. We say that a sequence $(A_m)_{m \in \mathbb{Z}}$ of invertible $p \times p$ matrices admits a strong nonuniform exponential dichotomy if there exist projections $P_m, m \in \mathbb{Z}$, satisfying (35), and there exist constants

$$a \leq \bar{a} < 0 < b < \bar{b}, \quad \epsilon \geq 0, \quad \text{and} \quad D \geq 1$$

such that for every $m, n \in \mathbb{Z}$ with $m \geq n$ we have

$$\|A(m,n)P_n\| \leq De^{\bar{a}(m-n)+\epsilon|n|}, \quad \|A(m,n)^{-1}Q_m\| \leq De^{-b(m-n)+\epsilon|m|}, \quad (67)$$
and for every \( m, n \in \mathbb{Z} \) with \( m \leq n \) we have

\[
\|A(m, n)P_n\| \leq De^{a(m-n)+\varepsilon|n|}, \quad \|A(m, n)^{-1}Q_m\| \leq De^{-b(m-n)+\varepsilon|m|},
\]

(68)

where \( Q_m = \text{Id} - P_m \) for each \( m \in \mathbb{Z} \). We also say that \( (A_m)_{m \in \mathbb{Z}} \) admits a strong uniform exponential dichotomy if it admits a strong nonuniform exponential dichotomy with \( \varepsilon = 0 \).

Clearly, if the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits a nonuniform exponential dichotomy and is bounded, then it also admits a strong nonuniform exponential dichotomy. The following example shows that an unbounded sequence may also admit a strong nonuniform exponential dichotomy.

**Example 4.** Given \( \omega < 0 \) and \( \varepsilon \geq 0 \), we consider the matrices \( A_m \) in (1), and the projections \( P_m \) and \( Q_m \) in (1). We know from Example 1 that the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits a nonuniform exponential dichotomy with the constants in (40). Furthermore, for every \( m \leq n \) it follows from (40) that

\[
\|A(m, n)P_n\| \leq e^{(\omega-\varepsilon/2)(m-n)+\varepsilon|m-n|/2+\varepsilon|n|+\varepsilon} = e^\varepsilon e^{(\omega-\varepsilon)(m-n)+\varepsilon|n|},
\]

and it follows from (40) that

\[
\|A(m, n)^{-1}Q_m\| \leq e^{(\omega+\varepsilon/2)(m-n)+\varepsilon|m|+\varepsilon|n-m|/2+\varepsilon} = e^\varepsilon e^{\omega(m-n)+\varepsilon|m|}.
\]

Therefore, \( (A_m)_{m \in \mathbb{Z}} \) admits a strong nonuniform exponential dichotomy with

\[
a = \omega - \varepsilon, \quad a = \omega, \quad b = -\omega - \varepsilon, \quad b = -\omega \quad \text{and} \quad D = e^\varepsilon
\]

provided that \( \varepsilon \) is sufficiently small so that \( \omega + \varepsilon \leq 0 \).

A simple example of a nonuniform exponential dichotomy which is not a strong nonuniform exponential dichotomy is the following.

**Example 5.** Consider the matrices

\[
A_m = \begin{pmatrix} e^{-(m+1/2)} & 0 \\ 0 & 2 \end{pmatrix}, \quad m \in \mathbb{Z}.
\]

One can easily verify that for each \( m, n \in \mathbb{Z} \) with \( m \geq n \) the first entry of \( A(m, n) \) is \( e^{(n^2-m^2)/2} \leq e^{-(m-n)} \). But it is impossible to choose constants \( a < 0, \varepsilon \geq 0, \) and \( D \geq 1 \) satisfying the first inequality in (68).

The following is a version of Theorem 2 for strong nonuniform exponential dichotomies (replacing eventually strict by strict Lyapunov sequences).
Theorem 5. Assume that:

1. there exists a strict Lyapunov sequence for \((A_m)_{m \in \mathbb{Z}}\) satisfying (12);
2. the subspaces \(E^u_m\) and \(E^s_m\) satisfy (43);
3. there exist \(\mu_u, \mu_s \geq \gamma\) with \(\mu_s < 1\) such that for every \(m \in \mathbb{Z}\) and \(x \in E^u_m\) we have

\[
V_{m+1}(A_m x) - V_m(x) \leq \mu_u V_m(x),
\]

and for every \(m \in \mathbb{Z}\) and \(x \in E^s_m\) we have

\[
V_{m+1}(A_m x) - V_m(x) \leq \mu_s |V_m(x)|.
\]

Then \((A_m)_{m \in \mathbb{Z}}\) admits a strong nonuniform exponential dichotomy.

