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Competition numbers of graphs with a small number of triangles

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Abstract

If D is an acyclic digraph, its competition graph is an undirected graph with the same vertex set and an edge between vertices x and y if there is a vertex a so that (x, a) and (y, a) are both arcs of D. If G is any graph, G together with sufficiently many isolated vertices is a competition graph, and the competition number of G is the smallest number of such isolated vertices. Roberts (1978) gives a formula for the competition number of connected graphs with no triangles. In this paper, we compute the competition numbers of connected graphs with exactly one or exactly two triangles.

Keywords: Competition graph; Competition number

1. Introduction

In this paper, we study the notion of competition graph which was introduced by Cohen [3] and has been widely studied since. If D is an acyclic digraph (V, A), then its *competition graph* is an undirected graph $G = (V, E)$ with the same vertex set and an edge between vertices x and y if there is a vertex a in V and arcs (x, a) and (y, a) in D. We say that a graph G is a competition graph if it arises as the competition graph of some acyclic digraph. (Sometimes the condition of acyclicity is weakened; see, for example, the papers by Dutton and Brigham [4] and Roberts and Steif [12]. However, we do not weaken the condition here.) Competition graphs arose in connection with an application in ecology and also have applications in coding, radio transmission, and modelling of complex economic systems. (See [10] for a summary of these applications and [6] for a sample paper on the modelling application.) The vast

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literature of competition graphs is summarized in the survey paper by Lundgren [8]. We shall study the notion of competition number which arose in connection with the attempts to characterize competition graphs.

If G and H are graphs with disjoint vertex sets, $G \cup H$ will denote the graph whose vertex and edge sets are the unions of those of G and H. We will not use the notation $\cup$ unless the graphs G and H have disjoint vertex sets. I_r will denote a graph with r isolated vertices and, hence, if G is a graph, $G \cup I_r$ is the graph obtained from G by adding r isolated vertices. Roberts [11] observed that if G is any graph, then $G \cup I_r$ is a competition graph of an acyclic digraph for r sufficiently large. He defined the *competition number* of G, $k(G)$, to be the smallest such r, and observed that characterization of competition graphs is equivalent to computation of competition number. The notion of competition number has since been widely studied, as have variants such as niche number and double competition number (see, for example, [2, 5, 7, 8, 13]). Roberts also proved the following simple theorem, which has proven to be widely useful, and is the starting point for this paper. In this theorem, as in the rest of the paper, $n(G)$ or simply n is the number of vertices of G and $q(G)$ or simply q is the number of edges.

Theorem 1 (Roberts [11]). *If G is a connected graph with no triangles, then $k(G) = q - n + 2$.*

Since Theorem 1 has proven to be so useful, we started out by asking what will happen if we had a connected graph with exactly one triangle. When we were able to answer that question, it seemed natural to try the case of two triangles. That case turned out to be more challenging. We have found the results useful because they give us precise formulas for the competition number of many interesting graphs, and so are very helpful in checking conjectures and testing algorithms about competition graphs – the same kinds of things which make a simple result like Theorem 1 so helpful.

Before closing this introductory section, we state two more results that will be useful in the following. If G is a graph, $\theta_e(G)$ is the smallest number of cliques in an *edge clique covering* of G, a collection of cliques that covers all edges of G.

Theorem 2 (Opsut [9]). *For any graph G, $\theta_e(G) \leqslant k(G) + n - 2$.*

We say that a graph G is *triangulated* if G does not have cycles of length greater than three as generated subgraphs.

Theorem 3 (Roberts [11]). *If G is triangulated, then $k(G) \leqslant 1$.*

In what follows, it will be useful to adopt some terminology that is commonly used in the literature of competition graphs and has its origins in the ecological applications of the subject. Specifically, if (x, y) is an arc of digraph D, we call y the *prey* of x, and if (x, a) and (y, a) are arcs, we say that a is a *common prey* of x and y. For undefined graph-theoretical terms, the reader is referred to [1].

2. Graphs with exactly one triangle

If T is a tree, then it is triangulated and so by Theorem 3, its competition number is at most 1. In fact, we have the following result. The algorithm given in the proof will be referred to later.

