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ON DOMINATION AND INDEPENDENT DOMINATION NUMBERS OF A GRAPH

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For a graph G, the definitions of domination number, denoted $\gamma(G)$, and independent domination number, denoted $i(G)$, are given, and the following results are obtained:

Theorem. *If G does not have an induced subgraph isomorphic to $K_{1,3}$, then $\gamma(G) = i(G)$.*

Corollary 1. *For any graph G, $\gamma(L(G)) = i(L(G))$, where $L(G)$ is the line graph of G.* (This extends the result $\gamma(L(T)) = i(L(T))$, where T is a tree. Hedetniemi and Mitchell, S. E. Conf. Baton Rouge, 1977.)

Corollary 2. *For any Graph G, $\gamma(M(G)) = i(M(G))$, where M is the middle graph of G.*

0. Introduction

In this paper we shall consider a graph $G = (V, E)$ as finite, undirected, with no multiple edges, and with no loops. All definitions not presented here can be found in [3].

0.1

A number of terms to be used are defined for a given graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_p\}$.

Definition 1. A set $D \subseteq V$ is a *dominating set* (of G), if $\forall v \in V - D$, $N(v) \cap D \neq \emptyset$.

Definition 2. A set $I \subseteq V$ is an *independent set* (of G), if $\forall u, v \in I$, $N(u) \cap \{v\} = \emptyset$.

Definition 3. A set $I \subseteq V$ is an *independent domination set* (of G) if I is both an independent and dominating set.

0.2

The following are two useful results which can be found in [6] and [1] respectively.

Proposition 1. *A dominating set D is minimal iff for each $d \in D$ either* (i) $N(d) \cap D = \emptyset$ *or* (ii) $\exists c \in (V-D)$ *such that* $N(c) \cap D = \{d\}$.

Proposition 2. (i) *I is a maximal independent set if and only if I is an independent dominating set.*

(ii) *If I is a maximal independent set then I is a minimal dominating set.*

0.3

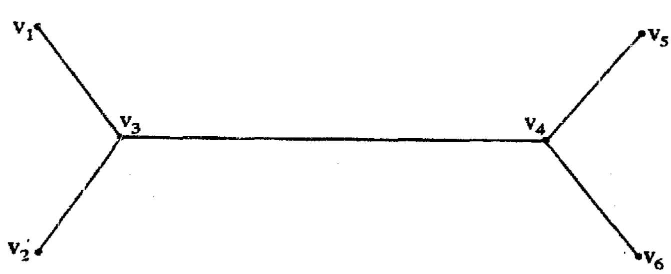
The two invariants of a graph G which are of interest in this paper are now defined as in [2].

Definition 4. The *domination number* (of G), denoted $\gamma(G)$, is the minimum cardinality taken over all minimal dominating sets.

Definition 5. In view of Proposition 2 we define the *independent domination number* (of G), denoted $i(G)$, as the minimum cardinality taken over all independent dominating sets of G.

Definitions 4 and 5, together with Proposition 2 imply the following result:

Proposition 3. *For any graph $G=(V,E)$, $\gamma(G) \leq i(G)$. We show by example that strict inequality may occur. Let G be determined by the following diagram:*



It is easy to see that $\gamma(G) = 2 < 3 = i(G)$.

We now give additional definitions and a result, all of which will be used later.

0.4

Definition 6. The *middle graph of a graph G*, denoted by $M(G)$, is the intersection graph $\Omega(F)$ on $V(G) = \{v_1, v_2, \ldots, v_p\}$ where $F = \{\{v_1\}, \{v_2\}, \ldots, \{v_p\}\} \cup E(G)$.

Definition 7. The graph G^+ is defined as follows: add to $V=\{v_1, v_2, \ldots, v_p\}$ p vertices $u_1, u_2, \ldots, u_p$ different from the elements of V and from each other. Add the p edges $u_iv_i=\{u_i, v_i\}$ $(i=1,2,\ldots,p)$ to E. The graph G^+ is the graph with vertex set $V\cup\{u_1, u_2, \ldots, u_p\}$ and edge set $E\cup\{u_1v_1, u_2v_2, \ldots, u_pv_p\}$. The following result is due to Hamada and Yoshimura and can be found in [4].