Proof. We already know from Theorem 2 that \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential dichotomy. It remains to establish the inequalities in (68). It follows from (70) that for each \(x \in E^s_m \setminus \{0\}\) we have

\[
\frac{|V_m(x)|}{|V_{m+1}(A_m x)|} \leq \frac{1}{1 - \mu_s}.
\]

Proceeding as in the proof of Lemma 1 we obtain

\[
k^s_{j,j+1} \leq \frac{1}{1 - \mu_s} \quad \text{for every} \ j \in \mathbb{Z},
\]

with \(k^s_{m,n}\) as in (22). Furthermore, it follows from (24) that for every \(m, n \in \mathbb{Z}\) with \(m \leq n\) we have

\[
k^s_{m,n} \leq \prod_{j=m}^{n-1} k^s_{j,j+1} \leq \left( \frac{1}{1 - \mu_s} \right)^{n-m}.
\]

Hence, by condition 2 in the notion of strict Lyapunov sequence and (8) we obtain

\[
\|A(m,n)x\| \leq |V_m(A(m,n)x)| \leq k^s_{m,n} |V_n(x)| \leq C \left( \frac{1}{1 - \mu_s} \right)^{n-m} e^{\delta |n|} \|x\|
\]

for every \(m \leq n\) and \(x \in E^s_n\) (note that (10) holds since \(V_m(x) \leq 0\) and \(V_{m+1}(A_m x) \leq 0\) for every \(x \in E^s_m\)). Thus, the first inequality in (68) holds with

\[
a = \log(1 - \mu_s), \quad \varepsilon = \delta, \quad \text{and} \quad D = C.
\]

Similarly, by (69), if \(x \in E^u_m \setminus \{0\}\) then

\[
\frac{V_m(x)}{V_{m+1}(A_m x)} \geq \frac{1}{1 + \mu_u}.
\]
Therefore,

\[ \kappa_{j,j+1}^u \geq \frac{1}{1 + \mu_u} \quad \text{for every } j \in \mathbb{Z}, \]

with \( \kappa_{m,n}^u \) as in (21). Furthermore, for every \( m, n \in \mathbb{Z} \) with \( m \leq n \) we have

\[ \kappa_{m,n}^u \geq \prod_{j=m}^{n-1} \kappa_{j,j+1}^u \geq \left( \frac{1}{1 + \mu_u} \right)^{n-m}. \]

Hence, by condition 2 in the notion of strict Lyapunov sequence and (8) we obtain

\[
\| A(m,n)x \| \geq \frac{1}{C} e^{-\delta |m|} V_m(A(m,n)x) \\
\geq \frac{1}{C} e^{-\delta |m|} \kappa_{m,n}^u V_n(x) \\
\geq \frac{1}{C} \left( \frac{1}{1 + \mu_u} \right)^{n-m} e^{-\delta |m|} \| x \|
\]

for every \( m \leq n \) and \( x \in E_n^u \) (note that (10) holds since \( V_m(x) \geq 0 \) and \( V_{m-1}(A_{m-1}^{-1}x) \geq 0 \) for every \( x \in E_m^u \)). Thus, the second inequality in (68) holds with

\[ \tilde{b} = \log(1 + \mu_u), \quad \epsilon = \delta, \quad \text{and} \quad D = C. \]

This completes the proof of the theorem. \( \square \)

We also establish a version of Theorem 3 for strong nonuniform exponential dichotomies.

**Theorem 6.** If the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a strong nonuniform exponential dichotomy, then it admits a strict Lyapunov sequence satisfying property 3 in Theorem 5.