Lemma 4. *For a tree T and a vertex v of T, there is an acyclic digraph D so that $T \cup \{v_0\}$ is the competition graph of D for v_0 not in T and so that v has only outgoing arcs in D.*

Proof. The following algorithm is essentially that used to prove Theorem 1. Let $T_1 = T$, $V(D_1) = V(T)$, and $A(D_1) = \phi$. Choose a vertex v_1 of degree 1 from T_1. If v^1 is adjacent to v_1 in T_1, let $T_2 = T - v_1$, $V(D_2) = V(D_1) \cup \{v_0\}$ for some vertex v_0 not in T, and $A(D_2) = \{(v_1, v_0), (v^1, v_0)\}$. Having defined T_i and D_i, choose a vertex v_i of degree 1 from T_i. If v^i is adjacent to v_i in T_i, then let $T_{i+1} = T_i - v_i$, $V(D_{i+1}) = V(D_i)$, and $A(D_{i+1}) = A(D_i) \cup \{(v_i, v_{i-1}), (v^i, v_{i-1})\}$. Repeat this last step until D_n has been defined. Let $D = (V(D_n), A(D_n))$. In the procedure, we may avoid selecting v until we select all other vertices since there are at least two vertices of degree 1 in a tree with more than one vertex. □

If D is an acyclic digraph, we can find a vertex labelling $\pi : V(D) \to \{1, 2, \ldots, |V(D)|\}$ so that whenever (x, y) is in $A(D)$, $\pi(y) < \pi(x)$. We call π an *acyclic labelling* of D. (Note that the proof of Lemma 4 produces an acyclic labelling of D if $\pi(v_i) = i$.) Conversely, if D is a digraph with an acyclic labelling, then D is acyclic.

Theorem 5. *If G is connected and has exactly one triangle, then $k(G) = q - n$ or $q - n + 1$.*

Proof. Since $\theta_e(G) = (q - 3) + 1 = q - 2$, the lower bound $k(G) \geqslant q - n$ now follows from Theorem 2.

To prove the upper bound $k(G) \leqslant q - n + 1$, let $\{x, y, z\}$ be the vertex set of the triangle and delete the edge $\{x, y\}$ from G. The resulting graph $G - \{x, y\}$ is connected and triangle-free, so Theorem 1 implies that $k(G - \{x, y\}) = (q - 1) - n + 2 = q - n + 1$. Let D' be an acyclic digraph whose competition graph is $G - \{x, y\} \cup I_{q-n+1}$. Since $\{y, z\}$ and $\{x, z\}$ belong to $E(G - \{x, y\})$, there are arcs (y, a) and (z, a) and also (x, b) and (z, b) in D' for vertices a and b of D'. Now $a \neq b$ since $\{x, y\} \notin E(G - \{x, y\})$, $\pi(a) \neq \pi(b)$ and we may assume that $\pi(a) < \pi(b)$, where π is an acyclic labelling for D'. Add an arc (x, a) to D' to obtain digraph D. This is acyclic because $\pi(x) > \pi(b)$ and so $\pi(x) > \pi(a)$. Moreover, the competition graph of D is $G \cup I_{q-n+1}$, and so $k(G) \leqslant q - n + 1$. $\square$

Theorem 6. *Suppose G is connected and has at least two cycles, including at least one triangle. Then $k(G) \leqslant q - n$.*

Proof. Let x, y, and z be the vertices of triangle S. Let T be a spanning tree of G that has exactly two edges of S. Now we delete those two edges from T. Then the resulting graph is a forest with exactly three tree components, say T_1, T_2, T_3. Clearly, each component contains exactly one of x, y, z. We may assume that x belongs to T_1. Since there is one more cycle than S, there is an edge $\{f, g\}$ in $E(G) - E(T)$. We may assume that T_1 does not contain f or g. Whether or not f and g belong to the same component, we may assume that g belongs to T_3. By Lemma 4, there are acyclic digraphs D_i and added vertices u_i so that the competition graph of D_i is $T_i \cup \{u_i\}$, $i = 1, 2, 3$. We may assume that the u_i have no outgoing arcs. By Lemma 4, we may assume that x has only outgoing arcs in D_1. In D_i, the vertices v and v' of the highest and second-highest indices, respectively, in an acyclic labelling can be assumed to have only outgoing arcs since the only possible incoming arc to either of these vertices is from v to v' and we can always delete this arc without changing the competition graph. Now let y_1 be another vertex that has only outgoing arcs in D_1 and let x_2 and y_2 be vertices having only outgoing arcs in D_2. Let D' be a digraph whose vertex set is $V(G) \cup \{u_1, u_2, u_3\}$ and whose arc set is

