Mechanism of singularity formation for quasilinear hyperbolic systems with linearly degenerate characteristic fields

Peng Qu
School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China

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1. Introduction

We discuss the formation of singularity of classical solutions to the following Cauchy problem for 1-D quasilinear strictly hyperbolic systems:

\[
\begin{cases}
\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = F(u), & x \in \mathbb{R}, \ t \geq 0, \\
t = 0: u = u_0(x), & x \in \mathbb{R},
\end{cases}
\]

(1)

(2)
where \( u = (u_1, \ldots, u_n)^T \) is an unknown vector function of \((t, x)\), the initial data \( u_0(x) \) is continuously differentiable with bounded \( C^1 \) norm

\[
\|u_0\|_{C^1} = \|u_0\|_{L^\infty} + \|u_0'\|_{L^\infty} < +\infty. \tag{3}
\]

\( A(u) \) is a \( C^2 \) \( n \times n \) matrix function with \( n \) distinct real eigenvalues

\[
\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{4}
\]

which implies that the left eigenvectors \( l_i(u) = (l_{i1}(u), \ldots, l_{in}(u)) \) \((i = 1, \ldots, n)\) and the right eigenvectors \( r_i(u) = (r_{1i}(u), \ldots, r_{ni}(u))^T \) \((i = 1, \ldots, n)\) form two bases of \( \mathbb{R}^n \) respectively, and \( \lambda_i(u), l_i(u), r_i(u) \) \((i = 1, \ldots, n)\) have the same regularity as \( A(u) \). Without loss of generality, we assume

\[
l_i(u)r_j(u) \equiv \delta_{ij} \quad \text{(i, j = 1, \ldots, n)}, \tag{5}
\]

where \( \delta_{ij} \) stands for Kronecker's symbol.

The local existence of \( C^1 \) solution to the Cauchy problem is well known (see [10]). If all the characteristics are linearly degenerate in the sense of P.D. Lax:

\[
\nabla \lambda_i(u)r_i(u) \equiv 0, \quad \forall u \in \mathbb{R}^n, \quad 1 \leq i \leq n, \tag{6}
\]

then Cauchy problem (1)–(2) with small and decaying initial data admits a unique global \( C^1 \) solution \( u = u(t, x) \) for all \( t \in \mathbb{R} \) (see [6,11,14]). However, for general \( C^1 \) initial data, the \( C^1 \) solution may blow up in a finite time (see [4] and [8] for examples). A. Majda gave the following conjecture in [12] (originally, for systems of conservation laws).

**Conjecture 1.** Under hypotheses (3)–(6), if the \( C^1 \) solution \( u = u(t, x) \) to Cauchy problem (1)–(2) blows up in a finite time \( T_* < +\infty \), then the \( C^0 \) norm of \( u \) must blow up at \( T_* \), i.e., \( \sup_{0 \leq t \leq T_*} \|u(t, \cdot)\|_{C^0} = +\infty. \)

This is a different kind of singularity from the shock formation, in which \( \|\delta_{ij}(t, \cdot)\|_{L^\infty} \) goes to the infinity in a finite time, while \( \|u(t, \cdot)\|_{L^\infty} \) remains bounded as for Burgers’ equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.
\]

Because the solution to the Riccati ordinary differential equation and the classical solution to the semilinear hyperbolic systems blow up in such a way, it is called the ODE singularity in [1] or the semilinear behavior in [2] and [3]. In this paper we will use the name “ODE singularity”.

To verify Conjecture 1 is still an open problem, but some results around it have been made in recent years (see [7] for review), among which [8] concerns the singularities caused by eigenvectors, and the property of ODE singularity for the so-called 2-step and 3-step completely reducible systems with constant eigenvalues is discussed, the ODE singularity is discussed in [9] for a series of multi-step completely reducible hyperbolic conservation laws, and the validity of Conjecture 1 is proved in [13] for the general inhomogeneous diagonal systems. All the results mentioned above are developed by the method of characteristics. On the other hand, the authors of [2] and [3] apply the method of energy integral to prove the property of ODE singularity for the generalized Kerr–Debye model in \( H^2 \) Sobolev space.

In this paper, we will show that if the singularity occurs, then all the \( W^{1,p} \) \((1 < p < +\infty)\) norms of the \( C^1 \) solution should blow up at the same time (see Theorem 2'). Thus, a new framework to verify the property of ODE singularity can be given by showing that the following conjecture is equivalent to Conjecture 1.
Conjecture 2. Under hypotheses (3)–(6) and the assumption that the initial data \( u_0 \) has a compact support, for any fixed \( T_0 > 0 \), if for any given \( T (0 < T < T_0) \), Cauchy problem (1)–(2) admits a unique \( C^1 \) solution \( u = u(t, x) \) on \( 0 \leq t \leq T \), which satisfies the uniform a priori estimate

\[
\left\| u(t, \cdot) \right\|_{C^0} = \left\| u(t, \cdot) \right\|_{L^\infty} \leq C_0, \quad \forall t \in [0, T],
\]

then we have the following uniform estimate: there exists a real number \( p \in (1, +\infty] \), such that

\[
\left\| \partial_x u(t, \cdot) \right\|_{L^p} \leq C_p, \quad \forall t \in [0, T],
\]

where \( C_0 = C_0(T_0) \) and \( C_p = C_p(T_0) \) denote positive numbers independent of \( T \), but possibly depending on \( T_0 \), and \( C_p \) is independent of the explicit form of \( u_0 \), but possibly depending on \( \| u_0 \|_{W^{1,1}} \) and \( \| u_0 \|_{C^1} \).

This equivalence comes from the following theorem.

**Theorem 1.** Under hypotheses (3)–(6) and the assumption that the initial data \( u_0 \) has a compact support, for any fixed \( T_0 > 0 \), if for any given \( T (0 < T < T_0) \), Cauchy problem (1)–(2) admits a unique \( C^1 \) solution \( u = u(t, x) \) on \( 0 \leq t \leq T \), which satisfies the uniform estimates (7) and (8), then it satisfies the following uniform a priori estimate:

\[
\left\| \partial_x u(t, \cdot) \right\|_{C^0} \leq C_1, \quad \forall t \in [0, T],
\]

where \( C_1 = C_1(T_0) \) denotes a positive number independent of \( T \) and the explicit form of \( u_0 \), but possibly depending on \( T_0 \), \( \| u_0 \|_{W^{1,1}} \) and \( \| u_0 \|_{C^1} \), and the constants \( C_0 \) and \( C_p \) have the same properties as in Conjecture 2.

