JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 52, 260–282 (1975)

Optimal Approximation of Convex Curves by Functions which Are Piecewise Linear*

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Submitted by Kenneth L. Cooke

In this paper an efficient method is presented for solving the problem of approximation of convex curves by functions that are piecewise linear, in such a manner that the maximum absolute value of the approximation error is minimized. The method requires the curves to be convex on the approximation interval only. The boundary values of the approximation function can be either free or specified. The method is based on the property of the optimal solution to be such that each linear segment approximates the curve on its interval optimally while the optimal error is uniformly distributed among the linear segments of the approximation function. Using this method the optimal solution can be determined analytically to the full extent in certain cases, as it was done for functions x^2 and $x^{1/2}$. In general, the optimal solution has to be computed numerically following the procedure suggested in the paper. Using this procedure, optimal solutions were computed for functions $\sin x$, $\operatorname{tg} x$, and $\operatorname{arc\,tg} x$. Optimal solutions to these functions were used in practical applications.

1. INTRODUCTION

The approximation of curves by chains of linear segments has great significance in science and technology. Thus, for example, this approximation is used in generating nonlinear functions in analog computers, in modeling nonlinear elements when analyzing a system, or for rapid computation of functions by table look-up and interpolation, etc. Various methods [1, 2, 3, 4] have been developed to solve this problem.

In this paper an efficient method is presented for solving the problem of approximation of convex curves by functions that are piecewise linear, by which the maximum absolute value of the approximation error is minimized.

The class of problems that can be solved by this method is extensive since the curve that is being approximated is required to be convex, particularly in the interval of approximation.

* This work was supported by Social and Rehabilitation Service, Department of H.E.W., Washington, D.C., under Grant 19-P-58393-F-01.

For some curves the method presented can be carried out completely analytically. Thus, for example, analytical solutions for optimal approximation of curves x^2 and $x^{1/2}$ are derived.

However, in the general case, the method must be carried out numerically. An efficient algorithm is described, which was realized as a BASIC program and with which the optimal solution for approximation of convex curves that have the continuous first and second derivatives can be determined. As an illustration the optimal solutions for approximation of the curves sin x, tg x, and arc tg x are computed.

2. Problem Statement

Let the curve $g(x)$ be strictly convex[1] on the segment $[a, b]$. The curve $g(x)$ on that segment is to be approximated by

$$l(x) = l_n(x) = y_n + \frac{y_{n+1} - y_n}{x_{n+1} - x_n}(x - x_n), \qquad x \in [x_n, x_{n+1}], \qquad (1)$$

where $n = 0, 1, 2,..., N$, and

$$a = x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant x_{N+1} = b,$$

whose plot represents a continuous polygonal line.

The maximum absolute value of the error of approximation has to be minimized.

Curve $l(x)$ presents a chain of straight linear segments.

Therefore, the problem of optimal approximation of curve $g(x)$ on the segment $[x_0, x_{N+1}]$ where this curve is convex, is reduced to determining the value of the parameters $x_1, x_2,..., x_N$ and $y_0, y_1, y_2,..., y_N, y_{N+1}$ for which the maximum absolute value of approximation error assumes a minimum value f_N,

$$f_N = \min_{\substack{x_1, x_2, \ldots, x_N \\ y_0, y_1, y_2, \ldots, y_N, y_{N+1}}} \max_{x \in [x_0, x_{N+1}]} |g(x) - l(x)|. \qquad (2)$$

[1] Curve $g(x)$ is called strictly convex on segment $[a, b]$ if and only if the inequality $g(\lambda x_1 + (1 - \lambda)x_2) < \lambda g(x_1) + (1 - \lambda) g(x_2)$ holds for all $x_1, x_2 \in [a, b]$, $x_1 \neq x_2$ for all real numbers $\lambda \in [0, 1]$, [5]. A geometrical interpretation of strict convexity of a curve $g(x)$ on the segment $[a, b]$ is that the line segment which connects the points $(a, g(a))$ and $(b, g(b))$ lies above the curve $g(x)$ in the interval $[a, b]$. The convex curve is continuous and has a left and a right derivative on $[a, b]$. Curve $g(x)$ is strictly concave if the curve $-g(x)$ is strictly convex. The text concerns convex curves, meaning strict convexity, and all the results obtained with the appropriate changes apply to strictly concave curves.

In addition to this problem in which the values of the approximation chain $l(x)$ are not specified at the ends of the chain, there are interesting problems where such values are specified in the left, right, or both end-points:

$$y_0 = g(x_0) \qquad \text{and/or} \qquad y_{N+1} = g(x_{N+1}). \tag{3}$$

If the parameters y_0 and y_{N+1} in expression (2), which might be specified and in which case they are excluded from the process of minimization, are marked by putting them in brackets, then the expression for f_N assumes the following general form:

$$f_N = \min_{\substack{x_1, x_2, \ldots, x_N \\ (y_0), y_1, y_2, \ldots, y_N, (y_{N+1})}} \max_{x \in [x_0, x_{N+1}]} | g(x) - l(x) | . \tag{4}$$

3. Approximation of a Convex Curve by a Linear Function on a Given Segment

Let us first analyze the approximation of the curve $g(x)$ on the segment $[x_n, x_{n+1}]$ by a linear function $l_n(x)$ whose values $y_{n+1} = l_n(x_{n+1})$ and $y_n = l_n(x_n)$ at the end-points are not specified in advance. The expression for the optimal error of approximation $f_0(x_n, x_{n+1})$ is formed using notation in accord with expression (4)

$$f_0(x_n, x_{n+1}) = \min_{y_n, y_{n+1}} \max_{x \in [x_n, x_{n+1}]} | g(x) - l_n(x) | . \tag{5}$$