$$
\begin{aligned}
&A(D_1) \cup A(D_2) \cup A(D_3) \cup \{(x, y_1), (y, y_1), (z, y_1)\} \\
&\quad - \{(v, u_2)\colon (v, u_2) \in A(D_2)\} \cup \{(v, x)\colon (v, u_2) \in A(D_2)\} \\
&\quad - \{(v, u_3)\colon (v, u_3) \in A(D_3)\} \cup \{(v, x_2)\colon (v, u_3) \in A(D_3)\}.
\end{aligned}
$$

We note that D' is acyclic. This is because every arc added here goes from D_i to D_j with $i > j$ except for (x, y_1). Since x and x_2 have only outgoing arcs in D_1 and D_2, respectively, no new cycles are created. By Lemma 4, we may assume that $f = x_2$ if f belongs to T_2. Now we add arcs (f, y_2), (g, y_2) to D' to obtain D''. This still leaves an acyclic digraph since the arc (g, y_2) goes from D_3 to D_2 and the arc (f, y_2) goes from D_3 to D_2 or from a vertex with no incoming arcs in D_2. Since u_2, u_3 have neither incoming nor outgoing arcs in D'' and x, x_2, y_1, y_2 have only outgoing arcs in the D_i, $D'' - \{u_2, u_3\}$ has competition graph $T \cup \{u_1\}$ together with the edges of the triangle and the edge $\{f, g\}$. There are $q - n - 1$ edges in $G - T$ other than $\{f, g\}$ that are not edges on triangles. Add to D'' a new isolated vertex i_e corresponding to each such edge e and arcs from the end vertices of e to i_e. The resulting digraph is still acyclic and has competition graph $G \cup I_{q-n}$. $\square$

Corollary 7. *Suppose G is connected and has exactly one triangle. Then $k(G) = q - n$ if G has a cycle of length at least 4 and $k(G) = q - n + 1$ otherwise.*

Proof. Suppose G has a cycle of length at least 4. Then by Theorem 6, $k(G) \leqslant q - n$ and so by Theorem 5, $k(G) = q - n$. Since G is connected, $k(G) \geqslant 1$, since every acyclic digraph has a vertex with no outgoing arcs. If G does not have any cycle other than the triangle, by Theorem 3, $k(G) = 1 = q - n + 1$. □

3. Graphs with exactly two triangles

Let $\mathscr{L}$ be the family of graphs that can be obtained from one of the graphs in Fig. 1 by subdividing edges except those on triangles. (Subdividing means adding any number (including zero) of vertices along the edges.) Let

$$\mathrm{VC}(G) = \{v \in V(G)\colon v \text{ is a vertex on a cycle of } G\}.$$

Lemma 8. *Suppose G is connected and* VC(G) *generates a subgraph in $\mathscr{L}$. Then $k(G) \geqslant 2$.*

Proof. Let $k(G) < 2$. Then $k(G) = 1$. Let D be an acyclic digraph whose competition graph is $G \cup I_1$. We can label the vertices of D as $v_1, \ldots, v_{n+1}$ so that $\pi(v_i) = i$ gives an acyclic labelling and where v_1 is the isolated vertex added to G. Let K be a subgraph in $\mathscr{L}$ generated by VC(G) and let v_r be the vertex of lowest index among the vertices of K. By construction, every vertex in K has a pair of nonadjacent neighbors and so v_r has at least two outgoing arcs and r is greater than 2. Since no vertex in $v_1, \ldots, v_{r-1}$ is in VC(G), the subgraph generated by these vertices is a forest. Suppose this forest has t tree components for some integer t. Then there are $r - 1 - t$ edges in the forest. Since all of these edges are maximal cliques of G, at least $r - 1 - t$ vertices are needed to serve as common prey of the end vertices of these edges, and these common prey each

