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Some Inequalities for Generalized Concave Functions

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1. INTRODUCTION

In a recent paper [7] discussing inverse Hölder inequalities, Nehari set forth the inequality

$$\prod_{\nu=1}^{n}\left[(p_\nu+1)\int_0^1 f_\nu^{p_\nu}(x)\,dx\right]^{1/p_\nu} \leqslant C_n\int_0^1\left[\prod_{\nu=1}^{n} f_\nu(x)\right]dx, \tag{1}$$

where $f_1,..., f_n$ are nonnegative, continuous and concave functions on [0, 1], and $p_1^{-1}+\cdots+p_n^{-1}=1$, $p_i>0$ for all i. Here $C_n=(n+1)!/([n/2]!)^2$.

The inequality (1) is somewhat misleading as it comes from two separate inequalities, viz., if $f_1,..., f_n$ belong to the class considered and are normalized by

$$\int_0^1 f_\nu(x)\,dx = \tfrac{1}{2}, \qquad \nu = 1,..., n,$$

then

$$\prod_{\nu=1}^{n}\left[(p_\nu+1)\int_0^1 f_\nu^{p_\nu}(x)\,dx\right]^{1/p_\nu} \leqslant 1 \tag{2}$$

and

$$C_n\int_0^1\left[\prod_{\nu=1}^{n} f_\nu(x)\right]dx \geqslant 1. \tag{3}$$

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Apparently, Nehari was unaware of this dichotomy. Moreover, his argument on behalf of (1) was inappropriate; a correct demonstration was provided by C. Borell [3] who also did not recognize that the two sides of (1) are essentially unrelated and that (2) and (3) are the basic inequalities. Actually, Borell offers the following more general result.

THEOREM. *Let $a_1, \ldots, a_n$ be real numbers, all $\geqslant 1$, and suppose*

$$\min_{I \subset \{1,\ldots,n\}} \left| \sum_{k \in I} a_k - \frac{1}{2} \sum_{k=1}^{n} a_k \right|$$

is attained for $I = I_0$. Set

$$a_0 = \sum_{k \in I_0} a_k \qquad \textit{and} \qquad a_{00} = \sum_{k \notin I_0} a_k .$$

Let $g_1, \ldots, g_n$ be nonnegative functions defined on the interval (0, 1) *such that the functions $g_1^{1/a_1}, \ldots, g_n^{1/a_n}$ are concave. Let $p_1, \ldots, p_n$ be real numbers $\geqslant 1$. Then*

$$C_2 \int_0^1 \left[\prod_{k=1}^{n} g_k(x) \right] dx \geqslant C_1 \prod_{k=1}^{n} \left[\int_0^1 g_k^{p_k}(x)\, dx \right]^{1/p_k}, \tag{1'}$$

where

$$C_1 = \left[\prod_{k=1}^{n} (1 + a_k p_k)^{1/p_k} \right], \qquad C_2^{-1} = B(1 + a_0, 1 + a_{00})$$

and $B(p, q)$ is the familiar Beta function. Equality occurs if

$$g_k(x) = x^{a_k}, \quad k \in I_0 \qquad \textit{and} \qquad g_k(x) = (1 - x)^{a_k}, \quad k \notin I_0 .$$

Again, (1′) also comes from two separate inequalities, and the number 1 can be interposed between the two sides of (1′) after making a normalization:

$$\int_0^1 g_k^{1/a_k}\, dx = \tfrac{1}{2}, \qquad k = 1, \ldots, n.$$

The verification of these inequalities parallels that of (2) and (3), and we shall confine full attention to their proofs. A weaker version of (3) involving two functions appeared earlier in Bellman [2].

In this paper, we establish (2) and (3) as stated. Furthermore, a formulation of (3) is developed encompassing generalized concave functions. More specifically, let $Lu = D_n D_{n-1} \cdots D_1 u$ be an nth order differential operator on

$C^{(n)}[0, 1]$ composed from the successive application of first order differential operators

$$D_i u = \frac{d}{dx} \frac{1}{w_i(x)} u(x), \qquad i = 1, 2, \ldots, n$$

where $w_i(x)$ are positive of class $C^{(n)}$ on $[0, 1]$. A function f is said to be L-concave if f is a pointwise limit on the open interval $(0, 1)$ of a sequence ϕ_m satisfying $L\phi_m \leqslant 0$. An equivalent definition expressed in terms of certain determinantal inequalities involving an extended complete Tchebycheff system (E.C.T.) is described in Karlin and Studden [5, Chap. 11] or Ziegler [8]. We will indicate some cases of (3) for certain L-concave functions satisfying boundary conditions.

2. An Upper Bound for Products of Powers of Concave Functions

In this section we deal with (2).