**Proof.** Choose \( \rho > 0 \) such that \( \rho < \min\{-\tilde{a}, \tilde{b}\} \). For each \( m \in \mathbb{Z} \) and \( x \in \mathbb{R}^p \) we set

\[ V_m(x) = -V_m^s(P_m x) + V_m^u(A_m^{-1} Q_m x), \]

where

\[
V_m^s(x) = \sum_{k \geq m} \| A(k,m)x \| e^{-(\tilde{a}+\rho)(k-m)} \\
+ \sum_{k \leq m-1} \| A(k,m)x \| e^{(\rho-\tilde{a})(m-k)}
\]

for \( x \in E_m^s \), and where
\[ V_m^u(x) = \sum_{k \leq m+1} \|A(m, k)^{-1}x\| e^{(b-\varrho)(m+1-k)} \]
\[ + \sum_{k \geq m+2} \|A(m, k)^{-1}x\| e^{-(b+\varrho)(k-m-1)} \]

for \(x \in F_m^n\). It follows readily from (67) and (68) that the four series converge. Writing \(y = P_m x\) and \(z = Q_m x\) we obtain

\[ \left| V_m(x) \right| \leq V_m^s(y) + V_m^u \left(A_{m-1}^{-1} z \right) \]
\[ \leq 2De^{\epsilon|m|} \|x\| \left( \sum_{k \geq m} e^{-\varrho(k-m)} + e^{b+\varrho} \sum_{k \leq m+1} e^{-\varrho(m-k)} \right) \]
\[ = Ce^{\epsilon|m|} \|x\| \]

for some constant \(C > 0\), and (8) holds with \(\delta = \epsilon\). Moreover,

\[-V_{m+1}^s(A_m y) + V_m^s(y) \]
\[= -\sum_{k \geq m+1} \|A(k, m+1)A_m y\| e^{-(\bar{\alpha}+\varrho)(k-m-1)} \]
\[+ \sum_{k \geq m} \|A(k, m)y\| e^{-(\bar{\alpha}+\varrho)(k-m)} \]
\[ - \sum_{k \leq m} \|A(k, m+1)A_m y\| e^{(\bar{\alpha}-\varrho)(m+1-k)} \]
\[+ \sum_{k \leq m-1} \|A(k, m)y\| e^{(\bar{\alpha}-\varrho)(m-k)} \]
\[= e^{\bar{\alpha}+\varrho} \|y\| + (1 - e^{\bar{\alpha}+\varrho}) \sum_{k \geq m} \|A(k, m)x\| e^{-(\bar{\alpha}+\varrho)(k-m)} \]
\[ - e^{\bar{\alpha}-\varrho} \|y\| + (1 - e^{\bar{\alpha}-\varrho}) \sum_{k \leq m-1} \|A(k, m)y\| e^{(\bar{\alpha}-\varrho)(m-k)} \]
\[= (e^{\bar{\alpha}+\varrho} - e^{\bar{\alpha}-\varrho}) \|y\| + (1 - e^{\bar{\alpha}+\varrho}) V_m^s(y) \]
\[+ (e^{\bar{\alpha}+\varrho} - e^{\bar{\alpha}-\varrho}) \sum_{k \leq m-1} \|A(k, m)y\| e^{(\bar{\alpha}-\varrho)(m-k)} \]
\[= (e^{\bar{\alpha}+\varrho} - e^{\bar{\alpha}-\varrho}) \left( \|y\| + \sum_{k \leq m-1} \|A(k, m)y\| e^{(\bar{\alpha}-\varrho)(m-k)} \right) \]
\[+ (1 - e^{\bar{\alpha}+\varrho}) V_m^s(y), \]

and
\[ V_m^u(z) - V_{m-1}^u(A_{m-1}z) \]
\[ = \sum_{k \leq m+1} \|A(m, k)^{-1}z\| e^{(b-\varrho)(m+1-k)} \]
\[ - \sum_{k \leq m} \|A(m-1, k)^{-1}A_{m-1}^{-1}z\| e^{(b-\varrho)(m-k)} \]
\[ + \sum_{k \geq m+2} \|A(m, k)^{-1}z\| e^{-(\beta+\varrho)(k-m-1)} \]
\[ - \sum_{k \geq m+1} \|A(m-1, k)^{-1}A_{m-1}^{-1}z\| e^{-(\beta+\varrho)(k-m)} \]
\[ = \|A_m z\| + (e^{b-\varrho} - 1) \sum_{k \leq m} \|A(m, k)^{-1}z\| e^{(b-\varrho)(m-k)} \]
\[ - \|A_m z\| + (e^{\beta+\varrho} - 1) \sum_{k \geq m+1} \|A(m, k)^{-1}z\| e^{-(\beta+\varrho)(k-m)} \]
\[ = (e^{b-\varrho} - 1)V_{m-1}^u(A_{m-1}^{-1}z) \]
\[ + (e^{\beta+\varrho} - e^{b-\varrho}) \sum_{k \geq m+1} \|A(m, k)^{-1}z\| e^{-(\beta+\varrho)(k-m)} \].