We will prove this theorem in Section 2 and then use it to show the equivalence of the former two conjectures in Section 3. With the help of this equivalence, we can verify the property of ODE singularity for the \( C^1 \) solution to Cauchy problem (1)–(2) by verifying Conjecture 2. Thus, we can directly use the energy method in the framework of \( C^1 \) solution so that there is no need of applying the higher order Sobolev norm estimation and embedding theory as in [2] and [3]. Noting that the proof of Theorem 1 in Section 2 uses the method of characteristics, our method is virtually a combination of energy method and characteristics method to accomplish this singularity analysis. In Section 5, as an application of this method, we will discuss the generalized Kerr–Debye model.

By Theorem 1, we can easily deduce the following result on the mechanism of singularity formation.

**Theorem 2.** Under hypotheses (3)–(6) and the assumption that the initial data \( u_0 \) has a compact support, the \( C^1 \) solution \( u = u(t, x) \) to Cauchy problem (1)–(2) blows up in a finite time \( t = T_* < +\infty \), if and only if, at least one of the following two conditions is satisfied:

\[
\text{(i)} \quad \sup_{0 \leq t < T_*} \left\| u(t, \cdot) \right\|_{C^0} = +\infty,
\]

\[
\text{(ii)} \quad \sup_{0 \leq t < T_*} \left\| \partial_x u(t, \cdot) \right\|_{L^p} = +\infty, \quad \forall 1 < p \leq +\infty.
\]

By means of Sobolev embedding theory, it is easy to see that Theorem 2 is equivalent to the following
Theorem 2’. Under hypotheses (3)–(6) and the assumption that the initial data $u_0$ has a compact support, the $C^1$ solution $u = u(t, x)$ to Cauchy problem (1)–(2) blows up in a finite time $t = T_* < +\infty$, if and only if all the $W^{1,p}$ norms of $u = u(t, x)$ blow up at the time $T_*:$

$$\sup_{0 \leq t < T_*} \| u(t, \cdot) \|_{W^{1,p}} = +\infty, \quad \forall 1 < p \leq +\infty.$$ 

We will generalize the above results in Section 4, but, generically speaking, as shown in Section 3 by two examples, the hypotheses of the strict hyperbolicity and of the linear degeneracy are essential for our results.

2. Proof of Theorem 1

In this section, we will give the proof of Theorem 1 in four steps with the method coming from [6,11] and especially [14]. Obviously, we only need to consider the case $1 < p < +\infty$.

First, setting

$$w_i = l_i(u) \frac{\partial u}{\partial x} \quad (i = 1, \ldots, n),$$

by (5), we have

$$\frac{\partial u}{\partial x} = \sum_{k=1}^{n} w_k r_k.$$  \hspace{1cm} (11)

Taking the partial derivative with respect to $x$ for system (1), and substituting (11) in it, we get the wave decomposition formulas (cf. [5] and [6])

$$\frac{\partial w_i}{\partial t} + \frac{\partial}{\partial x} (\lambda_i(u) w_i) = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n} H_{ij}(u) w_j \quad (i = 1, \ldots, n)$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2} (\lambda_j(u) - \lambda_k(u)) l_i(u) (\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)), \quad H_{ij}(u) = l_i(u) (\nabla F(u) r_j(u) - \nabla r_j(u) F(u)),$$  \hspace{1cm} (14) (15)

and

$$\gamma_{ijk}(u) = \Gamma_{ijk}(u) - \nabla \lambda_k(u) r_j(u) \delta_{ik}.$$  \hspace{1cm} (16)

Obviously, we have

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall 1 \leq i, j \leq n$$  \hspace{1cm} (17)
Lemma 1.

For any given \( T_0 > 0 \), noting (7), by the continuity of \( A(u), \lambda_i(u), l_i(u) \) and \( r_i(u) \), and the strict hyperbolicity condition (4), there exist three positive constants \( \delta_1 = \delta_1(T_0) > 0, M = M(T_0) > 0 \) and \( \widetilde{C}_0 = \widetilde{C}_0(T_0) > 0 \), depending only on \( T_0 \), such that

\[
\begin{align*}
\left| \lambda_i(u) \right| & \leq M, \quad \forall 1 \leq i \leq n, \forall x \in \mathbb{R}, \forall t \in [0, T], \\
\left| \lambda_i(u) - \lambda_j(u) \right| & > \delta_1, \quad \forall 1 \leq i \neq j \leq n, \forall x \in \mathbb{R}, \forall t \in [0, T]
\end{align*}
\]

and

\[
\sup_{x \in \mathbb{R}, t \in [0, T], 1 \leq i \leq n} \left\{ \sum_{j,k=1}^{n} \left| \Gamma_{ij}(u) \right|, \sum_{j,k=1}^{n} \left| \gamma_{ij}(u) \right|, \sum_{j=1}^{n} \left| H_{ij}(u) \right|, \sum_{j=1}^{n} \left| l_{ij}(u) \right|, 1 \right\} \leq \widetilde{C}_0.
\]

By (10), for \( 0 \leq t \leq T \), we have

\[
\sup_{1 \leq i \leq n} \left| w_i(t, x) \right| \leq \widetilde{C}_0 \left| \frac{\partial u}{\partial x} (t, x) \right|.
\]

By (8), for any given \( x_1 \in \mathbb{R} \) and \( t \in [0, T] \), we can use the Hölder inequality to obtain

\[
\int_{x_1 - L}^{x_1 + L} \left| \frac{\partial u}{\partial x}(t, x) \right| \, dx \leq \left\| \frac{\partial u}{\partial x} \right\|_{L^p[x_1 - L, x_1 + L]} \cdot \left\| 1 \right\|_{L^{p'}[x_1 - L, x_1 + L]} \leq C_p(2L)^{\frac{1}{p'}}.
\]

where \( L > 0 \) is a small positive number to be chosen later, \( p \ (1 < p < +\infty) \) is the constant given in (8), and \( p' \) is the constant satisfying

