Matrices of operators and regularized cosine functions ♠

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Abstract

This paper is concerned with matrices of abstract differential operators which are parabolic in the sense of Shilov or correct in the sense of Petrovskij. We show that they generate regularized cosine functions with suitable regularizing operators under sharper conditions. The results then are applied to matrices of partial differential operators on many function spaces. Finally, the wellposedness of the associated second-order systems is discussed.

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1. Introduction

Let $iA_j$ ($1 \leq j \leq n$) be generators of commuting bounded $C_0$-groups on a Banach space $X$, and write $A = (A_1, \ldots, A_n)$ and $A^\mu = A_1^{\mu_1} \cdots A_n^{\mu_n}$ for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In this paper, we consider the system of differential equations

$$
\begin{cases}
  u''(t) = P(A)u(t) + f(t) & \text{for } t \in \mathbb{R}^+ := [0, \infty), \\
  u(0) = x, & u'(0) = y
\end{cases}
$$

(1.1)
on $X^N$, where $P(\xi) = \sum_{|\mu| \leq m} P_\mu \xi^\mu$ ($\xi \in \mathbb{R}^n$) with $P_\mu \in M_N(\mathbb{C})$ (the ring of $N \times N$ matrices over $\mathbb{C}$), and $f \in C([0, \infty), X^N)$. Then $P(A)$ with maximal domain is closable on $X^N$ (cf. [1]).

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By choosing \( A = D := -i(\frac{\partial}{\partial t}, \ldots, \frac{\partial}{\partial x_n}) \) one finds that (1.1), in fact, gives an abstract form of the system of differential equations

\[
\begin{align*}
  u_t(t, x) &= P(D)u(t, x) + f(t, x) \quad \text{for} \ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\
  u(0, x) &= \phi(x), \quad u_t(0, x) = \psi(x) \quad \text{for} \ x \in \mathbb{R}^n
\end{align*}
\]

(1.2)
on different function spaces.

A classical method to treat the wellposedness of (1.2) is to show that \( P(D) \) is the generator of a strongly continuous cosine function. However, many \( P(D) \), e.g., the Laplacian \( \Delta \) on \( L^p(\mathbb{R}^n) \) \((n > 1, \ p \neq 2)\) are not such a generator [8]. Recently, a generalization of strongly continuous cosine functions, i.e., regularized cosine functions has received much attention (cf. [6,10,14]). It thus seems to be important to study their application to (1.2). Up to now, only is the case \( N = 1 \) considered by some authors [2,12].

The application of regularized semigroups to parabolic systems in the sense of Shilov and correct systems in the sense of Petrovskij has been studied extensively (cf. [1,5,13]). This paper is a continuation of [13]; its aim is to apply regularized cosine functions to such systems (1.1) and (1.2) is discussed in Section 3. In the special case \( N = 1 \), our results improve the corresponding results in [2,12].

Throughout the paper, \( B(X) \) will be the space of bounded linear operators on \( X \). By \( D(B), R(B), \) and \( \sigma(B) \) we denote the domain, range, and spectrum of the operator \( B \), respectively. We also denote by \( B \) the operator \( BL^N \) on \( X^N \) with domain \( (D(B))^N \), and by \( B|_Y \) the restriction of \( B \) to \( Y \). Moreover, set \(|A|^2 = \sum_{j=1}^n A_j^2 \) and \( C_\alpha = (1 + |A|^2)^{-\alpha/2} (\alpha \in \mathbb{R}) \) as fractional powers. Then \( C_\alpha \in B(X) \) for \( \alpha \geq 0 \).

The definitions of (exponentially bounded) regularized cosine functions can be given by using Laplace transforms (cf. [10]). Let \( C \in B(X) \) be injective. An exponentially bounded, strongly continuous family \( \{C(t)\}_{t \geq 0} \subset B(X) \) is called a \( C \)-regularized cosine function generated by \( B \) if \( C^{-1}BC = B \), for large \( \lambda \in \mathbb{R}, \lambda^2 - B \) is injective, \( R(C) \subset R(\lambda^2 - B) \), and \( \lambda(\lambda^2 - B)^{-1}C \) is the Laplace transform of \( \{C(t)\}_{t \geq 0} \). When \( C \) is the identity operator and \( B \) is densely defined, \( \{C(t)\}_{t \geq 0} \) is a strongly continuous cosine function.