Since \( a \leq \bar{a} < 0 \) (and thus \( e^{\beta+\varrho} - e^{\bar{a}-\varrho} > 0 \)), we obtain
\[ -V_m^s(A_m y) + V_m^s(y) \geq (1 - e^{\beta+\varrho}) V_m^s(y). \]

Similarly, since \( e^{\beta+\varrho} > e^{b-\varrho} \) we have
\[ V_m^u(z) - V_{m-1}^u(A_{m-1}^{-1}z) \geq (e^{b-\varrho} - 1)V_{m-1}^u(A_{m-1}^{-1}z). \]

This shows that
\[ V_{m+1}(A_m x) - V_m(x) \geq \min\{1 - e^{\beta+\varrho}, e^{b-\varrho} - 1\} \left[ V_m^s(y) + V_{m-1}^u(A_{m-1}^{-1}z) \right], \]
and we can take \( \gamma = \min\{1 - e^{\beta+\varrho}, e^{b-\varrho} - 1\} \) in (9).

Moreover, if \( V_m(x) \leq 0 \) and \( V_{m+1}(A_m x) \leq 0 \), taking into account that
\[ V_m^s(y) \geq \|y\| \quad \text{and} \quad V_{m-1}^u(A_{m-1}^{-1}z) \geq \|z\| \]
we obtain
\[ |V_m(x)| \geq |V_m(x) - V_{m+1}(A_m x)| \]
\[ = V_{m+1}(A_m x) - V_m(x) \]
\[ \geq \gamma \left[ V_m^s(y) + V_{m-1}^u(A_{m-1}^{-1}z) \right] \]
\[ \geq \gamma (\|y\| + \|z\|) \geq \gamma \|x\|. \]
Furthermore, if $V_m(x) \geq 0$ and $V_{m-1}(A_{m-1}^{-1}x) \geq 0$ we have

$$|V_m(x)| \geq V_m(x) - V_{m-1}(A_{m-1}^{-1}x) \geq \gamma \left[ V_{m-1}(A_{m-1}^{-1}y) + V_{m-2}(A_{m-2}^{-1}A_{m-1}^{-1}z) \right].$$

Taking $k = m - 1$ in the first sum in the definition of $V_{m-1}^s(A_{m-1}^{-1}y)$ and $k = m$ in the second sum in the definition of $V_{m-2}^u(A_{m-2}^{-1}A_{m-1}^{-1}z)$ we obtain respectively

$$V_{m-1}^s(A_{m-1}^{-1}y) \geq \|y\| \quad \text{and} \quad V_{m-2}^u(A_{m-2}^{-1}A_{m-1}^{-1}z) \geq e^{-(\tilde{\beta}+\varrho)}\|z\|.$$

Therefore,

$$|V_m(x)| \geq \gamma \left( \|y\| + e^{-(\tilde{\beta}+\varrho)}\|z\| \right) \geq \gamma e^{-(\tilde{\beta}+\varrho)}\|x\|.$$

This shows that $(V_m)_{m \in \mathbb{Z}}$ is a strict Lyapunov sequence for $(A_m)_{m \in \mathbb{Z}}$.

The sequence $(V_m)_{m \in \mathbb{Z}}$ has associated subspaces $E_m^u$ and $E_m^s$ for each $m \in \mathbb{Z}$, which satisfy property 3 in Theorem 1. On the other hand, it follows readily from (36) that the subspaces $F_m^u$ and $F_m^s$ also satisfy property 3 in Theorem 1. Proceeding as in the proof of Lemma 2 we find that $F_m^u = E_m^u$ and $F_m^s = E_m^s$ for every $m \in \mathbb{Z}$. Furthermore, since

$$\|y\| \leq \sum_{k \geq m} \|A(k, m)y\| e^{-(\tilde{\alpha}+\varrho)(k-m)},$$

we have

$$-V_{m+1}^s(A_my) + V_m^s(y) \leq (e^{\tilde{\alpha}+\varrho} - e^{\tilde{\alpha}-\varrho})V_m^s(x) + (1 - e^{\tilde{\alpha}+\varrho})V_m^s(y) = (1 - e^{\tilde{\alpha}-\varrho})V_m^s(y).$$