Proposition 1. *Let $f_1, \ldots, f_n$ be nonnegative concave functions on* $[0, 1]$ *normalized by*

$$\int_0^1 f_\nu(x)\, dx = \tfrac{1}{2}, \qquad \nu = 1, \ldots, n. \tag{4}$$

If $p_\nu > 0$, $\nu = 1, \ldots, n$ and

$$\sum_{\nu=1}^{n} \frac{1}{p_\nu} = 1, \tag{5}$$

then

$$\sum_{\nu=1}^{n} \left(1 + \frac{1}{p_\nu}\right) \int_0^1 f_\nu^{p_\nu}(x)\, dx \leqslant 1. \tag{6}$$

Proof. Recall first Favard's inequality [5, p. 411] which states that where f is a nonnegative concave function on $[0, 1]$ and

$$\bar{f} = \int_0^1 f(x)\, dx$$

then, for each ψ, convex on $[0, 2\bar{f}]$, the inequality

$$\frac{1}{2\bar{f}} \int_0^{2\bar{f}} \psi(y)\, dy \geqslant \int_0^1 \psi(f(x))\, dx \tag{7}$$

holds.

For the specification $f = f_\nu$, $\nu = 1,..., n$, we have by virtue of (4) $2\bar{f}_\nu = 1$, $\nu = 1,..., n$. Since $p_\nu > 1$, $\psi(x) = x^{p_\nu}$ is convex. Invoking Favard's inequality yields

$$\frac{1}{1+p_\nu} = \int_0^1 y^{p_\nu}\, dy \geqslant \int_0^1 f_\nu^{p_\nu}(x)\, dx.$$

Summing on ν and referring to (5), gives

$$\sum_{\nu=1}^{n} \left(1 + \frac{1}{p_\nu}\right) \int_0^1 f_\nu^{p_\nu}(x)\, dx \leqslant \sum_{\nu=1}^{n} \left(1 + \frac{1}{p_\nu}\right)\left(\frac{1}{1+p_\nu}\right) = \sum_{\nu=1}^{n} \frac{1}{p_\nu} = 1.$$

Q.E.D.

A direct consequence of (6) with appeal to the arithmetic-geometric mean inequality leads to

THEOREM 1. *Let $f_1,..., f_n$ be nonnegative concave functions on* [0, 1] *normalized as in* (4), *and let $p_\nu > 0$, $\nu = 1,..., n$, be real numbers satisfying* (5). *Then*

$$\prod_{\nu=1}^{n} \left[(1+p_\nu) \int_0^1 f_\nu^{p_\nu}(x)\, dx\right]^{1/p_\nu} \leqslant 1. \tag{8}$$

Observe that (8) *is sharp; equality obtains exclusively for the determinations*

$$\begin{aligned} f_\nu &= x, & \nu &\in I, \\ f_\nu &= 1 - x, & \nu &\in I', \end{aligned} \tag{9}$$

with I and I' denoting any disjoint sets of indices obeying $I \cup I' = \{1,..., n\}$.

3. A Lower Bound for Products of Concave Functions

We treat next the inequality (3), a bit more delicate.

THEOREM 2. *Let $f_1,..., f_n$ be nonnegative and concave functions on* [0, 1] *normalized as in* (4). *Then*

$$\int_0^1 \left[\prod_{\nu=1}^{n} f_\nu(x)\right] dx \geqslant \frac{([n/2]!)^2}{(n+1)!}. \tag{10}$$

Equality is attained only if

$$\begin{aligned} f_\nu(x) &= x, & \nu &\in I, \\ f_\nu(x) &= 1 - x, & \nu &\in I', \end{aligned}$$

where I and I' are sets of indices such that $I \cup I' = \{1, 2, \ldots, n\}$ and one of them contains $[n/2]$ elements.

Proof. It is familiar that the collection of all nonnegative and concave functions on [0, 1] comprises a pointed convex cone.

The normalization (4) delimits a section of this cone which is a convex set, spanned by the one parameter family of extreme points

$$g(x,t) = \begin{cases} x/t, & 0 \leqslant x \leqslant t, \\ (1-x)/(1-t), & t \leqslant x \leqslant 1, \end{cases} \qquad 0 < t < 1, \tag{11}$$

$g(x, 0) = 1 - x$, $g(x, 1) = x$ (see [6]). Since the functional $F(f_1, \ldots, f_n) = \int_0^1 [\prod_{\nu=1}^n f_\nu(x)]\, dx$ is multilinear, its minimum is attained at an extreme point.

Thus, our task reduces to the minimization of

$$G(s_1, \ldots, s_n) = \int_0^1 \left[\prod_{i=1}^n g(x, s_i)\right] dx, \qquad 0 \leqslant s_1, \ldots, s_n \leqslant 1. \tag{12}$$

Fix $0 \leqslant s_1 \leqslant \cdots \leqslant s_{n-1} \leqslant 1$ and let $s_n = t$ vary and accordingly consider the function

$$T(t) = \int_0^1 g(x,t)\, R(x)\, dx, \tag{13}$$

where $R(x) = \prod_{i=1}^{n-1} g(x, s_i)$.