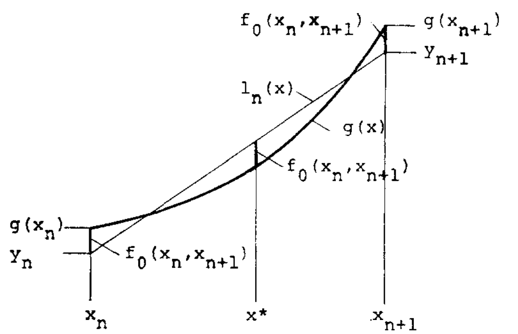

Figure 1

The optimal solution is presented in Fig. 1. Line $y = l_n(x)$ intersects the curve $y = g(x)$ at two points, and the errors of approximation are equal at the ends of the segment and are equal to the maximum error of approximation inside the interval (x_n, x_{n+1}). The optimal solution is described by the following set of equations:

$$\begin{aligned} f_0(x_n, x_{n+1}) &= | g(x_n) - y_n | \\ f_0(x_n, x_{n+1}) &= | g(x_{n+1}) - y_{n+1} | \\ f_0(x_n, x_{n+1}) &= \max_{x \in (x_n, x_{n+1})} \left| g(x) - y_n - \frac{y_{n+1} - y_n}{x_{n+1} - x_n} (x - x_n) \right| . \end{aligned} \tag{6}$$

Function $f_0(x_n, x_{n+1})$ has the following properties: $f_0(x_n, x_{n+1})$ decreases monotonously with x_n, increases monotonously with x_{n+1}, and has a zero value when $x_{n+1} = x_n$.

That is, if we translate the linear function $l_n(x)$ upwards for the value of $f_0(x_n, x_{n+1})$ it will intersect the curve $g(x)$ at the end-points of the segment $[x_n, x_{n+1}]$, and the maximum error of approximation will be $2f_0(x_n, x_{n+1})$. From the convexity of curve $g(x)$ we conclude that f_0 monotonously increases with x_{n+1} and monotonously decreases with x_n, while $f_0(x_n, x_n) = 0$ results from the continuity of curve $g(x)$.

Let us now consider the approximation of curve $g(x)$ on the segment $[x_0, x_1]$ by a linear function $l_0(x)$ whose value at the left end-point is specified in advance, $y_0 = g(x_0)$, and the value at the right end-point y_1 is free. From expression (4) the optimal error of approximation $f_0(x_0, x_1)$ can be written as

$$f_0(x_0, x_1) = \min_{y_1} \max_{x \in [x_0, x_1]} | g(x) - l_0(x) | \tag{7}$$

where $y_0 = g(x_0)$.

When an analysis, similar to the previous one, is undertaken, we find that the optimal solution to the problem defined by expression (7) is described by the following expressions:

$$\begin{aligned} 0 &= g(x_0) - y_0 \\ f_0(x_0, x_1) &= | g(x_1) - y_1 | \\ f_0(x_0, x_1) &= \max_{x \in (x_0, x_1)} \left| g(x) - y_0 - \frac{y_1 - y_0}{x_1 - x_0} (x - x_0) \right| . \end{aligned} \tag{8}$$

The optimal solution is shown in Fig. 2.

Let us analyze the approximation of curve $g(x)$ on the segment $[x_N, x_{N+1}]$ by a linear function $l_N(x)$ whose value in the right end-point of the segment is specified in advance, $y_{N+1} = g(x_{N+1})$, and the value of the left end-point y_N is free. From expression (4) the optimal error of approximation $f_0(x_N, x_{N+1})$ can be written as

$$f_0(x_N, x_{N+1}) = \min_{y_N} \max_{x \in [x_N, x_{N+1}]} | g(x) - l_N(x) | , \tag{9}$$

where $y_{N+1} = g(x_{N+1})$.

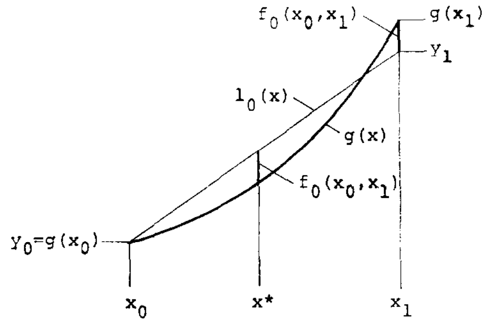

FIGURE 2

Analogous to the previous analysis, the optimal solution to the problem which is defined by expression (9) is

$$\begin{aligned} f_0(x_N, x_{N+1}) &= g(x_N) - y_N \\ 0 &= g(x_{N+1}) - y_{N+1} \\ f_0(x_N, x_{N+1}) &= \max_{x \in (x_N, x_{N+1})} \left| g(x) - y_N - \frac{y_{N+1} - y_N}{x_{N+1} - x_N}(x - x_N) \right| . \end{aligned} \tag{10}$$

A graphical interpretation of this optimal solution is given in Fig. 3.

The properties of the curve $f_0(x_n, x_{n+1})$ which have already been described could also be seen in curves $f_0(x_0, x_1)$ and $f_0(x_N, x_{N+1})$ which could easily be shown by analogy.

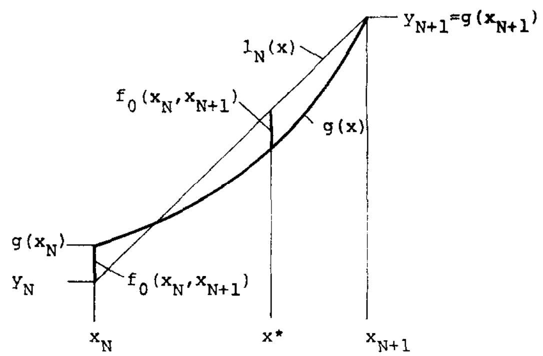

FIGURE 3

4. UNIFORM DISTRIBUTION OF THE OPTIMAL ERROR

The optimal solution to the approximation of a convex curve by a function which is piecewise linear, is defined by the following theorem.

THEOREM. *Let $g(x)$ be a convex curve on the segment $[a, b]$, and $l(x)$ a piecewise linear function which is defined by Eqs.* (1). *If we choose the parameters $(y_0), y_1, y_2, \ldots, y_N, (y_{N+1})$ so that for each segment $[x_n, x_{n+1}]$ (where $n = 0, 1, \ldots, N$), linear function $l_n(x)$ represents the optimal approximation of the curve $g(x)$ as given by Eqs.* (6) (*Eqs.* (8) *for the segment $[x_0, x_1]$ if $y_0 = g(x_0)$ is specified and Eqs.* (10) *for the segment $[x_N, x_{N+1}]$ if $y_{N+1} = g(x_{N+1})$ is specified*), *and we choose the parameters $x_1, x_2, \ldots, x_N$ in such a way that*

$$f_0(x_0, x_1) = \cdots = f_0(x_n, x_{n+1}) = \cdots = f_0(x_N, x_{N+1}) = f_N(x_0, x_{N+1}), \quad (11)$$

then the function $l(x)$ approximates the curve $g(x)$ on the segment $[a, b]$ in the optimal way according to expression (4). *The optimal error of approximation $f_N(x_0, x_{N+1})$ on segment $[a, b]$ equals the optimal error of approximation $f_0(x_n, x_{n+1})$ on individual segments $[x_n, x_{n+1}]$. This optimal approximation is unique.*

