A new existence proof for large sets of disjoint Steiner triple systems

L. Ji

Department of Mathematics, Suzhou University, Suzhou 215006, China

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Abstract

A Steiner triple system of order v (briefly STS(v)) consists of a v-element set X and a collection of 3-element subsets of X, called blocks, such that every pair of distinct points in X is contained in a unique block. A large set of disjoint STS(v) (briefly LSTS(v)) is a partition of all 3-subsets (triples) of X into v − 2 STS(v). In 1983–1984, Lu Jiaxi first proved that there exists an LSTS(v) for any v ≡ 1 or 3 (mod 6) with six possible exceptions and a definite exception v = 7. In 1989, Teirlinck solved the existence of LSTS(v) for the remaining six orders. Since their proof is very complicated, it is much desired to find a simple proof. For this purpose, we give a new proof which is mainly based on the 3-wise balanced designs and partitionable candelabra systems.

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1. Introduction

A Steiner triple system of order v (briefly STS(v)) consists of a v-element set X and a collection of 3-element subsets of X, called blocks, such that every pair of distinct points in X is contained in a unique block. It is well known that there exists an STS(v) if and only if v ≡ 1, 3 (mod 6) [17].

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E-mail address: jilijun@suda.edu.cn.
Two STS($v$) $(X, A)$ and $(X, B)$ are called disjoint if $A \cap B = \emptyset$. A set of more than two STS($v$) is called disjoint if each pair of them is disjoint. A set of $v-2$ disjoint STS($v$) is called a large set of disjoint STS($v$) and briefly denoted by LSTS($v$). It is clear that an LSTS($v$) is also a partition of all 3-subsets (triples) of $X$ into $v - 2$ STS($v$).

In 1850, Cayley showed that there are only two disjoint STS(7) [3], i.e., there does not exist an LSTS(7). In the same year, Kirkman showed that there exists an LSTS(9), which is the first nontrivial LSTS. Later, many people did some research on it. In 1973, Teirlinck [28] presented a recursive construction that there is an LSTS(3$v$) if there is an LSTS($v$). In 1975, Rosa [26] gave a recursive construction that there is an LSTS($2v + 1$) if there is an LSTS($v$) for $v \geq 7$. Besides, Schreiber (1973) [27], Wilson [31] and Denniston (1974) [5] successively got some small orders. Until 1982, the existence problem of LSTS($v$) had been still open. In 1983–1984, Lu [20,21] first determined the existence of LSTS($v$) with six possible exceptions and $v \neq 7$. In 1989, Teirlinck [29] found that the auxiliary design LD* introduced by Lu has a nice property. Using this property, he solved the existence of LSTS($v$) for the remaining six orders. Therefore the existence of spectrum for LSTS has been finally completed, which is stated below.

**Theorem 1.1** (Lu [20,21], Teirlinck [29]). For any integer $v \equiv 1, 3 \pmod{6}$ with $v > 7$, there is an LSTS($v$).

In Lu’s proof, auxiliary designs AD, AD*, AD** and their large sets LAD, LAD$_1$, LAD$_2$ and LAD$_3$ were introduced. All auxiliary designs and the related recursive constructions are rather complicated. It is much desired to find a simple proof. For this purpose, we shall make use of 3-wise balanced designs and partitionable candelabra systems to give an alternative proof of existence of LSTS($v$).

The remainder of this paper is arranged as follows. In Section 2, we present a recursive construction for LSTSs by use of partitionable candelabra systems (PCSs), where the uniform PCS has been introduced in [2]. In Section 3, we use CQS and special partitionable GDD to present a new construction for PCSs. We also construct a PCS($6^k : 3$) for any $k \geq 3$ which will lead to the existence of an LSTS($6k + 3$). In Section 4, we give a new proof of the tripling construction. In Section 5, we obtain the existence of a PCS($24^k : 13$) so as to determine the existence of an LSTS($24k + 13$). In Section 5, we obtain some non-uniform PCSs with a stem of size 13 in order to construct an LSTS($24k + 1$). In the last section, we give a new proof of the existence of an LSTS($v$).

2. A construction for LSTSs via PCSs

In this section we shall describe a construction to obtain LSTSs from partitionable candelabra systems (PCS) in Lemma 2.5.

Let $v$ be a non-negative integer, let $t$ be a positive integer and let $K$ be a set of positive integers. A candelabra $t$-system (or $t$-CS as in [24]) of order $v$, and block sizes from $K$ denoted by CS($t$, $K$, $v$) is a quadruple $(X, S, \Gamma, \mathcal{A})$ that satisfies the following properties:

1. $X$ is a set of $v$ elements (called points).
2. $S$ is a subset (called the stem of the candelabra) of $X$ of size $s$.
Theorem 2.2

Let $K = \{k\}$, we simply write $k$ for $K$. A candelabra system with $t = 3$ and $K = \{4\}$ is called a candelabra quadruple system and denoted by $\text{CQS}(g^1_1 g^2_2 \cdots g^r_r : s)$ (as in [13]).

In [19], Lenz stated an infinite class of CQS with three groups, with which he gave a new proof of the existence of a Steiner quadruple system (the concept is defined below).

Lemma 2.1 (Lenz [19]). A $\text{CQS}(g^3 : s)$ exists for all even $s$ and all $g \equiv 0, s \pmod{6}$ with $g \geq s$.

A candelabra system $\text{CS}(t, K, v)$ of type $(1^v : 0)$ $(X, S, G, B)$ is usually called a $t$-wise balanced design and briefly denoted by $\text{S}(t, K, v)$. As well, the stem and the group set are often omitted and we write a pair $(X, B)$ instead of a quadruple $(X, S, G, B)$. An $\text{S}(3, 4, v)$ is called a Steiner quadruple system and denoted by $\text{SQS}(v)$. It is well known that an $\text{SQS}(v)$ exists if and only if $v \equiv 2, 4 \pmod{6}$ [9].

In this paper, we mainly deal with candelabra system with $t = 3$ and $K = \{3\}$. For convenience, such a system will be denoted briefly by $\text{CS}(g^1_1 g^2_2 \cdots g^r_r : s)$. In fact, the block set of a $\text{CS}(g^1_1 g^2_2 \cdots g^r_r : s)$ consists of all triples that are not from the union of any group and the stem.

Let $v$ be a non-negative integer, let $t$ be a positive integer and $K$ be a set of positive integers. A group divisible $t$-design (or $t$-GDD) of order $v$ and block sizes from $K$ denoted by $\text{GDD}(t, K, v)$ is a triple $(X, G, B)$ such that

1. $X$ is a set of $v$ elements (called points),
2. $G = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets (called groups) of $X$ which partition $X$,
3. $B$ is a family of subsets of $X$ (called blocks) each of cardinality from $K$ such that each block intersects any given group in at most one point,
4. each $t$-set of points from $t$ distinct groups is contained in exactly one block.

The type of the GDD is defined to be the list $\{|G||G \in G\}$ of group sizes.

For $t = 2$, many results on it have been stated in [25]. For $t = 3$, Mills [23] almost completely determined the existence of a GDD$(3, 4, ng)$ of type $g^n$ (called an $H$ design as in [23]), which is stated below.

Theorem 2.2 (Mills [23]). For $n > 3$ and $n \neq 5$, an $H(n, g, 4, 3)$ exists if and only if $ng$ is even and $g(n - 1)(n - 2)$ is divisible by 3. For $n = 5$, an $H(5, g, 4, 3)$ exists if $g$ is divisible by 4 or 6.

In [18], the author introduced a large set of generalized Kirkman systems in order to construct large sets of Kirkman triple systems. Here, we do not need the resolvable property.
and this system can be reduced to the following concept of partitionable candelabra system with \( t = 3 \) and \( K = [3] \).

A \( CS(g_1^{a_1}g_2^{a_2}\cdots g_r^{a_r} : s) \) \((X, S, G, A)\) with \( s \geq 2 \) is called partitionable and denoted by \( PCS(g_1^{a_1}g_2^{a_2}\cdots g_r^{a_r} : s) \) if the block set \( A \) can be partitioned into \( A_i \) \((x \in G, G \in G)\) and \( A_1, A_2, \ldots, A_{s-2} \) with the following two properties: (i) for any \( x \in G \) and \( G \in G, A_i \) is the block set of a GDD\((2, 3, \sum_{1 \leq i \leq r} a_ig_i + s)\) of type 1 of order \( (|G| + s)^1 \) with \( G \cup S \) as the long group; (ii) \( \sum_{1 \leq i \leq r} a_ig_i \) \( PCS \) \( A \) \( \cup (\cup_{1 \leq i \leq s-2} A_i) \) is a PCS. In what follows, the block set \( A \) of a PCS is often given as a union of \( A_i \) \((x \in G, G \in G)\) and \( A_1, A_2, \ldots, A_{s-2} \).

The uniform PCS has been introduced in [2] and used to solve the existence of a large set of disjoint packings on \( 6k + 5 \) points. In [16], an infinite class of PCS\((6^k : 5)\) has also been used to solve Etzion’s conjecture [6]: there is a partition for triples of order \( 6k + 5 \) with \( 6k + 3 \) optimal packings and one packing of size \( 8k + 4 \).

Below is an example of PCS, which has been given in [16].