This establishes (70) when $x \in F_m^s = E_m^s$, taking $\mu_s = 1 - e^{\tilde{\alpha}-\varrho} \in [\gamma, 1)$. Similarly, since

$$\sum_{k \geq m+1} \|A(m, k)^{-1}z\| e^{-(\tilde{\beta}+\varrho)(k-m)} \leq V_{m-1}^u(A_{m-1}^{-1}z),$$

we have

$$V_m^u(z) - V_{m-1}^u(A_{m-1}^{-1}z) \leq (e^{\tilde{\beta}+\varrho} - 1)V_{m-1}^u(A_{m-1}^{-1}z).$$

This establishes (69) when $x \in F_m^u = E_m^u$, taking $\mu_u = e^{\tilde{\beta}+\varrho} - 1 > \gamma$. \qed
6. Quadratic Lyapunov sequences

We consider in this section the particular case of quadratic Lyapunov sequences, that is, Lyapunov sequences obtained from quadratic forms. Let $S_m$, $m \in \mathbb{Z}$, be symmetric invertible $p \times p$ matrices. For each $m \in \mathbb{Z}$ we consider the functions

$$H_m(x) = \langle S_m x, x \rangle \quad \text{and} \quad V_m(x) = -\text{sign} H_m(x) \sqrt{|H_m(x)|}.$$ \hfill (71)

Any Lyapunov sequence $(V_m)_{m \in \mathbb{Z}}$ obtained from quadratic forms $H_m$ as in (71) is called a quadratic Lyapunov sequence. Much attention has been given in other works to this particular class of Lyapunov sequences (although to the best of our knowledge never in relation to the study of nonuniform exponential dichotomies).

We emphasize that in general the existence of a strict Lyapunov sequence may not be sufficient to show that $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential dichotomy (see Section 3.2). Provided that an additional condition holds (see (43)), namely that the angles between the subspaces $E^s_m$ and $E^u_m$ in Theorem 1 decay at most exponentially in $m$, we show in Theorem 2 that the strictness property implies the existence of a nonuniform exponential dichotomy. But even if condition (43) holds it may be very difficult to verify (we note that in particular the subspaces $E^s_m$ and $E^u_m$ are given by the infinite intersections in (7)). It turns out that in the case of quadratic Lyapunov sequences the strictness property is sufficient to guarantee that $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential dichotomy for a large class of nonautonomous dynamics.

**Theorem 7.** For $X = \mathbb{R}^p$, if the sequence $(A_m)_{m \in \mathbb{Z}}$ satisfies

$$\limsup_{m \to \pm \infty} \frac{1}{|m|} \log \|A_m\| < \infty,$$ \hfill (72)

then the following properties hold:

1. if there exists a strict quadratic Lyapunov sequence for $(A_m)_{m \in \mathbb{Z}}$ such that

$$\frac{1 + \gamma}{1 - \gamma} > e^\delta$$ \hfill (73)

and

$$\limsup_{m \to \pm \infty} \frac{1}{|m|} \log \|S_m\| < \infty,$$ \hfill (74)

then $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential dichotomy;

2. if $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential dichotomy with a sufficiently small $\varepsilon > 0$, then there exists a strict quadratic Lyapunov sequence for $(A_m)_{m \in \mathbb{Z}}$ satisfying (73) and (74).

**Proof.** We start with an auxiliary statement.

**Lemma 3.** If $(V_m)_{m \in \mathbb{Z}}$ is a strict quadratic Lyapunov sequence for $(A_m)_{m \in \mathbb{Z}}$ satisfying (73), then

$$\|P_m\| = \|Q_m\| \leq \frac{2\|S_m\|}{\gamma^2 \min\{1, \|A_m^{-1}\|^{-2}\}}.$$ \hfill (75)
Proof. By Theorem 1, the existence of a strict Lyapunov sequence satisfying (73) ensures that for each $m \in \mathbb{Z}$ there exist projections $P_m$ and $Q_m$ obtained from the direct sum decomposition in (13) such that $P_m + Q_m = \text{Id}$. Furthermore, it follows from (20) that each function $V_m$ is positive in $E^u_m \setminus \{0\}$ and negative in $E^s_m \setminus \{0\}$. In view of (71) this implies that

$$V^2_m(P_m x) = \langle S_m P_m x, P_m x \rangle \quad \text{and} \quad V^2_m(Q_m x) = - \langle S_m Q_m x, Q_m x \rangle.$$  \hfill (76)