\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

Secondly, we estimate the interaction terms on the right-hand side of the wave decomposition formulas (12) and (13). For this purpose we introduce

\[
Q(t_1; t_0, x_0) = \sup_{i \neq j} \int_{t_0 - M(t-t_0)}^{t_1} \int_{x_0 - L + M(t-t_0)}^{x_0 + L - M(t-t_0)} \left| w_i(t, x) \right| \left| w_j(t, x) \right| \, dx \, dt
\]

to denote their integration, where \( 0 \leq t_0 \leq \bar{t}_0, t_0 \leq t_1 \leq \bar{t}_1, x_0 \in \mathbb{R}, \) and \( \bar{t}_0 = \max \{T - L/M, 0\}, \bar{t}_1 = \min \{t_1 + L/M, T\}. \) Obviously, \( 0 \leq t_1 - t_0 \leq \min \{T, L/M\}. \)

Lemma 1. (See [14], Lemma 3.2.) If \( \phi_i = \phi_i(t, x) \in C^1 \ (i = 1, 2) \) satisfies

\[
\begin{cases}
\frac{\partial \phi_i}{\partial t} + \frac{\partial}{\partial x} \left( \lambda_i(t, x) \phi_i \right) = E_i(t, x), & x \in \mathbb{R}, \ t \in [0, T], \\
\phi_i(0, x) = \phi_{i, 0}(x), & x \in \mathbb{R},
\end{cases}
\]

and, by the linear degeneracy (6),

\[
\gamma_{ij}(u) \equiv 0, \quad \forall 1 \leq i, j \leq n.
\]
respectively, where \( \lambda_i, E_i, \phi_{i,0} \in C^1 \) \((i = 1, 2)\), and there exists a constant \( \Lambda > 0 \), such that

\[
|\lambda_i(t, x)| \leq \Lambda, \quad \forall 0 \leq t \leq T, \forall x \in \mathbb{R} \quad (i = 1, 2)
\]

and a constant \( \delta_0 > 0 \), such that

\[
|\lambda_1(t, x) - \lambda_2(t, x)| \geq \delta_0, \quad \forall 0 \leq t \leq T, \forall x \in \mathbb{R},
\]

then for any two given real numbers \( \alpha \) and \( \beta \) with \( \beta > \alpha \), and any given \( t \in [0, T_1] \) with \( T_1 = \min\{T, (\beta - \alpha)/(2\Lambda)\} \), we have the following estimate

\[
\int_0^t \int_{\alpha + \Lambda \tau}^{\beta - \Lambda \tau} |\phi_1(\tau, x)| \left| \phi_2(\tau, x) \right| \, dx \, d\tau \leq \frac{2}{\delta_0} \prod_{i=1,2} \left( \int_{\alpha}^{\beta} |\phi_{i,0}(x)| \, dx + \int_0^t \int_{\alpha + \Lambda \tau}^{\beta - \Lambda \tau} |E_i(\tau, x)| \, dx \, d\tau \right).
\]

Noting (17) and (20), Lemma 1 can be applied to the wave decomposition formula (12) and we get

\[
Q(t_1; t_0, x_0) \leq \frac{2}{\delta_1} \sup_{i \neq j} \left( \int_{x_0 - L}^{x_0 + L} |w_i(t_0, x)| \, dx \right.
\]

\[
+ \int_{t_0}^{t_1} \int_{x_0 - \delta_0 M(t-t_0)}^{x_0 + \delta_0 M(t-t_0)} \left[ \sum_{a, b = 1, a \neq b} \Gamma_{iab}(u) w_a w_b + \sum_{a = 1}^n H_i a w_a \right] (t, x) \, dx \, dt
\]

\[
- \left( \int_{x_0 - L}^{x_0 + L} |w_j(t_0, x)| \, dx \right.
\]

\[
+ \int_{t_0}^{t_1} \int_{x_0 - \delta_0 M(t-t_0)}^{x_0 + \delta_0 M(t-t_0)} \left[ \sum_{a, b = 1, a \neq b} \Gamma_{jab}(u) w_a w_b + \sum_{a = 1}^n H_j a w_a \right] (t, x) \, dx \, dt
\].

Then, noting (21)–(23) and \( 0 \leq t_1 - t_0 \leq L/M \), we get

\[
Q(t_1; t_0, x_0) \leq \frac{2}{\delta_1} \left( \tilde{C}_0 C_p (2L)^{2_3} + \tilde{C}_0 Q(t_1; t_0, x_0) + \frac{L}{M} \tilde{C}_0^2 C_p (2L)^{1_3} \right)^2.
\]

Now, choosing \( L > 0 \) small enough and depending only on \( T_0 \), for example,

\[
L \leq \min \left\{ \frac{1}{2} \left( \frac{\delta_1}{12 C_p \tilde{C}_0^2} \right)^{\frac{1}{p'}}, \frac{M}{3 \tilde{C}_0} \right\}, \quad (25)
\]

we have

\[
Q(t_1; t_0, x_0) \leq \frac{2}{\delta_1} \left( \frac{\delta_1}{9 \tilde{C}_0} + \tilde{C}_0 Q(t_1; t_0, x_0) \right)^2, \quad \forall t_0, \forall t_1 \in [t_0, t_1], \forall x_0 \in \mathbb{R}.
\]
Solving this algebraic inequality, we get

\[ Q(t_1; t_0, x_0) \leq \frac{\delta_1}{18C_0^2} \quad \text{or} \quad Q(t_1; t_0, x_0) \geq \frac{2\delta_1}{9C_0^2}. \]

Noting that \( Q(t_1; t_0, x_0) \) is continuous with respect to \( t_1 \) and \( Q(t_0; t_0, x_0) = 0 \), finally we get

\[ Q(t_1; t_0, x_0) \leq \frac{\delta_1}{18C_0^2}, \quad \forall t_0 \in [0, \bar{t}_0], \quad \forall t_1 \in [t_0, \bar{t}_1], \quad \forall x_0 \in \mathbb{R}. \]  \hspace{1cm} (26)