**Example 2.3.** There exists a \( PCS(3^5 : 2) \).

**Proof.** Let \( S = \{\infty_1, \infty_2\} \) and \( X = Z_{15} \cup S \). We shall construct the desired design on \( X \) having group set \( G = \{i, i + 5, i + 10\} : 0 \leq i \leq 4 \} \) and a stem \( S \). The block set should be partitioned into 15 GDD\((2, 3, 17)\) of type 1\(^{12}_{5}\). We first construct an initial GDD\((2, 3, 17)\) of type 1\(^{12}_{5}\) with the long group \( \{0, 5, 10\} \cup S \) and the following blocks:

\[
\begin{align*}
0 & 1 2 & 0 & 3 4 & 0 & 6 7 & 0 & 8 11 & 0 & 9 14 & 0 & 12 13 \\
1 & 3 8 & 1 & 4 6 & 1 & 5 7 & 1 & 9 10 & 1 & 11 13 & 1 & 12 14 \infty_1 \\
1 & 14 10 & 2 & 3 10 & 1 & 2 4 8 & 2 & 5 11 & 2 & 6 12 & 2 & 7 9 \\
2 & 10 14 & 2 & 13 10 & 3 & 5 6 & 3 & 7 14 & 3 & 9 13 & 3 & 10 11 \\
3 & 12 10 & 4 & 5 10 & 4 & 7 10 & 4 & 10 12 & 4 & 11 9 \infty_2 & 4 & 13 14 \\
5 & 8 13 & 5 & 12 13 & 6 & 8 10 & 6 & 9 \infty_2 & 6 & 11 14 & 6 & 13 \infty_1 \\
7 & 8 10 & 7 & 10 13 & 7 & 11 12 & 8 & 9 12 & 8 & 14 \infty_1 & 9 & 11 \infty_1
\end{align*}
\]

Then, 15 GDD\((2, 3, 17)\) of type 1\(^{12}_{5}\) are generated from the initial modulo 15. It is readily checked that the above 15 GDDs of type 1\(^{12}_{5}\) do not contain any common block and therefore, they form the desired PCS\((3^5 : 2) \). \( \square \)

Below is a construction for PCSs from the known CQSs with three groups [2]. Here we give an outline of its proof.

**Lemma 2.4 (Cao et al. [2]).** If there exists a CQS\((g^3 : s)\) with \( s > 3 \), then there exists a PCS\((g^3 : s - 1)\).

**Proof.** Let \((X, S, G, T)\) be the given \( CS(3, 4, 3g + s)\) of type \((g^3 : s)\), where \( G = \{G_1, G_2, G_3\} \) and \( S = \{w_1, w_2, \ldots, w_s\} \). We shall convert this design into the desired design.
For each point \( x \in G_1 \), let 
\[
B_x = \{ B \setminus \{x\} : B \in \mathcal{T}, \; x \in B, \; w_s \not\subset B \} \cup \{ B \setminus \{w_s\} : x, w_s \subset B, \; B \in \mathcal{T} \}.
\]

For each point \( y \in G_2 \), let 
\[
B_y = \{ B \setminus \{y\} : B \in \mathcal{T}, \; y \in B, \; \{w_{s-1}, w_s\} \cap B = \emptyset \} \cup \{ B \setminus \{w_{s-1}\} : B \in \mathcal{T}, \; \{y, w_{s-1}\} \subset B \} \cup \{(B \setminus \{y, w_s\}) \cup \{w_{s-1}\} : B \in \mathcal{T}, \; \{y, w_s\} \subset B \}.
\]

For each point \( z \in G_3 \), let 
\[
B_z = \{ B \setminus \{z\} : B \in \mathcal{T}, \; z \in B, \; \{w_{s-2}, w_s\} \cap B = \emptyset \} \cup \{ B \setminus \{w_{s-2}\} : B \in \mathcal{T}, \; \{z, w_{s-2}\} \subset B \} \cup \{(B \setminus \{z, w_s\}) \cup \{w_{s-2}\} : B \in \mathcal{T}, \; \{z, w_s\} \subset B \}.
\]

For \( 1 \leq i \leq s - 3 \), denote \( F_i = \{ B \setminus \{w_i\} : B \in \mathcal{T}, \; w_i \in B \} \).

Let 
\[
\mathcal{F} = \left( \bigcup_{x \in G_1} B_x \right) \bigcup \left( \bigcup_{y \in G_2} B_y \right) \bigcup \left( \bigcup_{z \in G_3} B_z \right) \bigcup \left( \bigcup_{1 \leq i \leq s - 3} F_i \right).
\]

Then \((X \setminus \{w_s\}, S \setminus \{w_s\}, G, \mathcal{F})\) is the desired \(PCS(g^3 : s - 1)\). \(\square\)

In order to convert a \(PCS\) into an \(LSTS\), we need a holey large set. Let \(X\) be a \(v\)-element set and \(Y\) be a \(w\)-subset of \(X\) with \(w \geq 2\). A \textit{holey large set} of disjoint \(STS(v)\) on \(X\) with a hole \(Y\) (\(HLSTS(v, w)\)) is a partition of \(X(3) \setminus Y(3)\) into \(A_1, A_2, \ldots, A_{v-2}\) with the properties that (1) for \(1 \leq i \leq v - w\), each \((X, A_i)\) is an \(STS(v);\) (2) for \(v - w + 1 \leq i \leq v - 2\), each \(A_i\) is the block set of a \(GDD(2, 3, v)\) of type \(1^v - w^1\) with the long group \(Y\), where \(X(3)\) and \(Y(3)\) denote the sets of all triples of \(X\) and \(Y\), respectively.

Now, we are in a position to describe how to get an \(LSTS\) from a \(PCS\).

**Lemma 2.5 (Filling in holes).** Suppose there exists a \(PCS(g^1_0, g^2_1, \ldots, g^r_s)\) with \(s \geq 2\). If there is an \(HLSTS(g_i + s, s)\) for \(1 \leq i \leq r\), then there is an \(HLSTS(\sum 1 \leq i \leq r a_i g_i + g_0 + s, g_0 + s)\). Further, if there is an \(LSTS(g_0 + s)\), then there is an \(LSTS(\sum 1 \leq i \leq r a_i g_i + g_0)\).

**Proof.** Let \(a_0 = 1\) and \((X, S, G, A)\) be the given \(PCS(g^1_0, g^2_1, \ldots, g^r_s)\). By the definition, \(A\) can be partitioned into subsets \(A_y (y \in G \in G)\) and \(A_i (1 \leq i \leq s - 2)\) with the properties that each \(A_i\) is the block set of a \(GDD(2, 3, \sum 1 \leq i \leq r a_i g_i + g_0 + s)\) of type \(1^\sum 1 \leq i \leq r a_i (g_i + s - 1)^1\) with the long group \(G\) + \(S\) and that each \((X \setminus S, G, A_i)\) is a \(GDD(2, 3, \sum 1 \leq i \leq r a_i (g_i + s)\) of type \(g^1_0 g^2_1 g^3_2 \cdots g^r_s\).

Let \(G_0\) be a special group with \(|G_0| = g_0\). For each group \(G \in G \neq G_0\), suppose the given \(HLSTS(|G| + s, s)\) has \(|G| STS(|G| + s)\) with block sets \(B_y (y \in G)\) and \(|G| - 2\) \(GDD(2, 3, |G| + s)\) of type \(1^{|G|} s^1\) with the long group \(S\) and block sets \(B_i^G (1 \leq i \leq s - 2)\).

For any \(y \in G, G \in G \neq G_0\), let \(C_y = A_y \cup B_y\). For \(1 \leq i \leq s - 2\), let \(C_i = A_i \cup (\cup_{G \in G, G \neq G_0} B_i^G)\).

Then each \((X, C_y)\) is an \(STS(\sum 1 \leq i \leq r a_i g_i + g_0 + s)\) and each \(C_i, A_y (y \in G_0)\) is the block set of a \(GDD(2, 3, \sum 1 \leq i \leq r a_i g_i + g_0 + s)\) of type \(1^\sum 1 \leq i \leq r a_i (g_0 + s)^1\) with the
long group $G_0 \cup S$. It is easy to see that these block sets are pairwise disjoint. So, they form an HLSTS($\sum_{1 \leq i \leq r} a_i g_i + g_0 + s \cdot g_0 + s$).

Further, suppose the given LSTS($g_0 + s$) on $G_0 \cup S$ has $g_0 + s - 2$ disjoint STS($g_0 + s$) with block sets $B_y$ ($y \in G_0$) and $B_i$ ($1 \leq i \leq s - 2$). Then $(X, A_y \cup B_y)$ and $(X, C_i \cup B_i)$ are all STS($\sum_{1 \leq i \leq r} a_i g_i + g_0 + s$), and all $\sum_{1 \leq i \leq r} a_i g_i + g_0 + s - 2$ STSs form an LSTS($\sum_{1 \leq i \leq r} a_i g_i + g_0 + s$). □

We give a new construction of an LSTS(33).

Example 2.6. There is an LSTS(33).

Proof. We start with a CQS($10^3 : 4$), which exists by Lemma 2.1. Applying Lemma 2.4 gives a PCS($10^3 : 3$). For the union of a group and the stem, we construct an LSTS(13) which exists in [5]. For the union of each of the other groups and the stem, construct an HLSTS(13, 3) with the stem as a hole, which can be obtained by setting any triple of an LSTS(13) as a hole. By Lemma 2.5 we obtain an LSTS(33). □

Lemma 2.5 implies the PCSs are important in the construction of LSTSs. In the next section, we shall state a new construction for PCSs.

3. The existence of an LSTS($6k + 3$) and an LSTS($12k + 7$)

In this section, we shall use $s$-fan design to state a new construction for PCS in Lemma 3.4 and use this construction to produce a PCS($6^k : 3$) for any $k \geq 3$. Then we apply Lemma 2.5 to obtain an LSTS($6k + 3$) and use Rosa’s doubling construction to obtain an LSTS($12k + 7$).

Let $(X, S, G, A)$ be a CQS($3, K, v$) of type $(g_1, g_2^2, \ldots, g_r^a : s)$ with $s > 0$ and let $S = \{\infty_1, \ldots, \infty_s\}$. For $1 \leq i \leq s$, let $A_i = \{A \setminus \{\infty_i\} : A \in A, \infty_i \in A\}$ and $A_T = \{A \in A : A \cap S = \emptyset\}$. Then the $(s + 3)$-tuple $(X, G, A_1, A_2, \ldots, A_r, A_T)$ is called an $s$-fan design (as in [12]). If block sizes of $A_i$ and $A_T$ are from $K_i (1 \leq i \leq s)$ and $K_T$, respectively, then the $s$-fan design is denoted by $s$-FG($3, (K_1, K_2, \ldots, K_s, K_T), \sum_{i=1}^r a_i g_i$) of type $g_1, g_2^2, \ldots, g_r^a$.

