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The Newton method for solving the Theodorsen integral equation

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Abstract: The Newton method for the solution of the Theodorsen integral equation in conformal mapping is studied. One step of this method consists of solving a linear integral equation, the solution of which is given explicitly as the result of a Riemann–Hilbert problem. Quadratic convergence of the Newton method is established under certain assumptions. Whereas in other methods a so-called ϵ-condition with $\epsilon < 1$ is required to hold, our method converges also for $\epsilon \geqslant 1$. We will also present a numerical implementation in which the result of one step of the Newton method is approximated by a vector in $\mathbb{R}^{2N}$ which can be computed with $2N \log N + O(N)$ multiplications. In comparison, one step of the Newton method for the discrete Theodorsen equation requires $O(N^3)$ multiplications.

Keywords: Conformal mapping, Theodorsen integral equation, Newton method, Riemann–Hilbert problem.

1. Introduction

Let f be the conformal mapping of the unit disc $\mathbb{D}$ onto a domain bounded by a Jordan curve Γ which is starlike with respect to 0. Let $f(0) = 0$, $f'(0) > 0$, and let Γ be given in polar coordinates by $w = \rho(\Theta)e^{i\Theta}$. f can be extended continuously to $\overline{\mathbb{D}}$. $f(e^{i\varphi}) = \rho(\Theta(\varphi))e^{i\Theta(\varphi)}$ then defines the boundary correspondence function Θ: $[0, 2\pi] \to \mathbb{R}$. If Θ is known, then f is known.

Let K be the conjugation operator

$$K[h](\varphi) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta) \cot\frac{\varphi-\theta}{2} d\theta,$$

where the integral is taken in the Cauchy mean value sense. Theodorsen [15] showed in 1931 that Θ is a solution of the Theodorsen integral equation

$$\Theta(\varphi) = \varphi + K[\log \rho(\Theta(\theta))](\varphi) \tag{T}$$

(see Gaier [1, p. 65; 2], Hübner [10]). Gaier [1, p. 66] proved that (T) has exactly one solution which is continuous and strongly monotone. Grunsky improved this result in 1966 by showing that (T) has exactly one continuous solution. His proof together with a substantially new one has been published by von Wolfersdorf [19].

We assume for the entire paper that the 2π-periodic and positive function ρ fulfills a so-called ϵ-condition: ρ is absolutely continuous in $\mathbb{R}$, and $|\rho'/\rho| \leqslant \epsilon$ a.e. For abbrevation we use

$\sigma = \rho'/\rho$. If we put $\Psi(\varphi) := \Theta(\varphi) - \varphi$, equation (T) becomes

$$\Psi(\varphi) = K[\log \rho(\Psi(\theta) + \theta)](\varphi),$$

where the only continuous solution, always denoted by Ψ^*, is also 2π-periodic.

Let

$$\begin{aligned} L_\infty &= \{f : f \text{ is } 2\pi\text{-periodic and bounded}\}, \\ L_2 &= \{f : f \text{ is } 2\pi\text{-periodic and quadratically} \\ &\qquad \text{Lebesgue-integrable in } [0, 2\pi]\}, \\ H_\mu &= \{f : f \text{ is } 2\pi\text{-periodic and } |f(\varphi_1) - f(\varphi_2)| \leqslant L|\varphi_1 - \varphi_2|^\mu \\ &\qquad \text{for all } \varphi_1, \varphi_2 \in \mathbb{R}\}, \quad 0 < \mu \leqslant 1, \\ H &= \{f : f \in H_\mu \text{ for some } \mu \in (0, 1]\}, \\ W &= \{f : f \text{ is } 2\pi\text{-periodic and absolutely continuous, and } f' \in L_2\}. \end{aligned}$$

The following basic results can be found in the book of Gaier [1, p. 63 ff.]:

$$f \in L_2 \Rightarrow Kf \in L_2, \quad \|Kf\|_2 \leqslant \|f\|_2;$$

$$f \in W \Rightarrow Kf \in W, \quad (Kf)' = Kf'.$$

Here we have defined

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} f(x)^2 \mathrm{d}x\right)^{1/2}.$$

As usual we put $\|f\|_\infty = \max_{[0,2\pi]}|f(x)|$ for a function f continuous in $[0, 2\pi]$. It is easy to see that

$$\|f\| := \max(\|f\|_\infty, \|f'\|_2)$$

is a norm in the Sobolev space W, and that $(W, \|\cdot\|)$ is a Banach space. From Wegmann [18, Lemma 1] we learn that

$$\|Kf\|_\infty \leqslant \sqrt{\tfrac{1}{6}\pi}\left(\int_0^\pi (f'(x))^2 \mathrm{d}x\right)^{1/2} = C\|f\|_2 \quad \text{with } C = \tfrac{1}{3}\sqrt{3\pi} < 2,$$

and so $\|Kf\| \leqslant C\|f\|$ for all $f \in W$. It should be mentioned here that Wegmann uses a different L_2-norm.

To obtain the solution of (T) numerically, one usually first discretizes the integral equation and then applies iterative methods. For the convergence of these methods one always needs an ϵ-condition with $\epsilon < 1$, except in the case of certain symmetric curves Γ, as was shown by Gutknecht [7,8]; see also Hübner [10]. Warschawski and others (see Gaier [1, p. 68 ff.]) applied the Jacobi method directly to (T) and showed that $\epsilon < 1$, $\Psi_0 \in W$ implies $\|\Psi_n - \Psi^*\|_2 \to 0$ and $\|\Psi_n - \Psi^*\|_\infty \to 0$ as $n \to \infty$.

