A low-dimensional conjugacy for elliptic equations and symmetry breaking on rotated domains

Seth Armstrong\textsuperscript{a,*,1} and Renate Schaaf\textsuperscript{b}

\textsuperscript{a}Department of Computer Science and Mathematics, Arkansas State University, AR 72467-0070, USA
\textsuperscript{b}Department of Mathematics and Statistics, Utah State University, Logan, UT 84321-3900, USA

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Abstract

A topological conjugacy is established between certain elliptic PDEs with one unbounded time direction and a simple second-order differential equation, admitting the dynamics of such PDEs to be examined on a two-dimensional submanifold. By this means, periodic solutions can be obtained to elliptic equations as perturbations of those that are independent of time. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

This paper is an investigation of certain nonlinear elliptic PDEs on domains with one unbounded time-like direction. Of interest is obtaining solutions that vary in the time direction as perturbations of solutions that are independent of time. These problems have the form

$$\frac{d^2 U}{dt^2} + LU + G(U, \lambda) = 0$$  \hspace{1cm} (1)

*Corresponding author.

E-mail addresses: armstrong@suu.edu, armstrong@csm.astate.edu (S. Armstrong), schaaf@math.usu.edu (R. Schaaf).

1Current address: Department of Mathematics and Computer Science, Southern Utah University, 351 W. Center, Cedar City, UT 84720, USA.
on a suitable domain, where $L$ is an elliptic differential operator and $G$ is a nonlinearity that satisfies properties to be addressed later. The requirements on $L$ are relaxed enough that a variational formulation for (1) is not necessary; in fact, later we investigate a problem that is not variational. The main contribution of this work is the establishment of conditions under which a topological conjugacy exists between bounded solutions of (1) and a relatively simple second-order differential equation.

We begin with an example that serves to illustrate the motivation behind the reduction. Consider first the two-parameter normal form for a saddle node given by

\[ x^2 + \lambda - \varepsilon = 0, \quad \text{where } 0 < \lambda < \varepsilon. \]  

(2)

For fixed $\varepsilon$ there is a turning point at $(\lambda, x) = (\varepsilon, 0)$, with $x$-intercepts of $\pm \sqrt{\varepsilon}$. Now viewing $x$ and $x'$ as state variables and adjoining a second derivative term (in $t$, say) to (2) yields the equation

\[ x'' + x^2 + \lambda - \varepsilon = 0, \quad \text{where } 0 < \lambda < \varepsilon \text{ and } t = \frac{d}{dt}. \]  

(3)

The steady states of the system arising from (3) are $(x, x') = (\pm \sqrt{\varepsilon - \lambda}, 0)$, with $(-\sqrt{\varepsilon - \lambda}, 0)$ a saddle point and $(\sqrt{\varepsilon - \lambda}, 0)$ a center for periodic orbits.

If to the $(\lambda, x)$-plane we adjoin another axis representing the variable $x'$, we can graph both relations simultaneously in $(\lambda, x, x')$-space as shown in Fig. 1: For each choice of $\lambda$, $0 < \lambda < \varepsilon$, there is a “base curve” of steady-state solutions in the $(\lambda, x)$-plane, where on the positive $x$-branch of $x^2 + \lambda - \varepsilon = 0$ in that plane each point is a center for periodic orbits and on the negative $x$-branch each point is a saddle.

This idea may be generalized and compared with (1) in the following way. Suppose that it is known that the elliptic operator $L$ in (1) gives rise to an equation $LU + G(U, \lambda) = 0$ that has a turning point in $(\lambda, U)$-space. Then under reduction techniques used in [5] and employed in this work, the addition of the second time-like derivative in (1) gives rise to dynamical behavior similar near the turning point to that exhibited in the example above. Hence, (1) is an evolution equation in a suitable
domain. In fact, although $U$ inhabits an infinite-dimensional function space, it is shown that the dynamics of all bounded solutions of (1) occur on a two-dimensional submanifold. This result is obtained in Section 2.

An example that arises in [1] is studied in Section 3 of the paper. This is the semilinear elliptic PDE given by

$$\Delta u + \lambda f(u) = 0 \quad \text{in} \quad \Omega_\varepsilon, \quad \lambda \in \mathbb{R}^+, \quad u = 0 \quad \text{on} \ \partial \Omega_\varepsilon,$$

where $\Omega_\varepsilon \subset \mathbb{R}^{n+1}$ is a special type of noncontractible domain with a large hole, obtained by translating a domain $\Omega \subset \mathbb{R}^n$ a distance of $1/\varepsilon$ then rotating it about the $x_{n+1}$ axis. In [1] a connection is made between positive solutions of the semilinear elliptic equation

$$\Delta u + \lambda f(u) = 0 \quad \text{in} \quad \Omega, \quad \lambda \in \mathbb{R}^+,$$

and those of the corresponding problem on the annular domain $\Omega_\varepsilon$ introduced above. Briefly stated, in [1] it is shown that if a solution curve $\varepsilon \mapsto (u(\varepsilon), \lambda(\varepsilon))$ exists for (5) on the rotated domain, under certain conditions there exists a solution surface of rotationally invariant solutions of (4) (with respect to $\Omega_\varepsilon$) emanating from the solution curve as $\varepsilon \to 0$.

It is natural to ask whether any solutions of (4) exist aside from the rotationally invariant ones. It turns out that the answer to the question is affirmative in certain situations; existence of periodic solutions can be shown using a topological conjugacy established in Section 2. Emanation of these solutions from the curve of solutions of (5) is established. Thus Section 3 is devoted to describing this breaking from rotational symmetry as well as some asymptotic properties of these solutions as $\varepsilon$ approaches zero.

The existence of nonradial bifurcation from radial solutions has been studied in various settings. For example, Dancer [3] and Smoller and Wasserman [9,10] have examined the problem on ball domains, and for annular domains in $\mathbb{R}^2$ the bifurcation has been investigated by Lin [4]. We note that the treatise of nonradial bifurcation on annular domains in [4] will correspond closely to our result for a rotated one-dimensional domain; in fact, in that work only sufficiently thin domains are investigated, which coincides with the restriction imposed on our analysis: Only for small $\varepsilon$ do our conclusions apply. In the example of Section 4 the two results are compared and combined.

We add to the results in [4] in Section 3 by demonstrating approximately where bifurcations arise on the curve of rotationally invariant solutions. Moreover, besides holding for annular domains in $\mathbb{R}^2$ as in [4], our results are applicable to a domain in $\mathbb{R}^{n+1}$ that is a rotation of any smooth, bounded domain in $\mathbb{R}^n$ on which a turning point can be shown to exist. As an example, whenever $f'' > 0$, $f(0) > 0$, and $f$ exhibits
superlinear growth at infinity, the existence of a turning point is guaranteed (see [2]).
(These results can be generalized to nonlinearities which satisfy $f(0) = 0$; a good review of conditions required for the existence of a turning point is found in [8].)

In [1] the implicit function theorem is the mechanism used to establish the
existence of the surface of rotationally invariant solutions of (4) emanating from the
base curve (the curve of positive solutions to (5)). The only place where its existence
is difficult to establish is near turning points of the base curve; everywhere else the $u$-
linearization is invertible and there is no difficulty in a straightforward application of
the implicit function theorem. This is the reason that it is possible to have extra
solutions of (4) near a turning point.