Proof. We shall prove the theorem by mathematical induction. For $N = 1$ (the plot of function $l(x)$ consists of two straight linear segments) expression 4 for the optimal error of approximation is reduced to

$$f_1(x_0, x_2) = \min_{\substack{x_1 \\ (y_0), y_1, y_2}} \max_{x \in [x_0, x_2]} |g(x) - l(x)| . \quad (12)$$

Let us assume that the function $l(x)$ has a discontinuity at point x_1 (the straight linear segments l_0 and l_1 are not connected in a chain) and let

$$l(x_1) = l_0(x_1) = y_1',$$

$$\lim_{x \to x_1 + 0} l(x) = l_1(x_1) = y_1'' \qquad \text{and} \qquad y_1' \neq y_1'' .$$

Instead of expression (12) we could write

$$\begin{aligned} f_1(x_0, x_2) &= \min_{\substack{x_1 \\ (y_0), y_1', y_1'', y_2}} \max\{\max_{x \in [x_0, x_1]} |g(x) - l_0(x)|, \max_{x \in [x_1, x_2]} |g(x) - l_1(x)|\} \\ &= \min_{x_1} \max\{\min_{(y_0), y_1'} \max_{x \in [x_0, x_1]} |g(x) - l_0(x)|, \min_{y_1'', y_2} \max_{x \in [x_1, x_2]} |g(x) - l_1(x)|\} \\ &= \min_{x_1}\{\max[f_0(x_0, x_1), f_0(x_1, x_2)]\}. \end{aligned} \quad (13)$$

Since $f_0(x_0, x_1)$ increases, and $f_0(x_1, x_2)$ decreases with the increase of x_1, the minimum value of function $f_1(x_0, x_2)$ is assumed if the following condition is fulfilled:

$$f_0(x_0, x_1) = f_0(x_1, x_2) = f_1(x_0, x_2) \quad (14)$$

from which it follows that

$$y_1' = y_1'' = y_1 . \tag{15}$$

Thus the theorem is proved for $N = 1$.

From Eqs. (14) it can be seen that when the boundary value x_2 monotonously increases, the value of x_1 also increases. The properties that characterize curve f_0 also apply to curve $f_1(x_0, x_2)$, i.e., $f_1(x_0, x_2)$ tends to zero when x_2 tends to x_0, it decreases monotonously with the increase of x_0, and increases monotonously with the increase of x_2.

We shall now prove the theorem for an arbitrary value N, under the assumption that it holds for $N - 1$.

Let us denote the optimal value of the approximation error by $f_{N-1}(x_0, x_N)$,

$$f_{N-1}(x_0, x_N) = \min_{\substack{x_1, x_2, \ldots, x_{N-1} \\ (y_0), y_1, y_2, \ldots, y_{N-1}, y_N}} \max_{x \in [x_0, x_N]} |g(x) - l(x)| . \tag{16}$$

Let us also assume that $f_{N-1}(x_0, x_N)$ monotonously increases with x_N, decreases with x_0, and has a zero value for $x_N = x_0$ (which might be considered as a part of an inductive assumption since it has already been shown that it holds for $N = 1$).

Therefore, from expression (4) the following can be written:

$$f_N(x_0, x_{N+1}) = \min_{\substack{x_1, x_2, \ldots, x_N \\ (y_0), y_1, y_2, \ldots, y_N, (y_{N+1})}} \max_{x \in [x_0, x_{N+1}]} |g(x) - l(x)| . \tag{17}$$

In order to simplify the proof we shall separate the straight linear segment $l_N(x)$ from the segment $l_{N-1}(x)$ by introducing different values at the point x_N: $y_N' = l_{N-1}(x)$ and $y_N'' = l_N(x_N)$. Now we could replace expression (17) by

$$\begin{aligned} &f_N(x_0, x_{N+1}) \\ &= \min_{\substack{x_1, x_2, \ldots, x_N \\ (y_0), y_1, y_2, \ldots, y_N', y_N'', (y_{N+1})}} \max\{\max_{x \in [x_0, x_N]} |g(x) - l(x)| , \max_{x \in [x_N, x_{N+1}]} |g(x) - l(x)|\} \\ &= \min_{x_N} \max\{ \min_{\substack{x_1, x_2, \ldots, x_{N-1} \\ (y_0), y_1, y_2, \ldots, y_{N-1}, y_N'}} \max_{x \in [x_0, x_N]} |g(x) - l(x)| , \\ &\qquad \min_{y_N'', (y_{N+1})} \max_{x \in [x_N, x_{N+1}]} |g(x) - l_N(x)|\} \\ &= \min_{x_N} \max\{f_{N-1}(x_0, x_N), f_0(x_N, x_{N+1})\}. \end{aligned} \tag{18}$$

Since the assumption that $f_{N-1}(x_0, x_N)$ monotonously increases and $f_0(x_N, x_{N+1})$ monotonously decreases with the increase of x_N, the minimum value of function $f_N(x_0, x_{N+1})$ is assumed when

$$f_{N-1}(x_0, x_N) = f_0(x_N, x_{N+1}) = f_N(x_0, x_{N+1}), \tag{19}$$

that is, when

$$f_0(x_0, x_1) = f_0(x_1, x_2) = \cdots = f_0(x_N, x_{N+1}) = f_N(x_0, x_{N+1}) \tag{20}$$

from which follows that

$$y_N' = y_N'' = y_N .$$

Thus it is proved that the solution to the problem which is defined by expression (18), is also the solution to the previously stated problem which is defined by expression (17).

The increase of the value for x_{N+1} causes an increase of the optimal value x_N which follows from Eqs. (19) from which we could deduce that the properties of f_0 also apply to function $f_N(x_0, x_{N+1})$, e.g., it monotonously decreases with the increase of x_0, monotonously increases with the increase of x_{N+1}, and it has a zero value for $x_0 = x_{N+1}$.

Thus the theorem is proved for every N.