Recently Wegmann [17,18] proposed and analyzed a new method for solving our conformal mapping problem. At first sight this method has nothing to do with the Theodorsen integral equation. He proved quadratic convergence of his method if Γ is smooth enough, even if $\epsilon \geqslant 1$. We will show here the same for the Newton method applied to (T). Influenced by Wegmann's papers and by von Wolfersdorf's paper [19], we solve in each Newton step a Riemann–Hilbert problem.

More precisely, with

$$F(\Psi(\varphi)) = \Psi(\varphi) - (K[\log \rho(\Psi(\theta) + \theta)])(\varphi),$$

we have F: $W \to W$, and the Newton method for (T) becomes

$$\Psi_0 \in W,$$
$$F'(\Psi_n)[\Psi_{n+1} - \Psi_n] = -F(\Psi_n), \quad n = 0, 1, 2, \dots .$$

Here $F'(\Psi_n)$ is the F-derivative of F in $(\mathrm{W}, \|\cdot\|)$ at Ψ_n. We will show (under certain assumptions) that the operator $F'(\Psi)$ has an inverse for any $\Psi \in W$, and we will express this inverse operator in closed form with the help of the solution of a Riemann–Hilbert problem. Of course, we then have

$$\Psi_{n+1} = \Psi_n - (F'(\Psi_n))^{-1} F(\Psi_n). \tag{N}$$

It will turn out that the determination of Ψ_{n+1} from Ψ_N will require two applications of the conjungation operator K. This is the same amount of work as for one step of the Wegmann method. If $\sigma' \in H_1$, we will then show that the Newton method for (T) converges quadratically in $(W, \|\cdot\|)$ without any restriction on ϵ in the ϵ-condition. Moreover, we will prove global convergence in $(W, \|\cdot\|_2)$ if $\sigma' \in L_\infty$ and $\epsilon < \frac{1}{3}$. Wegmann has no counterpart to the last result. But his method works also for a non-starlike Γ.

For the numerical implementation of the method we propose discretization of (N) instead of (T). If one approximates Ψ_n by a vector $\in \mathbb{R}^{2N}$, one can compute an approximation of Ψ_{n+1} in the same space using FFTs with a total of $2N \log N + \mathrm{O}(N)$ multiplications. This is again the same amount of work as in one step of the discretized Wegmann method. Our example at the end shows that the Wegmann method and the Newton method for (T) are not identical.

2. The F-derivative of F and its inverse

First we compute the F-derivative of F in $(W, \|\cdot\|)$.

Lemma 1. *If $\sigma' \in H$ and $\Psi \in W$, then the F-derivative of F at Ψ in $(W, \|\cdot\|)$ exists and is given by*

$$(F'(\Psi)\Delta)(\varphi) = \Delta(\varphi) - (K[\sigma(\Psi(\theta) + \theta) \cdot \Delta(\theta)])(\varphi).$$

Proof. For the function defined by $\theta \mapsto \Psi(\theta) + \theta$ we write $\Psi + \mathrm{id}$. $\sigma' \in H$ implies $\sigma' \in L_\infty$. Let M be an upper bound for σ' and let $F'(\Psi)$ be defined by the last line of Lemma 1 for some $\Delta \in W$. Clearly $F'(\Psi)$ is linear (in Δ). It is also bounded in $(W, \|\cdot\|)$, since we have

$$\| F'(\Psi)\Delta \| \leqslant \| \Delta \| + 2 \| (\sigma(\Psi + \mathrm{id})) \cdot \Delta \|$$

and

$$\| \sigma \cdot \Delta \|_\infty \leqslant \epsilon \| \Delta \|_\infty \leqslant \epsilon \| \Delta \|,$$
$$\| (\sigma \cdot \Delta)' \|_2 = \| \sigma' \cdot (\Psi' + 1) \cdot \Delta \|_2 + \| \sigma \cdot \Delta' \|_2 \leqslant M \| \Delta \|_\infty \| \Psi' + 1 \|_2 + \epsilon \| \Delta' \|_2,$$

hence

$$\| F'(\Psi)\Delta \| \leqslant [1 + 2(M \| \Psi' + 1 \|_2 + \epsilon)] \| \Delta \|.$$

To show that $F'(\Psi)$ is the F-derivative, define

$$G := \log \rho(\Psi + \Delta + \mathrm{id}) - \log \rho(\Psi + \mathrm{id}) - (\sigma(\Psi + \mathrm{id})) \cdot \Delta.$$

Then

$$\|G\|_\infty = \|[\sigma(\Psi + \mathrm{id} + \tau\Delta) - \sigma(\Psi + \mathrm{id})] \cdot \Delta\|_\infty \leqslant M\|\Delta\|_\infty^2 \quad \text{with } 0 < \tau < 1,$$

$$\begin{aligned}\|G'\|_2 &= \|\sigma(\Psi + \Delta + \mathrm{id})\cdot(\Psi' + \Delta' + 1) - \sigma(\Psi + \mathrm{id})\cdot(\Psi' + 1)\\ &\qquad - \sigma'(\Psi + \mathrm{id})\cdot(\Psi' + 1)\cdot\Delta - \sigma(\Psi + \mathrm{id})\cdot\Delta'\|_2\\ &\leqslant \|\sigma(\Psi + \Delta + \mathrm{id}) - \sigma(\Psi + \mathrm{id})\|_\infty \|\Delta'\|_2\\ &\qquad + \|\Psi' + 1\|_2 \|\Delta\int_0^1(\sigma'(\Psi + \mathrm{id} + t\Delta) - \sigma'(\Psi + \mathrm{id}))\mathrm{d}t\|_\infty\\ &\leqslant M\|\Delta\|_\infty\|\Delta'\|_2 + \|\Psi' + 1\|_2\|\Delta\|_\infty \cdot L\|\Delta\|_\infty^\mu.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|F(\Psi + \Delta) - F(\Psi) - F'(\Psi)\Delta\| &= \|K[G]\| \leqslant 2\|G\|\\ &\leqslant \text{const}\cdot\|\Delta\|^{1+\mu} \quad \text{if } \|\Delta\| \leqslant 1. \qquad \square\end{aligned}$$