A main difference in this contribution is the local setting of the problem. In [1] the
analysis begins with a global branch of solutions of (5) from which the existence of
rotationally invariant solutions of (4) for nonzero $\varepsilon$ is shown. Here we are only
concerned with local properties of solutions of (4); that is, only near the turning
point. However, in Section 4 the two are combined for $f(u) = \varepsilon u^n$.

2. The conjugacy theorem

The main theorem in this work has its beginnings in a reduction principle found in
[5]. In that setting, $G$ is allowed to have a domain with a topology as strong as the power
$\gamma = 1$ of $X$ only in case $X$ is a Hilbert space. In our results here it is necessary that

$$G \in C^4(X^\gamma \times \mathbb{R} \times \mathbb{R}, X) \quad \text{with} \quad \gamma < 1,$$

whereas in [5], $G$ only need be of class $C^3$ for a general Banach space $X$. We require
the extra derivative for the proof of the conjugacy theorem (Theorem 2.1). However,
for applications of interest here, the perturbation in (1) involves second-order
derivative terms. So $\gamma = 1$ is required in order to extract enough regularity from $G$.

But the Hilbert space setting is problematic for nonlinearities which grow too
quickly. Fortunately, as is demonstrated in Section 4, such nonlinearities can be modified
using a cutoff function and elliptic regularity theory, so the Hilbert space framework is
sufficiently general for our purposes. So $\gamma = 1$ is used throughout this paper.

Now let

$$F(u, \lambda, \varepsilon) := Lu + G(u, \lambda, \varepsilon).$$

As mentioned in the Introduction, this analysis is local: specifically, in a
neighborhood of a turning point. So general conditions on $F$ under which a turning
point $(u^*, \lambda^*)$ exists need to be satisfied. Following [2], we require that

$$F(u^*, \lambda^*) = 0 \quad \text{and for} \quad T^* := D_uF(u^*, \lambda^*),$$

$$N(T^*) = \text{span}\{\rho\}, \quad \rho \notin R(T^*),$$

$$D_{uu}F(u^*, \lambda^*) (\rho, \rho) \notin R(T^*),$$

$$D_{\lambda_i}F(u^*, \lambda^*) \notin R(T^*) \quad \text{for some} \quad i, \quad i = 1, \ldots, n.$$
More specifically, suppose that $P$ is the orthogonal projection operator onto $\text{span}\{\rho\}$. Then (8) implies that $c \neq 0$ and $b = (b_1, \ldots, b_n)^\top \neq 0$, where

$$PD_{\omega}F(u^*, \lambda^*) (\rho, \rho) = 2c\rho,$$

$$PD_{\lambda^*} F(u^*, \lambda^*) = b_i\rho.$$

Now assume that the spectrum $\Sigma(T^*)$ of $T^*$ satisfies

$$\Sigma(T^*) \subset \{0\} \cup \{\sigma \in \mathbb{C} \mid \text{Re}(\sigma) < -k\} \text{ for some } k > 0,$$

and let $E^-$ be the direct sum of the generalized eigenspaces of $T^*$ for nonzero eigenvalues. Then for resolvent set elements $\sigma$ with $\text{Re}(\sigma) > k$, $T^*$ must satisfy the spectral property

$$\| (T^* - \sigma)^{-1} \|_{E^- \rightarrow E^-} \leq \frac{C}{|\sigma - w|},$$

where $w \in \mathbb{R}$, $w < 0$ and $C$ is constant.

With these preliminaries in hand we may state the conjugacy theorem, which investigates solutions $u \in C_{b,u}^2((\infty, \infty), X^1) \cap C_{b,u}^2((\infty, \infty), X)$ of (1) with $(u(x), u'(x), \lambda)$ in an $X^1 \times X^1 \times \mathbb{R}^n$-small neighborhood of $(u^*, 0, \lambda^*)$. (By $C_{b,u}$ we mean bounded and uniformly continuous functions.)

**Theorem 2.1 (Conjugacy Theorem).** Assume (6)–(11) are satisfied for $(u^*, \lambda^*)$. Then there exist neighborhoods $N_1(0) \subset \mathbb{R}^2$, $N_2(\lambda^*) \subset \mathbb{R}^n$, $N_3(u^*, 0) \subset X^1 \times X^1$ and a continuous map $p : N_2 \rightarrow \mathbb{R}$ such that for all $\lambda \in N_2$, the bounded flow of

$$u'' + F(u, \lambda) = 0$$

in $N_3$ is topologically conjugate to the bounded flow of

$$z'' + z^2 - p(\lambda) = 0$$

in $N_1$. That is, there exists a map $H \in C(N_1 \times N_2, N_3)$ such that $H(\cdot, \lambda)$ is topological for every $\lambda \in N_2$, and $H(\cdot, \lambda)$ establishes a conjugacy between (12) and (13): Namely, there exists $(u, u') \in C_{b,u}((\infty, \infty), N_3)$ solving (12) if and only if there exists $(z, z') \in C_{b,u}((\infty, \infty), N_1)$ solving (13) with $(u(x), u'(x)) = H(z(x), z'(x), \lambda)$. The maps $p$ and $H = (H_1, H_2)$ satisfy the properties

$$p(\lambda^*) = 0, \quad \nabla p(\lambda^*) = -c \cdot b,$$

where $b$ and $c$ are given by (9), and

$$H_1(x, y, \lambda) = H_1(x, -y, \lambda), \quad H_2(x, y, \lambda) = -H_2(x, -y, \lambda).$$

A few comments on the significance of the theorem are in order here before giving a proof.

1. The conjugacy gives several nice properties. It implies that the dynamical behavior of (12) and (13) is practically indistinguishable near the turning point: That
is, time is preserved under $H$, fixed points are mapped to fixed points, homoclinic orbits are mapped to homoclinic orbits and periodic orbits to periodic orbits with the same period under $H$.

2. By (14), $p$ has the form

$$p(\lambda) = -c \cdot b \cdot (\lambda - \lambda^*) + o(\lambda - \lambda^*). \quad (16)$$

3. $H$ is a homeomorphism between a neighborhood of $0 \in \mathbb{R}^2$ and a two-dimensional submanifold of $N_3$; although $U$ inhabits an infinite-dimensional space, the conjugacy implies that all dynamics of bounded solutions of (1) in the space occur on this two-dimensional submanifold.

4. It is not difficult (only tedious) to prove the theorem for a conjugacy of class $C^1$ instead. A function $s$ can be demonstrated such that if (13) is replaced by

$$z'' + (z^2 - p(\lambda))(1 + s(\lambda)z) = 0, \quad (17)$$

the mapping $H(\cdot, \lambda)$ would establish a $C^1$-conjugacy between (17) and (12). Because $s$ is a small function, no additional fixed points are contributed near the origin anyway by (17). However, in the applications of interest here, we only need the topological conjugacy. In any case, the only point at which that conjugacy in the theorem is not $C^1$ is at the homoclinic point of (12).