Given $m \in \mathbb{Z}$ and $x \in \mathbb{R}^p$ we write $x = y + z$ with

$$y = P_m x \in E^s_m \quad \text{and} \quad z = Q_m x \in E^u_m.$$

Take $\delta_m > 0$. For each $m \in \mathbb{Z}$ we define

$$V_m^+(y) = -V^2_m(y) + \delta_m \|y\|^2 = -\langle S_m y, y \rangle + \delta_m \|y\|^2.$$

For each $y \in E^s_m \setminus \{0\}$ we have

$$V_m(y) < 0 \quad \text{and} \quad V_{m+1}(A_m y) < 0$$

(since $A_mE_m^s = E_{m+1}^s$), and hence, by (9),

$$-V_m(y) = |V_m(y)| \geq |V_m(y)| - |V_{m+1}(A_m y)| = V_{m+1}(A_m y) - V_m(y) \geq \gamma \|y\|.$$

Therefore, $V^2_m(y) \geq \gamma^2 \|y\|^2$, and

$$V^+_m(y) \leq -\gamma^2 \|y\|^2 + \delta_m \|y\|^2 = (\delta_m - \gamma^2) \|y\|^2 \leq 0$$

if and only if $\delta_m \leq \gamma^2$. Similarly, we define

$$V_m^-(z) = V^2_m(z) - \delta_m \|z\|^2 = -\langle S_m z, z \rangle - \delta_m \|z\|^2.$$

For each $z \in E^u_m \setminus \{0\}$ we have

$$V_m(z) > 0 \quad \text{and} \quad V_{m-1}(A_{m-1}^{-1} z) > 0$$

(since $A_{m-1}^{-1}E^u_m = E^u_{m-1}$), and hence, again by (9),

$$V_m(z) \geq V_m(z) - V_{m-1}(A_{m-1}^{-1} z) \geq \gamma \|A_{m-1}^{-1} z\|.$$

Therefore, $V^2_m(z) \geq \gamma^2 \|A_{m-1}^{-1} z\|^2$, and

$$V^-_m(z) \geq \gamma^2 \|A_{m-1}^{-1} z\|^2 - \delta_m \|z\|^2 \geq (\gamma^2 \|A_{m-1}\|^2 - \delta_m) \|z\|^2 \geq 0.$$
if and only if $\delta_m \leq \gamma^2 \|A_{m-1}\|^{-2}$. We conclude that if

$$
\delta_m \leq \gamma^2 \min\{1, \|A_{m-1}\|^{-2}\},
$$

then

$$
-V_m^2(y) + \delta_m \|y\|^2 \leq 0 \quad \text{and} \quad V_m^2(z) - \delta_m \|z\|^2 \geq 0.
$$

Thus, it follows from (76) that

$$
-(S_m P_mx, P_mx) + \delta_m \|P_mx\|^2 \leq 0,
$$

and

$$
-(S_m Q_mx, Q_mx) - \delta_m \|Q_mx\|^2 \geq 0.
$$

Since $S_m$ is symmetric, subtracting the two inequalities we obtain

$$
0 \geq \delta_m \|P_mx\|^2 + \delta_m \|Q_mx\|^2 - (S_m P_mx, P_mx) + (S_m Q_mx, Q_mx)
= \delta_m \|P_mx\|^2 + \delta_m \|Q_mx\|^2 + (S_mx, x) - 2\langle S_m P_mx, x \rangle.
$$

Therefore,

$$
\delta_m \left\| P_mx - \frac{1}{2\delta_m} S_mx \right\|^2 + \delta_m \left\| Q_mx + \frac{1}{2\delta_m} S_mx \right\|^2
= \delta_m \left\| P_mx \right\|^2 + \frac{\|S_mx\|^2}{2\delta_m} + \delta_m \left\| Q_mx \right\|^2 + (S_mx, x) - 2\langle S_m P_mx, x \rangle
\leq \frac{\|S_mx\|^2}{2\delta_m},
$$

which is equivalent to

$$
\left\| P_mx - \frac{1}{2\delta_m} S_mx \right\|^2 + \left\| Q_mx + \frac{1}{2\delta_m} S_mx \right\|^2 \leq \frac{\|S_mx\|^2}{2\delta^2_m}.
$$