A generalized frame (as in [30]) $F(3, 3, n(g))$ is a GDD($3, 3, gn$) $(X, G, A)$ of type $g^n$ such that the block set $A$ can be partitioned into $gn$ subsets $A_y, y \in G$ and $G \in G$, each $(X \setminus G, G \setminus \{G\}, A_y)$ being a GDD($2, 3, g(n - 1)$) of type $g^{n-1}$. Teirlinck pointed out in [30] that an $F(3, 3, n(g))$ can be obtained from a GDD($3, 4, ng$) of type $g^n$. From the existence of such a GDD in Theorem 2.2, we have the following theorem.

Theorem 3.1 (Mills [23], Teirlinck [30]). For $n > 3$ and $n \neq 5$, an $F(3, 3, n(g))$ exists if and only if $gn$ is even and $g(n - 1)(n - 2)$ is divisible by 3. For $n = 5$, an $F(3, 3, 5(g))$ exists if $g$ is divisible by 4 or 6.

Let $g \geq 3$ and $(X, G, A)$ be a GDD($3, 3, g(n + 1) - 1$) of type $g^n(g - 1)^l$ and $G_0$ be the group of size $g - 1$. Such a GDD is shortly denoted by PGDD($g^n(g - 1)^l$) if the block set $A$ can be partitioned into $A_x (x \in G, G \in G$ and $G \neq G_0)$ and $A_1, \ldots, A_{g-3}$ with the
following two properties: (i) each $A_x$ is the block set of a GDD(2, 3, $g^n$) of type $g^n$ with group set $(G \setminus \{G_0, G\}) \cup \{G_0 \cup \{x\}\}$, (2) each $(X \setminus G_0, G \setminus \{G_0\}, A_i)$ is a GDD(2, 3, $g^n$) of type $g^n$.

We give two examples of PGDDs below.

**Lemma 3.2.** There is a PGDD($3^32^1$).

**Proof.** We construct the desired design on $X = Z_9 \cup \{x, y\}$ with group set $G = \{G_i = \{i, i + 3, i + 6\} : 0 \leq i \leq 2\} \cup \{\{x, y\}\}$. By the definition, it should contain nine GDD(2, 3, 9). Below are all blocks of an initial GDD(2, 3, 9) with group set $\{G_1, G_2, \{0, x, y\}\}$.

\[
0 1 2 0 4 8 0 5 7 1 5 x 2 4 x 7 8 x 1 8 y 4 5 y 2 7 y
\]

Developing this initial GDD modulo 9 generates the required nine pairwise disjoint GDD(2, 3, 9) of type $3^3$ and they form a PGDD($3^32^1$).

**Lemma 3.3.** There is a PGDD($3^52^1$).

**Proof.** We construct the desired design on $X = Z_{15} \cup \{x, y\}$ with group set $G = \{\{i, i + 5, i + 10\} : 0 \leq i \leq 4\} \cup \{\{x, y\}\}$. By the definition, it should contain 15 GDD(2, 3, 15). Below are all blocks of an initial GDD(2, 3, 15) with group set $(G \setminus \{\{0, 5, 10\}\}) \cup \{\{0, x, y\}\}$.

\[
0 1 2 0 3 4 0 6 9 0 7 14 0 8 11 0 1 2 13 1 3 1 2 1 4 7 1 8 1 4 1 9 x 1 1 3 y 2 3 1 1 2 4 8 2 6 y 2 9 1 3 2 1 4 x 3 6 1 4 3 7 x 3 9 y 4 6 1 2 4 1 1 y 4 1 3 x 6 7 1 3 6 8 x 7 8 y 7 9 1 1 8 9 1 2 1 1 1 2 13 1 1 1 3 1 4 1 2 1 4 y
\]

Developing this initial GDD modulo 15 generates the required 15 pairwise disjoint GDD(2, 3, 15) of type $3^5$ and they form a PGDD($3^52^1$).

**Lemma 3.4.** Suppose that there exists a 2-FG, $(K_1, K_2, K_T, \nu)$ of type $g_1^{a_1}g_2^{a_2} \cdots g_r^{a_r}$. Suppose that there exist a PGDD($m_k(1, m - 1)$) for any $k \in K_1$, a CQS($m_k^2 : \nu$) for any $k_2 \in K_2$ and an $F(3, 3, k \{m\})$ for any $k \in K_T$. Then there exists a PCS($(mg_1)^{a_1}(mg_2)^{a_2} \cdots (mg_r)^{a_r} : m + r - 1$).

**Proof.** Let $(X, G, A_1, A_2, T)$ be the given 2-FG. Let $S = \{\infty\} \times Z_s$, where $s = m + r - 1$. We shall construct the desired design on $X' = (X \times Z_m) \cup S$ with the group set $G' = G \times Z_m : G \in G\}$ and the stem $S$, where $(X \times Z_m) \cap S = \emptyset$. We shall describe its block set $F$ below.

Denote $G_x = \{x\} \times Z_m$ for $x \in X$. Denote $S_1 = \{\infty\} \times Z_{m-1}$ and $S_2 = S \setminus S_1$.

For each block $A \in A_1$, construct a PGDD($m|A|\{m - 1\}$) on $(A \times Z_m) \cup S_1$ having $\{G_x : x \in A\} \cup \{S_1\}$ as its group set. Such a design exists by assumption. Denote its block set by $D_A$. So, $D_A$ can be partitioned into $(m|A| + m - 3)$ disjoint block sets $D_A(x, i)$ ($x \in A, i \in Z_m$) and $D_A(\infty, d)$ ($2 \leq d \leq m - 2$) such that each $D_A(x, i)$ is the block set of a GDD(2, 3, $m|A|$) of type $m|A|$ with group set $\{G_y : y \in A, y \neq x\} \cup \{S_1 \cup \{(x, i)\}\}$, and such that each $(A \times Z_m, \{G_y : y \in A\}, D_A(\infty, d))$ is a GDD(2, 3, $m|A|$) of type $m|A|$.
For each block \( A \in \mathcal{A}_2 \), construct a CQS \((m^{\lceil |A| \rceil} : r)\) on \((A \times \mathbb{Z}_m) \cup S_2\) having \([G_x : x \in A]\) as its group set, \(S_2\) as its stem. Such a design exists by assumption. Denote its block set by \( B'_A \). For any \((x, i) \in (A \times \mathbb{Z}_m) \cup S_2\), let \( B_A(x, i) = \{B \setminus \{(x, i)\} : (x, i) \in B, B \in B'_A\} \) and \( B_A = \bigcup_{(x, i) \in (A \times \mathbb{Z}_m) \cup S_2} B_A(x, i) \). Clearly, \((A \times \mathbb{Z}_m) \cup S_2, S_2, \{G_x : x \in A\}, B_A\) is a \(CS(m^{\lceil |A| \rceil} : r)\), each \( B_A(x, i) \) \((x, i) \in A \times \mathbb{Z}_m\) is the block set of a GDD \((2, 3, m | A| + r - 1)\) of type \(1^{\lceil |A| - 1 \rceil}(m + r - 1)^1\) on \((A \times \mathbb{Z}_m) \setminus \{(x, i)\}) \cup S_2\) with the long group \((G_x \cup S_2) \setminus \{(x, i)\})\), and each \((A \times \mathbb{Z}_m), \{G_x : x \in A\}, B_A(x, i)\) \((x, i) \in S_2\) is a GDD \((2, 3, m | A|)\) of type \(m^{\lceil |A| \rceil}\).

For each block \( A \in \mathcal{T}\), construct a generalized frame \( F(3, 3, |A| \{m\})\) on \(A \times \mathbb{Z}_m\) having \( \Gamma_A = \{G_x : x \in A\}\) as its group set. Such a design exists by assumption. Denote its block set by \( D'_A\). Then \( D'_A\) can be partitioned into \(m|A|\) disjoint block sets \( D'_A(x, i) \) \((x \in A, i \in \mathbb{Z}_m)\) with the property that each \((A \setminus \{x\}) \times \mathbb{Z}_m, \Gamma_A \setminus \{G_x\}, D'_A(x, i)\) is a GDD \((2, 3, m | A| - 1)\) of type \(m^{\lceil |A| \rceil - 1}\).

For any \(x \in A\) and \(i \in \mathbb{Z}_m\), let
\[
\mathcal{F}(x, i) = \left( \bigcup_{x \in A, A \in \mathcal{A}_1} D_A(x, i) \right) \bigcup \left( \bigcup_{x \in A, A \in \mathcal{A}_2} B_A(x, i) \right) \bigcup \left( \bigcup_{x \in A, A \in \mathcal{T}} D'_A(x, i) \right).
\]
For any \(2 \leq i \leq m - 2\), let
\[
\mathcal{F}(\infty, i) = \bigcup_{A \in \mathcal{A}_1} D_A(\infty, i),
\]
For any \(m - 1 \leq i \leq m + r - 2\), let
\[
\mathcal{F}(\infty, i) = \bigcup_{A \in \mathcal{A}_2} B_A(\infty, i).
\]
Let
\[
\mathcal{F} = \left( \bigcup_{i \in \mathbb{Z}_m, x \in A} \mathcal{F}(x, i) \right) \bigcup \left( \bigcup_{2 \leq i \leq s - 1} \mathcal{F}(\infty, i) \right).
\]
It is left to show that \((X', S, G', \mathcal{F})\) is the desired PCS \(((m g_1)^{a_1}(m g_2)^{a_2} \cdots (m g_r)^{a_r} : m + r - 1)\). We shall prove this by three steps.