Thus given Ψ_n, to compute Ψ_{n+1} in (N) we have to solve the linear (but singular) integral equation

$$\Delta - K[\sigma(\Psi_n + \mathrm{id})\cdot\Delta] = r \tag{N'}$$

for Δ, where $\Delta = \Psi_{n+1} - \Psi_n$ and $r = -F(\Psi_n)$.

Theorem 1. *If $\sigma \in H$ and $\Psi_n \in H$, then (N′) has at most one continuous and 2π-periodic solution.*

Proof. If Δ_1 and Δ_2 are two such solutions, then $h := \Delta_1 - \Delta_2$ is continuous and 2π-periodic and solves the homogeneous integral equation $h - K[\sigma\cdot h] = 0$. By the Dirichlet principle, there are functions u, v which are harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ with $u(\mathrm{e}^{\mathrm{i}\varphi}) = \sigma(\Psi_n(\varphi) + \varphi)\cdot h(\varphi)$ and $v(\mathrm{e}^{\mathrm{i}\varphi}) = h(\varphi)$. Because of $h = K[\sigma\cdot h]$, the functions u and v are conjugates, and $H := u + \mathrm{i}v$ is continuous in $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$. Moreover, h being a conjugate implies $\int_0^{2\pi}h(\varphi)\mathrm{d}\varphi = 0$, and therefore

$$\operatorname{Im} H(0) = \operatorname{Im}\frac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\frac{H(\mathrm{e}^{\mathrm{i}\varphi})}{\mathrm{e}^{\mathrm{i}\varphi}}\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{2\pi}h(\varphi)\mathrm{d}\varphi = 0.$$

In addition, we have $H(z) = u + \mathrm{i}v = \sigma h + \mathrm{i}h = (\sigma + \mathrm{i})h$, and therefore H is a solution of the following Riemann–Hilbert problem:

Find H, continuous in $\overline{\mathbb{D}}$, analytic in $\mathbb{D}$, with $\operatorname{Re}\mathrm{e}^{\mathrm{i}\beta}H(z) = 0$ on $\partial\mathbb{D}$, $\operatorname{Im} H(0) = 0$, where

$$\mathrm{e}^{\mathrm{i}\beta(\varphi)} = \frac{1 + \mathrm{i}\sigma(\Psi_n(\varphi) + \varphi)}{|1 + \mathrm{i}\sigma(\Psi_n(\varphi) + \varphi)|}, \quad |\beta(\varphi)| < \tfrac{1}{2}\pi.$$

Because $\sigma \in H$ and $\Psi_n \in H$, the functions c and s defined by $c(\mathrm{e}^{\mathrm{i}\varphi}) := \cos\beta(\varphi)$ and $s(\mathrm{e}^{\mathrm{i}\varphi}) := \sin\beta(\varphi)$ are continuous on $\partial\mathbb{D}$ and satisfy a Hölder condition there; moreover $c^2 + s^2 = 1 \neq 0$. All solution of this problem are given in Muschelischwili [13, p. 155 ff.]. The number of solutions depends on the index of the Riemann–Hilbert problem, which is in this case

$$\kappa := \frac{1}{\pi}[\arg(c - \mathrm{i}s)]_L = -\frac{1}{\pi}[\tilde{\beta}]_L = 0.$$

Here $[\tilde{\beta}]_L$ denotes the change of $\tilde{\beta}(\mathrm{e}^{\mathrm{i}\varphi}) := \beta(\varphi)$ if one goes once around the unit circle. Since

$\kappa = 0$, all solutions are of the form

$$H(z) = C_0 e^{-i\alpha/2} e^{\Gamma(z)}, \quad \operatorname{Im} H(0) = 0$$

with an arbitrary real constant C_0,

$$\alpha = \frac{1}{2\pi} \int_0^{2\pi} \arg\left(- \frac{\cos\beta - i\sin\beta}{\cos\beta + i\sin\beta} \right) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} (\pi - 2\beta) d\varphi$$
$$= \pi - \frac{1}{\pi} \int_0^{2\pi} \beta \, d\varphi,$$

and

$$\Gamma(z) = \frac{1}{2\pi} \int_{|t|=1} \frac{\pi - 2\tilde{\beta}}{t - z} dt.$$

Because of $\Gamma(0) = i\alpha$, we have

$$\operatorname{Im} H(0) = 0 \quad \text{iff} \quad C_0 = 0 \quad \text{or} \quad \sin\tfrac{1}{2}\alpha = 0.$$

Now, since $|\beta(\varphi)| < \frac{1}{2}\pi$, $\sin\frac{1}{2}\alpha \neq 0$, and therefore $C_0 = 0$, $H = 0$, $h = 0$ and $\Delta_1 = \Delta_2$.