We prove first a lemma.

LEMMA 1.

$$\min_{0 \leqslant t \leqslant 1} T(t) = \min\{T(0), T(1)\}. \tag{14}$$

Proof. Observe that

$$g(x,t)\, t(1-t) = \begin{cases} x(1-t), & 0 \leqslant x \leqslant t, \\ (1-x)t, & x \leqslant t \leqslant 1, \end{cases} \qquad 0 \leqslant t \leqslant 1,$$

is a Totally Positive kernel (see [4, p. 33]) and consequently the induced integral transformation is variation diminishing, and the same property is endowed to $g(x, t)$.

We separate the discussion into two parts:

(I) $s_i = 0$, $i = 1, 2, \ldots, n-1$, or $s_i = 1$, $i = 1, 2, \ldots, n-1$. In these cases $R(x)$ is strictly monotone. Since $g(x, t)$ is variation diminishing, and $\int g(x, t)\, dx = 1$, it follows that $T(t)$ is monotone and thereby (14) is manifestly correct.

(II) There are at least two distinct s_i's. We will show in this case that $R(x)$ is unimodal with its mode located at an interior point. Suppose $s_0 = \cdots = s_j = 0$, $s_{n-k} = \cdots = s_n = 1$, where $0 \leqslant j \leqslant k + j \leqslant n - 1$.

Then $R(x)$ is continuous and composed of polynomial segments whose explicit form is

$$R(x) = a_i x^{n-1-i}(1-x)^i, \quad \text{on} \quad s_i \leqslant x \leqslant s_{i+1}, \quad j \leqslant i \leqslant n-1-k, \quad a_i > 0.$$

Consider the derivative of $R(x)$ on the interval (s_i, s_{i+1}). We have

$$\begin{aligned} R'(x) &= a_i x^{n-i-2}(1-x)^{i-1}[n-i-1)(1-x) - ix] \\ &= (n-1)\, a_i x^{n-i-2}(1-x)^{i-1}[1-x-i/(n-1)]. \end{aligned}$$

It is clear that the sign of this expression depends only on i such that the function $R(x)$ is increasing on the range $x < 1 - i/(n-1)$ and decreasing afterwards, verifying the unimodality property.

Observe on the basis of (4), that

$$T(t) - \tfrac{1}{2}c = \int_0^1 [R(x) - c]\, g(x, t)\, dx, \qquad 0 \leqslant t \leqslant 1, \tag{15}$$

for all real c. Since the kernel $g(x, t)$ is variation diminishing, we infer that the number of sign changes of $T(t) - \frac{1}{2}c$ as t traverses $[0, 1]$ does not exceed the number of sign changes of $R(x) - c$ (x varying in $[0, 1]$). Moreover, if the number of sign changes of $T(t) - \frac{1}{2}c$ and $R(x) - c$ agree then these functions exhibit the same arrangement of signs (see [4, p. 21]).

Because $R(x)$ is unimodal, we have

$$S^-[R(x) - c] \leqslant 2, \qquad \text{for all real } c,$$

where $S^-(f)$ denotes the number of sign changes of f. Furthermore, where $S^-[R(x) - c] = 2$, then the sequence of signs is $\{(-, +, -)\}$. The same properties are inherited by the functions $T_c(t) = T(t) - \frac{1}{2}c$ for each c. It follows that the minimum of $T(t)$ is achieved at an end point. The proof of the lemma is complete.

We now return to the theorem.

Recalling that $g(x, 0) = 1 - x$ and $g(x, 1) = x$, we can inductively replace each interior s_i by one of the end points, and calculate that

$$\begin{aligned} \min_{0 \leqslant s_i \leqslant 1} G(s_1, \ldots, s_n) &= \min_{0 \leqslant k \leqslant n} \int_0^1 [g(x, 0)]^k [g(x, 1)]^{n-k}\, dx \\ &= \min_{0 \leqslant k \leqslant n} \int_0^1 x^k (1-x)^{n-k}\, dx \\ &= \min_{0 \leqslant k \leqslant n} \left[\frac{1}{(n+1)\binom{n}{k}} \right] = \frac{([n/2]!)^2}{(n+1)!} \end{aligned}$$

and the minimum is taken only for $k = [n/2]$ or $n - k = [n/2]$. Q.E.D.

Remark 1. The following finding of Bellman [2] emanates as a corollary of Theorem 2.