**Proof.** For ease of notation, we may assume that $\lambda^* = 0 \in \mathbb{R}^n$ in the theorem. We first employ a time-dependent Liapunov–Schmidt reduction and split according to projection on the generalized eigenspaces of $T^*$, where $T^*$ is given by (8)

$$X^* = E^0 \oplus E^-, \quad E^0 = \text{span}\{\rho\},$$

$$P : X \to E^0, \quad Q = I - P : X \to E^-.$$

Let $P(u(t) - u^*) = a(t)\rho$ and $Q(u(t) - u^*) = v(t)$ so that $u(t) = u^* + a(t)\rho + v(t)$. Define $l \in X^*$ by

$$l(u) = \alpha, \quad \text{where } Pu = \alpha \rho.$$

Then expressing the $t$-dependence allows (12) to be split as

$$a''(t) + lF(u^* + a(t)\rho + v(t), \lambda) = 0,$$

$$v''(t) + QF(u^* + a(t)\rho + v(t), \lambda) = 0.$$
For use of the reduction principle in [5] we briefly rewrite this as a first order system:

\[
\begin{pmatrix}
  a' \\
  b'
\end{pmatrix} - \begin{pmatrix}
  0 & 1 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  a \\
  b
\end{pmatrix} = \begin{pmatrix}
  0 \\
  f_1(a, v, \lambda)
\end{pmatrix},
\]

\[
\begin{pmatrix}
  v' \\
  w'
\end{pmatrix} - \begin{pmatrix}
  0 & I_{|E^-} \\
  -T_{|E^-} & 0
\end{pmatrix}
\begin{pmatrix}
  v \\
  w
\end{pmatrix} = \begin{pmatrix}
  0 \\
  f_2(a, v, \lambda)
\end{pmatrix}.
\]

(18)

Here, \(f_1 = lf\) and \(f_2 = Qf\), where

\[
f(a, v, \lambda) = F(u^*, 0) - F(u^* + a\rho + v, \lambda)
\]

\[
= -D_2F(u^*, 0)\lambda + T^*(a\rho + v) + o(a, v, \lambda).
\]

(By assumption the “\(o\)” is with respect to the \(X^1\)-topology.)

We need to check that (18) admits reduction to a weak integral manifold

\[
\begin{pmatrix}
  v \\
  w
\end{pmatrix} = h\begin{pmatrix}
  a \\
  b
\end{pmatrix}, \lambda.
\]

According to [5], it must be verified that the operator

\[
B := \begin{pmatrix}
  0 & I_{|E^-} \\
  -T_{|E^-} & 0
\end{pmatrix}
\]

has the correct spectral properties; all other assumptions are easily satisfied. Following the notation there, let

\[
H_2 = (X \cap E^-) \times (X \cap E^-),
\]

\[
H_3 = (X^1 \cap E^-) \times (X^1 \cap E^-) = D(B)
\]

and

\[
V = \{0\} \times (X \cap E^-) \subset H_2.
\]

We then investigate the resolvent of \(B\) on \(H_2\) restricted to \(V\). Note that

\[
(B - x)^{-1} = -\begin{pmatrix}
  x(T^* + x^2)^{-1} & (T^* + x^2)^{-1} \\
  x^2(T^* + x^2)^{-1} - id & x(T^* + x^2)^{-1}
\end{pmatrix}
\]

and \(x \in \rho(B) \Leftrightarrow -x^2 \in \rho(T^*|_{E^-})\).

First, note that \(x \in \rho(T^*|_{E^-})\) for all \(x\) with \(|\text{Re}\, x| < \varepsilon\) for some small \(\varepsilon > 0\): Under \(-x^2\) the region \(|\text{Re}\, x| = \varepsilon\) is mapped into the parabolic region

\[
\text{Re}(-x^2) \geq (\text{Im}(-x^2))^2/\varepsilon^2 - \varepsilon^2.
\]
Since it is assumed that $T^*$ is sectorial with the spectrum $\Sigma^-$ contained in an angular region in the negative complex half-plane, the parabolic region will not meet $\Sigma^-$ for sufficiently small $\varepsilon$.

Next, the following resolvent estimates must be shown for $|\text{Re } x| \leq \varepsilon$:

\[
\| (B - x)^{-1} \|_{H^2 \to H^2} \leq \alpha e^{k|x|},
\]

\[
\| (B - x)^{-1} \|_{V \to H^2} \leq \frac{k}{1 + |x|}
\]

and

\[
\| (B - x)^{-2} \|_{V \to H^2} \leq \frac{k}{1 + |x|^2}
\]

for some $k > 0$. We claim that these follow directly from hypothesis (11) for $(T^*)^{-1}$. To see this, observe that

\[
\| (B - x)^{-1} \|_{H^2 \to H^2} \leq 1 + (1 + |x| + |x|^2) \| (T^* + x^2)^{-1} \|_{E^- \to E^-},
\]

and is better on $V$, namely

\[
\| (B - x)^{-1} \|_{V \to H^2} \leq 1 + (1 + |x|) \| (T^* + x^2)^{-1} \|_{E^- \to E^-}.
\]

Then

\[
\| (B - x)^{-1} \|_{H^2 \to H^2} \leq 1 + \frac{1 + |x| + |x|^2}{|x^2 + w|} \leq \varepsilon \leq \infty,
\]

verifying (19) for $|\text{Re } x| \leq \varepsilon$ if we pick $\varepsilon^2 < |w|$. Using a similar calculation establishes (20) as well.

Now for (21), consider

\[
(B - x)^{-2} \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} x(T^* + x^2)^{-1} & (T^* + x^2)^{-1} \\ x^2(T^* + x^2)^{-1} - id & x(T^* + x^2)^{-1} \end{pmatrix} \begin{pmatrix} (T^* + x^2)^{-1}g \\ x(T^* + x^2)^{-1}g \end{pmatrix} = \begin{pmatrix} 2x(T^* + x^2)^{-2}g \\ (2x^2 - 1)(T^* + x^2)^{-2}g \end{pmatrix},
\]
which yields
\[\| (B - x)^{-2} \|_{V \rightarrow H^2} \leq c(1 + |x| + |x|^2) \| (T^* + x^2)^{-2} \|_{E^* \rightarrow E^*} \]
\[\leq c \frac{1 + |x| + |x|^2}{|x^2 + w|^2},\]
and (21) is verified.

It has now been shown that solutions of (12) with \( \lambda \) in a neighborhood \( N_2 \) of \( \lambda^* = 0 \) and \( (u(t), u'(t)) \) in a neighborhood \( N_3 \) of \( (u^*, 0) \) are contained in an \( (n + 2) \)-dimensional manifold
\[(u(t), u'(t)) = (u^*, 0) + (a(t)\rho, a'(t)\rho) + h(a(t), a'(t), \lambda),\]
where \( (a(t), a'(t)) \in \mathcal{N}_1(0) \subset \mathbb{R}^2 \). Studying (12) in this set then reduces to studying Eq. (18) with \( v(t) \) replaced by \( h_1(a(t), a'(t)) \), which becomes a parameter-dependent scalar second-order ODE
\[a' = b, \quad b' = -f(a(t), b(t), \lambda), \quad (22)\]
where \( \lambda = d/dt \) and now \( f(a, b, \lambda) = IF(u^* + a\rho + h_1(a, b, \lambda), \lambda) \). Furthermore, we note that (22) has time reversibility symmetry.

Our goal is now to demonstrate an \( \epsilon > 0 \) and a map \( N_3(0) \ni \lambda \mapsto \rho(\lambda) \in \mathbb{R} \) such that (22) is topologically conjugate to
\[z'' + z^2 - \rho(\lambda) = 0. \quad (23)\]
This is accomplished by completing the following steps.

Step 1: Show that for \( |\lambda| < \epsilon \), (22) in \( N_1(0) \) either has no invariant set in \( N_1 \) or that the largest invariant set consists of a homoclinic orbit \( O \) enclosing a single steady state \( (s^+, 0) \) with the region between \( (s^+, 0) \) being filled with periodic orbits. (By results in [7], once this is established, we know that the periods are strictly increasing along any fixed ray emanating from \( (s^+, 0) \) that passes through the bounding homoclinic orbit.)