(1) Take any triple \(T = \{(x, a), (\beta, b), (\theta, c)\}\) not contained in any \(G' \cup S\) for \(G' \in G'\). We shall show that \(T \in \mathcal{F}\). We distinguish two cases.

(i) \(|T \cap (G' \cup S)| = 2\) for some \(G' \in G'\). If \(\infty \in \{x, \beta, \theta\}\), it is clear that only one element is \(\infty\) and the other two are from distinct groups of the 2-FG. Without loss of generality, let \(x = \infty\). Then \((x, a) \in S\). If \((x, a) \in S_1\), there is a unique block \(A \in \mathcal{A}_1\) containing \(\{\beta, \theta\}\) since \(\mathcal{A}_1\) is the block set of a GDD \((2, K_1, v)\) of type \(g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r}\). Since \(D_A\) is the block set of a PGDD \((m^{\lceil |A| \rceil}(m - 1)^1)\) on \((A \times \mathbb{Z}_m) \cup S_1, T \in D_A \subset \mathcal{F}\). Otherwise, \((x, a) \in S_2\). There is a unique block \(A \in \mathcal{A}_2\) containing \(\{\beta, \theta\}\) since \(\mathcal{A}_2\) is the block set of a GDD \((2, K_2, v)\). Since \(B_A\) is the block set of a \(CS(m^{\lceil |A| \rceil} : r)\) on \((A \times \mathbb{Z}_m) \cup S_2, T \in B_A \subset \mathcal{F}\).
Otherwise $\infty \notin \{\alpha, \beta, \theta\}$, there is a unique block $A$ containing $\{\alpha, \beta, \theta\}$, where $A \in \mathcal{A}_2$ if $|\{\alpha, \beta, \theta\}| = 2$ and $A \in \mathcal{T}$ if $|\{\alpha, \beta, \theta\}| = 3$. When $A \in \mathcal{A}_2$ (or $A \in \mathcal{T}$), $T$ is a block of $\mathcal{B}_A$ (or $\mathcal{D}'_A$). So, $T \in \mathcal{F}$.

(ii) $|T \cap (G' \cup S)| \leq 1$ for any $G' \in G'$. It is clear that $\infty \notin \{\alpha, \beta, \theta\}$ and $|\{\alpha, \beta, \theta\}| = 3$. Then there is a unique block $A \in (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{T})$ containing $\{\alpha, \beta, \theta\}$. When $A \in \mathcal{A}_1$ (or $A \in \mathcal{A}_2$, or $A \in \mathcal{T}$), $T$ is a block of $\mathcal{D}_A$ (or $\mathcal{B}_A$, or $\mathcal{D}'_A$). So, $T \in \mathcal{F}$. It follows that $(X', S, G', \mathcal{F})$ is a CS($(mg_1)^a (mg_2)^b \cdots (mg_r)^c : m + r - 1$).

(2) We shall prove that for any $i \in \mathbb{Z}_3$, $x \in G$ and $G \in G$, $\mathcal{F}(x, i)$ is the block set of a GDD(2, 3, $mv + s$) of type $1^{m(v-|G|)}(m|G| + s)^1$ with the long group $G' \cup S$. We need to show that any pair of points from distinct groups occurs in exactly one block of $\mathcal{F}(x, i)$. Suppose $P$ is such a pair, $P = (\{\alpha, \beta\}, (\beta, b))$ from $X'$, $P \notin G' \cup S$. We distinguish two cases.

Case 1: $\infty \notin \{\alpha, \beta\}$. Without loss of generality, let $x = \infty$. Then $(\alpha, a) \in S$, and $\beta \notin G$ since $P \notin G' \cup S$. Suppose $(\alpha, a) \in S$. There is a unique block $A \in \mathcal{A}_1$ containing $\{\alpha, \beta\}$. Since $\mathcal{D}_A(x, i)$ is the block set of a GDD(2, 3, $m|A|$) on $((\mathcal{A}_1 \{\{\alpha, \beta\}\} \times \mathbb{Z}_m) \cup S \cup \{(x, i)\})$, with groups $G, y \in G, y \neq x)$ and $(\{x, i\}) \cup S$, there is a unique block $B \in \mathcal{D}_A(x, i) \subset \mathcal{F}(x, i)$ such that $P \subset B$. It follows that there is a unique block $B \in \bigcup_{x \in A, a \in A_1} \mathcal{D}_A(x, i)$ such that $P \subset B$. Suppose that there is another block $B' \in \mathcal{F}(x, i)$ containing $P$. Since $(\alpha, a) \in S_1$, $B'$ must belong to $\bigcup_{x \in A_1, a \in A_1} \mathcal{D}_A(x, i)$. There must be some $A_1 \in \mathcal{A}_1$ such that $B' \in \mathcal{D}_A(x, i)$. Then $(\alpha, i) \subset A_1$. Since $\mathcal{A}_1$ is the block set of a GDD(2, $K_1$, $v$) and $(\alpha, \beta) \subset A$, we have $A_1 = A$ and $B' \in \bigcup_{x \in A, a \in A_1} \mathcal{D}_A(x, i)$. Since there is a unique block $B \in \bigcup_{x \in A_1, a \in A_1} \mathcal{D}_A(x, i)$ such that $P \subset B$, $P$ is contained in a unique block of $\mathcal{F}(x, i)$. Otherwise, $(\alpha, a) \in S_2$. Since $x \in G$, then there is a unique block $A \in \mathcal{A}_2$ containing $(\alpha, \beta)$. Since $B'\mathcal{A}_2(x, i)$ is the block set of a GDD(2, $3, m|A| + r - 1$) of type $1^{m(|A|-1)}(m + r - 1)^1$ on $((A \times \mathbb{Z}_m) \cup S_2 \setminus \{(x, i)\}$ with the long group $(G_1 \cup S_2) \setminus \{(x, i)\}$, there is a unique block $B \in B'\mathcal{A}_2(x, i)$ such that $P \subset B$. It follows that there is a unique block $B \in \bigcup_{x \in A_1, A \in A_2} B'(x, i)$ such that $P \subset B$. Similar to the above proof, $P$ is contained in a unique block of $\mathcal{F}(x, i)$.

Case 2: $\infty \notin \{\alpha, \beta\}$. If $x = \beta$, then $x \notin G$ since $P \notin G' \cup S$. Also, there is a block $A \in \mathcal{A}_2$ containing $(\alpha, \beta)$. It follows that there is a block $B \in B'(x, i) \subset \mathcal{F}(x, i)$ such that $P \subset B$. Suppose that there is another block $B' \in \mathcal{F}(x, i)$ containing $P$. Then $B'$ must belong to $\bigcup_{x \in A, a \in A_2} B'(x, i)$. There must be some $A_2 \in \mathcal{A}_2$ such that $B' \in B'(x, i)$. Then $A_2 = A$ and $B' = B$. So, $P$ is contained in a unique block of $\mathcal{F}(x, i)$. Otherwise, $x \notin \beta$. If $x \in (\alpha, \beta)$, without loss of generality, let $x = \alpha$. It follows that $\beta \notin G$ and there is a unique block $A_j \in \mathcal{A}_j$ ($j = 1, 2$) containing $(\alpha, \beta)$. Then there is a block $B$ such that $P \subset B$, where $B \in B'(x, i)$ if $a = i$ and $B \in B'\mathcal{A}_2(x, i)$ if $a \neq i$. Similarly, $P$ is contained in a unique block of $\mathcal{F}(x, i)$. If $x \notin (\alpha, \beta)$, then $(x, \beta, \beta)$ is contained in a block $A$ of the 2-FG. When $A$ is contained in $\mathcal{A}_1$ (or $\mathcal{A}_2$, or $\mathcal{T}$), there is a block $B \in \mathcal{D}_A(x, i)$ (or $B \in B'(x, i)$, or $B \in D'\mathcal{A}_2(x, i)$) such that $P \subset B$. Supposed that there is another block $B' \in \mathcal{F}(x, i)$ containing $P$. It is readily checked that $B' = B$. So, $P$ is contained in a unique block of $\mathcal{F}(x, i)$. This shows that $\mathcal{F}(x, i)$ is the block set of a GDD(2, $3, mv + s$) of type $1^{m(v-|G|)}(m|G| + s)^1$.

(3) We shall prove that for any $2 \leq i \leq s - 1$, $(X' \setminus S, G', \mathcal{F}(\infty, i))$ is a GDD(2, 3, $mv$) of type $(mg_1)^a (mg_2)^b \cdots (mg_r)^c$. We need to show that any pair of points $P = \{(\alpha, a), (\beta, b)\}$ from distinct groups is contained in a unique block of $\mathcal{F}(\infty, i)$. 

Lemma 3.5. There is a PCS($6^k : 3$) for any $k \geq 3$.

Proof. For $k \equiv 0, 1 \pmod{3}$ with $k \geq 3$, there is a 2-FG(3, (3, 3, 4), 2k) of type $2^k$, which can be obtained by deleting two points from the known SQS(2k + 2). Apply Lemma 3.4 with $m = 3$. The input designs are PGDD($3^22^1$) in Lemma 3.2, $F(3, 3, 4[3])$ in Theorem 3.1 and CQS($3^2 : 1$) in [9]. Then we obtain a PCS($6^k : 3$).

For $k \equiv 2 \pmod{3}$ with $k \geq 3$, there is a 2-FG(3, (3, 5), 4, 6), 2k) of type $2^k$, which can be obtained by deleting two points from two distinct groups of the known CQS($6^{(k+1)/3}$) in [22]. Apply Lemma 3.4 with $m = 3$. The input designs are PGDD($3^22^1$) ($l = 3, 5$) in Lemmas 3.2 and 3.3, $F(3, 3, 4[3])$ ($j = 4, 6$) in Theorem 3.1 and CQS($3^m : 1$) ($m = 3, 5$), where CQS($3^5 : 1$) exists in [1]. Then we obtain a PCS($6^k : 3$). □

Lemma 3.6. There is an LSTS(6k + 3) for any $k \geq 0$.

