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On power-boundedness of interval matrices

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Abstract: In a recent paper by G. Mayer [8] the convergence of the sequence $\{[A]^k\}$ of the powers of an interval matrix $[A]$ to the nullmatrix was investigated. In this note we give some conditions for the boundedness of the sequence $\{k^{-\alpha}[A]^k\}$, where α is a nonnegative number. The connection to the α-stability of the set $[A]$ is discussed.

Keywords: Power-boundedness, interval matrix, α-stability, Kreiss condition.

1. Introduction

Let $\mathbb{I}(\mathbb{R})$ be the set of the bounded, nonempty real intervals and let $M_n(\mathbb{I}(\mathbb{R}))$ be the set of the $n \times n$ matrices over $\mathbb{I}(\mathbb{R})$, called *interval matrices*. We represent intervals and interval matrices by their endpoints, e.g. we write $[a] := [\underline{a}, \bar{a}] = \{a \in \mathbb{R} \mid \underline{a} \leqslant a \leqslant \bar{a}\}$ for $[a] \in \mathbb{I}(\mathbb{R})$ and represent $[A] \in M_n(\mathbb{I}(\mathbb{R}))$, $[A] = ([a_{ij}])$, by $[A] = [\underline{A}, \bar{A}]$ with $\underline{A} = (\underline{a}_{ij})$, $\bar{A} = (\bar{a}_{ij})$. We consider the space of the real $n \times n$ matrices endowed with the natural (componentwise) partial ordering and define $[A] \geqslant 0$ and $[A] \leqslant 0$ if $\underline{A} \geqslant 0$ and $\bar{A} \leqslant 0$, respectively, for $[A] \in M_n(\mathbb{I}(\mathbb{R}))$.

The arithmetical operations + and $\cdot$ (subtraction and division will not be used in the sequel) on $\mathbb{I}(\mathbb{R})$ and $M_n(\mathbb{I}(\mathbb{R}))$ are defined as usual (e.g. [1, Chapters 1 and 10]):

$$[a] + [b] := [\underline{a} + \underline{b}, \bar{a} + \bar{b}],$$

$$[a] \cdot [b] := [\min\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}],$$

$$\text{for } [a] = [\underline{a}, \bar{a}], [b] = [\underline{b}, \bar{b}] \in \mathbb{I}(\mathbb{R});$$

$$[A] + [B] := ([a_{ij}] + [b_{ij}]),$$

$$[A] \cdot [B] := \left(\sum_m [a_{im}][b_{mj}]\right),$$

$$\text{for } [A] = ([a_{ij}]), [B] = ([b_{ij}]) \in M_n(\mathbb{I}(\mathbb{R})).$$

For $[A] \in M_n(\mathbb{I}(\mathbb{R}))$ the powers $[A]^k = ([a_{ij}^{(k)}]) = ([\underline{a}_{ij}^{(k)}, \bar{a}_{ij}^{(k)}])$, $k = 1, 2, 3, \ldots,$ are defined by $[A]^1 := [A]$, $[A]^k := [A]^{k-1} \cdot [A]$, $k = 2, 3, \ldots$. Note that the so defined powers differ in general form the powers ${}^k[A]$ defined by iterative multiplication by $[A]$ from the left: ${}^1[A] := [A]$,

${}^k[A] := [A] \cdot {}^{k-1}[A]$. The *width* d and the absolute value $|\cdot|$ of interval matrices are defined by

$$d([A]) := (\bar{a}_{ij} - \underline{a}_{ij}), \quad |[A]| := \left(\max_{a_{ij} \in [a_{ij}]} |a_{ij}| \right)$$

$$\text{for } [A] = ([a_{ij}]) = ([\underline{a}_{ij}, \bar{a}_{ij}]) \in M_n(\mathbb{I}(\mathbb{R})).$$

For properties of both functions consult for example [1, Chapters 2 and 10]. For later convenience we only state here the following properties:

$$|[A][B]| \leqslant |[A]|\,|[B]|, \qquad \text{for } [A], [B] \in M_n(\mathbb{I}(\mathbb{R})). \tag{1}$$

$$d([A])\,|[B]| \leqslant d([A][B]) \tag{2}$$

Let $\|\cdot\|$ be any norm on $M_n(\mathbb{I}(\mathbb{R}))$ which is *monotone*, i.e.