Step 2: Define \( \rho \) in such a way that the limiting period at \( (s^+, 0) \) of (22) is the same as the one of (23) at its center, or that \( \rho < 0 \) if (22) has no invariant set in \( N_2 \).

Step 3: Prove that the period of (22) is strictly increasing along any fixed ray with initial value \( (a_0, 0) \) through the bounding homoclinic orbit.

Once these are established we can exhibit the conjugacy between (22) and (23) in the following way. Let \( \gamma \) be the flow of (22) and \( \psi \) that of (23). Let \( P_{\gamma}(a, 0) \) be the period of \( t \mapsto \gamma_t(a, 0) \), where \( a \) is between \( \bar{s} \) and \( s^+ \) and let \( P_{\psi}(z, 0) \) be the period of \( t \mapsto \psi_t(x, 0) \), where \( x \) is between \( \sqrt{p(\lambda)} \) and \( \bar{p} \), both given in the figure. Then both \( P_{\gamma}\)
and $P_\psi$ are homeomorphisms with range $(P_0, \infty)$, say
\[ P_\gamma : (s, s^+) \times \{0\} \to (P_0, \infty), \]
\[ P_\psi : (\sqrt{p(\lambda)}, \bar{p}) \times \{0\} \to (P_0, \infty). \]
(In fact, it can be shown that $P_\gamma$ and $P_\psi$ are diffeomorphisms having differentiable extensions to the closed intervals, but we will not do so here.)

Let $I_\gamma$ be the region in $N_1(0)$ consisting of the bounding homoclinic orbit of (22) and its interior periodic orbits; define $I_\psi$ similarly. Then for $(x, y) \in I_\psi \setminus \{(\sqrt{p(\lambda)}, 0)\}$ we can find $t$ such that $\psi_{-t} \in (\sqrt{p(\lambda)}, \bar{p})$ but $\psi_{-t} \notin (\sqrt{p(\lambda)}, \bar{p})$ for $0 \leq \tau < t$. (Note that $t = 0$ if $(x, y)$ is already in the set; $t$ depends continuously on $(x, y)$.)

Now define
\[ \hat{H}(x, y) = \gamma_{-1}(P_\gamma(P_\psi(\psi_{-t}(x, y)))}. \]
Because the periods of $\psi$ and the conjugate orbit $\gamma$ have been matched, $\hat{H}$ will also be continuous in $(\sqrt{p(\lambda)}, \bar{p}) \times \{0\}$. It is easy to see that
\[ \hat{H} : (I_\psi \setminus \partial I_\psi) \setminus \{(\sqrt{p(\lambda)}, 0)\} \times \{0\} \to (I_\gamma \setminus \partial I_\gamma) \setminus \{(s, 0)\} \]
is continuous with continuous inverse, and that it has an extension to a homeomorphism
\[ \hat{H} : I_\psi \to I_\gamma, \]
preserving conjugacy. Now making $H(\cdot, \lambda)$ the same as $\hat{H}$ on $I_\psi$, but expressing the continuous $\lambda$ dependence guaranteed by $F$ in (12) for small $\lambda$ yields the conjugacy whose existence is claimed in the theorem.

In order to complete the proof of the theorem, Steps 1–3 must be carried out. First, a suitable blowup technique is employed to establish Step 1. When there is enough information about $f$, the map $p$ is given to complete Step 2. Step 3 is established in the last section (Section 5) due to the technical nature of its proof.

Combining (9) with the definition for $f$ in (22) gives the following values for derivatives of $f$ at $(0, 0, 0)$:

- $f = 0$ by time reversability,
- $\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = 0$ for some $i$, where we assume $i = 1$ without restriction,
- $\frac{\partial^2 f}{\partial a \partial b} = 2c \neq 0$, and we may assume $c > 0$ without restriction,
- $\frac{\partial^2 f}{\partial b^2} = 0$ since $\frac{\partial h}{\partial b} = 0$ and $\frac{\partial^2 h}{\partial b^2} \in E^-$,
- $\frac{\partial^2 f}{\partial a \partial b} = 0$ since $\frac{\partial f}{\partial b}(a, 0, \lambda) = 0$ for every $a$, $\lambda$ by time reversability, and
- $\nabla_\lambda \frac{\partial f}{\partial b} = 0$. 

Then letting \( \partial^2 f / \partial \lambda^2 = 2A \in \mathbb{R}^{n,n} \) and \( \tilde{d} = \nabla_\lambda (\partial f / \partial a) \), a Taylor expansion gives the second-order normal form

\[
f = b \cdot \lambda + ca^2 + (\tilde{d} \cdot \lambda)a + \langle A \lambda, \lambda \rangle + O((|a, b, \lambda|)^3)
\]

for \( f \) in (22), where \( \min f = 0 \) and \( f(a, 0, \lambda) = \partial f / \partial a(a, 0, \lambda) = 0 \).

Now assume that \( N_1 \) and \( N_2 \) are sufficiently small that \( D_{2i}^2 f > 0 \) and \( D_{ij} f > 0 \) throughout \( N_1 \times N_2 \). Because \( f = f_0 = 0 \) and \( f_{i\lambda} \neq 0 \) at 0, the implicit function theorem reveals an \((n - 1)\)-dimensional manifold

\[
(\lambda_2, \ldots, \lambda_n) = (\lambda) \mapsto (a^*(\lambda), A(\lambda))
\]

such that \( f(\cdot, 0, (A(\lambda), \lambda)) \) has a unique minimum of 0 at \( a^*(\lambda) \) and \( f \) has a negative minimum for \( \lambda_1 < A(\lambda) \), while \( f > 0 \) on \( N_1 \) for \( \lambda_1 > A(\lambda) \).

This implies that for \( \lambda_1 > A(\lambda) \), (22) has no invariant set in \( N_1 \), whereas for \( \lambda_1 < A(\lambda) \) there are precisely two steady states \((s^+(\lambda), 0)\) and \((s^-(\lambda), 0)\), \( s^- < s^+ \), with

\[
\frac{\partial f}{\partial a}(s^+(\lambda), 0, \lambda) > 0 \quad \text{and} \quad \frac{\partial f}{\partial a}(s^-(\lambda), 0, \lambda) < 0,
\]

so that \( (s^+(\lambda), 0) \) is a center and \( (s^-(\lambda), 0) \) is a saddle.

So note that once \( p(\lambda) > 0 \) has been defined suitably for \( \lambda_1 < A(\lambda) \) in order to establish the conjugacy, we may extend \( p \) and the conjugacy to \( \lambda_1 > A(\lambda) \) in a natural way by defining

\[
p(A(\lambda) + k) := -p(A(\lambda) - k).
\]

The conjugacy may then be a natural extension which requires no additional investigation. So henceforth we restrict the analysis to the region \( \lambda_1 < A(\lambda) \).