Proof. For $k = 0$, it is trivial. For $k = 1$, an LSTS(9) was constructed by Kirkman. For $k = 2$, an LSTS(15) was constructed by Denniston [5]. For $k \geq 3$, there is a PCS($6^k : 3$) by Lemma 3.5. Applying Lemma 2.5 with the known HLSTS(9, 3) yields an LSTS(6k + 3). □

Combining Lemma 3.6 and Rosa’s doubling construction [26], we have the following.

Lemma 3.7. There is an LSTS(12k + 7) for any $k \geq 1$.

In Sections 5 and 6, we shall further use Lemma 3.4 to obtain PCSs with a stem of size 13.

4. A new proof of the tripling construction

In the construction of an LSTS(24k + 13), we need an HLSTS(37, 13). For this purpose, we state a tripling construction in this section, whose proof is simpler than Lu’s. In the proof of the tripling construction, we need a special partition of a uniform candelabra system with block size 3 and stem size 1.

A $CS(g^n : 1)$ ($X, \mathcal{S}, \mathcal{G}, \mathcal{A}$) is denoted by $PICS(g^n : 1)$ if its block set $\mathcal{A}$ can be partitioned into $gn - 1$ subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{gn - 1}$ with the properties: (1) for $1 \leq i \leq g - 1$, each $(X \setminus \mathcal{S}, \mathcal{G}, \mathcal{A}_i)$ is a GDD(2, 3, $gn$) of type $g^n$; (2) for $g \leq i \leq gn - 1$, each $(X, \mathcal{A}_i)$ is an STS($gn + 1$).
Lemma 4.2. There exists a PICS$\langle g^n : 1 \rangle$ having $g - 1$ sets of blocks $B^i_j$, $1 \leq j \leq g - 1$, then each $(\bigcup_{G \in G} B^i_j) \cup A_j$ is an STS$(gn + 1)$, and $(\bigcup_{G \in G} B^i_j) \cup A_j$ $(1 \leq j \leq g - 1)$ and $A_i$ $(g \leq i \leq gn - 1)$ form an LSTS$(gn + 1)$. Also, such an LSTS$(gn + 1)$ has a subdesign LSTS$(g + 1)$.

Lemma 4.3. Suppose that there is a PICS$\langle g^n : 1 \rangle$. If there is an LSTS$(g + 1)$, then there are an HLSTS$(g + 1, g + 1)$ and an LSTS$(g + 1)$.

Lemma 4.2. There exists a PICS$\langle 4^3 : 1 \rangle$.

Proof. Let $G = (GF(4), +)$, where the primitive polynomial $f(x) = x^2 + x + 1$ is used to generate the field. Let $\zeta$ be a primitive element of the field. Denote the field elements $0, \zeta^0, \zeta^1, \zeta^2$ by 0, 1, 2, 3. We shall construct the design on $X = (G \times Z_3) \cup \{\infty\}$ with groups $G_j = G \times \{j\}, j \in Z_3$, and a stem $S = \{\infty\}$.

We first construct three GDD$(2, 3, 12)$ of type $4^3$ on $G \times Z_3$ with groups $G_j, j \in Z_3$. The block sets $B_i$ of these are generated by the following set $A_i$ of base blocks under $(G, -)$, $1 \leq i \leq 3$, where $a_b$ stands for $(a, b)$.

\[ A_1: \begin{array}{cccccc}
00 & 01 & 12 & 00 & 01 & 22 \\
00 & 10 & 11 & 01 & 12 & 10 \\
02 & 11 & 31 & 02 & 12 & 30
\end{array} \]

\[ A_2: \begin{array}{cccccc}
00 & 01 & 12 & 00 & 01 & 22 \\
00 & 10 & 11 & 01 & 12 & 10 \\
02 & 11 & 31 & 02 & 12 & 30
\end{array} \]

\[ A_3: \begin{array}{cccccc}
00 & 01 & 12 & 00 & 01 & 22 \\
00 & 10 & 11 & 01 & 12 & 10 \\
02 & 11 & 31 & 02 & 12 & 30
\end{array} \]

Then, we construct two $S(2, 3, 13)$ on $X$, each having 26 blocks.

\[ \begin{array}{cccccccccccc}
00 & 01 & 02 & 00 & 10 & 11 & 00 & 12 & 20 & 00 & 02 & 30 \\
01 & 10 & 21 & 01 & 11 & 32 & 01 & 12 & 30 & 01 & 21 & 30 \\
02 & 11 & 31 & 02 & 12 & 30 & 02 & 12 & 30 & 10 & 12 & 22
\end{array} \]

From the above $S(2, 3, 13)$, we can obtain eight $S(2, 3, 13)$ under $(G, -)$. It has been checked that all block sets of GDD$(2, 3, 12)$ and $S(2, 3, 13)$ are pairwise disjoint. So, they form a PICS$\langle 4^3 : 1 \rangle$. □

We mention some preliminary results that will be used in the tripling construction.

Lemma 4.3. For any integer $u > 2$ with $u \neq 6$, there exists a GDD$(2, 3, 3u)$ of type $u^3$ whose set of blocks can be partitioned into parallel classes.

Proof. Such a GDD is equivalent to a pair of orthogonal Latin squares of order $u$. □
Let $X$ be a set of cardinality $tu$, and $H$ be a partition of $X$ into $u$ subsets of size $t$ (elements of $H$ are called holes). Let $L$ be a square array of size $tu$, indexed by $X$, which satisfies the following properties:

(1) if $x, y \in H \in \mathcal{H}$, then $L(x, y)$ is empty, otherwise $L(x, y)$ contains a symbol of $X$;
(2) row or column $x$ of $L$ contains the symbols in $X \setminus H$, where $x \in H \in \mathcal{H}$.

$L$ is called a partitioned incomplete Latin square (or PILS) of type $t^u$. $L$ is said to be symmetric if $L(x, y) = L(x, y)$ for all $x, y$ not in the same hole.

The following result is proved in [7].

**Lemma 4.4** ([Fu [7]]). Suppose $t$ is even and $u \geq 3$. Then a symmetric PILS of type $t^u$ exists.

Now, we are in a position to state a tripling construction for PICS, which has some similarities to the construction used in [4].

**Lemma 4.5.** Suppose that there is a PICS$(m^{3} : 1)$. Then for any integer $u > 2$ with $u \neq 6$, there is a PICS$((mu)^{3} : 1)$.

**Proof.** We will construct a PICS$((mu)^{3} : 1)$ on the point set $Y = X \cup \{\infty\}$, $X = Z_{m} \times Z_{u} \times Z_{3}$, with the group set $G = \{Z_{m} \times Z_{u} \times \{i\} : i \in Z_{3}\}$ and the stem $S = \{\infty\}$. The construction proceeds in several steps.

**Step 1:** Let $(Z_{u} \times Z_{3}, G', A_{0})$ be a resolvable GDD$(2, 3, 3u)$ of type $u^{3}$ in Lemma 4.3, where $G' = \{Z_{u} \times \{i\} : i \in Z_{3}\}$. For each $h \in Z_{u}$, define $A_{h} = \{(a, 0), (b, 1), (c + h, 2) : (a, 0), (b, 1), (c, 2)\} \in A_{0}$. Then each $(X, G', A_{h})$ is a GDD$(2, 3, 3u)$, and $X = \{a, b, c\}$ is a partition of all triples from three distinct groups.

For $A = \{(a, 0), (b, 1), (c, 2)\} \in A_{h}$ with $1 \leq h \leq u - 1$, construct $m$ disjoint GDD$(2, 3, 3m)$ on $Z_{m} \times A$ with groups $Z_{m} \times \{x\}, x \in A$. The required $m$ GDD$(2, 3, 3m)$ have block set $B_{A}^{3}, i \in Z_{m}$, which contains blocks $\{(i_{0}, a, 0), (i_{1}, b, 1), (i_{2}, c, 2)\}, i_{0} + i_{1} + i_{2} \equiv i \pmod{m}, i_{0}, i_{1}, i_{2} \in Z_{m}$.

For $1 \leq h \leq u - 1$ and $i \in Z_{m}$, define $B_{ih} = \bigcup_{A \in A_{h}} B_{A}^{i}$. Then each $(X, G, B_{ih})$ is a GDD$(2, 3, 3mu)$ of type $(mu)^{3}$.

**Step 2:** Here, we make use of $A_{0}$. Since it is resolvable, we can partition $A_{0}$ into $u$ parallel classes, $P_{0}, \ldots, P_{u}$, each of which partitions the point set $Z_{u} \times Z_{3}$.

For any $A \in P_{h}$ with $h \in Z_{u}$, construct a PICS$(m^{3} : 1)$ on $(Z_{m} \times A) \cup S$ with groups $Z_{m} \times \{x\}, x \in B$, and stem $S$. Such a design exists by assumption. Denote its block set by $B_{A}$. Then $B_{A}$ can be partitioned into $3m - 1$ parts $B_{A}^{i}, 1 \leq i \leq 3m - 1$, such that each $B_{A}^{i}$, $(1 \leq i \leq 2m)$ is the block set of an STS$(3m + 1)$ and each $B_{A}^{i}$ $(2m + 1 \leq i \leq 3m - 1)$ is the block set of a GDD$(2, 3, 3m)$ of type $m^{3}$.

For $2m + 1 \leq i \leq 3m - 1$, let $B_{A}^{i} = \bigcup_{A \in A_{0}} B_{A}^{i}$. Then $(X, G, B_{A}^{i})$ is a GDD$(2, 3, 3mu)$ of type $(mu)^{3}$. For $1 \leq i \leq m$, $j \in Z_{2}$ and $h \in Z_{u}$, let $D_{ih}^{j} = \bigcup_{A \in P_{h}} B_{A}^{mj+i}$.