This implies that

$$
\|P_mx\| = \left\| P_mx - \frac{1}{2\delta_m} S_mx + \frac{1}{2\delta_m} S_mx \right\|
\leq \left\| P_mx - \frac{1}{2\delta_m} S_mx \right\| + \frac{1}{2\delta_m} \|S_mx\|
\leq \frac{1}{\sqrt{2\delta_m}} \|S_mx\| + \frac{1}{2\delta_m} \|S_mx\| \leq \frac{\sqrt{2}}{\delta_m} \|S_mx\|,
$$

and similarly,
\[ \|Q_m x\| = \left\| Q_m x + \frac{1}{2\delta_m} S_m x - \frac{1}{2\delta_m} S_m x \right\| \]
\[ \leq \left\| Q_m x + \frac{1}{2\delta_m} S_m x \right\| + \frac{1}{2\delta_m} \|S_m x\| \]
\[ \leq \frac{1}{\sqrt{2\delta_m}} \|S_m x\| + \frac{1}{2\delta_m} \|S_m x\| \leq \frac{\sqrt{2}}{\delta_m} \|S_m x\|. \]

Taking the best possible value for \(\delta_m\), that is,
\[ \delta_m = \gamma^2 \min \{1, \|A_{m-1}\|^{-2}\}, \]
we obtain the desired inequality. \(\square\)

Now we assume that there exists a strict quadratic Lyapunov sequence for \((A_m)_{m \in \mathbb{Z}}\) satisfying (73) and (74). By (72) there exist constants \(C, \delta > 1\) such that for every \(m \in \mathbb{Z}\) we have \(\|A_{m-1}\| \leq Ce^{\delta|m|}\), which yields
\[ \min \{1, \|A_{m-1}\|^{-2}\} \geq C^{-2} e^{-2\delta|m|}. \]

By (73), (74) and (75), it follows readily from Theorem 1 and Lemma 3 (using also (47) and (48)) that there exist constants as in (15) satisfying (36), that is, the inequalities in the notion of nonuniform exponential dichotomy. Alternatively, it follows from Lemma 3 together with (42) and (45) that (43) holds, and by Theorem 2 the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential dichotomy.

For the second property, if \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential dichotomy, then using Theorem 3 we can show that there exists a strict quadratic Lyapunov sequence, given by (55) with \(r = 2\). More precisely, using (52) we consider the quadratic form
\[ H_m(x) = U_m(x) = \langle S_m x, x \rangle, \]
where
\[ S_m = \sum_{k \geq m} \left( A(k, m) P_m \right)^* A(k, m) P_m e^{-2(\alpha+\varphi)(k-m)} \]
\[ - \sum_{k \leq m} \left( A(k, m) Q_m \right)^* A(k, m) Q_m e^{2(\beta-\varphi)(m-k)}. \quad (77) \]

Clearly, \(S_m\) is symmetric for each \(m\). It is also invertible. Indeed, since
\[ H_m\left( F_m^s \setminus \{0\} \right) > 0 \quad \text{and} \quad H_m\left( F_m^u \setminus \{0\} \right) < 0, \]
it follows from the identity \(F_m^s \oplus F_m^u = \mathbb{R}^p\) that \(S_m\) is invertible for each \(m\). Moreover,
\[ |H_m(x)| \leq \sum_{k \geq m} \|A(k, m)P_m x\|^2 e^{-2(\sigma+\varrho)(k-m)} + \sum_{k \leq m} \|A(m, k)^{-1}Q_m x\|^2 e^{2(\varrho-\varrho)(m-k)} \]
\[ \leq D^2 e^{2\varepsilon|m|\|x\|^2} 2 \left( \sum_{k \geq m} e^{-2\varrho(k-m)} + \sum_{k \leq m} e^{-2\varrho(m-k)} \right) \]
\[ = \frac{2D^2}{1 - e^{-2\varrho}} e^{2\varepsilon|m|\|x\|^2}. \] (78)

Since \( S_m \) is symmetric we obtain
\[ \|S_m\| = \sup_{x \neq 0} \frac{|H_m(x)|}{\|x\|^2} \leq \frac{2D^2}{1 - e^{-2\varrho}} e^{2\varepsilon|m|}, \] (79)
and this yields inequality (74). Furthermore, if \( \varepsilon \) is sufficiently small, then since \( \delta = \varepsilon \) (see (78)) we obtain inequality (73). This completes the proof of the theorem. \( \square \)

We also characterize uniform exponential dichotomies in terms of quadratic Lyapunov sequences.

**Theorem 8.** For \( X = \mathbb{R}^p \), if the sequence \((A_m)_{m \in \mathbb{Z}}\) is bounded, then the following properties are equivalent:

1. \((A_m)_{m \in \mathbb{Z}}\) admits a uniform exponential dichotomy;
2. there exists a strict quadratic Lyapunov sequence for \((A_m)_{m \in \mathbb{Z}}\) with \( \delta = 0 \) and \((S_m)_{m \in \mathbb{Z}}\) bounded.