As a first step we let

\[
\lambda_1 = A(\lambda) - q^2, \quad q \geq 0.
\]

This \( q \) will be the desired blowup parameter. Then replacing \( a^*(\lambda) + a \) by \( a \) and renaming \( f(a^*(\lambda) + a, b, (A(\lambda) - q^2, \lambda)) \) to \( f(a, b, q) \), regarding \( \lambda \) as fixed, \( |\lambda| < \varepsilon \), \( f \) has a domain which contains the set \( N_\varepsilon := \{|a| < \varepsilon, |b| < \varepsilon, q^2 < \varepsilon\} \) for some \( \varepsilon > 0 \). The new \( f \) has the following properties on its domain:

\[
\frac{\partial^2 f}{\partial a^2} \geq c > 0, \quad \frac{\partial f}{\partial q}(a, b, 0) = 0, \quad \frac{\partial^2 f}{\partial q^2} \leq -c < 0
\]

and

\[
f(0, 0, 0) = \frac{\partial f}{\partial a}(0, 0, 0) = \frac{\partial f}{\partial b}(a, 0, q) = 0.
\]
Then the transformed system for (22)
\[ a' = b, \quad b' = -f(a, b, q) \]
has exactly two steady states which can be written as \((qs^\pm(q), 0)\) with \(s^- < s^+\) satisfying
\[
\lim_{q \to 0} s^\pm(q) = \pm \sqrt{\frac{\partial^2 f}{\partial q^2}/\partial a^2} = \pm s_0
\]
at (0,0,0). We can then write \(f(a, 0, q)\) as
\[
f(a, 0, q) = f_1(a, b)(a - qs^+(q))(a - qs^-(q)),
\]
and because \(\frac{\partial f}{\partial b}(a, 0, q) = 0\),
\[
f(a, b, q) = f(a, 0, q) + b^2 f_2(a, b, q)
\]
for all fixed \(\lambda\). Properties satisfied by \(f_1\) and \(f_2\) are
\[
0 < c_1 \leq f_1(a, q) \leq c_2 < \infty,
\]
\[
|f_2(a, b, q)| \leq c < \infty,
\]
and if the original \(f \in C^3\), \(f_1\) and \(f_2\) are \(C^1\) functions in \(N_\varepsilon\).

Now for \(q > 0\) we make another transformation, namely
\[
a \rightarrow qa, \quad b \rightarrow q^{3/2}b \quad \text{and} \quad t = \tau / \sqrt{q}, \ \text{so that} \quad \frac{d}{dt} = \sqrt{q} \frac{d}{d\tau}
\]
to get
\[
q^{3/2} \frac{da}{d\tau} = q^{3/2}b,
\]
\[
q^2 \frac{db}{d\tau} = -f_1(qa, q)q^2(a - s^+(q))(a - s^-(q)) - f_2(qa, q^{3/2}b, q)q^3b,
\]
which yields the new system
\[
\frac{da}{d\tau} = b,
\]
\[
\frac{db}{d\tau} = -(d + O(q))(a - s^+(q))(a - s^-(q)) + O(1)qb^2,
\]
(25)
where \(d > 0\), \(d + O(q) > 0\), and \(s^\pm(q) \rightarrow \pm s_0\) as \(q \rightarrow 0\) where \(s_0 > 0\). With the assumption that \(f\) is \(C^3\), (25) has a differentiable right-hand side.
We are now ready to give the invariant set. For any fixed $q > 0$, the right-hand side of (25) is defined for $|a| < \epsilon/q$ and $|b| < \epsilon/q^{3/2}$, so certainly for $a, b = O(1)$ if $|q| < \epsilon$. As $q \to 0$, the right-hand side of (25) converges in $C^1$ to the right-hand side of

$$
\begin{align*}
\frac{da}{d\tau} &= b, \\
\frac{db}{d\tau} &= -d(a - s_0)(a + s_0) = d(a^2 - s_0^2).
\end{align*}
$$

(26)

This system has a maximal invariant set of the nature we wish to establish: Since (25) has its saddle $(s^-(q), 0)$ close to $(-s_0, 0)$ for $q$ small, we know that the local stable and unstable manifolds are close, and the stable and unstable directions of $(s^-(q), 0)$ converge to the ones of $(-s_0, 0)$.

For small $\delta > 0$, a solution of (26) starting at $a = -s_0 + \delta$ on its unstable manifold will reach a point $(a_0, b_0)$ with $b < -\delta$ in some finite time $T_\delta$. So if $q$ is small enough, a solution of (25) starting at the same $a$-value on the unstable manifold of (25) will be so close to $(a_0, b_0)$ after time $T_\delta$ that the right unstable manifold of $(s^-(q), 0)$ intersects the line $a > s^+(q)$, $b = 0$. By time reversibility symmetry of (25), the right unstable manifold coincides with the right stable manifold, such that there exists a homoclinic orbit originating in $(s^-(q), q)$.

This establishes a priori bounds for all orbits starting inside the homoclinic. The fact that $(s^+(q), 0)$ is the only steady state in this region, combined with the symmetry of the phase diagram with respect to $b = 0$ implies (by the Poincaré–Bendixson theorem) that the region inside the homoclinic orbit is filled with periodic orbits encircling $(s^+(q), 0)$, finishing Step 1.

Now for the second step, we note that

$$
p(\lambda) = \frac{1}{4}f_\lambda(s^+(\lambda), 0, \lambda)^2,
$$

ensures that the limiting period at $(s^+, 0)$ of (22) is the same as that of (23) at its center. Then since

$$
p_\lambda = -\frac{1}{2}
\left[
-\frac{d}{da}f_{\lambda}a + f_0a_{\lambda}
\right]
= -\frac{1}{2}(-b \cdot 2c)
$$

as $\lambda \to 0$, the second comment following the theorem is verified. Applying (24) to $p$, the function is defined in a neighborhood of $\lambda = 0$ and satisfies all of the desired properties. This completes Step 2.

We now proceed to the point of stating a lemma that establishes Step 3; however, due to the technical nature of the proof we postpone it until Section 5.

We first note that the rescalings applied to (22) to obtain (25) do not change the strict monotonicity of the period map as time is only scaled by a constant, so it suffices to investigate a system with a one-dimensional parameter
$q$ of the form

\[
\frac{da}{dt} = b, \\
\frac{db}{dt} = -f_1(a, q)(a - s^+(q))(a - s^-(q)) + f_2(a, b, q)qb^2.
\]

Because $|s^\pm(q)| \geq c > 0$ we can rescale the roots to become $-1$ and $0$ under

\[
-1 + \frac{a - s^-}{s^+ - s^-} \to a,
\]

so we get

\[
\frac{da}{dt} = b, \\
\frac{d^2a}{dt^2} = -f_1(a, q)(a^2 + a) + f_2(a, a', q)q(a')^2.
\] (27)

where $f_1(\cdot, q) \to d > 0$ in $C^1[-c, c]$ as $q \to 0$ for sufficiently small $q$ and $f_2$ is bounded on $[-c, c]^2 \times [0, c]$; without restriction we choose $d = 1$.

Now for fixed $q$ and for $0 < x < a^+(q)$, let $a(t) = A(t, x)$ be the solution of (27) with initial condition $a(0) = x$, $a'(0) = 0$. Then we can define the period map $P : (0, a^+) \to \mathbb{R}$, by

\[
P(x) := \{ \text{the first } t > 0 \text{ where } A(t, x) = x \}.
\] (28)

We are now ready to state the crucial lemma.

**Lemma 1.** $P$ is continuously differentiable on $(0, a^+)$, and $P'(x) > 0$ for sufficiently small $q$.

This is all that is needed to finish the proof of Theorem 2.1. As mentioned, its proof is found in Section 5.

### 3. An application: solutions that vary with rotation

The application has its origin in [1]. A brief overview of the setting is appropriate for the benefit of the reader and is provided next.