Thirdly, we estimate the integration of \(|w_i|\) along the characteristic curve with different index. For any given point \( A = (t_1, x_0) \in [0, T] \times \mathbb{R} \), let \( t_0 = \max\{t_1 - L/M, 0\} \) and \( C_j \) be the \( j \)th characteristic curve \( x = x_j(t) \) passing through \( A \):

\[ \frac{dx_j(t)}{dt} = \lambda_j(u(t, x_j(t))), \quad x_j(t_1) = x_0, \]

which intersects \( t = t_0 \) at the point \( B_j \). Correspondingly let \( C_i \) be the \( i \)th characteristic curve passing through \( A \) and intersecting \( t = t_0 \) at the point \( B_i \), where \( i \neq j \). Since the wave decomposition formula (12) can be transformed to the following exterior differential form:

\[ d(|w_i|)(dx - \lambda_i(u) \, dt)) = -\text{sgn}(w_i)G_i \, dx \wedge dt, \]  \hspace{1cm} (27)

where

\[ G_i = \sum_{a,b=1,\,a \neq b}^n \Gamma_{iab}w_a w_b + \sum_{a=1}^n H_{i\bar{a}}w_a, \]

we can apply the Stokes formula to (27) on the domain surrounded by \( C_i \), \( C_j \) and \( B_i B_j \), which gives

\[ \int_{C_j} |w_i| |\lambda_j(u) - \lambda_i(u)| \, dt \leq \int_{E_i E_j} |w_i(t_0, x)| \, dx + \int_{A_i B_j} |G_i(t, x)| \, dx \, dt. \]

Then, noting (19), (21), (22) and \( 0 \leq t_1 - t_0 \leq L/M \), we get

\[ \int_{C_j} |w_i| |\lambda_j(u) - \lambda_i(u)| \, dt \leq \tilde{C}_0 \left\| \frac{\partial u}{\partial x}(t_0, \cdot) \right\|_{L^1[|x_0 - L, x_0 + L]} + \tilde{C}_0 \sup_{a \neq b} \int_{A_i B_j} |w_a w_b| \, dx \, dt \]

\[ + \tilde{C}_0^2 \int_{t_0}^{t_1} \left\| \frac{\partial u}{\partial x}(t, \cdot) \right\|_{L^1[|x_0 - L, x_0 + L]} \, dt. \]

It is easy to see that \( 0 \leq t_0 \leq \bar{t}_0 \) and \( t_0 \leq t_1 \leq \bar{t}_1 \), then, noting (20), (23), (25), (26) and \( 0 \leq t_1 - t_0 \leq L/M \), we obtain

\[ \sup_{i \neq j} \int_{C_j} |w_i| \, dt \leq \frac{1}{\delta_1} \left[ \tilde{C}_0 C_p (2L)^{\frac{1}{p}} + \tilde{C}_0 Q(t_1; t_0, x) + \frac{L}{M} \tilde{C}_0^2 C_p (2L)^{\frac{1}{p}} \right] \leq \frac{1}{6C_0}. \]  \hspace{1cm} (28)
Finally, we estimate \( \| w_i \|_{L^\infty} \). Let

\[
W_\infty(t; t_1, x_0) = \sup_{t_0 \leq t \leq t_1} \max_{x_0 - M(t_1 - t) \leq x \leq x_0 + M(t_1 - t)} |w_i(t, x)|,
\]

where \( t_0 \leq t \leq t_1 \). Multiplying \( \text{sgn}(w_i) \) to the both sides of the second wave decomposition formula (13), integrating it along the \( i \)th characteristic curve, noting (18) and (21), we get

\[
|w_i(t, x)| \leq |w_i(t_0, \hat{x}^{(i)})| + \tilde{C}_0 \sup_{j \neq k} \int_{C_i^'} |w_j| w_k |dt + \tilde{C}_0 \sup_{1 \leq j \leq n} \int |w_j| dt,
\]

where \( t_0 \leq t \leq t_1, x_0 - M(t_1 - t) \leq x \leq x_0 + M(t_1 - t), C_i^' \) is the \( i \)th characteristic curve passing through \( (t, x) \), and \( \hat{x}^{(i)} \) is the intersection point of \( C_i^' \) with \( t = t_0 \). Then, taking the supremum to both sides with respect to \( (t, x) \), noting (25), (28) and \( t - t_0 \leq L/M \), we obtain

\[
\sup_{t_0 \leq t \leq t_1} W_\infty(t; t_1, x_0) \leq W_\infty(t_0; t_1, x_0) + \tilde{C}_0 \cdot \frac{1}{6\tilde{C}_0} \sup_{t_0 \leq t \leq t_1} W_\infty(t; t_1, x_0) + \frac{L}{M} \tilde{C}_0 \sup_{t_0 \leq t \leq t_1} W_\infty(t; t_1, x_0)
\]

\[
\leq W_\infty(t_0; t_1, x_0) + \frac{1}{2} \sup_{t_0 \leq t \leq t_1} W_\infty(t; t_1, x_0),
\]

namely,

\[
\sup_{t_0 \leq t \leq t_1} W_\infty(t; t_1, x_0) \leq 2W_\infty(t_0; t_1, x_0).
\]

Thus, for any given \( t_1 \in [0, T] \), we get

\[
\sup_{t_0 \leq t \leq t_1} \max_{t_0 \leq t \leq t_1, x \in \mathbb{R}} |w_i(t, x)| \leq 2 \sup_{x \in \mathbb{R}} \max_{1 \leq i \leq n} |w_i(t_0, x)|.
\]

(29)

in which, \( t_0 = \max(t_1 - L/M, 0) \) and \( 0 \leq t_1 \leq T \).