Moreover, if there exist functions \( V_m : \mathbb{R}^p \to \mathbb{R} \) for \( m \in \mathbb{Z} \) as in (71), and constants \( C > 0 \) and \( \gamma \in (0, 1) \) such that \( |V_m(x)| \leq C\|x\| \) and
\[ V_{m+1}(A_m x) - V_m(x) \geq \gamma \|x\| \] (80)
for every \( m \in \mathbb{Z} \) and \( x \in \mathbb{R}^p \), then these two properties hold.

**Proof.** The equivalence between the first two properties follows from the proof of Theorem 7. Indeed, if property 1 holds, then the sequence \((V_m)_{m \in \mathbb{Z}}\) defined by (71) with \( S_m \) as in (77) is a strict quadratic Lyapunov sequence for \((A_m)_{m \in \mathbb{Z}}\). Moreover, by (78) we have \( \delta = \varepsilon = 0 \), and by (79) the sequence \((S_m)_{m \in \mathbb{Z}}\) is bounded. This establishes property 2.

On the other hand, if property 2 holds, then it follows from Theorem 1 that there exist subspaces \( E_m^s \) and \( E_m^u \) satisfying (16) and (17) with \( \varepsilon = 0 \) (we note that when \( \delta = 0 \) inequality (73) is automatically satisfied). That is, there exist constants \( a < 0 < b \) and \( D \geq 1 \) such that
\[ \|A(m, n)|E_m^s\| \leq De^{\bar{\alpha}(m-n)}, \quad \|A(m, n)^{-1}|E_m^u\| \leq D e^{-\bar{b}(m-n)} \]
for every \( m, n \in \mathbb{Z} \) with \( m \geq n \). It follows from (75) that \( \|P_m\| = \|Q_m\| \) is a bounded sequence, and thus, by (42) we have

\[
\inf_{m \in \mathbb{Z}} \langle E^u_m, E^s_m \rangle > 0.
\]

It follows from Theorem 4 that the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a uniform exponential dichotomy. Now we assume that the last property holds. Then

\[
V_{m+1}(A_m x) - V_m(x) \geq \gamma \|x\| \geq \frac{\gamma}{C} |V_m(x)|.
\]

(81)

Moreover, if \( V_m(x) \leq 0 \) and \( V_{m+1}(A_m x) \leq 0 \), then

\[
|V_m(x)| \geq |V_m(x)| - |V_{m+1}(A_m x)|
= V_{m+1}(A_m x) - V_m(x) \geq \gamma \|x\|.
\]

(82)

We can easily verify that (81) and (82) are sufficient to repeat the proof of Theorem 1 to obtain subspaces \( E^s_m \) and \( E^u_m \) satisfying (13), (14), (16) and (17) with \( \delta = 0 \) (since inequality (73) is automatically satisfied when \( \delta = 0 \)). More precisely, the condition \( |V_m(x)| \geq \gamma \|x\| \) is not needed in the proof of Theorem 1 when \( V_m(x) \geq 0 \) and \( V_{m-1}(A_{m-1}^{-1} x) \geq 0 \). Moreover, since \( S_m \) is symmetric we have

\[
\|S_m\| \leq \sup_{x \neq 0} \frac{|V_m(x)|^2}{\|x\|^2} \leq C^2,
\]

and \((S_m)_{m \in \mathbb{Z}}\) is bounded. Therefore, it follows from (75) that \( \|P_m\| = \|Q_m\| \) is a bounded sequence. Together with (13), (14), (16) and (17) with \( \delta = 0 \) this shows that \((A_m)_{m \in \mathbb{Z}}\) admits a uniform exponential dichotomy. This completes the proof of the theorem. \( \square \)

We note that inequality (80) can be written in the form

\[
A^*_m S_{m+1} A_m - S_m \geq \gamma \text{Id}.
\]

A related approach to the one in Theorem 8 was described by Khatskevich and Zelenko in [11], for a sequence \((A_m)_{m \in \mathbb{N}}\) of bounded linear operators in an arbitrary Hilbert space. We note that they do not require the operators \( A_m \) to be invertible. On the other hand, they only consider uniform exponential dichotomies and quadratic Lyapunov sequences. They also use different methods.

References