$$|[A]| \leqslant |[B]| \Rightarrow \|[A]\| \leqslant \|[B]\|$$
$$\text{for all } [A], [B] \in M_n(\mathbb{I}(\mathbb{R})),$$

or equivalently [9],

$$\|[A]\| = \|\,|[A]|\,\| \quad \text{for all } [A] \in M_n(\mathbb{I}(\mathbb{R})).$$

Then it follows by [9] that

$$\|[A]\| = \sup_{A \in [A]} \|A\|. \tag{3}$$

This implies that all monotone norms on $M_n(\mathbb{I}(\mathbb{R}))$ are equivalent.

An interval matrix $[A]$ is called an *α-stable* set [3], $\alpha \geqslant 0$, if there exists a constant c such that

$$\|k^{-\alpha}A^k\| \leqslant c \quad \text{for all } A \in [A], \quad k = 1, 2, 3, \ldots, \tag{4}$$

hold; for $\alpha = 0$, $[A]$ is called a *stable* set [7]. The concept of stability of the numerical schemes for solutions of partial differential equations is closely connected with the concept of stable sets [7,10] and it seems that α-stable sets are related to the concept of weakly stable numerical schemes [2,7]. These necessary and sufficient conditions for stability were given by Kreiss in [7] and one necessary and sufficient condition for α-stability by Friedland in [3]. Their conditions, however, are hard to verify in the general case and hence difficult to apply. In this paper we give a simple sufficient and an easy necessary and sufficient condition for the α-stability of an interval matrix. We remark that the application of these conditions is not merely restricted to the interval matrices since each bounded, nonempty set S of real $n \times n$ matrices can be enclosed in an interval matrix, viz. the matrix [inf S, sup S]. We note that Lemma 1 and Theorems 1 and 2 of this paper remain true if matrices with complex discs as entries (cf. [1, Chapters 5, 6 and 10]) are considered. If each entry of such a matrix has the origin as midpoint and positive radius then the matrix satisfies the assumptions (i)–(iv) of [4].

We extend the concept of power-boundedness to the sequences of the powers of interval matrices. By (3) and the inclusion monotonicity of the interval arithmetical operations (cf. [1, p. 6]) we obtain the following lemma.

Lemma 1. *Let* $[A] \in M_n(\mathbb{I}(\mathbb{R}))$. *Then*

$$\|k^{-\alpha}[A]^k\| \leqslant c, \quad k = 1, 2, 3, \ldots, \tag{5}$$

implies that $\|k^{-\alpha}A^k\| \leqslant c$, $k = 1, 2, 3, \ldots,$ *for all* $A \in [A]$, *i.e.* $[A]$ *is an α-stable set.*

2. Results

Theorem 1. *Let* $[A] \in M_n(\mathbb{I}(\mathbb{R}))$. *Then* $\| k^{-\alpha} |[A]|^k \| \leqslant c$, $k = 1, 2, 3, \ldots$, *implies* $\| k^{-\alpha}[A]^k \| \leqslant c$, $k = 1, 2, 3, \ldots$.

Proof. By (1) and the monotonicity of the interval matrix norm

$$\| k^{-\alpha}[A]^k \| = \| k^{-\alpha} |[A]^k| \| \leqslant \| k^{-\alpha} |[A]|^k \|$$

holds from which the assertion follows. □

By the Perron–Frobenius theory (see [5]), $|[A]|$ has a nonnegative eigenvalue r equal to its spectral radius. If $r = 0$ then from the Cayley–Hamilton theorem it follows that $|[A]|^n = 0$. Hence a reverse statement of Theorem 1 is true in this case. Another reverse statement is given in the following theorem. Because all monotone norms on $M_n(\mathbb{I}(\mathbb{R}))$ are equivalent we may choose the interval matrix norm here and in the sequel as the following monotone norm

$$\| [A] \| = \max_{i=1(1)n} \sum_{j=1}^{n} |[a_{ij}]| \quad \text{for } [A] = ([a_{ij}]) \in M_n(\mathbb{I}(\mathbb{R})).$$