5. Analytical Determination of the Optimal Approximation

The analytical method for determining the optimal approximation can be carried out, in certain cases, analytically to full extent. The method is systematically presented using as examples the approximation of curves x^2 and $x^{1/2}$.

(a) $g(x) = x^2$ (y_0 and y_{N+1} are free).

From Eqs. (6) the following equations are deduced, for optimal approximation of curve x^2 by a linear function $l_n(x)$ on the interval $[x_n, x_{n+1}]$,

$$\begin{aligned} y_n &= x_n^2 - f_0(x_n, x_{n+1}) \\ y_{n+1} &= x_{n+1}^2 - f_0(x_n, x_{n+1}) \\ f_0(x_n, x_{n+1}) &= \left(\frac{x_{n+1} - x_n}{2 + 2^{1/2}}\right)^2, \qquad n = 0, 1, 2, \ldots, N. \end{aligned} \tag{21}$$

Introducing the expression for $f_0(x_n, x_{n+1})$ into Eqs. (11) and simplifying them, the following set of equations is deduced:

$$x_{n+1} - x_n = 2(2)^{1/2}(f_0)^{1/2}, \qquad n = 0, 1, 2, 3, \ldots, N. \tag{22}$$

Summing Eqs. (22) the expression for the optimal value of the error f_0 of approximation as a function of boundary values of the interval becomes

$$f_0 = f_N = \left(\frac{1}{2(2)^{1/2}} \frac{x_{N+1} - x_0}{N + 1}\right)^2. \tag{23}$$

From Eqs. (21) to (23), the equations for the optimal values of the parameters x_n and y_n, are derived:

$$x_n = x_0 + \frac{n}{N+1}(x_{N+1} - x_0)$$
$$y_n = x_n^2 - f_N, \qquad n = 0, 1, 2, \ldots, N+1. \tag{24}$$

(b) $g(x) = x^2$ ($x_0 = x_0^2$, $y_{N+1} = x_{N+1}^2$).

From Eqs. (8), (6), and (10) the following equations for the optimal approximation of curve x^2 by the linear function $l_n(x)$, are deduced,

$$f_0(x_0, x_1) = \left[\frac{x_1 - x_0}{1 + 2^{1/2}}\right]^2$$
$$f_0(x_n, x_{n+1}) = \left[\frac{x_{n+1} - x_n}{2(2)^{1/2}}\right]^2, \qquad n = 1, 2, \ldots, N-1 \tag{25}$$
$$f_0(x_N, x_{N+1}) = \left[\frac{x_{N+1} - x_N}{1 + 2^{1/2}}\right]^2.$$

When Eqs. (25) are introduced into Eqs. (11) and when these equations are rearranged, the following is obtained:

$$x_1 - x_0 = (1 + 2^{1/2})(f_0)^{1/2}$$
$$x_{n+1} - x_n = 2(2)^{1/2}(f_0)^{1/2}, \qquad n = 1, 2, \ldots, N-1 \tag{26}$$
$$x_{N+1} - x_N = (1 + 2^{1/2})(f_0)^{1/2}.$$

Summing Eqs. (26), the solution for f_0, i.e., f_N is derived

$$f_0 = f_N = \left[\frac{x_{N+1} - x_0}{2(2)^{1/2}(N + 1/2^{1/2})}\right]^2. \tag{27}$$

From Eqs. (26), (27), and (6) the equations for the optimal values of the required parameters are deduced to be

$$x_n = x_0 + \frac{n - 1/2 + 1/2(2)^{1/2}}{N + 1/2^{1/2}}(x_{N+1} - x_0)$$
$$y_n = x_n^2 - f_N, \qquad n = 1, 2, \ldots, N. \tag{28}$$

From Eq. (27) the optimal errors of approximation on the segment [0, 1] for $N = 1, 2, \ldots, 15$ are calculated and given in Table I.

TABLE I

Optimal Approximation of $X**2$

$X(0) = 0 \quad X(N+1) = 1$ Optimal errors	
N	F
1	$4.28932E-02$
2	$1.70568E-02$
3	$9.09577E-03$
4	$5.64159E-03$
5	$3.83776E-03$
6	$2.77869E-03$
7	$2.10439E-03$
8	$1.64877E-03$
9	$1.32657E-03$
10	$1.09035E-03$
11	$9.12034E-04$
12	$7.74135E-04$
13	$6.65302E-04$
14	$5.77904E-04$
15	$5.06661E-04$

TABLE II

Optimal Approximation of $X**2$

	$X(0) = 0 \quad X(N+1) = 1$	$N = 7$
J	$X(J)$	$Y(J)$
0	0	0
1	0.110748	$1.01609E-02$
2	0.240499	$5.57355E-02$
3	0.370249	0.13498
4	0.5	0.247895
5	0.62975	0.394481
6	0.759501	0.574737
7	0.889251	0.788663
8	1	1
$F = 2.10439E-03$		

From Eqs. (27) and (28) the optimal solution for approximation of the curve x^2 on the segment [0, 1] for $N = 7$ is calculated and given in Table II.

(c) $g(x) = x^{1/2}$ (y_0 and y_{N+1} are free).

From Eqs. (6) the following expressions for the optimal approximation of curve $x^{1/2}$ in the interval $[x_n, x_{n+1}]$ by a linear function $l_n(x)$, $n = 0, 1,..., N$ are derived

$$f_0(x_n, x_{n+1}) = \frac{1}{8} \frac{(g_{n+1} - g_n)^2}{g_{n+1} + g_n}, \qquad n = 0, 1, 2,..., N, \tag{29}$$

where $g_n = (x_n)^{1/2}$.