Now we consider the inhomogeneous integral equation (N′).

Theorem 2. *If r, $\Psi_n \in W$, and $\sigma' \in L_\infty$, then*

$$\Delta = r\cos^2\beta + e^{K\beta} \cos\beta \left(K\left[e^{-K\beta} r \sin\beta \right] + C \right)$$

with

$$C = \int_0^{2\pi} \left\{ r\sin^2\beta - e^{K\beta} \cos\beta \, K\left[e^{-K\beta} r \sin\beta \right] \right\} d\varphi \Big/ \int_0^{2\pi} e^{K\beta} \cos\beta d\varphi$$

is a solution of (N′) *in* W.

Proof. We consider the Riemann–Hilbert problem

$$\operatorname{Re}\left[e^{i\beta} H(z) \right] = r_1 \quad \text{on } \partial\mathbb{D}, \quad \operatorname{Im} H(0) = 0, \tag{H}$$

where $r_1 = r\sin\beta$. According to Gaier [1, p. 62] any solution $H = u + iv$ satisfies

$$v = v(0) = Ku \quad \text{on } \partial\mathbb{D}.$$

(H) implies

$$u\cos\beta - v\sin\beta = r\sin\beta \quad \text{or} \quad u = (v + r)\cdot\sigma \quad \text{and} \quad v(0) = 0.$$

So we have

$$v + r - r = K\left[(v + r)\cdot\sigma \right].$$

This means that $\Delta = v + r$ satisfies (N′).

Since $r \in W$, we have $\Delta \in W$ if $v \in W$. So we have to find a solution of the Riemann–Hilbert problem with $v \in W$. Let

$$H(z) := \left(r_1 e^{-K\beta} + iK\left[r_1 e^{-K\beta} \right] + iC \right) \exp\left[-i(\beta + iK\beta) \right]$$

for some constant $C \in \mathbb{R}$. Since $\Psi_n \in W$ and $\sigma' \in L_\infty$, we find $\beta \in W$ and therefore $K\beta \in W$.

Hence $r_1 e^{-K\beta}$ is again in W and conjugate to $K(r_1 e^{-K\beta})$, so $r_1 e^{-K\beta} + iK(r_1 e^{-K\beta})$ can be extended continuously to $\overline{\mathbb{D}}$ in such a way that the extended function is analytic in $\mathbb{D}$. The same applies to $\beta + iK\beta$. Therefore H is continuous in $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$. Moreover, H satisfies the boundary condition

$$\mathrm{Re}\left[e^{i\beta} H(z)\right] = \mathrm{Re}\left[r_1 + i\, e^{K\beta} K\left[r_1 e^{-K\beta}\right] + iCe^{K\beta}\right] = r_1.$$

Furthermore,

$$\begin{aligned}\mathrm{Im}\ H(0) &= \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} \mathrm{Im}\ H(e^{i\varphi})\,d\varphi \\ &= \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} \left(-r_1 \sin\beta + e^{K\beta}\cos\beta\, K\left[r_1 e^{-K\beta}\right] + Ce^{K\beta}\cos\beta\right)d\varphi = 0\end{aligned}$$

yields the constant C in the theorem. Because of $e^{K\beta} > 0$ and $\cos\beta = 1/\sqrt{1+\sigma^2} > 0$, the integral in the numerator of C is $\neq 0$. Finally, since products and sums of functions in W are again in W,

$$v = \mathrm{Im}\ H(e^{i\varphi}) = -r_1 \sin\beta + e^{K\beta}\cos\beta\, K\left(r_1 e^{-K\beta}\right) + Ce^{K\beta}\cos\beta \in W.$$

Hence $\Delta = v + r \in W$.

We study now the implication of the last two theorems for the operator $F'(\Psi)$ defined by $F'(\Psi)\Delta = \Delta - K(\sigma(\Psi + \mathrm{id})\Delta)$.

Theorem 3. *If $\sigma' \in L_\infty$, $\Psi \in W$, then $F'(\Psi)$ has an inverse which is a bounded linear operator in $(W, \|\cdot\|)$. $(F'(\Psi))^{-1}$ is locally uniformly bounded with respect to Ψ.*

Proof. $\sigma' \in L_\infty$, $\Psi \in W$ implies $\sigma \in H$, $\Psi \in H$. Theorem 1 tells us therefore (choosing $\Psi_n = \Psi$) that

$$F'(\Psi)\Delta = 0$$

has exactly one solution in W, namely $\Delta = 0$. Theorem 2 shows that for any $r \in W$, $F'(\Psi)\Delta = r$ is solvable. That suffices for $F'(\Psi)$ to have an inverse which is a bounded linear operator (see Gohberg and Goldberg [4, p. 221]).

We still have to show that $(F'(\Psi))^{-1}$ is locally uniformly bounded. This means that $(F'(\Psi))^{-1}$ has an operator norm which is bounded by a constant depending on $\Psi_1 \in W$ and $\rho > 0$ but not on Ψ, for all $\Psi \in \{\Psi: \|\Psi_1 - \Psi\| \leqslant \rho,\ \Psi \in W\}$. This can be proved by a somewhat lengthy but straightforward computation, with the result $\|(F'(\Psi))^{-1} r\| \leqslant C_1 \|r\|$ for all $r \in W$. □

Given Ψ_n, we need three applications of the operator K for the computation of the next Newton iteration $\Psi_{n+1} = \Psi_n + \Delta$ with the help of Theorem 2. This number can be reduced to two by the next theorem.