Theorem 2. *Let* $[A] \in M_n(\mathbb{I}(\mathbb{R}))$ *and let* $|[A]|$ *fulfill the following two conditions*:

(i) $|[A]|$ *has an eigenvector* x *corresponding to its Perron root* r *with a positive component* x_ν *such that there is an entry* $[a_{\mu\nu}^{(p)}]$ *of* $[A]^p$ *with* $d([a_{\mu\nu}^{(p)}]) > 0$;

(ii) *for every eigenvalue* λ *of* $|[A]|$ *with* $|\lambda| = r$ *all the elementary divisors of* $|[A]|$ *corresponding to* λ *are linear.*

Then there exists a constant c *such that*

$$\| k^{-\alpha}[A]^k \| \leqslant c, \quad k = 1, 2, 3, \ldots \tag{6}$$

if and only if there exists a constant c' *such that*

$$\| k^{-\alpha} |[A]|^k \| \leqslant c', \quad k = 1, 2, 3, \ldots . \tag{7}$$

Proof. It is no restriction that we assume in the following that $r > 0$.

By Theorem 1, it suffices to show that (6) implies (7). We suppose (6). By repeated application of (2) we obtain for $k = 1, 2, 3, \ldots$

$$\begin{aligned}((\sup[A]^{k+p} - \inf[A]^{k+p})x)_\mu &= (d([A]^{k+p})x)_\mu \\ &\geqslant (d([A]^p)|[A]|^k x)_\mu = r^k (d([A]^p)x)_\mu \geqslant r^k d([a_{\mu\nu}^{(p)}])x_\nu .\end{aligned}$$

Now, by (3), it follows that for $k = 1, 2, 3, \ldots$

$$\begin{aligned} r^k d([a_{\mu\nu}^{(p)}])x_\nu &\leqslant \| (\sup[A]^{k+p} - \inf[A]^{k+p})x \| \\ &\leqslant 2 \| [A]^{k+p} \| \, \| x \| \leqslant 2 \| [A]^p \| \, \| [A]^k \| \, \| x \| .\end{aligned}$$

Thus, by (6),

$$0 < r^k \leqslant \delta k^\alpha, \quad \text{where } \delta := 2c \| [A]^p \| \, \| x \| \left(x_\nu d([a_{\mu\nu}^{(p)}]) \right)^{-1} . \tag{8}$$

From assumption (ii) it follows by [6, Section 2.3] that there exists a (multiplicative) matrix norm $\|\cdot\|_*$ such that $\| |[A]| \|_* = r$, hence

$$\| |[A]|^k \|_* \leqslant \| |[A]| \|_*^k = r^k, \quad k = 1, 2, 3, \ldots. \tag{9}$$

Because all norms on the set of the real $n \times n$ matrices are equivalent, there is a constant $c'' > 0$ such that

$$c'' \| |[A]|^k \| \leqslant \| |[A]|^k \|_*, \quad k = 1, 2, 3, \ldots. \tag{10}$$

The assertion follows now by (8)–(10). □

Remark 1. Condition (ii) is fulfilled if $|[A]|$ is diagonalizable or possesses a positive Perron eigenvector (cf. [5, p. 104]), particularly if $|[A]|$ is irreducible. If $|[A]|$ has a positive Perron eigenvector condition (i) is always fulfilled for each $[A]$ with $d([A]) \neq 0$.

Remark 2. Condition (i) is always fulfilled if $[A]$ contains at least one nondegenerate interval in each column. In [8] a useful graph theoretical criterion is given which allows one to decide whether a power $[A]^p$ and a row index μ exist such that $d([a_{\mu\nu}^{(p)}]) > 0$ for fixed column index ν.

Remark 3. We recall two examples given in [8] to show that condition (i) can not be dropped.