To construct the regularized cosine function generated by \( \overline{P(A)} \), we need the following functional calculus [1,13]:

\[
u(A) = \int_{\mathbb{R}^n} \langle \mathcal{F}^{-1}u(s) \rangle e^{-i(s, A)} \, ds \quad \text{for} \ u \in \mathcal{F}L^1,
\]

where \( \mathcal{F}L^1 = \{ \mathcal{F}v; \ v \in L^1(\mathbb{R}^n) \} \) and \( \mathcal{F} \) denotes the Fourier transform. It is known that \( M_N(\mathcal{F}L^1) \) is a (noncommutative) Banach algebra under matrix pointwise multiplication and addition with norm \( \|u\|_{\mathcal{F}L^1} := \|\mathcal{F}^{-1}u\|_{L^1}, u \mapsto u(A) \) is an algebra homomorphism from \( M_N(\mathcal{F}L^1) \) into \( B(X^N) \), and \( \|u(A)\| \leq M\|u\|_{\mathcal{F}L^1} \) \((u \in M_N(\mathcal{F}L^1)) \) for some \( M > 0 \).

In the following lemma, (a) can be deduced by combining Bernstein’s theorem (see [11]) and [13, Lemma 1.1(c)], while (b) is due to [7].

**Lemma 1.1.** (a) Let \( u \in C^j(\mathbb{R}^n) \) \((j > \frac{n}{2})\). If there exist constants \( L, K, a > 0, b \in [-1, \frac{2a}{n} - 1) \) such that \(|D^k u(\xi)| \leq K^{[k]}|\xi|^b|\xi|^{-a} \) for \(|\xi| > L \) and \(|\xi| < L \), where \( k \in \mathbb{N}_0^n \) and \(|k| \leq j \), then \( u \in \mathcal{F}L^1 \) and \( \|u\|_{\mathcal{F}L^1} \leq M K^{n/2} \).
(b) Let $P$ be a polynomial, and $E = \{ \phi(A)x; \, \phi \in \mathcal{S} \text{ (the Schwartz space on } \mathbb{R}^n), \, x \in X \}$. Then $E \subset D(A^\mu)$ for $\mu \in \mathbb{N}_0^n$, $E = X$, $P(A)|_{E^N} = P(A)$, and $\phi(A)P(A) \subset P(A)\phi(A) = (P\phi)(A)$ for $\phi \in \mathcal{S}$.

2. Main results

For $P \in M_N(\mathbb{C})$, let $\sigma(P) = \{ \lambda_j \}_{j=1}^N$ and $\Lambda(P) = \sup_{1 \leq j \leq N} \Re \lambda_j$. Denote by $H(P)$ the convex hull of $\{ \lambda_j \}_{j=1}^N$. Then (see [3, p. 169])

$$
\| f(P) \| \leq \sum_{j=0}^{N-1} 2^j \| P \|^j \sup_{z \in H(P)} |f^{(j)}(z)|, \tag{2.1}
$$

where $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Moreover, set

$$
\Sigma_\omega = \begin{cases} 
\{ \lambda; \, (\Im \lambda)^2 \leq 4\omega^4 - 4\omega^2 \Re \lambda \} & \text{if } \omega > 0, \\
(-\infty, -\omega^2] & \text{if } \omega \leq 0
\end{cases}
$$

and for $t, l \geq 0$,

$$
g_l(t) = \begin{cases} 
(1 + t^{N-1+l})e^{\omega t} & \text{if } \omega > 0, \\
1 + t^{2(N-1+l)} & \text{if } \omega = 0, \\
1 + t^{N-1+l} & \text{if } \omega < 0.
\end{cases}
$$

Lemma 2.1. Let $f_l(z) = \sum_{j=0}^{\infty} r^{2j} z^j / (2j)!$ for $z \in \mathbb{C}$ and $t \geq 0$, and let $\sigma(P) \subset \Sigma_\omega$ for some $\omega \in \mathbb{R}$. Then for every $l \in \mathbb{N}_0$, there exists a constant $M_l$ such that

$$
\| f_l^{(l)}(P) \| \leq M_l g_l(t) \left( 1 + \| P \|^{N-1} \right) \text{ for } t \geq 0.
$$

If in addition $\Lambda(P) \leq -1$, then

$$
\| f_l^{(l)}(P) \| \leq M_l g_l(t) \sum_{j=0}^{N-1} \| P \|^j |A(P)|^{-(j+l)/2} \text{ for } t \geq 0.
$$