Theorem 4. *If $\sigma' \in L_\infty$, $\Psi_n \in W$, the next Newton iteration Ψ_{n+1} can be calculated by*

$$\Psi_{n+1} = -\tilde{r}\sin\beta\cos\beta + e^{K\beta}\cos\beta\left(K\left[\tilde{r}\cos\beta e^{-K\beta}\right] + \tilde{C}\right),$$

with

$$\tilde{C} = \int_0^{2\pi}\left\{\tilde{r}\sin\beta\cos\beta - e^{K\beta}\cos\beta\, K\left[e^{-K\beta}\tilde{r}\cos\beta\right]\right\}d\varphi \Big/ \int_0^{2\pi} e^{K\beta}\cos\beta\, d\varphi$$

and

$$\tilde{r} = -\sigma(\Psi_n + \mathrm{id}) \cdot \Psi_n + \log \rho(\Psi_n + \mathrm{id}).$$

Proof. From (N′) we get

$$\Psi_{n+1} - \Psi_n - K[\sigma \cdot (\Psi_{n+1} - \Psi_n)] = -\Psi_n + K \log \rho(\Psi_n + \mathrm{id})$$

or

$$\Psi_{n+1} - K[\sigma \cdot \Psi_{n+1} + \tilde{r}] = 0.$$

By the proof of Theorem 2 we know that the Riemann–Hilbert problem (H) is solvable for any $r_1 \in W$. We choose $r_1 = \tilde{r} \cos \beta \in W$. Then (H) implies for $H = u + \mathrm{i}v$ that $u \cos \beta - v \sin \beta = \tilde{r} \cos \beta$, and again by Gaier [1, p. 62] we have

$$v = v(0) = Ku.$$

These two facts and $v(0) = 0$ give

$$v = K[\sigma \cdot v + \tilde{r}].$$

Thus we have $v = \Psi_{n+1}$, and we can take v from the proof of Theorem 2 with $r_1 = \tilde{r} \cos \beta$. □

Usually a direct numerical application of the Newton method as treated so far will not be possible. But the method has numerical implementations. One runs as follows.

(i) Choose $\Psi_0 \in W$, and compute the vector $P_1\Psi_0 = (\Psi_0(\varphi_k))_{k=0}^{2N-1}$, $\varphi_k = k\pi/N$ ($k = 0, 1, \ldots, 2N-1$).

(ii) Compute by interpolation the trigonometric polynomial $P_2\beta$ of degree N (with zero coefficient of the sin $N\varphi$-term) for which

$$(P_2\beta)(\varphi_k) = \arcsin \frac{\sigma(\Psi_0(\varphi_k) + \varphi_k)}{\sqrt{1 + \sigma^2(\Psi_0(\varphi_k) + \varphi_k)}}.$$

(iii) $KP_2\beta$ is then available by a simple and well-known procedure (see Gaier [1, p. 63]).

(iv) Compute the vector $P_1KP_2\beta = ((KP_2\beta)(\varphi_k))_{k=0}^{2N-1}$. The main work involved is trigonometric interpolation and the evaluation of the trigonometric polynomial at equidistant points. Both can be done rapidly with an FFT at the cost of $\frac{1}{2}N \log_2 N$ complex multiplications each, if N is a power of 2, which we assume from now on (see Gaier and Hübner [3], Gutknecht [6] and Hübner [10]).

The application of the FFT in this context was first suggested by Henrici [9] and Ives [11]. Continuing in the way described and using the formulae in Theorem 4, one can compute a vector approximation of Ψ_1 at the cost of $2N \log N + O(N)$ complex multiplications. Of course, here the evaluations of log ρ and σ have not been counted.

3. Convergence of the Newton method

We turn now to (N), i.e. to the exact application of the Newton method to (T). We want to apply a Newton–Kantorovich theorem for a mapping of one Banach space into another, for example, in the form of Gragg and Tapia [5], see also Ortega and Rheinboldt [14, p. 421 and

428]. Here, both Banach spaces are $(W, \|\cdot\|)$. We know F: $W \to W$, and from Lemma 1 we also know that F is F-differentiable in W if $\sigma' \in H$.

As before, Ψ^* is the only continuous solution of (T). We first show that $\sigma \in H$ implies $\Psi^* \in W$. For any $\delta > 0$ we then define $D_\delta = \{\Psi: \|\Psi - \Psi^*\| < \delta, \Psi \in W\}$. The local convergence of the Newton method is proven if we can show

$$\|F'(\Phi) - F'(\Psi)\| \leqslant C_2 \|\Phi - \Psi\| \quad \text{for all } \Phi, \Psi \in D_\delta,$$

and that there is a δ_1 with $0 < \delta_1 \leqslant \delta$ such that for any starting point $\Psi \in D_{\delta_1}$,

$$h := 2C_2 \|F'(\Psi_0)^{-1}\|^2 \|F(\Psi_0)\| \leqslant 1 \quad \text{and}$$

$$\{\Psi: \|\Psi - \Psi_0\| < t^*, \Psi \in W\} \subset D_\delta \quad \text{if } t^* = \frac{2}{h}(1 - \sqrt{1-h}) \|F'(\Psi_0)^{-1} F(\Psi_0)\|.$$