Example 1. Let

$$[A] := A := \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then A is irreducible and $A^2 = 0$. Hence the degenerate interval matrix $[A]$ fulfills (5) for all $\alpha \geqslant 0$. Taking absolute values,

$$|A| = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Because of $\| |A|^k \| = 2^k$ the sequence $\{k^{-\alpha} |A|^k\}$ is not uniformly bounded for any $\alpha \geqslant 0$.

Example 2. Let

$$[A] := \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & [0, 0.5] \end{pmatrix}.$$

Then

$$[A]^k = 2^{-k} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [0, 1] \end{pmatrix}, \quad k = 2, 3, 4, \ldots,$$

which shows that $[A]$ satisfies (5) for all $\alpha \geqslant 0$. Again, we have $\| |[A]|^k \| = 2^k$. The eigenvalues of $|[A]|$ are 0, 0.5, 2, hence $|[A]|$ is diagonalizable and (ii) is fulfilled. The eigenvectors corresponding to the Perron root are the vectors $(x, x, 0)^{\mathrm{T}}$. Because the only nondegenerate entry of $[A]^k$ is the entry $[a_{33}^{(k)}]$, condition (i) is not met.

Corollary. *Let* $n = 2$, $[A] \in M_2(\mathbb{I}(\mathbb{R}))$ *and let condition* (i) *of Theorem* 2 *be fulfilled. Then* (6) *and* (7) *are equivalent.*

Proof. It suffices to show that (6) implies (7). W.l.o.g. we may assume that $[a_{21}] \cdot [a_{12}] = 0$ since otherwise Theorem 2 applies. Furthermore, it suffices to consider only the case $[a_{21}] = 0$ and $[a_{12}] \neq 0$. If $|[a_{11}]| \neq |[a_{22}]|$ then Theorem 2 applies, so we may restrict ourselves on interval matrices $[A]$ with

$$|[A]| = \begin{pmatrix} r & \beta \\ 0 & r \end{pmatrix} \quad \text{with } \beta > 0.$$

It suffices to consider only $r = 1$. Thus, we get

$$\| \, |[A]|^k \| = 1 + \beta k. \tag{11}$$

From assumption (i) it follows that $d([a_{11}]) > 0$. In the sequel we choose an interval matrix $[B]$,

$$[B] = \begin{pmatrix} [b_1] & b_2 \\ 0 & b_3 \end{pmatrix},$$

in such a way that $[B] \subseteq [A]$ and the asymptotic behaviour of $\|[B]^k\|$ is the same as (11). Since by (3)

$$\|[B]^k\| \leqslant \|[A]^k\| \leqslant \| \, |[A]|^k \|,$$

the assertion will then follow. Because of $d([a_{11}]) > 0$ there is a positive number $\epsilon < 1$ such that the interval matrix $[B]$ with

$$\text{(I)} \;\; [b_1] = [\epsilon, 1] \quad \text{or} \quad \text{(II)} \;\; [b_1] = [-1, -\epsilon],$$
$$\text{(a)} \;\; b_3 = 1 \quad \text{or} \quad \text{(b)} \;\; b_3 = -1,$$

is contained in $[A]$. As it will be clear from the following there is no restriction to assume that $b_2 = \beta$. The powers of $[B]$ are given by

$$[B]^k = \begin{pmatrix} [b_1]^k & [b_2^{(k)}] \\ 0 & b_3^k \end{pmatrix}.$$

We first consider the case (I, a): Here we have, setting $[\epsilon, 1]^0 := 1$,

$$[b_2^{(k)}] = \beta \sum_{m=0}^{k-1} [\epsilon, 1]^m = \beta \left[\frac{1-\epsilon^k}{1-\epsilon}, k \right],$$

hence

$$\|[B]^k\| = 1 + \beta k = \| \, |[A]|^k \|.$$

In the case (I, b) we obtain

$$[b_2^{(k)}] = \beta \sum_{m=0}^{k-1} (-1)^{k-1-m} [\epsilon, 1]^m,$$

therefore for $k \geqslant 4$

$$\|[B]^k\| = 1 + \beta \max\left\{\left|-k_1 + \epsilon^2 \frac{1-\epsilon^{2k_2}}{1-\epsilon^2}\right|, \left|k_3 - \epsilon \frac{1-\epsilon^{2k_4}}{1-\epsilon^2}\right|\right\},$$

$$\text{where } k_1 := \left[\frac{k-2}{2}\right], \quad k_2 := \left[\frac{k-1}{2}\right], \quad k_3 := \left[\frac{k+1}{2}\right], \quad k_4 := \left[\frac{k}{2}\right].$$