Proof. We first note that for every $t \geq 0$ and $l \in \mathbb{N}_0$,

$$
f_l^{(l)}(z) = \sum_{j=0}^{\infty} \frac{(j+l)!r^{2j+2l}}{(2j+2l)!j!} z^j \quad \text{for } z \in \mathbb{C} \tag{2.2}
$$

and

$$
f_l^{(l)}(z) = \sum_{j=0}^{l} C_j t^j U_j(t \sqrt{z}) z^{-l+j/2} \quad \text{for } z \in \mathbb{C} \setminus \{ 0 \}, \tag{2.3}
$$

where $C_j \,(0 \leq j \leq l)$ are constants independence of $t$, $\sqrt{z}$ is taken as the principle branch, and

$$
U_j(x) = \begin{cases} 
\cosh(x) & \text{for even } j, \\
\sinh(x) & \text{for odd } j.
\end{cases}
$$

If $\omega > 0$, then $|U_j(t \sqrt{z})| \leq e^{\omega t}$ for $z \in H(P)$, where we notice that $H(P) \subset \Sigma_\omega$ because of the convexity of $\Sigma_\omega$. In the case $z \in H(P)$ with $|z| \geq \omega^2/2$, by (2.3) we have
\[ |f_t^{(l)}(z)| \leq \sum_{j=0}^{l} |C_j|t^j e^{\omega t}(\omega^2/2)^{-l+j/2} \leq M_l(1+t^l)e^{\omega t}. \]

In the case \( z \in H(P) \) with \(|z| < \omega^2/2 \), by (2.2) we have
\[ |f_t^{(l)}(z)| \leq t^{2l} \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!}(\omega^2/2)^j \leq M_l e^{\omega t}. \]

If \( \omega = 0 \), then \( a := t\sqrt{-z} \geq 0 \) for \( z \in H(P) \). In the case \( 0 \leq a < 1 \), by (2.2) we have
\[ 0 \leq f_t^{(l)}(z) = \sum_{j=0}^{\infty} (-1)^j \frac{(j+l)!t^{2l}a^{2j}}{(2j+2l)!} \leq \frac{l!t^{2l}}{(2l)!}. \]

In the case \( a \geq 1 \), by (2.3) we have
\[ |f_t^{(l)}(z)| \leq \sum_{j=0}^{l} |C_j|t^{2l}a^{-2l+j} \leq M_l(1+t^{2l}). \]

If \( \omega < 0 \), it then follows from (2.3) that for \( z \in H(P) \),
\[ |f_t^{(l)}(z)| \leq \sum_{j=0}^{l} |C_j|t^l|\omega|^{-2l+j} \leq M_l(1+t^l). \]

If \( \Lambda(P) \leq -1 \), then for \( z \in H(P) \), \(|z| \geq -\Lambda(P)\) and \(|U_j(t\sqrt{z})| \leq e^{\omega t} \), where \( \omega' = \max\{\omega, 0\} \), and thus by (2.3)
\[ |f_t^{(l)}(z)| \leq \sum_{j=0}^{l} |C_j|t^j e^{\omega' t} |\Lambda(P)|^{1+j/2} \leq M_l(1+t^l)e^{\omega' t} |\Lambda(P)|^{-1/2}. \]

Combining these estimates with (2.1) yields to the claim. \( \square \)

Let \( r \in (0, m] \), a polynomial \( P \) of degree \( m \) is said to be \( r \)-parabolic in the sense of Shilov [4] if there exist constants \( \omega > 0 \) and \( \omega' \in \mathbb{R} \) such that \( \Lambda(P(\xi)) \leq -\omega|\xi|^{r} + \omega' \) for \( \xi \in \mathbb{R}^n \). It is correct in the sense of Petrovskij [4] if \( r = 0 \), i.e., sup\{\( \Lambda(P(\xi)) ; \xi \in \mathbb{R}^n \} < \infty \). Note that if \( m = 1 \) then \( P \) cannot be \( r \)-parabolic in the sense of Shilov for any \( r \in (0, 1] \). For convenience, we make the following condition \( (H_r) \) \((0 \leq r \leq m)\):

\( (H_r) \) There exists a constant \( \omega \in \mathbb{R} \) such that \( \sigma(P(\xi)) \subset \Sigma_{\omega} \) for \( \xi \in \mathbb{R}^n \) and, in the case \( 0 < r \leq m \), \( P \) is \( r \)-parabolic in the sense of Shilov.