By introduction of these expressions into Eqs. (11) the following equation is deduced,

$$g_{n+1} = g_n \left[1 + 2 \frac{g_n - g_{n-1}}{g_n + g_{n-1}}\right], \qquad n = 0, 1, 2,..., N, \tag{30}$$

from which follows

$$g_n = \frac{1}{2} \frac{1}{g_1 + g_0} [(n + 1) g_1 - (n - 1) g_0] [ng_1 - (n - 2) g_0], \qquad n = 0, 1, 2,..., N, N + 1. \tag{31}$$

Equation (31) could easily be proved by mathematical induction. When $n = N + 1$ is introduced into Eq. (31) and then is solved for $g_1 = (x_1)^{1/2}$, the following is obtained,

$$g_1 = - g_0 + \frac{1}{(N + 1)(N + 2)} \{[2(N + 1)^2 - 1] g_0 + g_{n+1} + [\{[2(N + 1)^2 - 1] g_0 + g_{N+1}\}^2 - 4N(N + 1)^2 (N + 2) g_0^2]^{1/2}\}. \tag{32}$$

The optimal values of parameters x_n, $n = 1, 2,..., N$ are calculated by squaring the values that are obtained from Eqs. (31) and (32), the optimal value $f_N(x_0, x_{N+1})$ is calculated from Eq. (29) by taking any of the given values for n. The optimal values of parameter y_n are calculated from the first two equations of Eqs. (6).

(d) $g(x) = x^{1/2}$ ($y_0 = (x_0)^{1/2}$, $y_{N+1} = (x_{N+1})^{1/2}$).

The expressions for the optimal errors of approximation of curve $x^{1/2}$ are derived: for the interval $[x_0, x_1]$ from Eqs. (8), for the intervals $[x_n, x_{n+1}]$, $n = 1, 2,..., N - 1$ from Eqs. (6), and for the interval $[x_N, x_{N+1}]$ from Eqs. (10):

$$\begin{aligned} f_0(x_0, x_1) &= \frac{1}{2(1 + 2^{1/2})} \frac{(g_1 - g_0)^2}{g_1 + 2^{1/2} g_0} \\ f_0(x_n, x_{n+1}) &= \frac{1}{8} \frac{(g_{n+1} - g_n)^2}{g_{n+1} + g_n}, \qquad n = 1, 2,..., N - 1 \\ f_0(x_N, x_{N+1}) &= \frac{1}{2(1 + 2^{1/2})} \frac{(g_{N+1} - g_N)^2}{2^{1/2} g_{N+1} + g_N}, \end{aligned} \tag{33}$$

where $g_n = (x_n)^{1/2}$.

When Eqs. (33) are introduced into the set of Eqs. (11) the following set of equations is obtained:

$$\begin{aligned} g_2 &= g_1\left(1 + 2(2)^{1/2}\frac{g_1 - g_0}{g_1 + 2^{1/2}g_0}\right) \\ g_{n+1} &= g_n\left(1 + 2(2)^{1/2}\frac{g_n - g_{n-1}}{g_n + g_{n-1}}\right), \qquad n = 2, 3,..., N-1 \\ g_{N+1} &= g_N\left(1 + \frac{2+2^{1/2}}{2}\frac{g_N - g_{N-1}}{g_N + g_{N-1}}\right). \end{aligned} \tag{34}$$

The following is deduced from the first two equations of Eqs. (34):

$$\begin{aligned} g_n = \frac{2^{1/2}-1}{g_1 + 2^{1/2}g_0}&[(2^{1/2}n+1)g_1 - 2^{1/2}(n-1)g_0] \\ &\cdot[(2^{1/2}(n-1)+1)g_1 - 2^{1/2}(n-2)g_0], \qquad n = 1, 2,..., N, \end{aligned} \tag{35}$$

which could be easily proved by induction.

When Eqs. (35) are introduced into the third equation of Eqs. (34) and this one is solved for $g_1 = (x_1)^{1/2}$ the following is obtained:

$$\begin{aligned} g_1 = -2^{1/2}g_0 + \frac{1+2^{1/2}}{2(2^{1/2}N+1)(2^{1/2}N+2)}&\{[2(2^{1/2}N+1)^2 - 1]g_0 + g_{N+1} \\ &+ [\{[2(2^{1/2}N+1)^2 - 1]g_0 + g_{N+1}\}^2 \\ &- 4(2^{1/2})N(2^{1/2}N+1)^2(2^{1/2}N+2)g_0)^2]^{1/2}\}. \end{aligned} \tag{36}$$

The rest of the values for $g_n = (x_n)^{1/2}$, $n = 2, 3,..., N$ are calculated from Eqs. 35.

The optimal values of parameters x_n, $n = 1, 2,..., N$ are calculated by squaring the derived values for $g_n = (x_n)^{1/2}$. The optimal value of the error of approximation $f_N = f_0$ is calculated from one of Eqs. (33). The optimal values of parameters y_n are calculated from the first two equations of Eqs. (6).

When the lower limit of the approximation interval is equal to zero, $x_0 = 0$, then the above expressions are simplified to a great extent. Thus Eq. (36) is reduced to

$$g_1 = \frac{1+2^{1/2}}{(2^{1/2}N+1)(2^{1/2}N+2)}g_{N+1} \tag{37}$$

and Eqs. (35) to

$$g_n = \frac{(2^{1/2}n+1)[2^{1/2}(n-1)+1]}{(2^{1/2}N+1)(2^{1/2}N+2)}g_{N+1}, \qquad n = 1, 2,..., N. \tag{38}$$

When Eq. (37) is introduced into the first equation of Eqs. (33), the following is obtained for optimal error of approximation:

$$f_N(0, x_{N+1}) = \frac{1}{2} \frac{1}{(2^{1/2}N + 1)(2^{1/2}N + 2)} g_{N+1} . \tag{39}$$

TABLE III

Optimal Approximation of SQRT(X)

$X(0) = 0 \quad X(N + 1) = 1$
Optimal errors

N	F
1	$6.06601E - 02$
2	$2.70485E - 02$
3	$1.52774E - 02$
4	$9.80958E - 03$
5	$6.82937E - 03$
6	$5.02736E - 03$
7	$3.85509E - 03$
8	$3.04987E - 03$
9	$2.47300E - 03$
10	$2.04560E - 03$
11	$1.72017E - 03$
12	$1.46665E - 03$
13	$1.26533E - 03$
14	$1.10278E - 03$
15	$9.69670E - 04$

TABLE IV

Optimal Approximation of SQRT(X)

	$X(0) = 0 \quad X(N + 1) = 1$	$N = 7$
J	$X(J)$	$Y(J)$
0	0	0
1	$3.46483E - 04$	$2.24691E - 02$
2	$5.07834E - 03$	$7.51176E - 02$
3	$2.39481E - 02$	0.158606
4	$7.24049E - 02$	0.272936
5	0.171605	0.418107
6	0.348411	0.594119
7	0.635393	0.80097
8	1	1
$F = 3.85509E - 03$		

The optimal values of the parameters x_n, $n = 1, 2, \ldots, N$ are calculated by squaring the values $g_n = (x_n)^{1/2}$ which are calculated from expressions (37) and (38). The optimal values of parameters y_n are calculated from the first two equations of Eqs. (6).