The cases (II, a) and (II, b) can be reduced to (I, b) and (I, a), respectively, and the proof is completed. □

Remark 4. As it can be seen by choosing

$$[A] := \begin{pmatrix} 1 & 1 \\ 0 & [-1, -0.5] \end{pmatrix},$$

condition (i) can not be dropped in the Corollary.

We have already noted that the existence of a constant c with (5) implies that $[A]$ is an α-stable set. However, the converse is not true in general (for a partial relaxation see below) even if $|[A]|$ is irreducible: Let

$$[A] := \begin{pmatrix} [0.5, 1] & -1 \\ 1 & -1 \end{pmatrix}.$$

By Theorem 2 and the above Example 1 there is no constant c such that $[A]$ fulfills (5) for any $\alpha \geqslant 0$. However, each $A \in [A]$ has a spectral radius less than 1. Hence $\lim_{k\to\infty} A^k = 0$ for all $A \in [A]$ which implies that $[A]$ is an α-stable set for all $\alpha \geqslant 0$.

Now, we consider a class of interval matrices for which (6) is equivalent to α-stability. We say that an interval matrix $[A]$ *has D-sign pattern*, if D is a signature matrix, i.e. $D = \operatorname{diag}(\delta_1, \ldots, \delta_n)$ with $\delta_i = \pm 1$ for $i = 1(1)n$, and if $D[A]D \geqslant 0$. [1] Then we define for two real $n \times n$ matrices B, C the $n \times n$ matrix $B \& C$ by

$$(B\&C)_{ij} := \begin{cases} b_{ij} & \text{if } \delta_i = \delta_j, \\ c_{ij} & \text{if } \delta_i \neq \delta_j, \end{cases} \qquad i, j = 1(1)n.$$

For example, if $[A] \geqslant 0$ (choose D as the identity matrix) then $\underline{A}\&\overline{A} = \underline{A}$ and $\overline{A}\&\underline{A} = \overline{A}$. If $[A]$ or $-[A]$ possesses a D-sign pattern then the endpoints of $[A]^k$ can be calculated only by using powers of $\underline{A}\&\overline{A}$ and $\overline{A}\&\underline{A}$:

Lemma 2. *Let* $[A] \in M_n(\mathbb{I}(\mathbb{R}))$. *If* $[A]$ *has D-sign pattern then*

$$[A]^k = \left[(\underline{A}\&\overline{A})^k \& (\overline{A}\&\underline{A})^k, (\overline{A}\&\underline{A})^k \& (\underline{A}\&\overline{A})^k\right], \quad k = 1, 2, 3, \ldots;$$

if $-[A]$ *has D-sign pattern then for* $k = 1, 2, 3, \ldots$

$$[A]^{2k} = \left[(\overline{A}\&\underline{A})^{2k} \& (\underline{A}\&\overline{A})^{2k}, (\underline{A}\&\overline{A})^{2k} \& (\overline{A}\&\underline{A})^{2k}\right],$$

$$[A]^{2k+1} = \left[(\underline{A}\&\overline{A})^{2k+1} \& (\overline{A}\&\underline{A})^{2k+1}, (\overline{A}\&\underline{A})^{2k+1} \& (\underline{A}\&\overline{A})^{2k+1}\right].$$

[1] Because $(D[A])D = D([A]D)$ we may suppress brackets.