The problem of interest is

\[
\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega_\varepsilon, \quad \lambda \in \mathbb{R}^+, \\
u = 0 \quad \text{on } \partial \Omega_\varepsilon,
\] (29)
where $\Omega_\varepsilon \subset \mathbb{R}^{n+1}$ is a translation of a domain $\Omega$ by $1/\varepsilon$ with a rotation about the $x_{n+1}$-axis: namely

$$
\Omega_\varepsilon := \{(x_1', \ldots, x_{n+1}') \mid (x_1', x_{n+1}') = \left( x_1 + \frac{1}{\varepsilon} \right) (\cos \theta, \sin \theta),
\quad x_2' = x_2, \ldots, x_n' = x_n, (x_1, \ldots, x_n) \in \Omega, \ 0 \leq \theta < 2\pi \}. \tag{30}
$$

Problem (29) is obtained by applying the change of variables (30) to

$$
\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad \lambda \in \mathbb{R}^+,
$$

$$
u = 0 \quad \text{on } \partial \Omega. \tag{31}
$$

Under (30) the Laplacian becomes

$$
\Delta_\varepsilon := \Delta + \frac{1}{\left( x_1 + \frac{1}{\varepsilon} \right)^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{\left( x_1 + \frac{1}{\varepsilon} \right)} \frac{\partial}{\partial x_1}, \tag{32}
$$

where $\Delta$ is the normal Laplacian in $x_1, \ldots, x_n$. Then (31) together with (30) and (32) becomes an equation on $\Omega \times S^1$ that needs to be written in the form of (1) in order to satisfy the theorem. So let $\theta = \varepsilon \phi$ and define $v(x, \theta) := u(x, \varepsilon^{-1}\theta)$. Then substituting $v$ in for $u$ and renaming it back to $u$, we get all solutions as $2\pi/nc$-periodic solutions of

$$
\Delta u + \frac{1}{(\varepsilon x_1 + 1)^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\varepsilon}{\varepsilon x_1 + 1} \frac{\partial u}{\partial x_1} + \lambda f(u) = 0 \quad \text{in } \Omega \times \mathbb{R}, \quad \lambda \in \mathbb{R},
$$

$$
u = 0 \quad \text{on } \partial(\Omega \times \mathbb{R}). \tag{33}
$$

Now we compare (33) with (1) by letting $U(\phi)(x) := u(x, \phi)$, with $U : \mathbb{R} \to X := L^2(\Omega)$. Then $X^1 = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$, where the second space implies zero boundary conditions. Define

$$
G(U, \lambda, \varepsilon)(x) := (\varepsilon^2 x_1^2 + 2\varepsilon x_1)(A_x U)(x)
$$

$$
- \varepsilon(\varepsilon x_1 + 1)(\partial_1 U)(x) - (\varepsilon x_1 + 1)^2 \lambda f(U(x)).
$$

Because of the perturbation involving second-order derivative terms in (33), in order for $G$ to be sufficiently regular we require $\gamma = 1$. As previously stated, in this case the reduction in [5] only applies to Hilbert spaces. Since $LU := A_x U$, we see that $L : D(L) \subset X \to X$ is unbounded, and $D(L)$ is dense in $X$. Furthermore, $L^\gamma$ is defined for every $\gamma \in [0, 1]$, with $D(L^\gamma) = X^\gamma$.

Much is known about (31). For example, under minimal conditions on $f$ named below, a parameterized curve of positive solutions $\alpha \mapsto (u(\alpha), \lambda(\alpha))$ is known to exist (where $\alpha = \max_{\Omega} u$, say) if $\Omega$ is partially convex for $n = 2$ or a ball for any $n$. In [1] it is shown that a surface of positive rotationally invariant solutions $(\alpha, \varepsilon) \mapsto (\tilde{u}(\alpha, \varepsilon), \tilde{\lambda}(\alpha, \varepsilon))$ emanates from the base curve of (31) in (29), with
\((\hat{u}(x), \hat{\lambda}(x, 0)) = (u(x), \lambda(x))\), as long as \(f\) satisfies

\[
f : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+ \text{ is in } C^2(\mathbb{R}^+, \mathbb{R}^+),
\]

has at most isolated zeros and \(f(0) \geq 0\). \hspace{1cm} (34)

(We shall later require more differentiability for \(f\).) Omitting the \(\theta\)-derivative term in (32) allows investigation of rotationally invariant terms in (29); putting it back in allows use of Theorem 1 to find periodic solutions. Once the conditions of the theorem have been verified for (29), we may use the conjugacy to get some additional information about them.

Using reference to the solution surface in [1], let \((u, \lambda, \varepsilon) = (u_0, \lambda_0, 0)\) be a solution to the \(\varepsilon = 0\), \(\theta\)-constant problem (which is merely (31)). Let the \(u'\) in (8) be \(u_0\) and let \(\lambda^*\) become \((\lambda_0, 0)\). Note that (11) is a natural property for elliptic operators whose leading eigenvalue is zero at the turning point. This follows from the Lumer–Phillips theorem (see e.g. [6]) by taking \(w \in (-k, 0)\). Hence for the problem at hand, the \(u\)-linearization \(T_0\) of \(F\) at the point \((u_0, \lambda_0, 0)\), given by \(T_0 = A + \lambda_0 f'(u_0)\cdot\), satisfies (11). In [1], it is shown that \(\ker(T_0)\) is one dimensional at a turning point on the types of domains mentioned previously; call that kernel element \(\rho\). Moreover, 0 is the leading eigenvalue of \(T_0\). Then following notation in (8) and (9), since

\[
D_{uu}F(u_0, \lambda_0, 0)(\rho, \rho) = \lambda_0 f''(u_0)\rho^2
\]

and

\[
D_{\xi}F(u_0, \lambda_0, 0) = f(u_0) \text{ while } D_{\xi}F(u_0, \lambda_0, 0) = \partial_1 u_0,
\]

under \(P\) we see that

\[
c = \int\int_{\Omega} \lambda_0 f''(u_0)\rho^3\, dx
\]

and

\[
b_1 = \int\int_{\Omega} f(u_0)\rho\, dx \quad \text{and} \quad b_2 = \int\int_{\Omega} (\partial_1 u_0)\rho\, dx
\]

because of the Hilbert space setting.

The additional assumptions on \(f\) (over those in (34)) needed for our specific problem to match the abstract setup can now be stated.

(i) \(f\) is globally Lipschitz and \(C^4\) with all derivatives bounded, and

(ii) there exists a solution \((u, \lambda, \varepsilon) = (u_0, \lambda_0, 0)\) to (29) such that 0 is an eigenvalue of the \(u\)-linearization there with eigenfunction \(\rho\), and with \(c\) and \(b_1\) nonzero in (35) and (36).

Property (i) is sufficient to guarantee (6). Although this condition may seem restrictive, we see later how these properties can be localized; the result ends up being
local with respect to a Hölder norm. Also note that property (ii) gives no conditions on $b_2$. The reason is that both $u_0$ and $\rho$ are symmetric, and hence $b_2$ will be zero. All these properties are verified in [1].

Since the requirements of the theorem are satisfied a conjugacy is guaranteed between (33) and (13) in a neighborhood of $(u, \lambda, \varepsilon) = (u_0, \lambda_0, 0)$. Let $\varepsilon \neq 0$ (along with its coordinate triple) be any value for which the conjugacy exists. It is clear that a solution $v(x, \phi)$ of (33) is either $\phi$-invariant or has a primitive period of $2\pi/ne$ in $\phi$ for some $n \in \mathbb{Z}^+$. Also, for each $\phi$ Theorem 1 guarantees that $v(\cdot, \phi) \in W^{2,2}(\Omega)$; in fact, it is of class $C^2_{b,a}((-\infty, \infty), W^{2,2}(\Omega))$. This carries over naturally to a solution of (29). Of course, an $x$-scaling may be used to show the same for any sufficiently thin domain. Hence, the conjugacy between (13) and (33) may be applied to (29) to investigate solution properties.