When \( T \leq L/M \), by specially taking \( t_1 = T \), we have \( t_0 = 0 \), then by (29) and (11), we get the desired conclusion (9). When \( T > L/M \), taking \( t_1 = T \), we have \( t_0 = T - L/M \), then by (29) we get

\[
\sup_{T - L/M \leq t \leq T} \max_{t \in \mathbb{R}} |w_i(t, x)| \leq 2 \sup_{x \in \mathbb{R}} \max_{1 \leq i \leq n} |w_i(T - L/M, x)|.
\]

(30)

Since \( L \) and \( M \) depend only on \( T_0 \), successively repeating this procedure and noting the result obtained in the case \( T \leq L/M \), we get (9). \( \square \)

### 3. Proof of the main results

In this section, we will use Theorem 1 to prove the equivalence between Conjecture 1 and Conjecture 2.

First, as shown in [7–9], Conjecture 1 can be stated in the following equivalent form.
Conjecture 3. Under hypotheses (3)–(6), for any fixed $T_0 > 0$, if for any given $T$ $(0 < T < T_0)$, Cauchy problem (1)–(2) admits a unique $C^1$ solution $u = u(t, x)$ on $0 \leq t \leq T$, whose $C^0$ norm satisfies the uniform a priori estimate (7), then the uniform a priori estimate (9) holds, where $C_0 = C_0(T_0)$ and $C_1 = C_1(T_0)$ are positive numbers independent of $T$, but possibly depending on $T_0$.

Next, Conjecture 3 can be equivalently rewritten only for compactly supported initial data as follows.

Conjecture 4. Under hypotheses (3)–(6) and the assumption that the initial data $u_0$ has a compact support, for any fixed $T_0 > 0$, if for any given $T$ $(0 < T < T_0)$, Cauchy problem (1)–(2) admits a unique $C^1$ solution $u = u(t, x)$ on $0 \leq t \leq T$, whose $C^0$ norm satisfies the uniform a priori estimate (7), then the uniform a priori estimate (9) holds, where the positive numbers $C_0$ and $C_1$ have the same properties as in Conjecture 3, and $C_1$ is independent of the explicit form of $u_0$, but possibly depending on $\|u_0\|_{W^{1,1}}$ and $\|u_0\|_{C^1}$.

As a matter of fact, Conjecture 3 obviously implies Conjecture 4. On the other hand, since (7) implies the finite speed of propagation for the solution $u = u(t, x)$, by a suitable truncation and noting that $C_1$ is independent of the explicit form of $u_0$, it is easy to get Conjecture 3 from Conjecture 4.

Now, we need only to show the equivalence of Conjecture 2 with Conjecture 4. This fact can be easily obtained by taking $p = +\infty$ and by Theorem 1, respectively.

In the remainder of this section, we will show the necessity of the strict hyperbolicity and the linear degeneracy by examples. First, we point out that, generically speaking, the conclusions mentioned before are not valid for systems without linear degeneracy.

Consider the following Cauchy problem

$$
\begin{cases}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \\
t = 0: u = u_0(x),
\end{cases}
$$

where $u_0(x) \in C^1$ has a compact support, satisfying

$$
u_0(x) = \begin{cases} 
-\sqrt{x+1} + 1, & 0 \leq x \leq 3, \\
\sqrt{-x+1} - 1, & -3 \leq x < 0,
\end{cases}
$$

and $|u'(x)| \leq 1/2$ for $|x| \geq 3$. It is easy to see that the $C^1$ solution to this Cauchy problem blows up at the point $(t, x) = (2, 0)$ and the solution on the determinate domain $D = \{0 \leq t < 2, \ t - 3 \leq x \leq 3 - t\}$ is given by

$$
u(t, x) = \begin{cases} 
-\frac{t+\sqrt{t^2-4(t-x-1)}}{2} + 1, & 0 \leq x \leq 3 - t, \\
\frac{t+\sqrt{t^2-4(t+x+1)}}{2} - 1, & t - 3 \leq x < 0.
\end{cases}
$$

So $\|u(t, x)\|_{L^\infty(D)} < +\infty$. While, since

$$
\frac{\partial u}{\partial x}(t, x) = \begin{cases} 
\frac{1}{\sqrt{t^2-4(t-x-1)}}, & 0 \leq x \leq 3 - t, \\
\frac{1}{\sqrt{t^2-4(t+x+1)}}, & t - 3 \leq x < 0,
\end{cases}
$$
we have \( \| \partial_x u(t, x) \|_{L^\infty(D)} = +\infty \), but
\[
\lim_{t \to 2^-} \left\| \partial_x u(t, \cdot) \right\|_{L^{3/2}([-1, 1])} = 2 < +\infty.
\]
Thus, the \( L^{3/2} \) norm of \( \partial_x u \) is bounded, but the \( L^\infty \) norm blows up. This means that the conclusions of Theorem 1 and Theorem 2 are not valid for this system without linear degeneracy.

A similar consideration for the following Cauchy problem
\[
\begin{align*}
\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} &= 0,
\end{align*}
\]
t = 0: \( u = v = u_0(x) \), where \( u_0(x) \) is the same as in Example 1, shows the same invalidity for systems without strict hyperbolicity.

4. Generalization

The results given in the previous sections can be generalized in the following two ways.

First, we generalize the previous results to systems without strict hyperbolicity but having some weaker properties. In this case, we do not require (4), but we still assume the hyperbolicity of the system, i.e., the coefficient matrix \( A(u) \) possesses \( n \) real eigenvalues \( \lambda_i(u) \) \((i = 1, \ldots, n)\), and the related left eigenvectors \( l_i(u) \) \((i = 1, \ldots, n)\) and right eigenvectors \( r_i(u) \) \((i = 1, \ldots, n)\) suitably chosen form two bases of \( \mathbb{R}^n \) respectively. Moreover, we assume that the eigenvalues and eigenvectors have the same regularity as the coefficient matrix \( A(u) \), i.e., \( \lambda_i(u), l_i(u), r_i(u) \in C^2 \) \((i = 1, \ldots, n)\). Under this assumption, the linear degeneracy means that (6) is still satisfied. Without loss of generality, we still assume to have (5).