THEOREM A. *Let u, v be nonnegative concave functions on* $[0, 1]$ (*designate this class by $\mathscr{C}$*) *satisfying the constraints*

$$\begin{aligned} &\int_0^1 u^2(x)\,dx = \int_0^1 v^2(x)\,dx = 1, \\ &u(0) = v(0) = u(1) = v(1) = 0. \end{aligned} \tag{16}$$

Then

$$\int_0^1 u(x)v(x)\,dx \geqslant \tfrac{1}{2}. \tag{17}$$

Proof. We employ Favard's inequality (consult the proof of Proposition 1) under the identification $f(x) = u(x)$, $\psi(x) = x^2$ to obtain

$$\frac{4}{3}\left(\int_0^1 u(x)\,dx\right)^2 = \frac{(2\bar{f})^2}{3} = \frac{1}{2\bar{f}}\int_0^{2\bar{f}} y^2\,dy \geqslant \int_0^1 u^2(x)\,dx.$$

Hence, the minimum of (17) evaluated with respect to the set of functions belonging to $\mathscr{C}$ satisfying (16) is not less than the minimum of $\int_0^1 u_1 u_2\,dx$ with respect to the functions of $\mathscr{C}$ obeying the convex constraints

$$\begin{aligned} &\int_0^1 u_i\,dx \geqslant \frac{\sqrt{3}}{2}, \\ &u_i(0) = u_i(1) = 0. \end{aligned} \tag{18}$$

But the minimum of $\int u_1 u_2$ over (18) is necessarily attained for functions fulfilling

$$\int_0^1 u_i(x)\,dx = \frac{\sqrt{3}}{2}, \qquad i = 1, 2. \tag{18a}$$

The latter problem is clearly recognized as a special case of Theorem 2, ($n = 2$).

By virtue of (10), adjusting for the altered normalization constant we find

$$\int_0^1 u_1(x)u_2(x)\,dx \geqslant \frac{1}{3!}\cdot\sqrt{3}\cdot\sqrt{3} = \frac{1}{2},$$

and Theorem A is proved.

Notice that the boundary conditions in (16) are superfluous.

The same reasoning proves:

Consider the class of nonnegative concave functions $u_\nu(x)$ satisfying

$$\int_0^1 [u_\nu(x)]^2\, dx = 1, \qquad \nu = 1,..., n.$$

Then

$$\int_0^1 \left(\prod_{\nu=1}^{n} u_\nu(x) \right) dx \geqslant (\sqrt{3})^n \cdot \frac{([n/2)!)^2}{(n+1)!}. \tag{19}$$

Remark 2. If concavity is replaced by convexity then by similar but far more elementary means we find Theorem B.

THEOREM B. *Let f_ν, $\nu = 1,..., n$, be nonnegative convex functions on* $[0, 1]$, *vanishing at* 0, *and satisfying* (4). *Then*

$$\int_0^1 \left(\prod_{\nu=1}^{n} f_\nu(x) \right) dx \geqslant \frac{1}{n+1} \tag{20}$$

with equality achieved for $f_\nu(x) = x$, $\nu = 1,..., n$.

The extreme points for the collection of all convex nonnegative functions vanishing at 0 normalized as in (4) are

$$h_t(x) = \begin{cases} 0, & 0 \leqslant x \leqslant t, \\ (x-t)/(1-t)^2, & t \leqslant x \leqslant 1, \end{cases} \qquad h_0(x) = x,$$

and the result follows quickly as before. Theorem B is due to Anderson [1].

4. GENERALIZED SECOND ORDER CONCAVITY

In this section we generalize Theorem 2 to functions that are nonnegative and concave with respect to a second order extended complete Tchebycheff system (E.C.T.).

More specifically, consider the differential operator

$$Lf = D_2 D_1 f,$$

where

$$D_i u = (d/dx)(1/w_i)\, u(x), \qquad i = 1, 2$$

and $w_1(x)$, $w_2(x)$ are twice continuously differentiable (for $Lf = f''$ then $w_1(x) \equiv w_2(x) \equiv 1$) strictly positive functions defined on $[0, 1]$.

The functions

$$u_1(x) = w_1(x), \qquad u_2(x) = w_1(x) \int_0^x w_2(\xi)\, d\xi \tag{21}$$

constitute an ECT-system on [0, 1]. The role of the functions x and $1 - x$ in the present context will be played by

$$\begin{aligned} v_1(x) &= w_1(x) \int_0^x w_2(\xi)\, d\xi, \\ v_2(x) &= w_1(x) \int_x^1 w_2(\xi)\, d\xi. \end{aligned} \tag{22}$$

Let $\mathscr{C}^-(u_1, u_2)$ denote the set of functions f which are concave with respect to (u_1, u_2), meaning that f satisfies

$$\begin{vmatrix} u_1(x_1) & u_1(x_2) & u_1(x_3) \\ u_2(x_1) & u_2(x_2) & u_2(x_3) \\ f(x_1) & f(x_2) & f(x_3) \end{vmatrix} \leqslant 0 \qquad \begin{array}{l} \text{for all choices,} \\ 0 \leqslant x_1 < x_2 < x_3 \leqslant 1. \end{array}$$

This agrees with the concept of L-concavity referred to in the introduction.