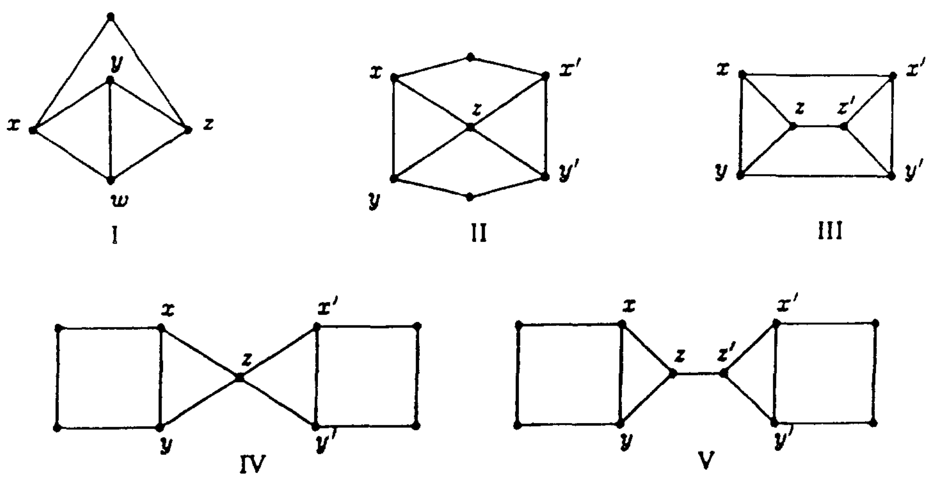

Fig. 1. Some graphs having two triangles.

have index less than $r-2$. Except for the component consisting of the isolated vertex v_1, there is at least one vertex in each of the components that is joined to a vertex of index higher than $r-1$, since G is connected. Let X be the set of edges of G having exactly one end vertex $v_i, i \leqslant r-1$. Then X has at least $t-1$ edges. All of the edges in X are maximal cliques since $v_1, \dots, v_{r-1}$ are not in VC(G). Thus, each such edge corresponds to a different common prey of its end vertices and these common prey all have indices at most $r-2$. It follows that there are at least $(r-1-t)+(t-1)=r-2$ common prey used for the edges in the forest and in X, and thus every vertex in $v_1, \dots, v_{r-2}$ is used as such a prey. Now v_r is adjacent to nonadjacent vertices v_x and v_y in K. The only available common prey for v_r and v_x is v_{r-1}, and similarly for v_r and v_y. This is impossible since v_x and v_y could not have a common prey. We have reached a contradiction. □

Let H be the first graph in Fig. 1 and $\mathscr{H}$ be the family of graphs obtained from H by subdividing edges not on the triangles.

Theorem 9. *Suppose a connected graph G has exactly two triangles and these share one of their edges. Then*
(a) $k(G)=q-n$ *if G is triangulated or* VC(G) *generates a subgraph in* $\mathscr{H}$.
(b) $k(G)=q-n-1$ *otherwise.*

Proof. Theorem 6 implies that $k(G) \leqslant q-n$. We next show that $k(G) \geqslant q-n-1$. Note that $\theta_e(G)=q-5+2=q-3$ and by Theorem 2,

$$k(G) \geqslant (q-3)-n+2=q-n-1.$$

Suppose that G is triangulated. Then by Theorem 3, $k(G)=1$. There can be no cycles other than the two triangles and so we can choose two edges to remove to get a tree. It follows that $q=n+1$ and so $k(G)=q-n$. Suppose next that VC(G) generates a subgraph in $\mathscr{H}$. Then by Lemma 8, $k(G) \geqslant 2$. Every graph in $\mathscr{H}$ has $q-n=2$. Thus, $k(G) \geqslant q-n$ and so $k(G)=q-n$.

To prove part (b), suppose that G is not triangulated and that VC(G) generates a subgraph that does not belong to $\mathscr{H}$. Let the two triangles have vertices labelled as in the first graph of Fig. 1. We are going to build an acyclic digraph whose competition graph is $G \cup I_{q-n-1}$. Since $k(G) \geqslant q-n-1$, the result will follow. We can easily show that G has a spanning tree with exactly three edges of the triangles.