We introduce regions that will allow us to study the multiplicity of periodic solutions for a fixed small $\varepsilon$. It is necessary to say more about the dynamical setup of (13) before proceeding.

Extracting the sets $N_2(\lambda_0, 0)$ and $N_3(u_0, 0)$ from Theorem 2.1, let $(\lambda, \varepsilon) \in N_2(\lambda_0, 0)$. Recalling (14), there are three possibilities for solutions of (29) depending on where the coordinate $(\lambda, \varepsilon)$ lies with respect to the curve $p(\lambda, \varepsilon) = 0$. If $p(\lambda, \varepsilon) < 0$ for the given pair, then there are no steady states for (13) and $(\lambda, \varepsilon)$ lies in a region which gives no solutions to (29). If $p(\lambda, \varepsilon) > 0$, then we have steady states

$$(z, z') = (-\sqrt{p(\lambda, \varepsilon)}, 0) =: P_0(\lambda, \varepsilon)$$

and

$$(z, z') = (\sqrt{p(\lambda, \varepsilon)}, 0) =: P_1(\lambda, \varepsilon)$$

of (13). $P_0$ is a homoclinic point, while $P_1$ is a center for periodic orbits. The periods of the orbits are strictly increasing from

$$P(\lambda, \varepsilon) := \frac{\sqrt{2\pi}}{\sqrt{p(\lambda, \varepsilon)}}$$

(the period of solutions to the linearized problem) to infinity. This implies that $2\pi/ne$-periodic solutions of (29) will exist for $n = 1, 2, \ldots$ as long as $n$, $\lambda$ and $\varepsilon$ satisfy

$$\frac{2\pi}{ne} > P(\lambda, \varepsilon).$$

(37)

From this it is clear that in the final case $p(\lambda, \varepsilon) = 0$, $(z, z') = (0, 0)$ is the only steady state of (13) and there is a rotationally invariant solution at the given pair. Hence $p = 0$ is a curve of fold points for the rotationally invariant problem.

Now from (37) comes the $(\lambda, \varepsilon)$-regions $R_n$ of multiple periodic solutions of (29) defined next. They are given by

$$R_n := \left\{ (\lambda, \varepsilon) \in N_2 \mid \frac{2\pi}{ne} > P(\lambda, \varepsilon) \right\}.$$
Note that by definition $R_1 \supset R_2 \supset \cdots$. Since the boundaries of the $R_n$ are given by $4\pi^2 p = n^4 \epsilon^4$, by (38) they are $C^0$-close to the functions $4\pi^2 b_1(\lambda_0 - \lambda) = n^4 \epsilon^4$ by (16). Taking into consideration that $b_1 > 0$, we have sketched the regions in Fig. 2 with their approximate shapes.

Note that $\lambda$ is a function of $\epsilon$. (This can be seen by applying the implicit function theorem to the function $P$ and using the fact that $\partial r / \partial \lambda \neq 0$.) For fixed $\epsilon^*$ we define $\lambda_n(\epsilon^*)$ to be the root of (37) with equality replacing the inequality, or the $\lambda$-intersection point with the boundary of $R_n$ for fixed $\epsilon^*$, where $(\lambda_n(\epsilon^*), \epsilon^*) \in N_3$. For the given $\epsilon^*$, the collection of images of steady states $(P_1(\lambda, \epsilon^*), \lambda, \epsilon^*)$ of (13) under $H$ will be called the one branch, while the collection of homoclinic points $(P_0(\lambda, \epsilon^*), \lambda, \epsilon^*)$ mapped into under $H$ will be called the zero branch. Since $r$ and $H$ are both smooth functions, both of these branches will be smooth, and hence there is a $C^1$-curve of steady states of (29) consisting of the zero branch, the one branch, and a fold point.

Having verified all of the previous results, we are ready to state the existence of periodic solutions of (29) as a theorem. The $b_1$ and $c$ given in the theorem are those constants given in (35) and (36).

**Theorem 3.1.** Suppose that $f$ satisfies (i) and (ii) and $(u_0, \lambda_0)$ is a solution of (31). Then there is a $\delta > 0$ independent of $\epsilon$ such that there exist solutions of (29) satisfying

(a) $| (\lambda, \epsilon) - (\lambda_0, 0) | < \delta$ and
(b) $\| u - u_0 \|_{L^\infty(\Omega_\epsilon)} < \delta$

that can be enumerated in the following way:

There exist regions $R_n$, $n = 1, 2, \ldots$, defined in $(\lambda, \epsilon)$-space, such that if $(\lambda, \epsilon) \in R_K \setminus R_{K+1}$, there exist $2K + 2$ solutions $(u, \lambda, \epsilon)$ in $C^{2,h}(\Omega_\epsilon)$ to (29) satisfying (a) and (b). For each $n$, the boundary $\partial R_n$ of $R_n$ is smooth, and $\partial R_n$ is $o(|\epsilon| + |\lambda - \lambda_0|)$-
close (in the graph norm) to the function

\[ n^4 \varepsilon^4 = 4\pi^2 cb_1(\lambda - \hat{\lambda}_0). \]

For a given \( \varepsilon \), there exists a curve \( t \mapsto (u(t), \lambda(t), \varepsilon) \) of rotationally invariant solutions of (29) satisfying (a) and (b) so that if \( (\lambda(t), \varepsilon) \) lies on the boundary of \( \mathbb{R}_K \), then \( (u(t), \lambda(t), \varepsilon) \) is a \( C^0 \) pitchfork bifurcation of solutions with \( \theta \)-period \( 2\pi/K \) from the one branch of the curve of solutions of (31) in \( C^{2,\alpha}(\Omega_e) \).

We depict the situation in Fig. 3.

4. A final example

We wish to mention how the results we have obtained extend those in [4]. We will do this with the nonlinearity \( f(u) = e^u \). The results are derived for an annulus in \( \mathbb{R}^2 \), which corresponds in our setup to a base domain \( \Omega \in \mathbb{R} \) that is translated and rotated to obtain \( \Omega_e \). A result in [4] states that for \( f(u) = e^u \), there is a sequence of points \( \lambda_n \to 0 \) as \( n \to \infty \) such that \( \lambda_n \) is a bifurcation for nonradial solutions of (29) for \( \varepsilon \) small enough. We can examine the existence of rotationally variant solutions near the first turning point \( (u_0, \lambda_0, 0) \) since the leading eigenvalue of the linearization is zero there. We do this for (29) on a rotated ball domain in \( \mathbb{R}^n \), so that \( \Omega_e \subset \mathbb{R}^{n+1} \). When \( n = 1 \), there results an annulus in \( \mathbb{R}^2 \) and results can be compared with those in [4].