We start with

Lemma 2. *$\sigma \in H$ implies $\Psi^* \in W$.*

Proof. We have $\Psi^*(\varphi) = \Theta(\varphi) - \varphi$, where Θ is defined by $f(e^{i\varphi}) = \rho(\Theta(\varphi))e^{i\Theta(\varphi)}$, f being the normalized conformal mapping from $\mathbb{D}$ onto the interior of Γ. Let $x(\Theta)$ be the angle between the normal to Γ at $f(e^{i\varphi})$ and the radius vector leading from 0 to $f(e^{i\varphi})$. Then $\sigma(\Theta) = \tan x(\Theta)$. Further, let $\theta(s)$ be the angle between the real axis and the tangent to Γ at $f(e^{i\varphi})$, where $s = \int_0^\Theta \sqrt{\rho'(t)^2 + \rho(t)^2}\, dt$ is the arc length. An easy geometric consideration shows that $\sigma \in H$ implies $\theta \in H$. According to Kellog and Warschawski (see Gaier [1, p. 263]), this suffices for the existence of

$$\lim_{\substack{z \to z_0 \\ |z| \leqslant 1}} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0), \quad z_0 \in \overline{\mathbb{D}}.$$

Using the continuity of Θ we find

$$\begin{aligned}
&\frac{\Theta(\varphi + h) - \Theta(\varphi)}{h} \\
&= \frac{\left[\dfrac{f(e^{i(\varphi+h)}) - f(e^{i\varphi})}{e^{i(\varphi+h)} - e^{i\varphi}} \cdot \dfrac{e^{i(\varphi+h)} - e^{i\varphi}}{ih} \cdot i\right]}{\left[\dfrac{\rho(\Theta(\varphi+h)) - \rho(\Theta(\varphi))}{\Theta(\varphi+h) - \Theta(\varphi)} e^{i\Theta(\varphi+h)} + i\rho(\Theta(\varphi)) \dfrac{e^{i\Theta(\varphi+h)} - e^{i\Theta(\varphi)}}{i(\Theta(\varphi+h) - \Theta(\varphi))}\right]} \\
&\to \frac{f'(e^{i\varphi})e^{i\varphi} i e^{-i\Theta(\varphi)}}{\rho'(\Theta(\varphi)) + i\rho(\Theta(\varphi))}
\end{aligned}$$

for $h \to 0$. This means that Θ is differentiable and Θ' is continuous; therefore $\Psi^* \in W$. □

Next we show that F' satisfies a Lipschitz condition.

Lemma 3. If Φ, Ψ, $\Psi^* \in W$, $\sigma' \in H_1$, *then*

$$\|F'(\Phi) - F'(\Psi)\| \leqslant C_2 \|\Phi - \Psi\| \quad \textit{for all } \Phi, \Psi \in D_\delta.$$

C_2 depends only on Ψ^* and δ.

Proof. We have to estimate the operator norm

$$\|F'(\Phi)-F'(\Psi)\| = \sup_{\substack{\Delta\neq 0\\ \Delta\in W}} \frac{\|(F'(\Phi)-F'(\Psi))\Delta\|}{\|\Delta\|}$$
$$= \sup \frac{\|\Delta - K[\sigma(\Phi+\mathrm{id})\cdot\Delta] - \Delta + K[\sigma(\Psi+\mathrm{id})\cdot\Delta]\|}{\|\Delta\|}$$
$$\leqslant \sup \frac{2\|(\sigma(\Phi+\mathrm{id})-\sigma(\Psi+\mathrm{id}))\cdot\Delta\|}{\|\Delta\|}.$$

Now with $|\sigma(\Theta_1)-\sigma(\Theta_2)| \leqslant L_0|\Theta_1-\Theta_2|$, we have

$$\|(\sigma(\Phi+\mathrm{id})-\sigma(\Psi+\mathrm{id}))\cdot\Delta\|_\infty \leqslant L_0\|\Phi-\Psi\|_\infty\|\Delta\|_\infty,$$

and with $|\sigma'(\Theta_1)-\sigma'(\Theta_2)| \leqslant L|\Theta_1-\Theta_2|$, we get

$$\begin{aligned}
&\|\{(\sigma(\Phi+\mathrm{id})-\sigma(\Psi+\mathrm{id}))\cdot\Delta\}'\|_2\\
&\quad= \|\sigma'(\Phi+\mathrm{id})\cdot(\Phi'+1)\cdot\Delta - \sigma'(\Psi+\mathrm{id})\cdot(\Psi'+1)\cdot\Delta\\
&\qquad+(\sigma(\Phi+\mathrm{id})-\sigma(\Psi+\mathrm{id}))\cdot\Delta'\|_2\\
&\quad\leqslant \|\sigma'(\Phi+\mathrm{id})\cdot\Phi' - \sigma'(\Phi+\mathrm{id})\cdot\Psi'\|_2\|\Delta\|_\infty\\
&\qquad+\|\sigma'(\Phi+\mathrm{id})\cdot\Psi' - \sigma'(\Psi+\mathrm{id})\cdot\Psi'\|_2\|\Delta\|_\infty\\
&\qquad+\|\sigma'(\Phi+\mathrm{id})-\sigma'(\Psi+\mathrm{id})\|_2\|\Delta\|_\infty\\
&\qquad+\|\sigma(\Phi+\mathrm{id})-\sigma(\Psi+\mathrm{id})\|_\infty\|\Delta'\|_2\\
&\quad\leqslant M\|\Phi'-\Psi'\|_2\|\Delta\|_\infty + L\|\Phi-\Psi\|_\infty\|\Psi'\|_2\|\Delta\|_\infty\\
&\qquad+L\|\Phi-\Psi\|_\infty\|\Delta\|_\infty + L_0\|\Phi-\Psi\|_\infty\|\Delta'\|_2\\
&\quad\leqslant (M+L\|\Psi\|+L+L_0)\|\Phi-\Psi\|\,\|\Delta\|\\
&\quad\leqslant (M+L(\delta+\|\Psi^*\|+1)+L_0)\|\Phi-\Psi\|\,\|\Delta\|.
\end{aligned}$$