From Eq. (39) the values of the optimal error of approximation of curve $x^{1/2}$ on the segment [0, 1] for $N = 1, 2, \ldots, 15$ are calculated and given in Table III.

From Eqs. (38), (39), and (6), the optimal solution of the approximation of curve $x^{1/2}$ on the segment [0, 1] for $N = 7$ is calculated and presented in Table IV.

6. Numerical Determination of the Optimal Approximation

The determination of the optimal approximation reduces to determining the piecewise linear function which satisfies the conditions defined by the theorem in Section 4. In this section the method of numerical determination of the optimal approximation of convex curves which have continuous first and second derivatives will be described.

The optimal error of approximation $f_0(x_0, x_{N+1})$ is a function which tends to zero when x_{N+1} tends to x_0, it monotonously decreases with the increase of x_0, and monotonously rises with the increase of x_{N+1}. We can deduce from there, that the inverse function $x_{N+1}(x_0, f_0)$ tends to x_0, when f_0 tends to zero, and it monotonously rises with the increase of x_0 and f_0. The set of optimal values of the upper boundary x_{N+1} corresponds to the set of values f_0, for the given lower boundary $x_0 = a$. Therefore, from the set of optimal solutions $x_{N+1}(x_0, f_0)$ we have to select the one whose boundaries of the approximation interval coincide with the given one,

$$b - x_{N+1}(a, f_0) = 0. \tag{40}$$

Since curve $g(x)$ is convex and has continuous first and second derivatives, Eq. (40) could be easily solved numerically using the Newton–Raphson method. The block diagram for that method is presented in Fig. 4. The computation of the required values $x_{N+1}(a, f_0)$ and $\partial x_{N+1}(a, f_0)/\partial f_0$ is done sequentially starting from the values $x_1(x_0, f_0)$, $\partial x_1(x_0, f_0)/\partial f_0$, and then computing $x_2(x_1, f_0)$, $\partial x_2(x_1, f_0)/\partial f_0$, etc. The method for computing these values will be presented first for the interval $[x_n, x_{n+1}]$ for which the boundary values y_n and y_{n+1} of the linear function are free. The optimal position of that linear segment in respect to the curve $g(x)$ is described by the set of Eqs. (6). If we denote the value on the x-axis which has the maximum error of

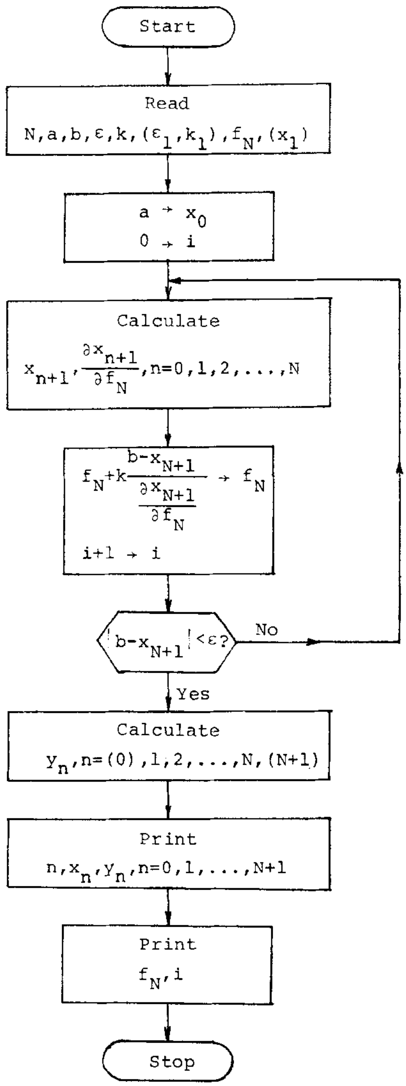

FIGURE 4

approximation inside the interval (x_n, x_{n+1}), with x^*, then the optimal solution can be described by the set of equations,

$$\begin{aligned} f_0(x_n, x_{n+1}) &= \pm [g(x_n) - y_n] \\ f_0(x_n, x_{n+1}) &= \pm [g(x_{n+1}) - y_{n+1}] \\ f_0(x_n, x_{n+1}) &= \mp [g(x^*) - y_n - g'(x^*)(x^* - x_n)] \\ 0 &= g'(x^*) - \frac{y_{n+1} - y_n}{x_{n+1} - x_n}, \end{aligned} \tag{41}$$

where the upper sign is taken when curve $g(x)$ is convex and the lower one when it is concave.

Since the optimal values of parameters y_n and y_{n+1} can be easily computed from the first two equations of the set (41), assuming that the optimal value for the error of approximation has been found previously, Eqs. (41) can be reduced to two equations by eliminating the variables y_n and y_{n+1},

$$\begin{aligned} g(x_{n+1}) - g(x_n) - g'(x^*)(x_{n+1} - x_n) &= 0 \\ g(x^*) - g(x_n) \pm 2f_0 - g'(x^*)(x^* - x_n) &= 0. \end{aligned} \tag{42}$$

In the general case, Eqs. (41) could not be solved explicitly for $x_{n+1} = x_{n+1}(x_n, f_0)$. However, since the curve $g(x)$ is convex and has continuous first and second derivative, Eqs. 42 could be easily solved numerically using the Newton–Raphson method. In Eqs. (42) the parameters x_n and f_0 are the constants to be specified. However, in the iterative method which is shown in Fig. 4, the value f_0 is the current value and value x_n is the one that was obtained for the previous interval $[x_{n-1}, x_n]$.

To compute the derivative $\partial x_{n+1}/\partial f_0$, Eqs. (42) are differentiated in respect to f_0. Thus we get the set of linear equations for $\partial x_{n+1}/\partial f_0$ and for $\partial x^*/\partial f_0$. When we solve that system for $\partial x_{n+1}/\partial f_0$ we obtain the following expression:

$$\begin{aligned} \frac{\partial x_{n+1}}{\partial f_0} = \frac{1}{g'(x_{n+1}) - g'(x^*)} \Big\{ &\Big(\frac{x_{n+1} - x_n}{x^* - x_n} - 1 \Big) \\ &\cdot [g'(x^*) - g'(x_n)] \frac{\partial x_n}{\partial f_0} \pm 2 \frac{x_{n+1} - x_n}{x^* - x_n} \Big\}, \end{aligned} \tag{43}$$

which is used for direct computation of the required derivative.