For simplicity, we introduce the set \( \mathcal{A} \) of index pairs to mark the eigenvalues violating the strict hyperbolicity:
\[
\mathcal{A} = \{(i, j) \in \mathbb{Z} \times \mathbb{Z}: 1 \leq i, j \leq n, \exists \alpha \in \mathbb{R}^n \text{ s.t. } \lambda_i(\alpha) = \lambda_j(\alpha)\},
\]
so
\[
(i, j) \notin \mathcal{A} \iff \lambda_i(u) \neq \lambda_j(u), \quad \forall u \in \mathbb{R}^n,
\]
and obviously, for any \( i = 1, \ldots, n, \) we have
\[
(i, i) \in \mathcal{A}.
\]

With the help of this notation, we can express our generalized version of Theorem 1 as

\[\textbf{Theorem 3.} \text{ Under hypotheses (3) and (5), suppose furthermore that the initial data has a compact support and}\]
\[
\Gamma_{ijk}(u) \equiv \gamma_{ijk}(u) \equiv 0, \quad \forall u \in \mathbb{R}^n, \forall 1 \leq i \leq n, \forall (j, k) \in \mathcal{A}.
\]
Then, conclusion of Theorem 1 is still valid, i.e., for any fixed $T_0 > 0$, if for any given $T$ $(0 < T < T_0)$, Cauchy problem (1)–(2) admits a unique $C^1$ solution $u = u(t, x)$ on $0 \leq t \leq T$, which satisfies both (7) and (8), then (9) holds, where the positive numbers $C_0$, $C_p$ and $C_1$ have the same properties as in Theorem 1.

**Proof.** We only need to replace the definition of $Q(t_1; t_0, x_0)$ by

$$
\tilde{Q}(t_1; t_0, x_0) = \sup_{(i, j) \in A} \int_{t_0}^{t_1} \int_{x_0 - L + M(t - t_0)}^{x_0 + L - M(t - t_0)} |w_i(t, x)| |w_j(t, x)| \, dx \, dt
$$

and repeat the whole procedure in Section 2.

**Remark 1.** Condition (32) can be regarded as a generalization of the strict hyperbolicity, because (32) means the eigenvalues that have the same quantity at some points never interact each other in the wave decomposition formulas.

**Remark 2.** Condition (32) implies the linear degeneracy (6). In fact, if we take $i = j = k$ in (32), (6) follows directly from (16).

By Theorem 3, a similar discussion as in Section 3 can be used to obtain the equivalence of the following two conjectures.

**Conjecture 1’.** Under hypotheses (3), (5) and (32), the singularity of the $C^1$ solution $u = u(t, x)$ to Cauchy problem (1)–(2) must be of the ODE type, in other words, if $u = u(t, x)$ blows up in a finite time $T_* < +\infty$, then its $C^0$ norm must blow up at $T_*$.  

**Conjecture 2’.** Under hypotheses (3), (5) and (32), suppose furthermore that the initial data has a compact support, for any fixed $T_0 > 0$, if for any given $T$ $(0 < T < T_0)$, Cauchy problem (1)–(2) admits a unique $C^1$ solution $u = u(t, x)$ on $0 \leq t \leq T$, which satisfies (7), then (8) holds, where the positive numbers $C_0$ and $C_p$ have the same properties as in Conjecture 2.

Thus, we establish a framework to get the property of ODE singularity for systems with property (32) by verifying the validity of Conjecture 2’. Moreover, we can obtain the following result on the mechanism of the singularity formation.

**Theorem 4.** Under hypotheses (3), (5) and (32), suppose furthermore that the initial data has a compact support, the $C^1$ solution $u = u(t, x)$ to Cauchy problem (1)–(2) blows up in a finite time $t = T_* < +\infty$, if and only if $u = u(t, x)$ satisfies at least one of the following two conditions

(i) $\sup_{0 \leq t < T_*} \|u(t, \cdot)\|_{C^0} = +\infty$,  

(ii) $\sup_{0 \leq t < T_*} \|\partial_t u(t, \cdot)\|_{L^p} = +\infty, \quad \forall 1 < p \leq +\infty$,  

in other words, if and only if $u = u(t, x)$ satisfies

$$
\sup_{0 \leq t < T_*} \|u(t, \cdot)\|_{W^{1, p}} = +\infty, \quad \forall 1 < p \leq +\infty.
$$

**Remark 3.** Noting that the strict hyperbolicity of the system is equivalent to

$$(i, j) \in A \iff i = j,$$
it is easy to see from (17) and (18) that the linearly degenerate strictly hyperbolic system always satisfies (32), then all the results mentioned in the previous sections can be regarded as special cases of the results in this section.

**Remark 4.** Since the condition of eigenvalues with constant multiplicity (see [9] for definition) can be expressed as

$$\lambda_i(u) \equiv \lambda_j(u), \quad \forall u \in \mathbb{R}^n, \forall (i, j) \in A,$$

it is easy to see from (14) and (16) that (32) holds for any linearly degenerate hyperbolic system with eigenvalues with constant multiplicity, then all the results mentioned above are valid for this kind of system.

Now we generalize the results from another side. In the proof of Theorem 1, all we need is to use the uniform a priori estimate (8) to show that for any given $\varepsilon > 0$, there exists $L > 0$ depending only on $T_0$ and $\varepsilon$, such that

$$\left| \int_{x_1-L}^{x_1+L} \left| \frac{\partial u}{\partial x}(t, x) \right| \, dx \leq \varepsilon, \quad \forall x_1 \in \mathbb{R}, \forall t \in [0, T], \right.$$  

(34)

hence it is possible to replace (8) by another condition which still implies (34). For example, instead of (8), we use the following uniform estimation of the $C^1$ solution $u = u(t, x)$: There exist $n$ continuous functions $g_i : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\lim_{y \to +\infty} g_i(y) = +\infty \ (i = 1, \ldots, n)$, such that

$$\int \sum_{i=1}^{n} \left| \frac{\partial u_i}{\partial x}(t, x) \right| \cdot g_i\left( \left| \frac{\partial u_i}{\partial x}(t, x) \right| \right) \, dx \leq C_g, \quad \forall t \in [0, T],$$

(35)

where $C_g = C_g(T_0)$ is a positive number independent of both $T$ and the explicit form of $u_0$, but possibly depending on $T_0$, $\|u_0\|_{W^{1,1}}$ and $\|u_0\|_{C^1}$.