Together these results yield

$$\|F'(\Phi)-F'(\Psi)\| \leqslant C_2\|\Phi-\Psi\|$$

for all Φ, $\Psi\in D_\delta$ with $C_2 = 2[M+L(\delta+\|\Psi^*\|+1)+L_0]$. □

For $\Psi\in D_\delta$ and $\sigma\in H$, we find

$$\begin{aligned}
\|F(\Psi)\| &= \|F(\Psi)-F(\Psi^*)\| = \|\Psi-\Psi^* - K\log\rho(\Psi+\mathrm{id}) + K\log\rho(\Psi^*+\mathrm{id})\|\\
&\leqslant \|\Psi-\Psi^*\| + 2\|\log\rho(\Psi+\mathrm{id})-\log\rho(\Psi^*+\mathrm{id})\|,
\end{aligned}$$

$$\|\log\rho(\Psi+\mathrm{id})-\log\rho(\Psi^*+\mathrm{id})\|_\infty \leqslant \epsilon\|\Psi-\Psi^*\|_\infty,$$

$$\begin{aligned}
&\|(\log\rho(\Psi+\mathrm{id})-\log\rho(\Psi^*+\mathrm{id}))'\|_2\\
&\quad= \|\sigma(\Psi+\mathrm{id})\cdot(\Psi'+1)-\sigma(\Psi^*+\mathrm{id})\cdot(\Psi^{*\prime}+1)\|_2\\
&\quad\leqslant \|\sigma(\Psi+\mathrm{id})\cdot\Psi' - \sigma(\Psi+\mathrm{id})\cdot\Psi^{*\prime}\|_2 + \|\sigma(\Psi+\mathrm{id})\cdot\Psi^{*\prime} - \sigma(\Psi^*+\mathrm{id})\cdot\Psi^{*\prime}\|_2\\
&\qquad+\|\sigma(\Psi+\mathrm{id})-\sigma(\Psi^*+\mathrm{id})\|_2\\
&\quad\leqslant \epsilon\|\Psi'-\Psi^{*\prime}\|_2 + L_0\|\Psi-\Psi^*\|_\infty\|\Psi^{*\prime}\|_2 + L_0\|\Psi-\Psi^*\|_\infty\\
&\quad\leqslant (\epsilon+L_0(\|\Psi^{*\prime}\|_2+1))\|\Psi-\Psi^*\|.
\end{aligned}$$

Altogether we have

$$\| F(\Psi) \| \leqslant C_3 \| \Psi - \Psi^* \| \quad \text{with} \quad C_3 = 1 + 2\epsilon + 2L_0(\| \Psi^{*\prime} \|_2 + 1).$$

By Theorem 3 we know $\| F'(\Psi)^{-1} \| \leqslant C_1$ for all $\Psi \in D_\delta$ if $\sigma' \in L_\infty$. If we now choose $\Psi_0 \in W$ with $\| \Psi^* - \Psi_0 \| \leqslant \delta_1$, where δ_1 satisfies

$$h = 2C_1^2 C_2 C_3 \delta_1 \leqslant 1$$

and

$$\delta_1 + t^* \leqslant \delta_1 + \frac{2}{1 + \sqrt{1-h}} C_1 C_3 \delta_1 \leqslant (1 + 2C_1C_3)\delta_1 \leqslant \delta,$$

all conditions required for the Newton–Kantorovich theorem are fulfilled. So we have proven

Theorem 5. *If $\sigma' \in H_1$, then the Newton method for the Theodorsen integral equation converges locally and quadratically in $(W, \| \cdot \|)$.*

Remarks. Theorem 5 also gives the uniform convergence of Ψ_n to Ψ^* locally.

There are variants of the Newton–Kantorovich theorem where for example the assumption $\sigma' \in H_\mu$ with $0 < \mu < 1$ suffices and where the rate of convergence is less then quadratic (see Keller [12]). Also instead of $(W, \| \cdot \|)$, other spaces could be taken. We do not elaborate all these possibilities.

Instead we will show that sometimes global convergence of the Newton method in $(W, \| \cdot \|_2)$ takes place.

Theorem 6. *If $|\sigma| < \frac{1}{3}$ and $\sigma' \in L_\infty$, then the Newton method for the Theodorsen integral equation converges globally in $(W, \| \cdot \|_2)$.*

Proof. Let $\Psi_0 \in W$, and let Ψ_{n+1} be defined by (N'), i.e.

$$\Psi_{n+1} - \Psi_n - K[\sigma(\Psi_n + \mathrm{id}) \cdot (\Psi_{n+1} - \Psi_n)] = -\Psi_n + K \log \rho(\Psi_n + \mathrm{id}).$$