**Step 3:** For $h \in Z_{u}, g \in Z_{3}$ and $j \in Z_{2}$, we define a permutation of $Z_{u}$ as follows:

$$x_{h,j}^{g}(x) = y \text{ if the pair } \{(x, g + j), (y, g + 1 - j)\} \text{ is contained in some block of } P_{h}. $$
Since there is a PICS\((m^3 : 1)\), we have that \(m\) is even. By Lemma 4.4, there is a symmetric PILS of type \(m^u\). Let \(L\) be such a square having holes \(Z_m \times \{h\}, h \in Z_u\).

For \(h \in Z_u, i \in Z_m\) and \(j \in Z_2\), define
\[
\varepsilon_{ih}^j = \{(a, b, g), (a', b', g), (a'' + i, z_h^g - j (b''), g + 1 + j) : a, a' \in Z_m, b, b' \in Z_u, g \in Z_3, (a'', b'') = L((a, b), (a', b'))\}.
\]

Let \(C_{ih}^j = D_{ih}^j \cup \varepsilon_{ih}^j\). Then each \(C_{ih}^j\) is the block set of an STS\((3mu + 1)\).

From Steps 1–3, we obtain \(2mu\) STS\((3mu + 1)\) and \(mu - 1\) GDD\((2, 3, 3mu)\) of type \((mu)^3\).

It has been checked that all these sets are disjoint. So, we obtain a PICS\(((mu)^3 : 1)\). \(\square\)

We use the above lemma to give a new proof of the tripling construction in [21].

**Theorem 4.6.** If there is an LSTS\((4u + 1)\), then there is an LSTS\((12u + 1)\). Further, if \(u \neq 2, 6\), there is an HLSTS\((12u + 1, 4u + 1)\).

**Proof.** For \(u = 2\), an LSTS\((12u + 1)\) is constructed by Denniston [5].

For \(u = 6\), from the proof of Example 2.6 there is an HLSTS\((33, 13)\). Since there is a CQS\((20^3 : 14)\) in Lemma 2.1, similar to the proof of Example 2.6 we obtain an LSTS\((12u + 1)\).

For \(u \neq 2, 6\) and there is a PICS\((4^3 : 1)\) in Lemma 4.2, by Lemma 4.5 there is a PICS\(((4u)^3 : 1)\). The conclusion then follows by Lemma 4.1 since there is an LSTS\((4u + 1)\) by assumption. \(\square\)

Since there is an LSTS\((12k + 9)\) by Lemma 3.6, we apply Theorem 4.6 to have the following.

**Lemma 4.7.** There exists an LSTS\((36k + 25)\) for any integer \(k \geq 0\).

By Lemmas 3.6, 3.7 and 4.7, we need to consider the case \(v \equiv 1, 13\ (\text{mod } 36)\). For \(v \equiv 1\ (\text{mod } 36)\), an LSTS\((v)\) can be obtained by applying Theorem 4.6 with an LSTS\(((v - 1)/3 + 1)\). So, the existence of an LSTS\((v)\) can be determined if we can prove the existence of an LSTS\((v)\) for \(v \equiv 13\ (\text{mod } 36)\). We had wanted to construct an LSTS\((36k + 13)\) from a PCS\((36^k : 13)\). However, we could not construct an HLSTS\((49, 13)\). Applying Theorem 4.6 with the known LSTS\((13)\) can yield an HLSTS\((37, 13)\). For these two reasons, we first construct an LSTS\((24k + 13)\) and an HLSTS\((24k + 13, 13)\) in Section 5, then construct an LSTS\((24k + 1)\) in Section 6.

**5. The existence of an LSTS\((24k + 13)\)**

In this section, we shall obtain a PCS\((24^k : 13)\) for any integer \(k \geq 3\) so as to construct an LSTS\((24k + 13)\).
Firstly, we give the existence of $\text{PGDD}(12^k11^1)$ and $\text{CQS}(12^k : 2)$.

The following is a construction for 3-CSs which is a special case of the fundamental construction of Hartman [12].

**Theorem 5.1.** Suppose that there exists a 1-FG$(3, (K_1, K_T), v)$ of type $g^n$. If there exists a CS$(3, L, bk_1 + s)$ of type $(b^{k_1} : s)$ for any $k_1 \in K_1$ and a GDD$(3, L, bk)$ of type $b^k$ for any $k \in K_T$, then there exists a CS$(3, L, vb + s)$ of type $(bg)^n : s)$.

In [10], it was proved that there is an $S(3, \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27, 29, 31\}, v)$ for $v \geq 4$. Later, this result was improved as follows.

**Lemma 5.2 (Hanani [10], Ji [14]).** For $v \geq 4$, there is an $S(3, \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27\}, v)$.

**Lemma 5.3 (Granville and Hartman [8]).** A $\text{CQS}(g^4 : s)$ exists for all even $g$, $s$ with $g \geq s$.

**Lemma 5.4.** There is a $\text{CQS}(12^k : 2)$ for any $k \geq 3$.

**Proof.** For each given $k$, there is an $S(3, \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27\}, k + 1)$ by Lemma 5.2. Deleting one point yields a 1-FG$(3, (\{3, 4, 5, 6, 8, 10, 12, 14, 18, 22, 26\}, \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27\}), k)$ of type $t^k$. Applying Theorem 5.1 with $b = 12$ gives the desired design. The input designs $\text{GDD}(3, 4, 12m')$ of type $12m' (m' \in \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27\})$ exist by Theorem 2.2. The other input designs $\text{CQS}(12m' : 2)$ are constructed below, where $m \in \{3, 4, 5, 6, 8, 10, 12, 14, 18, 22, 26\}$.

For $m \in \{3, 4, 6, 12, 18, 22\}$, it exists from the proof of [12, Theorem 5.6].

For $m = 5, 8$, there is an $S(3, 5, 3m + 2)$ in [11]. Deleting two points gives a 2-FG$(3, (4, 4, 5), 3m)$ of type $3^m$, which is also a 1-FG$(3, (4, \{4, 5\}), 3m)$ of type $3^m$. Apply Theorem 5.1 with $b = 4$ and the known CQS$(4^4 : 2)$ in Lemma 5.3, GDD$(3, 4, 4i)$ of type $4^i$ ($i = 4, 5$) by Theorem 2.2. Then we obtain a CQS$(12^m : 2)$.

For $m = 10$, a CQS$(36^3 : 14)$ exists by Lemma 2.1. Let $S$ be the stem, $\{x, y\} \subset S$ and let $S' = S \setminus \{x, y\}$. For each group $G$, construct a CQS$(12^4 : 2)$ on $G \cup S$ with $S'$ as a group and $\{x, y\}$ as a stem. Then we obtain a CQS$(12^m : 2)$.

For $m = 26$, start with a 1-FG$(3, (4, \{4, 5\}), 15)$ of type $3^5$. Apply Theorem 5.1 with $b = 20$ and the known CQS$(20^4 : 14)$ in Lemma 5.3, and GDD$(3, 4, 20i)$ of type $20^i$ ($i = 4, 5$) in Theorem 2.2. Then we obtain a CQS$(60^5 : 14)$, similar to the above, we can obtain a CQS$(12^m : 2)$ since a CQS$(12^6 : 2)$ exists.

For $m = 14$, there is an inversive plane of order 13, i.e., there is an $S(3, 14, 170)$. Fix two points $x, y$. For any block $B$ with $\{x, y\} \not\subset B$, construct an SQS$(14)$. Then we obtain a CQS$(12^m : 2)$, where all blocks containing $x$ and $y$, with these two points deleted, form the groups, and $\{x, y\}$ is a stem. This completes the proof. □

**Lemma 5.5.** Suppose that there exists a GDD$(3, K, g(n + 1))$ of type $g^{n+1}$. If there is an $F(3, 3, k\{m\})$ and a PGDD$(m^{k-1}(m - 1)^{1})$ for any $k \in K$, then there is a PGDD$((mg)^{n} (mg - 1)^{1})$. 

Let $(X, G, A)$ be the given GDD. Take a group $G_0 \in G$ and a point $z \in G_0$. Denote $G_x = \{x\} \times Z_m$ for $x \in X \setminus \{z\}$ and $G_z = \{z\} \times Z_{m-1}$. We shall construct the desired design on $X' = \bigcup_{x \in X} G_x$ with the group set $G' = \{G' = \bigcup_{x \in G} G_x : G \in G\}$. Its block set $\mathcal{F}$ is described below.

For each block $A \in \mathcal{A}$ with $z \in A$, suppose the given PGDD$(m|A|^{-1}(m-1)^{\frac{3}{2}})$ on $\bigcup_{x \in A} G_x$ with group set $\{G_x : x \in A\}$ has block set $D_A$. So, $D_A$ can be partitioned into $(m|A| - 3)$ disjoint block sets $D_A(x, i) (x \in A \setminus \{z\}, i \in Z_m)$ and $D_A(z, d) (2 \leq d \leq m - 2)$ such that each $D_A(x, i)$ is the block set of a GDD$(2, 3, m|A| - m)$ of type $m|A|^{-1}$ with the group set $\{G_y : y \neq z\} \cup \{G_z \cup \{(x, i)\}\}$, and such that each $((A \setminus \{z\}) \times Z_m, \{G_y : y \in A, y \neq z\}, D_A(z, d))$ is a GDD$(2, 3, m|A| - m)$ of type $m|A|^{-1}$.