In fact, if the $C^1$ solution $u = u(t, x)$ satisfies (35), then for any given $\varepsilon > 0$ and $i = 1, \ldots, n$, noting $\lim_{y \to +\infty} g_i(y) = +\infty$, there exists a positive number $M_{i, \varepsilon} > 0$, such that for any given $y \geq M_{i, \varepsilon}$, we have

$$g_i(y) \geq \frac{2C_g}{\varepsilon}.$$ 

Set $M_{\varepsilon} = \sum_{i=1}^{n} M_{i, \varepsilon}$, $L = \varepsilon/(4M_{\varepsilon}) > 0$, and for any given $t \in [0, T]$, denote

$$\mathcal{D}_{i, t} = \left\{ x \in \mathbb{R} : g_i\left( \left| \frac{\partial u_i}{\partial x}(t, x) \right| \right) \geq \frac{2C_g}{\varepsilon} \right\},$$

then

$$\mathbb{R} \setminus \mathcal{D}_{i, t} = \left\{ x \in \mathbb{R} : g_i\left( \left| \frac{\partial u_i}{\partial x}(t, x) \right| \right) < \frac{2C_g}{\varepsilon} \right\} \subseteq \left\{ x \in \mathbb{R} : \left| \frac{\partial u_i}{\partial x}(t, x) \right| < M_{i, \varepsilon} \right\}.$$
Thus, by (35), for any given \( t \in [0, T] \) and \( x_1 \in \mathbb{R} \), we have

\[
\int_{x_1 - L}^{x_1 + L} \frac{\partial u}{\partial x}(t, x) \, dx \leq \sum_{i=1}^{n} \left( \int_{[x_1 - L, x_1 + L] \cap D_{i,t}} + \int_{[x_1 - L, x_1 + L]} \frac{\partial u_i}{\partial x}(t, x) \, dx \right)
\]

\[
\leq 2LM \epsilon + \int_{\mathbb{R}} \sum_{i=1}^{n} \left| \frac{\partial u_i}{\partial x}(t, x) \right| \cdot g_i \left( \left| \frac{\partial u_i}{\partial x}(t, x) \right| \right) \cdot \frac{\epsilon}{2Cg} \, dx
\]

\[
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

then (34) holds. Therefore, when (8) is replaced by (35), all the results are still valid, for example, we have

**Theorem 4'.** Under hypotheses (3), (5) and (32), suppose furthermore that the initial data \( u_0 \) has a compact support, then the \( C^1 \) solution \( u = u(t, x) \) to Cauchy problem (1)–(2) blows up in a finite time \( t = T_\ast < +\infty \), if and only if at least one of the following two conditions is satisfied.

(i) \( \sup_{0 \leq t < T_\ast} \| u(t, \cdot) \|_{C^0} = +\infty \),

(ii) \( \sup_{0 \leq t < T_\ast} \| \partial_x u(t, \cdot) \|_{L^\infty} = +\infty \), and

\[
\sup_{0 \leq t < T_\ast} \int_{\mathbb{R}} \sum_{i=1}^{n} \left| \frac{\partial u_i}{\partial x}(t, x) \right| \cdot g_i \left( \left| \frac{\partial u_i}{\partial x}(t, x) \right| \right) \, dx = +\infty,
\]

\( \forall (g_1, \ldots, g_n) \in \left\{ g : C(\mathbb{R}^+ \to \mathbb{R}^+) : \lim_{y \to +\infty} g(y) = +\infty \right\}^n \).

Obviously, all the previous results could be treated as the special case that \( g_i(y) = y^{p-1} (1 < p < +\infty, i = 1, \ldots, n) \).

### 5. Application

As mentioned in Sections 1 and 4, the previous results provide us a way to directly use the energy method in the framework of \( C^1 \) solution to verify the ODE singularity. Now, as an application, we consider the generalization of the following Kerr–Debye model

\[
\begin{align*}
(1 + \chi) \partial_t e + \partial_x h &= -\frac{1}{\epsilon} e(e^2 - \chi), \\
\partial_t h + \partial_x e &= 0, \\
\partial_t \chi &= \frac{1}{\epsilon} (e^2 - \chi)
\end{align*}
\]

(cf. [3]). We now prove the property of ODE singularity for the Cauchy problem of the system

\[
\begin{align*}
A_0(\chi) \partial_t v + A_1 \partial_x v &= \phi(u), \\
\partial_t \chi &= \psi(u),
\end{align*} \quad x \in \mathbb{R}, \ t \geq 0
\]
with the initial data
\[ t = 0: u = u_0(x) = \begin{pmatrix} \psi_0(x) \\ \chi_0(x) \end{pmatrix}, \quad x \in \mathbb{R}, \tag{37} \]

where \( u = (y^T \chi) \in \mathbb{R}^{n-r} \times \mathbb{R}^r \) is the unknown vector function, \( A_0(\chi) \in \mathbb{C}^2 \) is a symmetric and positive definite matrix, \( A_1 \) is a reversible constant symmetric matrix, \( \Phi(u) = (\psi(u) \ \chi(u)) \in \mathbb{C}^2 \), and \( u_0(x) \) is a compactly supported \( C^1 \) vector function.