We notice that if \( \sigma(P(\xi)) \subset \Sigma_{\omega} \) for \( \xi \in \mathbb{R}^n \) then sup\{\( \Lambda(P(\xi)) ; \xi \in \mathbb{R}^n \} \leq \omega^2 \), in particular \( P \) is correct in the sense of Petrovskij. In the sequel, \( M \) will denote a general constant independent of \( t \) and \( \xi \).

**Theorem 2.2.** Let \( P \) satisfy the condition \( (H_r) \) for some \( r \in [0, m] \), and let \( \alpha > (m - r/2) \times (N - 1 + n/2) \). Then \( \widehat{P}(A) \) generates a \( C_\alpha \)-regularized cosine function \( \{C(t)\}_{t \geq 0} \) on \( X^N \). Moreover, \( \|C(t)\| \leq M_{gn/2}(t) \) for \( t \geq 0 \) and \( C(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^N) \).
Proof. Let $t \geq 0$ and $|\mu| \leq \left[ \frac{d}{2} \right] + 1$ ($\mu \in \mathbb{N}_0^d$). Then an induction on $|\mu|$ leads to

$$D^\mu f_1(P(\xi)) = \sum_{l=0}^{[\mu]} f_1^{(l)}(P(\xi)) Q_l(\xi) \quad \text{for } \xi \in \mathbb{R}^n,$$  \hspace{1cm} (2.4)

where $f_1$ was defined as in Lemma 2.1, and $Q_l \in M_N(C)$ with degree $\leq l(m - 1)$. If $0 < r \leq m$, by the condition $(H_r)$, there exists a constant $L \geq 1$ such that $\lambda(P(\xi)) \leq -M|\xi|^r \leq -1$ for $|\xi| \geq L$. It thus follows from (2.4) and Lemma 2.1 that

$$\|D^\mu f_1(P(\xi))\| \leq M \sum_{l=0}^{[\mu]} g_l(t) \sum_{j=0}^{N-1} \|P(\xi)\| |\lambda(P(\xi))|^{-j/2} \|Q_l(\xi)\|$$

$$\leq M \sum_{l=0}^{[\mu]} g_l(t) \sum_{j=0}^{N-1} |\xi|^{mj-r(j+1)/2+l(m-1)}$$

$$\leq Mg_{[\mu]}(t)|\xi|^{m(N-1)+(m-1-r)\mu} \quad \text{for } |\xi| \geq L.$$  \hspace{1cm} (2.5)

If $r = 0$, by (2.4) and Lemma 2.1, we get

$$\|D^\mu f_1(P(\xi))\| \leq M \sum_{l=0}^{[\mu]} g_l(t) (1 + \|P(\xi)\|^{N-1}) \|Q_l(\xi)\|$$

$$\leq Mg_{[\mu]}(t)|\xi|^{m(N-1)+(m-1)\mu} \quad \text{for } |\xi| \geq L.$$  \hspace{1cm} (2.6)

Set $h_1(\xi) = (1 + |\xi|^2)^{-\alpha/2} f_1(P(\xi))$ for $\xi \in \mathbb{R}^n$. Then by Leibniz’s formula

$$\|D^\mu h_1(\xi)\| \leq Mg_{[\mu]}(t)|\xi|^{(m-r/2)(N-1)+(m-1-r)\mu} \quad \text{for } |\xi| \geq L.$$  \hspace{1cm} (2.7)

Also, by (2.4) and Lemma 2.1 one has that $\|D^\mu h_1(\xi)\| \leq Mg_{[\mu]}(t)$ for $|\xi| < L$. Thus Lemma 1.1(a) leads to $h_1 \in M_N(FL^1)$ and

$$\|h_1\|_{FL^1} \leq Mg_{n/2}(t) \quad \text{for } t \geq 0.$$  \hspace{1cm} (2.8)