According to Theorem 2, $\Psi_{n+1} \in W$. We subtract

$$\Psi^* = K \log \rho(\Psi^* + \mathrm{id})$$

from the above equation and obtain

$$\Psi_{n+1} - \Psi^* - K[\sigma(\Psi_n + \mathrm{id}) \cdot (\Psi_{n+1} - \Psi^*)] - K[\sigma(\Psi_n + \mathrm{id}) \cdot (\Psi^* - \Psi_n)]$$
$$= K[\log \rho(\Psi_n + \mathrm{id}) - \log \rho(\Psi^* + \mathrm{id})],$$
$$\| \Psi_{n+1} - \Psi^* \|_2 \leqslant \epsilon \| \Psi_{n+1} - \Psi^* \|_2 + \epsilon \| \Psi_n - \Psi^* \|_2 + \epsilon \| \Psi_n - \Psi^* \|_2$$
$$\| \Psi_{n+1} - \Psi^* \|_2 \leqslant \frac{2\epsilon}{1-\epsilon} \| \Psi_n - \Psi^* \|_2 \leqslant \left(\frac{2\epsilon}{1-\epsilon} \right)^n \| \Psi_0 - \Psi^* \|_2 \to 0$$

if $\frac{2\epsilon}{1-\epsilon} < 1$ or $\epsilon < \frac{1}{3}$. □

We finish with the question of whether for boundary curves given in polar coordinates, the method of Wegmann and the Newton method for (T) are identical. The answer is no. As an example we take $\dot{\rho}(\Theta) \equiv 1$ and $\Psi_0 = \sin \varphi$. This gives $F(\Psi_0(\varphi)) = \Psi_0 - K[1] = \sin \varphi$. In Theorem 4 we have $r = 0$ and therefore $\Psi_1 = 0$. This is the exact solution of (T). With some further computations one can show that in this example the Wegmann method does not give the exact solution after one step.

However, after studying this manuscript, Gutknecht has noticed a close connection between the Wegmann method and our method. Let Γ be given in polar coordinates. If one applies the idea of the Wegmann method to the auxiliary function $h(z) = \log f(z)/z$ (instead of the mapping function $f(z)$ as in Wegmann's work), one gets our method as presented in Theorem 4. It will require some numerical experiments to decide whether our method has advantages over the original Wegmann method. In addition, it is worthwhile to check whether or not discretizing the formula in Theorem 2 gives a more stable method than doing this with the formula in Theorem 4.

Gutknecht has also pointed out that Vertgejm [16] has previously applied the Newton method even to a generalized Theodorsen integral equation. However, Vertgejm used the modified Newton method

$$\Psi_{n+1} = \Psi_n - (F'(\Psi_0))^{-1} F(\Psi_n)$$

and proved linear convergence in H_μ, whereas we have quadratic convergence in W.

References

[1] D. Gaier, Konstruktive Methoden der konformen Abbildung (Springer, Berlin, Göttingen, Heidelberg, 1964).

[2] D. Gaier, Numerical aspects in conformal mapping, in: *Computational Aspects of Complex Analysis* (Reidel, Dordrecht, Boston, Lancaster 1983) 51–78.

[3] D. Gaier and O. Hübner, Schnelle Auswertung von Ax bei Matrizen A zyklischer Bauart, Toeplitz- und Hankel-Matrizen, *Mitt. Math. Sem. Gießen* **121** (1976) 27–38.

[4] I. Gohberg and S. Goldberg, *Basic Operator Theory* (Birkhäuser, Boston, Basel, Stuttgart, 1981).

[5] W.B. Gragg and R.A. Tapia, Optimal error bounds for the Newton-Kantorovich theorem, *SIAM J. Num. Anal.* **11** (1974) 10–13.

[6] M.H. Gutknecht, Fast algorithms for the conjugate periodic function, *Computing* **22** (1979) 79–91.

[7] M.H. Gutknecht, Solving Theodorsen's integral equation for conformal maps with the fast Fourier transform and various nonlinear iterative methods, *Num. Math.* **36** (1981) 405–429.

[8] M.H. Gutknecht, Numerical experiments on solving Theodorsen's integral equation for conformal maps with the fast Fourier transform and various nonlinear iterative methods, *SIAM J. Sci. Stat. Comput.* **4** (1983) 1–30.

[9] P. Henrici, Einige Anwendungen der schnellen Fouriertransformation, in: *Moderne Methoden der Numerischen Mathematik*, ISNM **32** (Birkhäuser, Basel, Stuttgart, 1976) 111–124.

[10] O. Hübner, Zur Numerik der Theodorsenschen Integralgleichung in der konformen Abbildung, *Mitt. Math. Sem. Gießen* **140** (1979) 1–32.

[11] D.C. Ives, A modern look at conformal mapping including multiply connected regions, *AIAA J.* **14** (1976) 1006–1011.

[12] H.B. Keller, Newton's method under mild differentiability conditions, *J. Comput. System Sci.* **4** (1970) 15–28.

[13] N.T. Muschelischwili, *Singuläre Integralgleichungen* (Akademie-Verlag, Berlin, 1965).

[14] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables* (Academic Press, New York, London, 1970).

[15] T. Theodorsen, Theory of wing sections of arbitrary shape, *NACA Report* **411** (1931).

[16] B.A. Vertgejm, Approximate construction of some conformal mappings (Russian), *Dokl. Akad. Nauk SSSR* **119** (1958) 12–14.

[17] R. Wegmann, Ein Iterationsverfahren zur konformen Abbildung, *Numer. Math.* **30** (1978) 453–466; translated as: An iterative method for conformal mapping, *J. Comput. Appl. Math.* **14** (1986) (this volume) 7–18.

[18] R. Wegmann, Convergence proofs and error estimates for an iterative method for conformal mapping, *Numer. Math.* **44** (1984) 435–461.

[19] L. v. Wolfersdorf, Zur Unität der Lösung der Theodorsenschen Integralgleichung der konformen Abbildung, *Z. Anal. Anwendungen* **3** (1984) 523–526.