First, we verify that system (36) satisfies the generalized strict hyperbolicity condition (32). Noting the special form of the coefficient matrix
\[ A(u) = \begin{pmatrix} A_0^{-1}(\chi)A_1 & 0 \\ 0 & 0 \end{pmatrix} \]
and the reversibility of \( A_0 \) and \( A_1 \), we have
\[ \lambda_i(u) \neq 0, \quad \nabla \lambda_i(u) = 0, \quad \forall 1 \leq i \leq n-r, \quad \forall u \in \mathbb{R}^n, \]
\[ \lambda_j(u) = 0, \quad \forall n-r + 1 \leq j \leq n, \quad \forall u \in \mathbb{R}^n. \]

Hence, if there exists \( \alpha \in \mathbb{R}^n \), such that \( \lambda_i(\alpha) = \lambda_j(\alpha) \), it is easy to see that \( 1 \leq i, j \leq n-r \) or \( n-r + 1 \leq i, j \leq n \), namely, \( \mathcal{A} \subseteq \{(i, j): 1 \leq i, j \leq n-r\} \cup \{(i, j): n-r + 1 \leq i, j \leq n\} \). Moreover, the corresponding right eigenvectors can be chosen as
\[
\begin{align*}
    r_i(u) &= (r_{i1}(\chi), r_{i2}(\chi), \ldots, r_{i(n-r)}(\chi), 0, \ldots, 0)^T, \quad 1 \leq i \leq n-r, \\
    r_j(u) &= e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T, \quad n-r + 1 \leq j \leq n.
\end{align*}
\]

Thus, if \( (i, j) \in \mathcal{A} \), then we have
\[ \nabla \lambda_j(u) \cdot r_k(u) \equiv 0, \quad \forall u \in \mathbb{R}^n. \]

Then, substituting above properties into (14) and (16), we get (32).

As shown in Section 4, we can show the property of ODE singularity for Cauchy problem (36)–(37) by verifying Conjecture 2’. For this purpose, for any fixed \( T_0 > 0 \), if for any given \( T (0 < T < T_0) \), Cauchy problem (36)–(37) admits a unique \( C^1 \) solution \( u = u(t, x) \) on \( 0 \leq t \leq T \), satisfying
\[
\| u(t, \cdot) \|_{C^0} \leq \hat{C}_0, \quad \forall t \in [0, T], \tag{38}
\]
then we need to show that \( u = u(t, x) \) also satisfies
\[
\| \partial_x u(t, \cdot) \|_{L^2} \leq \hat{C}_2, \quad \forall t \in [0, T], \tag{39}
\]
where \( \hat{C}_0 = \hat{C}_0(T_0) \) and \( \hat{C}_2 = \hat{C}_2(T_0) \) are positive numbers independent of \( T \), but possibly depending on \( T_0 \), and \( \hat{C}_2 \) is independent of the explicit form of \( u_0 \), but possibly depending on \( \| u_0 \|_{W^{1,1}} \) and \( \| u_0 \|_{C^1} \).

For \( H^2 \) initial data, the boundedness of the \( H^2 \) norm of the solution has been shown in [3]. Now we need only to get the estimate (39) for \( C^1 \) initial data, and we repeat this part here. Taking the partial derivative with respect to \( t \) for (36) gives
where \( \xi = \partial_t \nu, \eta = \partial_t X, \zeta = \partial_t u = (\xi, 0, 0) \). On the other hand, by (36)-(37) we have the initial data of \( \zeta \) as follows:

\[
\begin{align*}
\xi(0, x) &= (A_0(\chi_0(x)))^{-1}(\phi(u_0(x)) - A_1 v_0'(x)), \\
\eta(0, x) &= \psi(u_0(x)).
\end{align*}
\]

Taking the inner product of (40) with \( \zeta \) and integrating by parts with respect to \( x \) on both sides, by the finite speed of propagation, we get

\[
\frac{1}{2} \frac{d}{dt} \int \int (A_0(\chi) \xi \cdot \xi + |\eta|^2) \, dx = \int \int -\frac{1}{2} (\nabla_x A_0(\chi) \psi(u)) \xi \cdot \xi + \nabla_u \phi(u) \zeta \cdot \xi + \nabla_u \psi(u) \zeta \cdot \eta \, dx.
\]

Noting (38), by the continuity and the positive definite property, \( A_0 \) is uniformly positive definite, i.e., there exists a positive number \( \hat{C}_+ \) depending only on \( T_0 \), such that

\[ A_0(\chi) \theta \cdot \theta \geq \hat{C}_+ \| \theta \|^2, \quad \forall \theta \in \mathbb{R}^{n-1}, \quad \forall t \in [0, T], \quad \forall x \in \mathbb{R}. \]

By (38) and the continuity, there exists a positive number \( \hat{M} \) depending only on \( T_0 \), such that

\[
\max \left\{ \sum_{ijk} |(\nabla_x A_0(\chi) \cdot \psi(u))_{ij}|, \sum_{ij} |(\nabla_u \Phi(u))_{ij}| \right\} \leq \hat{M}.
\]

Hence we have

\[
\| \zeta(t, \cdot) \|_{L^2}^2 \leq (1 + \hat{C}_+^{-1}) \int \int (A_0(\chi) \xi \cdot \xi + |\eta|^2)(t, x) \, dx \\
\leq (1 + \hat{C}_+^{-1}) \left[ \int \int (A_0(\chi_0) \xi_0 \cdot \xi_0 + |\eta_0|^2)(x) \, dx + 5\hat{M} \int_0^t \int |\xi|^2(t, x) \, dx \, dt \right].
\]

Then, by Gronwall’s inequality, we get \( \| \zeta \|_{L^2} \leq \hat{C}_2(T_0) \), here and hereafter the notation as \( \hat{C}_2(T_0) \) stands for a positive number independent of \( T \), but possibly depending on \( T_0 \). Next, we use

\[ \partial_x v = (A_0)^{-1}(\phi(u) - A_0(\chi) \xi) \]

to get \( \| \partial_x v \|_{L^2} \leq \hat{C}_{2,v}(T_0) \). Moreover, we have

\[ \partial_t \partial_x \chi = \nabla_u \psi(u) \partial_x u, \]

taking the inner product of it with \( \partial_x \chi \) and integrating over \( \mathbb{R} \) on both sides, we get \( \| \partial_x \chi \|_{L^2} \leq \hat{C}_{2,\chi}(T_0) \), which finally gives us the uniform estimate (39) and then the property of ODE singularity.

**Remark 5.** The result here is slightly different from that given in [3]. By means of Theorem 3, we do not need the estimate on \( H^2 \) norm and we only need \( u_0 \in C^1 \) instead of \( H^2 \). On the other hand, we need the hypothesis that the eigenvalues and eigenvectors can be chosen to have the \( C^2 \) smoothness.
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