$G-T$ has $q-n+1$ edges, with $q-n-1$ of them not on the triangles. Now we delete the three edges of the triangles from T. The resulting graph F is a forest with four components in which the four vertices x, y, z, w are disconnected from each other. We denote the components of F by T_1, T_2, T_3, T_4. Each contains exactly one of the vertices x, y, z, w.

Case 1: *There is an edge* $\{f, g\}$ *in* $G-T$ *that is not a triangle edge and a component* T_i *of* F *such that either* x *or* z *belongs to* T_i *and* f, g *do not.*

We may assume that T_1 is such a component and we may also assume that x belongs to T_1. There is another component, say T_2, that does not contain f and g. Let x_2 be a vertex in T_2 on one of the triangles. If f and g belong to the same component, we can assume it is T_4. If they belong to different components, we can assume that f belong to T_3 and g to T_4. By Lemma 4, there are acyclic digraphs D_i and added vertices u_i so that the competition graph of D_i is $T_i \cup \{u_i\}$, $i = 1, \ldots, 4$. We may assume that the u_i have no outgoing arcs. By Lemma 4, we may also assume that x and x_2 have only outgoing arcs in D_1 and D_2, respectively. We may assume that there are at least two vertices of indegree 0 in D_i. Let x_3 and y_3 be vertices of T_3 having only outgoing arcs in D_3; by Lemma 4, we may assume that $f = x_3$ if f is in T_3. Similarly, we may assume that there are other vertices y_1 and y_2 having only outgoing arcs in D_1 and D_2, respectively. Let D' be a digraph whose vertex set is $V(G) \cup \{u_1, u_2, u_3, u_4\}$ and whose arc set is

$$A(D_1) \cup A(D_2) \cup A(D_3) \cup A(D_4) \cup \{(x, y_1), (y, y_1), (w, y_1)\} \cup \{(z, y_2)(y, y_2), (w, y_2)\}$$
$$- \{(v, u_2)\colon (v, u_2) \in A(D_2)\} \cup \{(v, x)\colon (v, u_2) \in A(D_2)\}$$
$$- \{(v, u_3)\colon (v, u_3) \in A(D_3)\} \cup \{(v, x_2)\colon (v, u_3) \in A(D_3)\}$$
$$- \{(v, u_4)\colon (v, u_4) \in A(D_4)\} \cup \{(v, x_3)\colon (v, u_4) \in A(D_4)\}.$$

As in the proof of Theorem 6, we can easily check that D' is acyclic.

Now we add arcs (f, y_3) and (g, y_3) to D' to obtain D''. Then it can be easily checked that $D'' - \{u_2, u_3, u_4\}$ is acyclic and has competition graph $T \cup \{u_1\}$ together with the edges of the triangles and the edge $\{f, g\}$. There are $q - n - 2$ edges in $G - T$ other than $\{f, g\}$ that are not on triangles. Add to D'' a new isolated vertex i_e corresponding to each such an edge e and arcs from the end vertices of e to i_e. The resulting digraph D is still acyclic and has competition graph $G \cup I_{q-n-1}$, as desired.

Case 2: Every edge of $G - T$ that is not a triangle edge has one end in the component with x in F and the other end in the component with z.

Let E_0 be the set of all edges of $G - E(T)$ that are not triangle edges. Since G is not triangulated, $E_0 \neq \phi$. Since VC(G) does not generate a subgraph in $\mathscr{H}$, $|E_0| \geqslant 2$. Consider two distinct $\alpha, \beta \in E_0$. Obviously, $F + \alpha + \beta$ has a cycle C. Since G has exactly two triangles, C has length at least four. Let $G' = G - \{x, y\} - \{x, w\}$. Then G' has exactly one triangle, it is easy to show that it is connected, and it contains cycle C of length $\geqslant 4$. Thus, we can apply Corollary 7 to G'. We conclude that $k(G') = q' - n = q - n - 2$. Let D' be an acyclic digraph whose competition graph is $G' \cup I_{q-n-2}$. Add to D' a vertex v_0 and arcs $(x, v_0), (y, v_0), (w, v_0)$. Then the resulting digraph is still acyclic and its competition graph is $G \cup I_{q-n-2+1} = G \cup I_{q-n-1}$. □