For each block $A \in \mathcal{A}$ with $z \notin A$, construct a generalized frame $F(3, 3, |A||m|)$ on $\bigcup_{x \in A} G_x$ having $\Gamma_A = \{G_x : x \in A\}$ as its group set. Such a design exists by assumption. If we denote its block set by $D'_A$, then $D'_A$ can be partitioned into $m|A|$ disjoint block sets $D'_A(x, i) (x \in A, i \in Z_m)$ with the property that each $\{G_y \times Z_m, \Gamma_A \setminus \{G_x\}, D'_A(x, i)\}$ is a GDD$(2, 3, m(|A| - 1))$ of type $m|A|^{-1}$.

For any $x \in X \setminus G_0$ and $i \in Z_m$, let

$$\mathcal{F}(x, i) = \left( \bigcup_{\{x, z\} \subseteq A, A \in \mathcal{A}} D_A(x, i) \right) \bigcup \left( \bigcup_{A \cap \{x, z\} = \{x\}, A \in \mathcal{A}} D'_A(x, i) \right).$$

For any $2 \leq i \leq m - 2$, let

$$\mathcal{F}(z, i) = \bigcup_{z \in A, A \in \mathcal{A}} D_A(z, i).$$

For any $x \in G_0 \setminus \{z\}$ and $i \in Z_m$, let

$$\mathcal{F}(x, i) = \bigcup_{x \in A, A \in \mathcal{A}} D'_A(x, i).$$

Let

$$\mathcal{F} = \left( \bigcup_{i \in Z_m, x \in X \setminus \{z\}} \mathcal{F}(x, i) \right) \bigcup \left( \bigcup_{2 \leq i \leq m - 2} \mathcal{F}(z, i) \right).$$

It has been checked that $(X', G', \mathcal{F})$ is the desired PGDD$((mg)^n(mg - 1)^{\frac{3}{2}})$. This completes the proof. \hfill \Box

Lemma 5.6. There is a PGDD$(12^k 11^k)$ for any $k \geq 3$.

Proof. From the proof of [15, Lemma 4.10], there is a GDD$(3, \{4, 6\}, 4(k + 1))$ of type $4^{k+1}$ for each given $k$. Apply Lemma 5.5 with $m = 3$, the known $F(3, 3, i\{3\}) (i = 4, 6)$ and PGDD$(3^j 2^j)$ ($j = 3, 5$). Then we obtain a PGDD$(12^k 11^k)$. \hfill \Box

Now, we give an infinite class of PCSs with stem size 13.
Lemma 5.7. There is a PCS$(24^k : 13)$ for any $k \geq 3$.

Proof. From the proof of Lemma 3.5, there is a 2-FG(3, $\{3, 5\}$, $\{3, 5\}$, $\{4, 6\}$, 2)$ of type $2^k$ for any $k \geq 3$. Apply Lemma 3.4 with $m = 12$. The input designs are $F(3, 3, k\{12\})$ $(k = 4, 6)$ in Theorem 3.1, PGDD$(12^k 11^1)$ $(k = 3, 5)$ in Lemma 5.6, and CQS$(12^k : 2)$ $(k = 3, 5)$ in Lemma 5.4. Then we obtain a PCS$(24^k : 13)$. □

Lemma 5.8. There is an LSTS$(24k + 13)$ for any $k \geq 0$ and an HLSTS$(24k + 13, 13)$ for any $k \geq 1$ with $k \neq 2$.

Proof. For $k = 0$, an LSTS(13) was constructed in [5]. For $k = 1$, applying Theorem 4.6 with the known LSTS(13) gives an LSTS(37) and an HLSTS(37, 13). For $k = 2$, an LSTS(61) exists by Theorem 4.6 since there is an LSTS(21) in Lemma 3.6. For $k \geq 3$, there is a PCS$(24^k : 13)$ by Lemma 5.7. Applying Lemma 2.5 with the known HLSTS(37, 13) gives an HLSTS$(24k + 13, 13)$. Further, inputting an HLSTS(37, 13) gives an HLSTS$(24k + 13, 37)$. While inputting an LSTS(37) gives an LSTS$(24k + 13)$. □

6. The existence of an LSTS$(24k + 1)$

In this section, we shall construct some non-uniform PCSs with stem size 13 in order to construct an LSTS$(24k + 1)$.

Let $P = \{k : \exists$ a 2-FG(3, $(N, N, N)$, $k$) of type $1^1 g_1^{a_1} \cdots g_r^{a_r}$, all $g_i$ be even with $g_i \neq 4$ $\}$, where $N = \{k \geq 4 : k$ is an integer}. We shall show that $k \in P$ for any odd $k \geq 37$ with $k \notin \mathcal{L} = \{m : m$ is an odd, $m \in [41, 55] \cup [75, 79] \cup [159, 175], m \neq 43\}$, where $[c, d]$ denotes the set $\{h : h$ is an integer and $c \leq h \leq d\}$. This result is obtained by deleting some points from the known inverse planes.

The next lemma is the well-known result on $S(3, k, \nu)$.

Lemma 6.1 (Hanani [11]). Let $q$ be a prime power. There is an $S(3, q + 1, q^2 + 1)$.

Lemma 6.2. $k \in P$ for any odd $k \geq 37$ with $k \notin \mathcal{L}$.

Proof. We first prove the fact that for any odd prime power $q \geq 7$, $k \in P$ for any odd $k \in [(q - 1)(q + 5)/2 + 1, (q - 1)q + 1]$ with a possible exception $k = 41$ and $q = 7$.

Let $(X, \mathcal{G}, B_1, B_2, \mathcal{T})$ be a 2-FG(3, $(q, q, q + 1)$, $q^2 - 1$) of type $(q - 1)^{q + 1}$, which can be obtained by deleting two points from an $S(3, q + 1, q^2 + 1)$ in Lemma 6.1. Let $\mathcal{G} = \{G_0, \ldots, G_q\}$. Firstly, delete all but one point from the group $G_0$. Then delete all but $a_i$ points from $G_i$ for $1 \leq i \leq (q - 5)/2$, where $a_i \in \{0, 2, 6, 8, \ldots, q + 1\}$. Clearly, for any $0 \leq j \leq q$, $|B \cap G_j| \leq 2$ for any block $B \in \mathcal{T}$, and $|B \cap G_j| \leq 1$ for any block $B \in B_1 \cup B_2$. So, the truncated design is a 2-FG(3, $\{(q + 3)/2, q\}$, $\{(q + 3)/2, q\}$, $\{q, q + 1\}$, $1 + (q - 1)(q - 5)/2 + \sum_{1 \leq i \leq (q - 5)/2} a_i$) of type $1^1(q - 1)^{(q + 5)/2}a_1^1a_2^1 \cdots a_{(q - 5)/2}^1$, where the blocks in $B_i$ $(i = 1, 2)$ with those points deleted form the block set of the sub-design GDD$(2, \{(q + 3)/2, q\}, 1 + (q - 1)(q - 5)/2 + \sum_{1 \leq i \leq (q - 5)/2} a_i)$. Since $q \geq 7$, we have that $(q + 3)/2 \geq 4$ and $\{(q + 3)/2, q\} \subset N$. So, $k \in P$ for any odd $k \in [(q - 1)(q + 5)/2 + 1, (q - 1)q + 1]$ with a possible exception $k = 41$ and $q = 7$. 
Then for each \( k \in [37, 1333] \), by the above fact we can prove \( k \in P \). We only give the appropriate \( q \) such that \( k \in [(q - 1)(q + 5)/2 + 1, q(q - 1) + 1] \). We list the \( k \) and \( q \) as follows:

<table>
<thead>
<tr>
<th>( k )</th>
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<th>( k )</th>
<th>( q )</th>
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</thead>
<tbody>
<tr>
<td>[37, 43) ( \backslash ) {41}</td>
<td>7</td>
<td>[217, 343]</td>
<td>19</td>
</tr>
<tr>
<td>[57, 73]</td>
<td>9</td>
<td>[309, 507]</td>
<td>23</td>
</tr>
<tr>
<td>[81, 111]</td>
<td>11</td>
<td>[477, 813]</td>
<td>29</td>
</tr>
<tr>
<td>[177, 273]</td>
<td>17</td>
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For \( k \geq 1335 \), our proof of \( k \in P \) is based on induction. For each prime \( q \geq 33 \), by the above fact we have that \([(q - 1)(q + 5)/2 + 1, q(q - 1) + 1] \subset P \). Since there exists a prime between \( x \) and \( 8x/7 \) for \( x \geq 33 \) in [10], there always exists a prime \( q' \) such that \( q + 1 \leq q' \leq 8(q + 1)/7 \). By the above fact we also have that \([(q' - 1)(q' + 5)/2 + 1, (q' - 1)q' + 1] \subset P \). Since \( q \geq 33 \) and \( q' \leq 8(q + 1)/7 \), then \((q' - 1)(q' + 5)/2 + 1 < (q - 1)q + 1 \). So these two consecutive intervals intersect. Since for \( q = 37 \), \([(q - 1)(q + 5)/2 + 1, q(q - 1) + 1] \subset P \) and \((q - 1)(q + 5)/2 + 1 < 1333 \), the consecutive intervals obtained continuously cover all odd \( k \geq 1333 \). The conclusion then follows. \( \square \)

From the above result on 2-FGs, we have the following results on PCSs.

**Lemma 6.3.** For any odd \( k \geq 37 \) with \( k \not\in \mathcal{L} \), there is a PCS\((12^1(12g_1)^{a_1} \cdots (12g_r)^{a_r} : 13)\) with \( k = 1 + \sum_{1 \leq i \leq r} a_i g_i \), all \( g_i \)'s even and \( g_i \not= 4 \).

**Proof.** For each given \( k \), there is a 2-FG\((3, (N, N, N), k)\) of type \( 1^1 g_1^{a_1} \cdots g_r^{a_r} \) with \( g_i \) being even and \( g_i \not= 4 \) by Lemma 6.2. Apply Lemma 3.4 with \( m = 12 \). Since there are a PGDD\((12^n11^1)\) by Lemma 5.6, a CQS\((12^n : 2)\) by Lemma 5.4 and \( F(3, 3, n\{12\}) \) by Theorem 3.1 for any \( n \geq 4 \), we have the result. \( \square \)

**Lemma 6.4.** For any odd \( k \geq 37 \) with \( k \not\in \mathcal{L} \), there is an HLSTS\((12k + 13, 25)\) and an LSTS\((12k + 13)\).