Define $C(t) = h_1(A)$ for $t \geq 0$. Then $\|C(t)\| \leq Mg_{n/2}(t)$ for $t \geq 0$, and by Lemma 1.1(b) $C^{-1}_\alpha \bar{P}(A)C_\alpha = \bar{P}(A)$. Observing the proof of (2.5) and using Lebesgue’s dominated convergence theorem, one finds that $u_t \ (t \geq 0)$ is continuous in the norm $\| \cdot \|_{FL^1}$, and thus $C(\cdot) \in C([0, \infty), B(X))$. Finally, let $L_\lambda (\lambda > \omega)$ be the Laplace transform of $\{|C(t)|_{t \geq 0}\}$. Lemma 1.1(b) and Fubini’s theorem shows that

$$(\lambda^2 - \bar{P}(A))L_\lambda \phi(A) = L_\lambda (\lambda^2 - \bar{P}(A)) \phi(A) = \left(\int_0^\infty e^{-\lambda t} h_t d\lambda \right) \phi(A)$$

$$= \lambda C_\alpha \phi(A) \quad \text{for } \phi \in S.$$  \hspace{1cm} (2.9)

In view of $\bar{P}(A)|_{E_N} = \bar{P}(A)$, we obtain that $R(C_\alpha) \subset R(\lambda^2 - \bar{P}(A))$ and $L_\lambda = \lambda (\lambda^2 - \bar{P}(A))^{-1} C_\alpha$. Thus $\bar{P}(A)$ generates the norm-continuous, $C_\alpha$-regularized cosine function $\{C(t)\}_{t \geq 0}$ on $X^N$. \hspace{1cm} \(\square\)

Let $X$ be a function space on which all translations are uniformly bounded and strongly continuous, such as $L^p(\mathbb{R}^n) \ (1 \leq p < \infty)$ and some spaces of continuous functions (cf. [13]). We now turn to the application to $P(D)$ with maximal domain in the distributional sense in $X^N$. \hspace{1cm} (2.10)
The following theorem, except $\mathcal{X} = L^p(\mathbb{R}^n)$ ($1 < p < \infty$), is a direct consequence of Theorem 2.2. In the remainder case, it can be deduced by using the Riesz–Thorin convexity theorem and Miyachi’s multiplier theorem $G$ in [9]. Moreover, set

$$n\mathcal{X} = \left\{ n\left| \frac{1}{2} - \frac{1}{p} \right| \right\}$$

for $\mathcal{X} = L^p$ ($1 < p < \infty$), and for remainder $\mathcal{X}$.

**Corollary 2.3.** Let $P$ satisfy the condition $(H_r)$ for some $r \in [0, m]$. Then $P(D)$ generates an $(1 - \Delta)^{-\alpha}$-regularized cosine function $\{C(t)\}_{t \geq 0}$ on $\mathcal{X}^N$, where $\alpha = (2m - r)(N - 1 + n\mathcal{X})/4$. Moreover, $\|C(t)\| \leq M_{g_n\mathcal{X}}(t)$ for $t \geq 0$.

To obtain, from Corollary 2.3, the strongly continuous cosine function $\{C(t)\}_{t \geq 0}$ on $\mathcal{X}^N$ we have to assume that $N = 1$ and $\mathcal{X} = L^2$. In the case $P(D) = \Delta$, this result is due to Littman [8]. By Corollary 2.3 one knows that $\Delta$ generates a $(1 - \Delta)^{-n\mathcal{X}/2}$-regularized cosine function on $\mathcal{X}$.

**Example 2.4.** We first consider the following matrix of polynomials (cf. [13]):

$$P(\xi) = \begin{pmatrix} -2a\xi^2 & ib\xi + ic\xi^3 \\ i\xi & 0 \end{pmatrix}$$

for $\xi \in \mathbb{R}$, where $a > 0$ and $b, c \in \mathbb{R}$. Since the eigenvalues of $P$ are $-a\xi^2 \pm \sqrt{a^2\xi^4 - b^2\xi^2 - c^2\xi^4}$, $P$ is 2-parabolic in the sense of Shilov, and thus $P$ satisfies the condition $(H_2)$ if and only if $a^2 \geq c$. In this case, by Corollary 2.3 $P(D)$ generates a $(1 - \Delta)^{-\alpha}$-regularized cosine function on $(L^1(\mathbb{R}))^2$, where $\alpha > 3/2$. Next, let

$$P(\xi) = \begin{pmatrix} 0 & 1 \\ -b|\xi|^2 & -2a|\xi|^2 \end{pmatrix}$$

for $\xi \in \mathbb{R}^n$. Then $P$ satisfies the condition $(H_0)$. It thus follows from Corollary 2.3 that $P(D)$ generates a $(1 - \Delta)^{-\alpha}$-regularized cosine function on $\mathcal{X}^2$, where $\alpha = 1 + n\mathcal{X}$. Finally, consider the strongly elliptic polynomial