Theorem 10. *Suppose a connected graph G has exactly two triangles and they are edge-disjoint. Then*
(a) $k(G) = q - n$ *if G is triangulated.*

(b) $k(G) = q - n - 1$ *if* G *has exactly only one cycle of length* $\geqslant 4$ *that is a generated subgraph or* $\mathrm{VC}(G)$ *generates a subgraph in* $\mathscr{L}$.
(c) $k(G) = q - n - 2$ *otherwise.*

Proof. If G is triangulated, then $k(G) = 1$, by Theorem 3. As in the proof of Theorem 9, $q = n + 1$ and so $k(G) = q - n$. This proves (a).

To prove (b), suppose that either (i) $\mathrm{VC}(G)$ generates a subgraph in $\mathscr{L}$ or (ii) G has exactly one cycle of length $\geqslant 4$ that is a generated subgraph. In case of (i), Lemma 8 implies that $k(G) \geqslant 2$. Every graph in $\mathscr{L}$ except those obtained by subdividing nontriangle edges of the first graph in Fig. 1 has $q = n + 3$. However, since the two triangles of G are edge-disjoint, $\mathrm{VC}(G)$ is obtained by subdividing nontriangle edges of graphs other than the first one in Fig. 1, and so G has $q = n + 3$. Thus, $k(G) \geqslant 2 = q - n - 1$. In case of (ii), we must have $q = n + 2$ since removing one edge from each triangle and one edge from the generated cycle of length $\geqslant 4$ must give rise to a spanning tree. Since $k(G) \geqslant 1$ for any connected graph, $k(G) \geqslant q - n - 1$. This gives the lower bound in part (b).

To show that $k(G) \leqslant q - n - 1$, we build an acyclic digraph D whose competition graph is $G \cup I_{q-n-1}$. Delete one edge $\{x, y\}$ of a triangle $\{x, y, z\}$ from G. Then the resulting graph G' is still connected, has exactly one triangle and at least one cycle of length $\geqslant 4$. By Corollary 7, $k(G') = q' - n = q - n - 1$. Let D' be an acyclic digraph whose competition graph is $G' \cup I_{q-n-1}$. Label the vertices of D' as $v_1, v_2, \ldots$ so that $\pi(v_i) = i$ is an acyclic labelling. Let v_p be the prey of x and z and v_q be the prey of y and z. If there is an arc from any other vertex to v_p or v_q, then G has edge-shared triangles, which contradicts the hypothesis. We may assume that $p < q$. Now add arc (y, v_p) to D' to obtain another digraph D. This is acyclic since $\pi(y) > \pi(v_q) > \pi(v_p)$. Moreover, D has competition graph $G \cup I_{q-n-1}$, as required.

To prove part (c), suppose that G is not triangulated, has at least two distinct cycles of length $\geqslant 4$ as generated subgraphs, and $\mathrm{VC}(G)$ does not generate a subgraph in $\mathscr{L}$. Proceeding as in the proof of Theorem 9, we note that $\theta_e(G) = q - 6 + 2 = q - 4$ and so, by Theorem 2, $k(G) \geqslant q - n - 2$. Thus, it suffices to find an acyclic digraph whose competition graph is $G \cup I_{q-n-2}$.

Let us suppose that the triangles have vertices $\{x, y, z\}$ and $\{x', y', z'\}$. It could be that the two triangles have a common vertex. In that case, we assume $z = z'$. Let T be a spanning tree for G. We may find such a T with exactly four edges on the triangles (it cannot have more). To obtain such a spanning tree, a total of $q - n - 1$ edges not on the triangles are deleted from G. Then delete all the edges of the triangles from T, obtaining a forest F. F has five tree components since each edge deletion breaks a component into two. If the two triangles are joined at a common vertex z, then the vertices of the triangles are in different components. If the two triangles are vertex disjoint, then four components have exactly one vertex from the triangles and the remaining component has two vertices, say z and z', of the triangles.