Proof. By induction noting that $(B\&C)\&(C\&B)=B$. □

Theorem 3. *Let* $[A]\in M_n(\mathbb{I}(\mathbb{R}))$ *have D-sign pattern. Then the following three statements are equivalent*

(i) $\|k^{-\alpha}[A]^k\|\leqslant c, \quad k=1,2,3,\ldots,$

(ii) $\|k^{-\alpha}A^k\|\leqslant c, \quad k=1,2,3,\ldots, \quad$ *for all* $A\in[A]$,

(iii) $\|k^{-\alpha}(\bar{A}\&\underline{A})^k\|\leqslant c, \quad k=1,2,3,\ldots.$

Proof. By Lemma 1, (i) ⇒ (ii) and because $\bar{A}\&\underline{A}\in[A]$ it follows that (ii) ⇒ (iii). Thus it suffices to show that (iii) ⇒ (i). One shows by induction that

$$|\bar{A}\&\underline{A}|^k=|(\bar{A}\&\underline{A})^k|, \quad k=1,2,3,\ldots, \tag{12}$$

holds. We assume (iii). Then by (12)

$$\begin{aligned}\|k^{-\alpha}(\bar{A}\&\underline{A})^k\| &= \|k^{-\alpha}|(\bar{A}\&\underline{A})^k|\| = \|k^{-\alpha}|\bar{A}\&\underline{A}|^k\|\\ &= \|k^{-\alpha}|[A]|^k\|\leqslant c.\end{aligned}$$

The assertion (i) follows now by Theorem 1. □

Remark 5. An analogous statement holds if $-[A]$ has D-sign pattern. Then (iii) has to be replaced by $\|k^{-\alpha}(\underline{A}\&\bar{A})^k\|\leqslant c$, $k=1,2,3,\ldots$.

We conclude with two remarks on the uniform boundedness of the sequence $\{k^{-\alpha}\,{}^k[A]\}$. Example 4 in [8] shows that there may be a constant c' such that $\|k^{-\alpha}\,{}^k[A]\|\leqslant c'$, but no constant c satisfying (5) for $\alpha\geqslant 0$ and vice versa. Observing that $({}^k[A])^{\mathrm{T}}=([A]^{\mathrm{T}})^k$, we state a result which is analogous to Corollary 4 in [8].

Theorem 4. *Let* $[A]\in M_n(\mathbb{I}(\mathbb{R}))$. *Then there is a constant* c *such that* $\|k^{-\alpha}\,{}^k[A]\|\leqslant c$, $k=1,2,3,\ldots,$ *if there exists a constant* c' *such that* $\|k^{-\alpha}([A]^{\mathrm{T}})^k\|\leqslant c'$, $k=1,2,3,\ldots$.

As can readily be confirmed, we have ${}^k[A]=[A]^k$, $k=1,2,3,\ldots,$ if $[A]$ has D-sign pattern.

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References

[1] G. Alefeld and J. Herzberger, *Introduction to Interval Computations* (Translation from the German, Bibliographisches Institut, Mannheim, 1974). (Academic Press, New York, 1983).

[2] G.E. Forsyth and W.R. Wasow, *Finite-Difference Methods for Partial Differential Equations* (Wiley, New York, 1960).
[3] S. Friedland, A generalization of the Kreiss matrix theorem, *SIAM J. Math. Anal.* **12** (1981) 826–832.
[4] S. Friedland and E. Tadmor, Optimality of the Lax–Wendroff condition, *Linear Algebra Appl.* **56** (1984) 121–129.
[5] F.R. Gantmacher, *Applications to the Theory of Matrices* (Interscience, New York, 1959).
[6] A.S. Householder, *The Theory of Matrices in Numerical Analysis* (Blaisdell, New York-Toronto-London, 1964).
[7] H.O. Kreiss, Über die Stabilitätsdefinition für Differenzengleichungen, die partielle Differentialgleichungen approximieren, *BIT* **2** (1962) 153–181.
[8] G. Mayer, On the convergence of powers of interval matrices, *Linear Algebra Appl.* **58** (1984) 201–216.
[9] O. Mayer, Algebraische und metrische Strukturen in der Intervallrechnung und einige Anwendungen, *Computing* **5** (1970) 144–162.
[10] R.D. Richtmyer and K.W. Morton, *Difference Methods for Initial-Value Problems* (Interscience, New York, 2nd ed., 1967).