$$P(\xi) = \sum_{|\mu| \leq m} a_\mu\xi^\mu$$

for $\xi \in \mathbb{R}^n$,

where $n \geq 2$ and $a_\mu \in \mathbb{R}$ for $|\mu| > \frac{m}{2}$. Then $P$ satisfies the condition $(H_m)$. By Corollary 2.3 $P(D)$ generates a $(1 - \Delta)^{-\alpha}$-regularized cosine function on $\mathcal{X}$, where $\alpha = mn\mathcal{X}/4$.

**3. Wellposedness of the associated systems**

In this section, we will apply Theorem 2.2 to the system (1.1). Denote by $Y_\alpha$ ($\alpha \geq 0$) the Banach space $R(C_\alpha)$ with the norm $\| \cdot \|_{\alpha} := \|C_\alpha\|$. We also denote by $\| \cdot \|_{\alpha}$ the norm of $Y_\alpha^N$, and set $S(t) = \int_0^t C(s)\, ds$ ($t \geq 0$).

**Theorem 3.1.** Let $P$ satisfy the condition $(H_r)$ for some $r \in [0, m]$, and let $\alpha > (m - r/2)(N - 1 + n/2)$. If $f \in C(\mathbb{R}^+, Y_{\alpha+m/2}^N)$ or $f \in C(\mathbb{R}^+, Y_N^N)$, then for every pair $(x, y) \in Y_{\alpha+m}^N \times Y_{\alpha+m/2}^N$ (1.1) has a unique solution $u \in C^2(\mathbb{R}^+, X_N)$ such that

$$\|u(t)\| \leq M_{gn/2}(t)\left(\|x\|_\alpha + t\|y\|_\alpha + t^2 \sup_{0 \leq s \leq t} \|f(s)\|_\alpha \right)$$

for $t \geq 0$. (3.1)
Proof. By Theorem 2.2 and its proof, \( \overline{P}(\mathcal{A}) \) generates a \( C_\alpha \)-regularized cosine function \( (C(t))_{t \geq 0} \) given by \( C(t) = h_t(A) \) for \( t \geq 0 \). Let \( k_t = v_t(P)(1 + | \cdot |^2)^{-\frac{m}{2}}(t \geq 0) \), where \( v_t(z) = \sum_{j=1}^{\infty} t^{2j-1}z^j/(2j-1)! \). Similarly to \( h_t \) (see the proof of Theorem 2.2), we can show that \( k_t (t \geq 0) \) is continuous in \( \| \cdot \|_{FL^1} \)-norm. Consequently

\[
\left\| \frac{1}{\varepsilon}(C(t+\varepsilon) - C(t))C_{m/2} - k_t(A) \right\| \leq M \left\| \int_0^{t+\varepsilon} (k_s - k_t) \, ds \right\|_{FL^1}
\]

for \( t, t + \varepsilon \geq 0 \). Letting \( \varepsilon \to 0 \) yields that \( C'(t)C_{m/2} = k_t(A) \) for \( t \geq 0 \), which implies that \( S(t)y \in R(C_\alpha) \) for \( t \geq 0 \) and \( C_\alpha^{-1}S(\cdot)y \in C^2(\mathbb{R}^+, X_N) \). Similarly, \( C(t)x \in R(C_\alpha) \) for \( t \geq 0 \) and \( C_\alpha^{-1}C(\cdot)x \in C^2(\mathbb{R}^+, X_N) \). Define \( w(t) = C_\alpha^{-1} \int_0^t S(t-s)f(s) \, ds \) for \( t \geq 0 \). Then by the assumption on \( f, w \) is well defined and \( w \in C^2(\mathbb{R}^+, X_N) \). Indeed,

\[
w''(t) = \begin{cases} f(t) + \int_0^t v_{t-s}(A)C_{\alpha+m/2}f(s) \, ds & \text{if } f \in C(\mathbb{R}^+, Y_N^N), \\ C(t)C_{\alpha}^{-1}f(0) + \int_0^t C(s)C_{\alpha}^{-1}f'(t-s) \, ds & \text{if } f \in C(\mathbb{R}^+, Y_N^N). \end{cases}
\]