**Proof.** For each given \( k \), by Lemma 6.3 there is a PCS\((12^1(12g_1)^{a_1} \cdots (12g_r)^{a_r} : 13)\) with \( k = 1 + \sum_{1 \leq i \leq r} a_i g_i \), all \( g_i \) being even and \( g_i \not= 4 \). Since there is an HLSTS\((12g_i + 13, 13)\) by Lemma 5.8, there is an HLSTS\((12k + 13, 25)\) by Lemma 2.5. Further, inputting an LSTS\((25)\) in [5] gives an LSTS\((12k + 13)\). \( \square \)

By Lemmas 3.6, 3.7, 4.7, 5.8 and 6.4, we need to consider the orders \( v = 12k + 13 \), \( k \in (\mathcal{L} \cup \{m : m \text{ is odd and } 3 \leq m \leq 35\}) \backslash \{7, 13, 19, 25, 31, 49, 55, 79, 163, 169, 175\} \).

**Lemma 6.5.** For \( k \in [27, 51, 171] \), there is an LSTS\((12k + 13)\).

**Proof.** Start with a 2-FG\((3, (3, 3, 4), (k - 3)/4)\) of type \( 2^{(k - 3)/8} \), which is obtained by deleting two points from an SQS\(((k - 3)/4 + 2)\). Apply Lemma 3.4 with the \( F(3, 3, 4\{48\}) \).
CQS\((48^3 : 2)\) and PGDD\((48^3 47^1)\). The last input design can be obtained by applying Lemma 5.5 with the known GDD\((3, 4, 16)\) of type \(4^4\), \(F(3, 3, 4\{12\})\) and PGDD\((12^3 11^1)\). Then we obtain a PCS\((96^{(k-3)/8} : 49)\). Since there is an LSTS\((49)\) in [5], by Theorem 4.6 there is an HLSTS\((145, 49)\) and an LSTS\((145)\). The result then follows from Lemma 2.5. □

Lemma 6.6. There is a PGDD\((5^3 4^1)\).

Proof. The desired design is constructed on \(X = (Z_5 \times Z_3) \cup S\) with groups \(G_i = Z_5 \times \{i\}\), \(i \in Z_3\), and \(S\), where \(S = \{\infty\} \times \{1, 2, 3, 4\}\). We simply write \(a_i\) for \((a, i)\).

Let 
\[
D(0) = \{[0_0, 0_1, 3_2], [0_0, 1_1, 0_2], [0_0, 2_1, 2_2], [0_0, 3_1, 4_2], [0_0, 4_1, 1_2]\},
\[
D(1) = \{[0_0, 0_1, 4_2], [0_0, 1_1, 1_2], [0_0, 2_1, 3_2], [0_0, 3_1, 0_2], [0_0, 4_1, 2_2]\}.
\]

For \(j \in Z_2\), let \(C(j) = \{D + i_0 : i \in Z_5, D \in D(j)\}\). Then each \(C(j)\) is the block set of a GDD\((2, 3, 15)\) of type \(5^3\) with groups \(G_0, G_1\) and \(G_2\).

Let 
\[
A(0, 0) = \{[0_0, 0_2], [1_1, 2_2], [2_1, 4_2], [3_1, 1_2], [4_1, 3_2]\},
\[
A(0, 1) = \{[0_0, 1_2], [4_0, 2_2], [3_0, 3_2], [2_0, 4_2], [1_0, 0_2]\},
\[
A(0, 2) = \{[0_0, 4_1], [3_0, 1_2], [1_0, 2_1], [4_0, 1_1], [2_0, 0_1]\}.
\]

For \(j \in Z_3\), let \(B(0, j) = \{[0_0] \cup A : A \in A(0, j)\} \cup \{\infty_4\} \cup (A + k_0) : A \in A(0, j), 1 \leq k \leq 4\). For \(i \in Z_5\) and \(j \in Z_3\), let \(B(i, j) = \{A + i_0 : A \in B(0, j)\}\). Then each \(B(i, j)\) is the block set of a GDD\((2, 3, 15)\) with groups \(G_{i+1}, G_{i-1}\) and \((i, j) \cup S\).

It is easy to see that these \(17\) GDD\((2, 3, 15)\) of type \(5^3\) are pairwise disjoint and they form a PGDD\((5^3 4^1)\). □

Lemma 6.7. For \(k \in \{5, 15, 35, 45, 75, 165\}\), there is an LSTS\((12k + 13)\).

Proof. For the given \(k\), there is a 2-FG\((3, 3, 4, 6k/5)\) of type \(2^{3k/5}\), which is obtained by deleting two points from an SQS\((2 + 6k/5)\). Apply Lemma 3.4 with the known \(F(3, 3, 4\{10\})\), CQS\((10^3 : 4)\) and PGDD\((10^{3^1})\). The last one can be obtained by Lemma 5.5 with the known GDD\((2, 4, 8)\) of type \(2^4\), \(F(3, 3, 4\{5\})\) and PGDD\((5^3 4^1)\) in Lemma 6.6. We then obtain a PCS\((20^{3k/5} : 13)\). Since there is an HLSTS\((33, 13)\) in the proof of Example 2.6 and an LSTS\((33)\), applying Lemma 2.5 gives an LSTS\((12k + 13)\). □

Lemma 6.8. For \(k \in \{3, 9, 11, 17, 21, 23, 29, 33, 41, 47, 53, 77, 159, 161, 167, 173\}\), there is an LSTS\((12k + 13)\).

Proof. For \(k = 3\), it is constructed in [5].

For \(k = 9\), by Rosa’s doubling construction there is an HLSTS\((15, 7)\). From Teirlinck’s tripling construction of an LSTS\((45)\) from an LSTS\((15)\) there is an HLSTS\((45, 15)\). So, there is an HLSTS\((45, 7)\). Since there is a CQS\((38^3 : 8)\) in Lemma 2.1, there is a PCS\((38^3 : 7)\). Further applying Lemma 2.5 with the known LSTS\((45)\) in Lemma 3.6 gives an LSTS\((12k + 13)\).
For \( k = 21 \), since there is an LSTS(33) in Lemma 3.6, by Theorem 4.6 there is an HLSTS(97, 33). By replacing the hole with an HLSTS(33, 13) in the proof of Example 2.6, we obtain an HLSTS(97, 13). Since there is a CQS(84\(^3\) : 14) in Lemma 2.1, by Lemma 2.4 there is a PCS(84\(^3\) : 13). Then an LSTS(12\(k + 13\)) follows from Lemma 2.5 with the known LSTS(97) in Lemma 4.7.

For \( k = 33 \), by Lemmas 2.4 and 2.1, there is a PCS(58\(^3\) : 3) and a PCS(116\(^3\) : 61). Apply Lemma 2.5 with the known LSTS(61) gives an HLSTS(177, 61). Since there is a CQS(12\(^n\) : 3) in Lemma 2.1, by Lemma 2.4 there is a PCS(12\(^n\) : 13). Then an LSTS(12\(k + 13\)) follows from Lemma 2.5 with the known LSTS(97) in Lemma 4.7.

For \( k = 159 \), deleting nine points from a group in a 2-FG\((3, (N, N, N), k)\) of type 12\(^{14}\) gives a 2-FG\((3, (N, N, N), k)\) of type 12\(^{13}3^1\). Apply Lemma 3.4 with \( m = 12 \). Since there is a PGDD(12\(^n\) : 11\(^1\)) by Lemma 5.6 and a CQS(12\(^n\) : 2) by Lemma 5.4 for any \( n \geq 4 \), we have a PCS(144\(^3\) : 36\(^1\) : 13). Since there is an HLSTS(157, 13) in Lemma 5.8 and an LSTS(49), there is an LSTS(12\(k + 13\)).

For the other given \( k \), since there is an LSTS(4\(k + 5\)) in the above, by Theorem 4.6 there is an LSTS(12\(k + 13\)). This completes the proof.

Combining Lemmas 4.7, 6.4, and 6.7–6.8, we have the following.

**Lemma 6.9.** There is an LSTS(2\(4k + 1\)) for any \( k \geq 1 \).

### 7. Conclusion

Combining Lemmas 3.6, 3.7, 5.8, and 6.9, we obtain a new proof of Theorem 1.1.

**Remark.** We use the existence of an SQS\((v)\) and CQS\((6^k : 0)\) to obtain a 2-FG\((3, (3, 5, 3, 5, 4, 6), 2k)\) of type 2\(^k\). When we apply Lemma 3.4 with \( m = 36 \) and the known PGDD(36\(^3\) : 35\(^1\)), CQS(36\(^i\) : 14) \((i = 3, 5)\) and \( F(3, 3, l[36]) \) \((l = 4, 6)\), we obtain a PCS(72\(^k\) : 49). Although we have spent much time, we could not find an HLSTS(121, 49). If such a design exists, then the proof of the existence of an LSTS(24\(k + 1\)) in Section 6 can be reduced much more. When an HLSTS(121, 49) exists, we can use the known PCS(72\(^k\) : 49) to construct an LSTS(72\(k + 49\)). Applying Theorem 4.6 with the known LSTS(24\(k + 9\)) in Lemma 3.6 gives an LSTS(72\(k + 25\)). We then apply Theorem 4.6 repeatedly to solve the case 72\(k + 1\). So, it is worth constructing an HLSTS(121, 49).

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[16] L. Ji, Partition of triples of order $6k + 5$ into $6k + 3$ optimal packings and one packing of size $8k + 4$, Graphs Combin., to appear.