Let us now compute $x_1(x_0, f_0)$ and $\partial x_1(x_0, f_0)/\partial f_0$. The set of Eqs. (8) reduces to

$$\begin{aligned} 0 &= g(x_0) - y_0 \\ f_0(x_0, x_1) &= \pm [g(x_1) - y_1] \\ f_0(x_0, x_1) &= \mp [g(x^*) - y_0 - g'(x^*)(x^* - x_0)] \\ 0 &= g'(x^*) - \frac{y_1 - y_0}{x_1 - x_0}. \end{aligned} \tag{44}$$

Eliminating parameters y_0 and y_1, Eqs. (44) reduce to the following system of equations:

$$\begin{aligned} g(x^*) - g(x_0) \pm f_0 - g'(x^*)(x^* - x_0) &= 0 \\ g(x_1) - g(x_0) \mp f_0 - g(x^*)(x_1 - x_0) &= 0, \end{aligned} \tag{45}$$

from which we compute, for the given values x_0 and f_0, the boundary x_1 numerically in the general case. Equations (45) are solved using the Newton–Raphson methods just as efficiently as Eqs. (42). From Eqs. (45) we derive the following expression for derivative $\partial x_1/\partial f_0$,

$$\frac{\partial x_1}{\partial f_0} = \pm \frac{1}{g'(x_1) - g'(x^*)} \left\{ \frac{x_1 - x_0}{x^* - x_0} + 1 \right\}. \tag{46}$$

Let us calculate functions $x_{N+1}(x_N, f_0)$ and $\partial x_{N+1}(x_N, t_0)/\partial f_0$ when the value of the linear function $l_N(x)$ on the right end of the interval is specified, $y_{N+1} = g(x_{N+1})$. The system of Eqs. (10) reduces to the system of equations

$$\begin{aligned} f_0(x_N, x_{N+1}) &= \pm [g(x_N) - y_N] \\ 0 &= g(x_{N+1}) - y_{N+1} \\ f_0(x_N, x_{N+1}) &= \mp [g(x^*) - y_N - g'(x^*)(x - x_N)] \\ 0 &= g'(x^*) - \frac{y_{N+1} - y_N}{x_{N+1} - x_N}. \end{aligned} \tag{47}$$

We compute for given x_N and f_0 the right boundary x_{N+1}, by solving the following system of equations,

$$\begin{aligned} g(x_{N+1}) - g(x_N) \pm f_0 - g'(x^*)(x_{N+1} - x_N) &= 0 \\ g(x^*) - g(x_N) \pm 2f_0 - g'(x^*)(x^* - x_N) &= 0, \end{aligned} \tag{48}$$

to which Eqs. (47) are reduced. Eqs. (48) are solved numerically in the same manner as are Eqs. (42) and (45). We derive the following expression for $\partial x_{N+1}/\partial f_0$ from the system of Eqs. (48)

$$\begin{aligned} \frac{\partial x_{N+1}}{\partial f_0} = \frac{1}{g'(x_{N+1}) - g'(x^*)} & \left\{ \left(\frac{x_{N+1} - x_N}{x^* - x_N} - 1 \right) \right. \\ & \left. \cdot [g'(x^*) - g'(x_N)] \frac{\partial x_N}{\partial f_0} \pm 2 \frac{x_{N+1} - x_N}{x^* - x_N} \mp 1 \right\}. \end{aligned} \tag{49}$$

The block diagram of the method for solving the values, x_{n+1} and $\partial x_{n+1}/\partial f_0$ for $n = 0, 1, 2, \ldots, N$ is presented in Fig. 5. The left sides of the set of Eqs. (42) for $n = 1, 2, \ldots, N - 1$ (Eqs. (45) for $n = 0$, and Eqs. (48) for $n = N$) are denoted by $u(x_{n+1}, x^*)$ and $v(x^*)$.

The method described for numerical determination of the optimal solution was realized using the BASIC programming language on computer Varian 620-i. The optimal solutions for approximation of some standard curves are computed using that program.

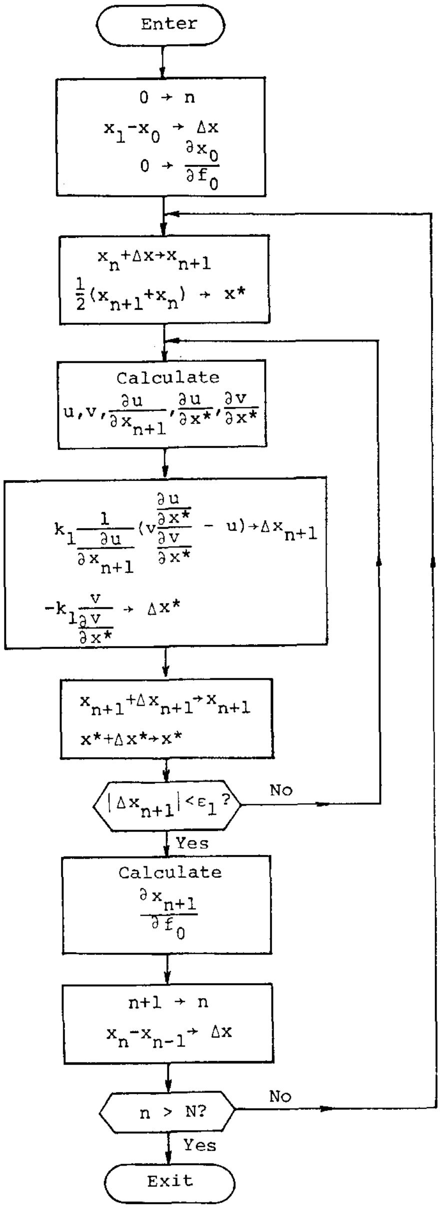

FIGURE 5

(a) $g(x) = \sin x (x_0 = 0,\ y_0 = 0;\ x_{N+1} = \pi/2,\ y_{N+1} = 1)$.

The optimal errors of approximation of curve $\sin x$ in the interval $[0, \pi/2]$ with the boundary values of the approximating function $y_0 = 0$, $y_{n+1} = 1$

are computed for various values of parameters N and the values are given in Table V.