THEOREM 3. *Let f_ν, $\nu = 1,..., n$, be nonnegative functions of $\mathscr{C}^-(u_1, u_2)$, obeying the normalization*

$$\int_0^1 f_\nu(x)\, dx = 1, \qquad \textit{for all } \nu. \tag{23}$$

This convex set is designated as $\mathscr{D}(u_1, u_2)$. Then

$$\int_0^1 \left(\prod_{i=1}^n f_i(x) \right) \frac{dx}{w_1^{n-1}(x)} \geqslant \min_{0 \leqslant k \leqslant n} \int_0^1 [v_1(x)]^k [v_2(x)]^{n-k} \frac{dx}{w_1^{n-1}(x)}. \tag{24}$$

Proof. Consider the boundary value problem

$$Lf = h \qquad (h \text{ square integrable}),$$
$$f(0) = f(1) = 0,$$

and let $\tilde{g}(x, t)$ be the corresponding Green's function. Its explicit expression has the form

$$\tilde{g}(x, t) = \begin{cases} a(t)\, v_1(x)\, v_2(t), & 0 \leqslant x \leqslant t, \\ a(t)\, v_2(x)\, v_1(t), & t \leqslant x \leqslant 1, \end{cases} \qquad 0 < t < 1,$$

where $a(t)$ is the reciprocal of the Wronskian of v_1 and v_2.

In place of $\tilde{g}(x, t)$, analogous to (11), we consider

$$g^*(x, t) = (c(t)/a(t))\, \tilde{g}(x, t) \qquad 0 \leqslant t \leqslant 1$$

where

$$c(t) = 1\Big/\Big(v_2(t) \int_0^t v_1(\xi)\, d\xi + v_1(t) \int_t^1 v_2(\xi)\, d\xi\Big).$$

Observe that

$$\lim_{t \downarrow 0} g^*(x, t) = v_2(x) \qquad \text{and} \qquad \lim_{t \uparrow 1} g^*(x, t) = v_1(x). \tag{25}$$

The elements of the collection $\{ g_t(x) = g^*(x, t), 0 \leqslant t \leqslant 1\}$ fulfill the normalization condition (23) and constitute the extreme points of $\mathscr{D}(u_1, u_2)$ (see [6]). Paralleling the proof of Theorem 2 we will prove that

$$\begin{aligned} \min_{f_i \in \mathscr{D}} \int \prod_{i=1}^n f_i(x) \frac{dx}{[w_1(x)]^{n-1}} &= \min_{0 \leqslant s_i \leqslant 1} \int \left[\prod_{i=1}^n g_{s_i}(x) \right] \frac{1}{[w_1(x)]^{n-1}}\, dx \\ &= \min_{0 \leqslant s_i \leqslant 1} G(s_1, ..., s_n). \end{aligned} \tag{26}$$

Fix now $0 \leqslant s_1 \leqslant \cdots \leqslant s_{n-1} \leqslant 1$ and let $s_n = t$ vary. The right integral in (2.6) reduces to the function

$$T(t) = \int_0^1 g^*(x, t)\, R(x)\, dx, \tag{27}$$

where $R(x) = \prod_{i=1}^{n-1} [g^*(x, s_i)/w_1(x)]$.

We prove now the counterpart of Lemma 1.

LEMMA 2. *The function $T(t)$ defined in* (27) *satisfies*

$$\min_{0 \leqslant t \leqslant 1} T(t) = \min[T(0), T(1)].$$

Proof. Observe that $g^*(x, t)$ is Totally Positive (see [4, p. 33]) and consequently the integral transformation (27) is variation diminishing.

Case I. All the s_i's are equal to one end point. Then

$$R(x) = \left[\int_0^x w_1(t)\, dt\right]^{n-1} = \left[\frac{v_1(x)}{w_1(x)}\right]^{n-1} \qquad (s_i = 0,\ 1 \leqslant i \leqslant n-1),$$

or

$$R(x) = \left[\int_x^1 w_1(t)\,dt\right]^{n-1} = \left[\frac{v_2(x)}{w_1(x)}\right]^{n-1}, \qquad \text{for} \quad (s_i = 1,\ 1 \leqslant i \leqslant n-1).$$

In either case, $R(x)$ is monotone. Since $g^*(x, t)$ is variation diminishing and normalized such that $\int g^*(x, t)\,dx \equiv 1$ it follows that $T(t)$ is monotone in the same direction, and the assertion of the lemma is validated in this case.

Case II. There are at least two distinct s_i's.