Claim. *There are distinct edges $\{f, g\}$ and $\{f', g'\}$ in $G - T$ that are not triangle edges and a component in F that does not contain any of f, g, f', g', and z.*

Let E_0 be the set of all edges of $G - E(T)$ that are not triangle edges. Since G is not triangulated, $E_0 \neq \phi$. Since G does not have only one cycle of length $\geqslant 4$, $|E_0| \neq 1$. If $|E_0| \geqslant 3$, then it is not difficult to see that the claim holds. If $|E_0| = 2$ and the claim does not hold, then VC(G) generates a subgraph in $\mathscr{L}$, which is a contradiction. This verifies the claim.

Let T_1 be the component called for in the claim. We may assume that x is in T_1. Later in the proof we will pay attention to the subscripts of the remaining components. However, for now, let us assume the subscripts 2, 3, 4, 5 have been assigned to them. Let D_i be acyclic digraphs whose competition graphs are $T_i \cup \{u_i\}$, $i = 1, \dots, 5$. We may assume that the u_i have no outgoing arcs in D_i. By Lemma 4, we may assume that x has only outgoing arcs in D_1. We may also assume that the two vertices in D_i having the highest indices in an acyclic labelling have only outgoing arcs. Let y_1 be the other vertex of D_1 having only outgoing arcs and let x_i and y_i be the vertices of T_i having only outgoing arcs, for $i = 2, \dots, 5$. By Lemma 4, we can make any desired vertex of T_i the vertex x_i, and we will do so below.

There are at least two components other than T_1 which have at most one of f, g, f', g'. Let T_2 and T_3 be such components. We may assume that T_2 is the component containing y. By renaming things, we may assume that g does not belong to T_2, f' does not belong to T_2 or T_3, and g' belongs to T_5. Since x, y, x', y', z' are in different components, one of x', y', z' is in T_3, one in T_4, and one in T_5. By Lemma 4, we may assume that $x_2 = f$ if f belongs to T_2, x_3 is x', y', or z', whichever one is in T_3, and $x_4 = f'$ if f' belongs to T_4.

Let D' be a digraph whose vertex set is $V(G) \cup \{u_1, \dots, u_5\}$ and whose arc set is

$$\begin{aligned} &A(D_1) \cup A(D_2) \cup A(D_3) \cup A(D_4) \cup A(D_5) \cup \{(x, y_1), (y, y_1), (z, y_1)\} \\ &\quad - \{(v, u_2) \colon (v, u_2) \in A(D_2)\} \cup \{(v, x) \colon (v, u_2) \in A(D_2)\} \\ &\quad - \{(v, u_3) \colon (v, u_3) \in A(D_3)\} \cup \{(v, x_2) \colon (v, u_3) \in A(D_3)\} \\ &\quad - \{(v, u_4) \colon (v, u_4) \in A(D_4)\} \cup \{(v, x_3) \colon (v, u_4) \in A(D_4)\} \\ &\quad - \{(v, u_5) \colon (v, u_5) \in A(D_5)\} \cup \{(v, x_4) \colon (v, u_5) \in A(D_5)\}. \end{aligned}$$

As in the proof of Theorem 6, we easily check that D' is acyclic.

Now we add to D' the arcs (f, y_2), (g, y_2), (x', y_3), (y', y_3), (z', y_3), (f', y_4), (g', y_4). Then we can easily check that the resulting digraph D'' is still acyclic. Since $u_2, \dots, u_5$ have no incoming or outgoing arcs in D'' and $x, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ have only outgoing arcs in the respective D_i, $D'' - \{u_2, \dots, u_5\}$ has competition graph $T \cup \{u_1\}$ together with the edges of triangles and the edges $\{f, g\}$ and $\{f', g'\}$. Now there are $q - n + 1$ edges in $G - T$ and these include two triangle edges and the distinct edges $\{f, g\}$ and $\{f', g'\}$, which are not triangle edges by hypothesis. For each of the other $q - n + 1 - 4 = q - n - 3$ edges e in $G - T$, add an isolated vertex i_e and arcs from the end vertices of e to i_e. This gives us an acyclic digraph D whose competition graph is $G \cup I_{q-n-2}$, where the isolated vertices are u_1 and the new vertices i_e just added. □

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