TABLE V

Optimal Approximation of SIN(X)

$X(0) = 0$ $X(N+1) = 1.5708$ $E = 1.00000E - 04$

Optimal errors

N	F
1	$3.25547E - 02$
2	$1.26675E - 02$
3	$6.69129E - 03$
4	$4.12798E - 03$
5	$2.79862E - 03$
6	$2.02133E - 03$
7	$1.52814E - 03$
8	$1.19583E - 03$
9	$9.61079E - 04$
10	$7.89192E - 04$

The optimal solution for the approximation is computed for the case when $N = 7$. The parameters which describe this solution are given in Table VI.

TABLE VI

Optimal Approximation of SIN(X)

$X(0) = 0$ $X(N+1) = 1.5708$ $N = 7$ $S = 1$

$E = 1.00000E - 04$ $E1 = 1.00000E - 04$ $T = 1$ $T1 = 1$

Initial values: $X(1) = 0.67$ $F = 1.27000E - 02$

J	$X(J)$	$Y(J)$
0	0	0
1	0.333017	0.328423
2	0.569716	0.540921
3	0.7683	0.696442
4	0.948101	0.813837
5	1.11689	0.900268
6	1.27896	0.959244
7	1.43715	0.992611
8	1.57077	0.999999

$F = 1.52814E - 03$ $I = 5$

(b) $g(x) = \text{tg}\, x (x_0 = 0, y_0 = 0;\ x_{N+1} = \pi/4, y_{N+1} = 1)$.

The optimal values of the approximation error method for the interval of approximation $[0, \pi/4]$ when the boundary values of the approximating function are $y_0 = 0$, $y_{N+1} = 1$, and for different numbers of breaking points N, are computed for curve tg x. The values of optimal errors of approximation are given in Table VII.

TABLE VII

Optimal Approximation of TAN(X)

$X(0) = 0$ $X(N + 1) = 0.7854$ $E = 1.00000E - 04$
Optimal errors

N	F
1	$1.44745E - 02$
2	$5.65688E - 03$
3	$2.99182E - 03$
4	$1.84703E - 03$
5	$1.25247E - 03$
6	$9.04862E - 04$
7	$6.84128E - 04$
8	$5.35257E - 04$
9	$4.30220E - 04$
10	$3.53293E - 04$

TABLE VIII

Optimal Approximation of TAN(X)

$X(0) = 0$ $X(N + 1) = 0.7854$ $N = 7$ $S = -1$
$E = 1.00000E - 04$ $E1 = 1.00000E - 04$ $T = 1$ $T1 = 1$
Initial values: $X(1) = 0.21$ $F = 9.00000E - 04$

J	$X(J)$	$Y(J)$
0	0	0
1	0.200371	0.202412
2	0.336042	0.348606
3	0.442712	0.473414
4	0.532361	0.588408
5	0.609925	0.698123
6	0.67814	0.804907
7	0.738781	0.910173
8	0.785401	1
$F = 6.84128E - 04$	$I = 4$	

The parameters of the optimal solution of approximation are computed for the case when $N = 7$ and the results are given in Table VIII.

(c) $g(x) = \text{arc tg}\, x \quad (x_0 = 0,\ y_0 = 0;\ x_{N+1} = 1,\ y_{N+1} = \pi/4)$.

The optimal values for the errors of approximation in the interval [0, 1] when the boundary values of the approximating function are $y_0 = 0$, $y_{N+1} = \pi/4$ are computed for curve arc tg x. The optimal value of approximation errors are computed for various numbers of breaking points N and they are listed in Table IX.

TABLE IX

Optimal Approximation of ATAN(X)

$X(0) = 0 \quad X(N+1) = 1 \quad E = 1.00000E - 04$

Optimal errors

N	F
1	1.08099E − 02
2	4.19885E − 03
3	2.21666E − 03
4	1.36716E − 03
5	9.26933E − 04
6	6.68449E − 04
7	5.06173E − 04
8	3.95990E − 04
9	3.18073E − 04
10	2.61445E − 04

The parameters of the optimal solution for the case when $N = 7$ are listed in Table X.

TABLE X

Optimal Approximation of ATAN(X)

$X(0) = 0 \quad X(N+1) = 1 \quad N = 7 \quad S = 1$

$E = 1.00000E - 04 \quad E1 = 1.00000E - 04 \quad T = 1 \quad T1 = 1$

Initial values: $X(1) = 0.11 \quad F = 6.50000E - 04$

J	$X(J)$	$Y(J)$
0	0	0
1	0.18399	0.182461
2	0.319067	0.309362
3	0.437449	0.412874
4	0.550158	0.50347
5	0.662042	0.5853
6	0.775887	0.660371
7	0.894062	0.730031
8	1.00007	0.785436
$F = 5.06104E - 04$	$I = 3$	

7. Conclusion

This paper treats the problem of optimal approximation of a given curve by a function which is piecewise linear on the segment in which the given curve is convex, for which the value of the approximating function at the boundaries of the approximation segment can be either free or specified.

It was proved that the optimal solution, in the sense that it minimizes the maximum value of the approximation errors, is the one for which each of the linear functions optimally approximates the given curve in its interval, and that the intervals are chosen in such a manner that all the optimal errors in the intervals of approximation are equal among themselves. The efficiency of the presented procedure is based on the fact that the search is done among optimal solutions in order to select the one that satisfies the specified boundary conditions.

In certain cases the problem of optimal approximation could be solved analytically to full extent using the method presented. To illustrate the analytical method, the solutions of optimal approximation of curves x^2 and $x^{1/2}$ were derived.

In general, the problem of optimal approximation, is solved using the method presented, but numerically. A method for numerical solution of optimal approximation of convex curves which have continuous first and and second derivatives is described in this paper. To determine the optimal positions of the linear segments in respect to its intervals of approximation and to obtain the optimal approximating function which satisfies the specified boundary conditions, the Newton–Raphson method is chosen. The method described was realized using the BASIC programming language. Using this program the optimal solutions for approximation of curves sin x, tg x, and arc tg x, were computed for the purpose of illustration. However, the approximation of convex curves, in the general case, requires a different method instead of the Newton–Raphson method, which does not require the continuity of the first and second derivatives of the curve to be approximated.

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