We now claim that $R(x)$ is unimodal with its maximum located at an interior point. The function $R(x)$ is of the form

$$R(x) = \frac{a_i[v_1(x)]^{n-i-1}[v_2(x)]^i}{w_1^{n-1}(x)}, \qquad \text{on} \qquad s_i \leqslant x \leqslant s_{i+1}, \qquad 0 < a_i .$$

On the segment (s_i, s_{i+1}), we have

$$\begin{aligned} R'(x) &= a_i \left[\frac{v_1(x)}{w_1(x)}\right]^{n-i-2}\left[\frac{v_2(x)}{w_1(x)}\right]^{i-1} \\ &\quad \times \left\{(n-i-1)\frac{v_2(x)}{w_1(x)}\, w_2(x) - \frac{i w_2(x)}{w_1(x)}\, v_1(x)\right\} \\ &= a_i(n-1)\, w_2(x) \left[\frac{v_1(x)}{w_1(x)}\right]^{n-i-2}\left[\frac{v_2(x)}{w_1(x)}\right]^{i-1} \\ &\quad \times \left\{\int_x^1 w_2(t)\,dt - \frac{i}{n-1}\int_0^1 w_2(t)\,dt\right\}. \end{aligned}$$

Thus, $R(x)$ is increasing over the range where x satisfies

$$\int_x^1 w_2(t)\,dt \Big/ \int_0^1 w_2(t)\,dt > i/n - 1$$

and decreasing subsequently verifying our claim that $R(x)$ is unimodal.

Note, in view of (23) the identity

$$T(t) - c = \int_0^1 [R(x) - c]\, g^*(x, t)\,dx, \qquad 0 \leqslant t \leqslant 1, \tag{28}$$

for all real c.

By virtue of the variation diminishing property endowed to the kernel $g^*(x, t)$, and paraphrasing the analysis of Theorem 2, we readily deduce

$$\min_{0\leqslant s_i\leqslant 1} G^*(s_1, \ldots, s_n) = \min_{0\leqslant k\leqslant n} \int_0^1 [g^*(x, 0)]^k [g^*(x, 1)]^{n-k} [dx/w_1^{n-1}(x)].$$

Q.E.D.

5. Some Generalizations to Higher Order Differential Operators

We presently generalize the inequality (24) to certain classes of functions satisfying higher order differential inequalities. This is accomplished only for the case of products of two functions.

Consider the sequence of first order differential operators

$$D_i u = \frac{d}{dx}\frac{u}{w_i}, \qquad D_i^* u = \frac{1}{w_i(x)}\frac{du(x)}{dx}, \qquad i = 0, 1, \ldots, k-1$$

where $w_i(x) > 0$, $w_i \in \mathscr{C}^{(k)}$, $i = 0, 1, \ldots, k-1$.

Consider the $2k$-order formally self-adjoint differential operator

$$Mu(x) = (-1)^k D_0{}^* \cdots D_{k-1}^* D_{k-1} \cdots D_0 u(x) = (-1)^k L_{k-1}^* L_{k-1} \tag{29}$$

and associated boundary conditions

$$\begin{aligned}
&(D_1{}^* \cdots D_{k-1}^* D_{k-1} \cdots D_0 u)(y_i) + (-1)^k c_{i,1} u(y_i) = 0,\\
&(D_2{}^* \cdots D_{k-1}^* D_{k-1} \cdots D_0 u)(y_i) + (-1)^{k+1} c_{i,2} D_0 u(y_i) = 0,\\
&\qquad\vdots\\
&(D_k{}^* D_{k-1} \cdots D_0 u)(y_i) + (-1)^{2k-2} c_{i,k-1} (D_{k-3} \cdots D_0 u)(y_i) = 0,\\
&(D_{k-1} \cdots D_0 u)(y_i) + (-1)^{2k-1} c_{i,k} (D_{k-2} \cdots D_0 u)(y_i) = 0,\\
&\qquad\qquad i = 1, 2,
\end{aligned} \tag{30}$$

where $y_1 = 0$, $y_2 = 1$, $0 \leqslant c_{i,j} \leqslant \infty$, $c_{1,j} + c_{2,j} > 0$, $1 \leqslant j \leqslant k$.

The interpretation of $c_{i,j} = \infty$ is the conventional one, namely, the other term vanishes. Under these stipulations M is a self-adjoint operator, and in [4, p. 545], the following theorem is proved.

Theorem C. *Let dF denote a measure of bounded variation on* [0, 1]. *Then the equation*

$$Mu = dF \tag{31}$$

has a unique solution u satisfying (30). *This solution* $u(x)$ *admits the representation*

$$u(x) = \int_0^1 g(x, s)\, dF(s), \tag{32}$$

where $g(x, s)$ *is the Green's function for the differential operator* M *with boundary conditions* (30). *Moreover* $g(x, s)$ *is a totally positive kernel.*

The interpretation of (31) is standard, viz.,

$$(-1)^k D_1^* \cdots D_{k-1}^* D_{k-1} \cdots D_0 u = \tilde{F},$$

where $\tilde{F}(t) = \int_0^t w_0(x)\, dF(x)$.