Summarizing these statements one sees that \( v(t) \in R(C) \) (\( t \geq 0 \)) and \( C^{-1}v \in C^2(\mathbb{R}^+, X_N) \) where

\[
v(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s) \, ds \quad \text{for } t \geq 0,
\]

(3.2)

Now, by [10, Theorem 3.1], (1.1) has a unique solution \( u := C^{-1}v \) for every pair \((x, y) \in Y_N^{\alpha+m} \times Y_N^{\alpha+m/2}\), while (3.1) follows from (3.2) and the fact \( \| C(t) \| \leq M_{g_{n/2}} \) for \( t \geq 0 \). \( \square \)

We remark that Theorem 3.2 in [10] and the fact \( Y_N^N \subset L(D(\overline{P}(A))) \) for \( \beta > m \) (cf. [7, Lemma 4.4]) imply a similar result as Theorem 3.1. But \( y \in Y_N^{\alpha+m} \) and \( f \in C(\mathbb{R}^+, Y_N^{\alpha+m}) \) (or \( f \in C^2(\mathbb{R}^+, Y_N^N) \)) are required, where \( \alpha \) is chosen as in Theorem 3.1.

Let \( W_{\alpha, \chi}(\alpha \geq 0) \) be the completion of \( \mathcal{S} \) under the norm

\[
\| u \|_{\alpha, \chi} := \| u \|_{\chi} + \| F^{-1}(1 + | \cdot |^2)^{\alpha/2} Fu \|_{\chi} \quad \text{for } u \in \mathcal{S},
\]

where \( \chi \) was given in Section 2. We also denote by \( \| \cdot \|_{\alpha, \chi} \) the norm of \( W_N^{\alpha, \chi} \). When \( \chi = L^p (1 < p < \infty) \), \( W_{\alpha, \chi} \) is the Sobolev space.

Corollary 3.2. Let \( P \) satisfy the condition (H) for some \( r \in [0, m] \), and let \( \alpha = (m-r/2)(N-1+n_{\chi}) \). If \( f \in C(\mathbb{R}^+, W_N^{\alpha+m/2, \chi}) \) or \( f \in C^1(\mathbb{R}^+, W_N^{\alpha, \chi}) \), then for every pair \((\varphi, \psi) \in W_N^{\alpha+m, \chi} \times W_N^{\alpha+m/2, \chi} \), (1.2) has a unique solution \( u \in C^2(\mathbb{R}^+, \chi^N) \) such that

\[
\| u(t, \cdot) \|_{\chi} \leq M_{g_n, \chi}(t) \left( \| \varphi \|_{\alpha, \chi} + t \| \psi \|_{\alpha, \chi} + t^2 \sup_{0 \leq s \leq t} \| f(s, \cdot) \|_{\alpha, \chi} \right) \quad \text{for } t \geq 0.
\]

Example 3.3. We first consider (1.2) with \( P \) given by (2.6) in which \( a^2 \geq c \). If \( f \in C(\mathbb{R}^+, W_{\alpha+3/2, \chi}) \) or \( f \in C^1(\mathbb{R}^+, W_{\alpha, \chi}) \), where \( \chi = L^1(\mathbb{R}) \) and \( \alpha > 3 \), then Corollary 3.2 implies that the corresponding system has a unique solution for every pair \((\varphi, \psi) \in W_{\alpha+3, \chi} \times W_{\alpha+3/2, \chi} \). Next, if \( f \in C(\mathbb{R}^+, W_{\alpha+1, \chi}) \) or \( f \in C^1(\mathbb{R}^+, W_{\alpha, \chi}) \), then the system (1.2) with
$P$ given by (2.7) has a unique solution for every pair $(\varphi, \psi) \in W^{2}_{\alpha+2,\mathcal{X}} \times W^{2}_{\alpha+1,\mathcal{X}}$, where $\alpha = 2(1 + n \chi)$. Finally, if $f \in C(\mathbb{R}^{+}, W_{\alpha+m/2,\mathcal{X}})$ or $f \in C^{1}(\mathbb{R}^{+}, W_{\alpha,\mathcal{X}})$, then Eq. (1.2) with $P$ given by (2.8) has a unique solution for every pair $(\varphi, \psi) \in W^{m}_{\alpha+m,\mathcal{X}} \times W^{m/2}_{\alpha+m/2,\mathcal{X}}$, where $\alpha = mn \chi/2$.

References