Proposition 4. *For any graph G, $L(G^+)$ is isomorphic to $M(G)$, where $L(G^+)$ denotes the line graph of G^+.*

1. Results

Theorem. *If $G=(V,E)$ is a graph which does not have an induced subgraph isomorphic to $K_{1,3}$, then $\gamma(G)=i(G)$.*

Proof. By Proposition 3, $\gamma(G)\leqslant i(G)$. We will show that $i(G)\leqslant\gamma(G)$ and hence the desired equality holds.

Let $m=\gamma(G)$ and let $D_{-1}=\{w_0, w_1, \ldots, w_{m-1}\}\subseteq V$ be a dominating set. Also, for any non-empty $V'\subseteq V$ let $a(V')$ denote the number of edges in the subgraph induced by V'. Clearly $0\leqslant a(D_{-1})\leqslant\binom{m}{2}$. If $a(D_{-1})=0$ then D_{-1} is an independent set, and by proposition 2 $i(G)\leqslant m=\gamma(G)$. Therefore without loss of generality we may assume that $w_0w_1\in E$.

Now by Proposition 1, the set $N_0=\{u\in V-D_{-1}\mid N(u)\cap D_{-1}=\{w_0\}\}$ is not empty. Let u and w be any two distinct elements of N_0 and consider $\{w_0, w_1, u, w\}\subseteq V$. The subgraph induced by this set certainly contains $\{w_0w_1, w_0u, w_0w\}$. By hypothesis $\{w_1u, w_1w, uw\}\cap E\neq\emptyset$, but since $N(u)\cap D_{-1}=\{w_0\}=N(w)\cap D_{-1}$ it must be that $uw\in E$. Now we see that any two distinct elements of $N_0\cup\{w_0\}$ are adjacent.

Take $u_0\in N_0$ and consider $D_0=\{u_0, w_1, \ldots, w_{m-1}\}$. Let $z\in V-D_0=M\cup K$, where $M=(N_0-\{u_0\})\cup\{w_0\}$ and $K=V-(N_0\cup D_{-1})$. If $z\in M$ then $zu_0\in E$ and if $z\in K$ then $N(z)\cap D_{-1}\supseteq\{w_i\}$, where $1\leqslant i\leqslant m-1$, which says $zw_i\in E$. Hence D_0 is a dominating set such that $|D_0|=m$. Now $N(u_0)\cap D_0=\emptyset$ and hence $0\leqslant a(D_0)\leqslant\binom{m-1}{2}$. If $a(D_0)=0$ then $i(G)\leqslant m=\gamma(G)$ as before. If $a(D_0)>0$ we can repeat the process used to obtain D_0 to obtain, without loss of generality, a dominating set $D_1=\{u_0, u_1, w_2, \ldots, w_{m-1}\}$ such that $N(u_i)\cap D_1=0$ for $i=0,1$. We then have $0\leqslant a(D_1)\leqslant\binom{m-2}{2}$. Again if $a(D_1)=0$ then we are done and if $a(D_1)>0$ then we repeat the process used to obtain D_0 and then D_1.

Clearly the repetitions of the process must terminate in at most $m-1$ steps with a dominating set D_k, $-1\leqslant k\leqslant m-2$, such that $|D_k|=m=\gamma(G)$ and $a(D_k)=0$. Hence D_k is an independent dominating set, which by Proposition 2 implies that D_k is a maximal independent set. Whence $i(G)\leqslant|D_k|=m=\gamma(G)$, and the proof is complete.

Since for $K_{1,3}$, $\gamma(K_{1,3})=1=i(K_{1,3})$, we see that the hypothesis of the theorem is not a necessary condition; however, we do have the following:

Corollary 1. *For any graph G, $\gamma(L(G)) = i(L(G))$, where $L(G)$ denotes the line graph of G.*

Proof. The result is immediate from Theorem 1, since $L(G)$ does not have an induced subgraph isomorphic to $K_{1,3}$. See [3].

This corollary extends the result $\gamma(L(T)) = i(L(T))$, where T is a tree which was established in [5].

Corollary 2. *For any graph G, $\gamma(M(G)) = i(M(G))$ where $M(G)$ denotes the middle graph of G.*

Proof. By Corollary 1, $\gamma(L(G^+)) = i(L(G^+))$ and by Proposition 4 we have $\gamma(M(G)) = i(M(G))$.

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