Define now

$$\gamma(t) = \int_0^1 g(x, t)\, dx,$$
$$R_1(x) = \lim_{t \to 0} \frac{g(x, t)}{\gamma(t)}, \qquad R_2(x) = \lim_{t \to 1} \frac{g(x, t)}{\gamma(t)}, \qquad 0 \leqslant x \leqslant 1 \tag{33}$$

and the convergence is uniform.

We can now state Theorem 4.

THEOREM 4. *Let* u_ν, $\nu = 1, 2$ *be functions of class* C^{2k} *satisfying the differential inequalities*

$$Mu_\nu \leqslant 0$$

and obeying the boundary conditions (30). *If*

$$\int_0^1 u_\nu\, dx = 1, \qquad \nu = 1, 2, \tag{34}$$

then

$$F(u_1, u_2) = \int_0^1 u_1 u_2\, dx \geqslant \min \left[\int_0^1 R_1 R_2\, dx, \int_0^1 R_1^2\, dx, \int_0^1 R_2^2\, dx \right]. \tag{35}$$

Remark. The continuity conditions on u_ν can be relaxed to read: Let $u_\nu \in C^{2k-2}$ be such that Mu_ν are finite measures on [0, 1].

Proof. In view of the representation theorem (32), and the multilinear character of the functional $F(u_1, u_2) = \int u_1 u_2$ the minimum of $F(u_1, u_2)$ is attained where u_1 and u_2 are multiples of the corresponding Green's functions.

Denoting

$$\frac{g(x, t)}{\gamma(t)} = \bar{g}(x, t), \quad 0 < t < 1, \qquad \bar{g}(x, 0) = R_1(x), \qquad \bar{g}(x, 1) = R_2(x).$$

We conclude on the basis of (33), that the minimum of $F(u_1, u_2)$ reduces to

$$\min_{0 \leqslant s_1, s_2 \leqslant 1} G(s_1, s_2),$$

where

$$G(s_1, s_2) = \int_0^1 \bar{g}(x, s_1)\, \bar{g}(x, s_2)\, dx.$$

Observe that $\bar{g}(x, s)$ is unimodal as a function of x for each $0 < s < 1$.

Indeed, consider

$$\bar{g}_n(x, s) \equiv \int_0^1 \bar{g}(x, t)\, K_n(t, s)\, dt \quad \text{converging to} \quad \bar{g}(x, s),$$

where $[K_n(t, s)]_1^\infty$ is a sequence of "approximating kernels" (an approximate identity) peaking at s and $K_n(t, s)$ can be chosen unimodal, as a function of t for each s.

Notice now that, in view of (33),

$$\bar{g}_n(x, s) - c = \int_0^1 \bar{g}(x, t)[K_n(t, s) - c]\, dt.$$

Since $\bar{g}(x, t)$ induces a variation diminishing integral transformation, we infer

$$S_x^-(\bar{g}_n(x, s) - c) \leqslant S_t^-[K_n(t, s) - c], \qquad \text{for all real } c.$$

In particular, $\bar{g}_n(x, s) - c$ exhibits at most two sign changes, and if two, in the arrangement $-$, $+$, $-$. It follows that $\bar{g}_n(x, s)$ is unimodal in x and consequently the limit $\bar{g}(x, s)$ is likewise unimodal.

Hence, fixing s_1, and paraphrasing the previous reasoning, we find that

$$G(s_1, t) = \int_0^1 \bar{g}(x, s_1)\, \bar{g}(x, t)\, dx,$$

is unimodal as a function of t and accordingly attains its minimum at an end point $t = 0$ or $t = 1$. Finally, we deduce

$$G(s_1, s_2) \geqslant \min[G(0, 1), G(1, 1), G(0, 0)].$$

Q.E.D.

We close this section with some concrete examples of Theorem 4 and a number of variations.

THEOREM 5. *Let $f_i \in C^4[0, 1]$, $i = 1, 2$ satisfy,*

$$f_i^{(4)}(x) \geqslant 0,$$
$$f_i(0) = f_i(1) = f_i''(0) = f_i''(1) = 0, \qquad i = 1, 2, \tag{36}$$
$$\int_0^1 f_i \, dx = 1.$$

Then

$$\int_0^1 f_1 f_2 \, dx \geqslant \frac{124}{105} \tag{37}$$

and the inequality is sharp.