Starting with a base domain \( \Omega \) that is a ball in \( \mathbb{R}^n \), we can get turning points with leading eigenvalue zero in the case \( n < 9 \). We translate this domain and rotate it to obtain a torus \( \Omega_e \). We wish to apply Theorem 3.1 to solutions on \( \Omega_e \). The problem is that \( f \) is not globally Lipschitz; because of excessively rapid growth, the function \( G \)
in (1) is not even in $C^3(X^1 \times \mathbb{R} \times \mathbb{R}, X)$ as required unless $\Omega$ is of dimension less than three. However, results here are local. So we replace $f$ by $\hat{f}$, where $\hat{f}(t) = f(t)$ for some $t_0 > t$, and $\hat{f}$ is bounded and Lipschitz on $(t_0, \infty)$, where $t_0$ depends on a priori bounds of $(u^*, \lambda^*)$. (For example, the choice of $t_0 = 2 \max_\Omega u_0$ would be appropriate in this setting.)

With the bound on solutions near the turning point, it is easy to see that conditions on (35) and (36) are satisfied in any case since $\hat{f}$, $\hat{f}''$, and $\rho$ are positive in the bounded region. Since $\hat{f}$ is globally Lipschitz, with $\hat{f}$ replacing $f$ in (29) Theorem 2 yields rotationally invariant solutions for pairs $\lambda'$, $\epsilon'$ where $\epsilon'$ and $\lambda_0 - \lambda'$ are sufficiently small.

In this way we get multiplicity results of periodic solutions for (29) with $f(u) = e^u$ near the first turning point on $\Omega$, where $\epsilon$ is arbitrarily close to zero. It is clear from Theorem 2 (specifically, the properties of the regions $R_n$) that the smaller $\epsilon$ becomes, the more rotationally invariant solutions there will be bifurcating from the invariant curve, and the closer the rotationally invariant bifurcation points become to the first turning point.

The addition to results in [4] is that we can say approximately where the bifurcation points are located on the curve of rotationally invariant solutions. Furthermore, we can say more of the nature of the points: Since the rotationally invariant solution set consists of all rotations of the solutions found in the theorem, the bifurcation is shown to be smooth and two dimensional.

5. Proof of a fundamental lemma

We finish by proving Lemma 1. That $P$ can indeed be written as a function of $x$ and that $P'(x)$ exists is a simple application of the implicit function theorem to $A_t$, given that $P(x)$ is also equal to the second $t > 0$ for which $A_t(t, x) = 0$. We wish to exhibit a suitable representative of $P$. Let $g(a) := a^2 + a$ (as in (27)). Then note that for $f_1 = 1$ and $f_2 = 0$, (27) would be integrable with a first integral

$$\frac{1}{2} (a')^2 + G(a) \equiv C_1,$$

where $G(a) = \int g(a) \, da = \frac{1}{3} a^3 + \frac{1}{2} a^2$.

We make a transformation that would map the orbits of the simplified problem into circles in the following way.

Define a function $h : (-1, 1) \rightarrow (-\frac{1}{6}, 0)$ by

$$G(h(x)) = \frac{1}{2} x^2, \quad \text{sign } h(x) = \text{sign } x.$$

It is easy to see that $h$ is a smooth function with $h'(x) > 0$. We now make the transformation

$$a = h(x), \quad a' = y$$
and obtain

\[ h'(x)x' = y, \]
\[ h'(x)y' = -f_1(h(x), q)x - f_2(h(x), y, q)qy^2, \]
\[ x(0) = h^{-1}(a), \quad y(0) = 0 \] (39)

since \( g(h(x))h'(x) = \frac{dx}{dt} G(h(x)) = x. \)

This suggests introducing polar coordinates

\[ x = r(t) \cos t, \quad y = r(t) \sin t. \]

Now

\[
\frac{dr}{dt} = \frac{1}{r}(x'x + y'y) = \frac{1}{r} \left( xy - f_1(h(x), q)xy - f_2(h(x), y, q)qy^2 \right) h'(x) = \frac{1 - f_1(h(x), q) r \cos \theta \sin \theta - f_2(h(x), y, q) qr \cos \theta \sin^2 \theta}{h'(x)},
\]

and

\[
\frac{d\theta}{dt} = \frac{xy' - yy'}{r^2} = \frac{-f_1(h(x), q)x^2 - f_2(h(x), y, q)qxy^2 - y^2}{r^2 h'(x)} = \frac{(1 - f_1(h(x), q)) \cos^2 \theta - 1 - f_2(h(x), y, q) qr \cos \theta \sin^2 \theta}{h'(x)},
\]

so for small \( q \), \( d\theta/dt < 0 \) and we can regard \( t \) and \( r \) as depending on the independent variable \( \theta \). Using the previous calculations gives

\[
\frac{dr}{d\theta} = (dr/dt)(dt/d\theta) = \left[ (1 - f_1(h(x), q)) r \cos \theta \sin \theta - f_2(h(x), y, q) qr \cos \theta \sin^2 \theta \right] \frac{dt}{d\theta} / h'(x); \]
\[
\frac{dt}{d\theta} = - h'(x) / [1 + (f_1(h(x), q) - 1) \cos^2 \theta + f_2(h(x), y, q) qr \cos \theta \sin^2 \theta] = - h'(x) + O(q),
\]
so
\[
\frac{dr}{d\theta} = f_3(r, \theta, q)r,
\]
\[
\frac{dt}{d\theta} = -h'(x) + f_4(r, \theta, q),
\]
where both \(f_3, f_4\) in \(C^2\) as \(q \to 0\). Our initial conditions become
\[r(0) = h^{-1}(x), \quad t(0) = 0.\]
Now it is easy to see that
\[P(x) = t(-2\pi)\]
and we get the formula
\[P(x) = \int_{-2\pi}^{0} [h'(r \cos \theta) + f_4(r, \theta, q)] \, d\theta,
\]
where
\[
\frac{dr}{d\theta} = f_3(r, \theta, q)r,
\]
\[r(0) = h^{-1}(x) = \beta,
\]
where \(\beta\) is a new parameter. Then we may associate \(P(x)\) with \(P(\beta)\) and \(r\) with \(R(\theta, \beta)\).

So we have that
\[P'(\beta) = \int_{-2\pi}^{0} \left[ h''(R \cos \theta) \frac{\partial R}{\partial \beta} \cos \theta + \frac{\partial f_4}{\partial r}(r, \theta, q) \frac{\partial R}{\partial \beta} \right] \, d\theta,
\]
and \(P'(0) = 0\). Using the regularity of \(f\) to differentiate \(f_4\) one more time, we get
\[P''(\beta) = \int_{-2\pi}^{0} \left[ h''(R \cos \theta) \left( \frac{\partial R}{\partial \beta} \cos \theta \right)^2 + h''(R \cos \theta) \frac{\partial^2 R}{\partial \beta^2} \cos \theta \right.
\]
\[+ \frac{\partial f_4}{\partial r}(r, \theta, q) \frac{\partial^2 R}{\partial \beta^2} + \frac{\partial^2 f_4}{\partial r^2}(r, \theta, q) \left( \frac{\partial R}{\partial \beta} \right)^2 \left] \, d\theta.
\]
Following [7] (specifically the proof of Theorem 1.4.2), we have that \(h'' \geq c > 0\). Also from the fact that
\[\frac{\partial}{\partial \beta} R(0, \beta) = 1 \quad \text{and} \quad \frac{\partial^2 R}{\partial \beta \partial \theta} \to 0
\]
as \(q \to 0\), we get that
\[\frac{\partial R}{\partial \beta} \geq c > 0, \quad \text{while} \quad \frac{\partial^2 R}{\partial \beta^2} \to 0
\]
as $q \to 0$ follows in a similar way. Hence the sign of $h''$ dominates the integral, and the monotonicity of the period map is established.

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References