Proof. Straightforward computations produce

$$g(x, t) = \begin{cases} \dfrac{t(1-t)(2-t)}{6} x + \dfrac{t-1}{6} x^3, & 0 \leqslant x \leqslant t, \\ \dfrac{t(1-t)(1+t)}{6}(1-x) + \dfrac{t}{6}(x-1)^3, & t \leqslant x \leqslant 1, \end{cases}$$

$$\gamma(t) = \int_0^1 g(x, t)\, dx = \frac{t(1-t)}{24}[1 + t(1-t)],$$

$$R_1(x) = \lim_{t \to 0} \frac{g(x, t)}{\gamma(t)} = 4x(1-x)(2-x),$$

$$R_2(x) = \lim_{t \to 1} \frac{g(x, t)}{\gamma(t)} = 4x(1-x)(1+x) = 4x(1-x^2).$$

Note that $R_2(x) = R_1(1-x)$. Hence, we are left with the task of computing

$$G(1, 1) = G(0, 0) = 16 \int [x(1-x^2)]^2 \, dx = \frac{128}{105}$$

and

$$G(1, 0) = 16 \int x^2(1-x)^2(1+x)(2-x)\, dx = \frac{124}{105}.$$

Comparing these values, (36) is validated.

THEOREM 6. *Let* $f_i \in C^4[0, 1]$, $i = 1, 2$, *satisfy*

$$\begin{gathered} f_i^{(4)}(0) \geqslant 0, \\ f_i(0) = f_i(1) = f_i'(0) = f_i'(1) = 0, \qquad i = 1, 2, \\ \int_0^1 f_i \, dx = 1. \end{gathered} \tag{38}$$

Then

$$\int_0^1 f_1 f_2 \, dx \geqslant \frac{36}{35}. \tag{39}$$

and the inequality is sharp.

Proof. Here we have

$$g(x, t) = \begin{cases} \dfrac{x^2(1-t)^2}{6} [3t - x(2t+1)], & 0 \leqslant x \leqslant t, \\ \dfrac{(x-1)^2 t^2}{6} [(3-2t)x - t], & t \leqslant x \leqslant 1, \end{cases}$$

so that

$$\gamma(t) = \int_0^1 g(x, t) \, dx = \frac{t^2(1-t)^2}{24},$$

$$R_1(x) = 12x(1-x)^2, \qquad R_2(x) = 12x^2(1-x).$$

Again $R_1(x) = R_2(1-x)$; so that we have only to compare

$$144 \int x^2(1-x)^4 \, dx = \frac{48}{35}$$

and

$$144 \int x^3(1-x)^3 \, dx = \frac{36}{35}.$$

and the required result manifestly follows.

The last two theorems can be extended embracing a wider class of functions preserving the same inequalities.

To wit, consider the class of functions

$$\begin{aligned} \mathscr{E} = \Big\{ & f; f^{(4)} \in C(0, 1), f(0) = f(1) = 0, f^{(4)}(x) \geqslant 0, \\ & f'(0) \geqslant 0 \geqslant f'(1), \int^1 f \, dx = 1 \Big\}, \end{aligned}$$

which strictly contains the functions delineated by the conditions in (38).

We have

THEOREM 7. *Let f_1 and f_2 be functions of $\mathscr{E}$. Then* (39) *holds.*

Proof. Let $\tilde{f} = f - P$ where P is a third degree polynomial, determined such that $\tilde{f}$ satisfies

$$\tilde{f}(0) = \tilde{f}(1) = \tilde{f}'(0) = \tilde{f}'(1) = 0.$$

Clearly, $\tilde{f}^{(4)} = f^{(4)} \geqslant 0$. It follows that $\tilde{f} \geqslant 0$, by virtue of the representation (32) since the appropriate Green's function is positive.

Since $P(0) = f(0) = P(1) = f(1) = 0$, a direct calculation reveals that

$$P_3 = f'(0)\, x(1-x)^2 + (-f'(1))\, x^2(1-x).$$

Since $f'(0) > 0, f'(1) < 0$ we secure

$$f = \alpha[12x(1-x)^2] + \beta[12x^2(1-x)] + \gamma \cdot \tilde{f}/\gamma$$

with $\alpha > 0, \beta > 0, \gamma = \int_0^1 \tilde{f}\, dx > 0$.

Note that $\tilde{f}/\gamma$ belongs to the set singled out in Theorem 6, and the functions in the square brackets coincide with $R_1(x)$ and $R_2(x)$, respectively. Since $\alpha + \beta + \gamma = 1$, we have a convex linear combination of the three functions, and the minimum is the same as in Theorem 6.

Similar considerations lead us to the following class, containing the set defined by (36)

$$\mathscr{C}_1 = \Big\{ f; f^{(4)} \in C(0,1), f^{(4)} \geqslant 0, f(0) = f(1) = 0,$$
$$f''(0) \leqslant 0, f''(1) \leqslant 0, \int_0^1 f\, dx = 1 \Big\}.$$

We have here Theorem 8.

THEOREM 8. *Let f_1, f_2 be functions of $\mathscr{C}_1$. Then* (37) *holds.*

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