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Weak* dentability index of spaces $C([0,\alpha])$ ☆

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ABSTRACT

We compute the weak*-dentability index of the spaces $C(K)$ where K is a countable compact space. Namely $\mathrm{Dz}(C([0,\omega^{\omega^\alpha}])) = \omega^{1+\alpha+1}$, whenever $0 \leqslant \alpha < \omega_1$. More generally, $\mathrm{Dz}(C(K)) = \omega^{1+\alpha+1}$ if K is a scattered compact whose height $\eta(K)$ satisfies $\omega^\alpha < \eta(K) \leqslant \omega^{\alpha+1}$ with an α countable.

1. Introduction

The Szlenk index has been introduced in [20] in order to show that there is no universal space for the class of separable reflexive Banach spaces. The general idea of assigning an isomorphically invariant ordinal index to a class of Banach spaces proved to be extremely fruitful in many situations. We refer to [16] for a survey with references. In the present note we will give an alternative geometrical description of the Szlenk index (equivalent to the original definition whenever X is a separable Banach space not containing any isomorphic copy of ℓ_1 [12]), which stresses its close relation to the weak*-dentability index. The later index proved to be very useful in renorming theory [12–14].

Let us proceed by giving the precise definitions. Consider a real Banach space X and K a weak*-compact subset of X^*. For $\varepsilon > 0$ we let $\mathcal{V}$ be the set of all relatively weak*-open subsets V of K such that the norm diameter of V is less than ε and $s_\varepsilon K = K \setminus \bigcup\{V\colon\ V \in \mathcal{V}\}$. Then we define inductively $s_\varepsilon^\alpha K$ for any ordinal α by $s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K)$ and $s_\varepsilon^\alpha K = \bigcap_{\beta<\alpha} s_\varepsilon^\beta K$ if α is a limit ordinal. We denote by B_{X^*} the closed unit ball of X^*. We then define $\mathrm{Sz}(X,\varepsilon)$ to be the least ordinal α so that $s_\varepsilon^\alpha B_{X^*} = \emptyset$, if such an ordinal exists. Otherwise we write $\mathrm{Sz}(X,\varepsilon) = \infty$. The *Szlenk index* of X is finally defined by $\mathrm{Sz}(X) = \sup_{\varepsilon>0} \mathrm{Sz}(X,\varepsilon)$. Next, we introduce the notion of *weak*-dentability index*. Denote $H(x,t) = \{x^* \in K,\ x^*(x) > t\}$, where $x \in X$ and $t \in \mathbb{R}$. Let K be again a weak*-compact. We introduce a weak*-slice of K to be any non-empty set of the form $H(x,t) \cap K$ where $x \in X$ and $t \in \mathbb{R}$. Then we denote by $\mathcal{S}$ the set of all weak*-slices of K of norm diameter less than ε and $d_\varepsilon K = K \setminus \bigcup\{S\colon\ S \in \mathcal{S}\}$. From this derivation, we define inductively $d_\varepsilon^\alpha K$ for any ordinal α by $d_\varepsilon^{\alpha+1} K = s_\varepsilon(d_\varepsilon^\alpha K)$ and $d_\varepsilon^\alpha K = \bigcap_{\beta<\alpha} s_\varepsilon^\beta K$ if α is a limit ordinal. We then define $\mathrm{Dz}(X,\varepsilon)$ to be the least ordinal α so that $d_\varepsilon^\alpha B_{X^*} = \emptyset$, if such an ordinal exists. Otherwise we write $\mathrm{Dz}(X,\varepsilon) = \infty$. The weak*-dentability index is defined by $\mathrm{Dz}(X) = \sup_{\varepsilon>0} \mathrm{Dz}(X,\varepsilon)$.

Let us now recall that it follows from the classical theory of Asplund spaces (see for instance [10,9,6] and references therein) that for a Banach space X, each of the following conditions: $\mathrm{Dz}(X) \neq \infty$ and $\mathrm{Sz}(X) \neq \infty$ is equivalent to X being an Asplund space. In particular, if X is a separable Banach space, each of the conditions $\mathrm{Dz}(X) < \omega_1$ and $\mathrm{Sz}(X) < \omega_1$ is

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equivalent to the separability of X^*. In other words, both of these indices measure "quantitatively" the "Asplundness" of the space in question. Moreover, these indices are invariant under isomorphism.

It is immediate from the definition, that $\mathrm{Dz}(X) \geqslant \mathrm{Sz}(X)$ for every Banach space X. Relying on tools from descriptive set theory, Bossard (for the separable case, see [4,5]) and the second named author [14], proved non-constructively that there exists a universal function $\psi : \omega_1 \to \omega_1$, such that if X is an Asplund space with $\mathrm{Sz}(X) < \omega_1$, then $\mathrm{Dz}(X) \leqslant \psi(\mathrm{Sz}(X))$.

Recently, Raja [17] has obtained a concrete example of such a ψ, by showing that $\mathrm{Dz}(X) \leqslant \omega^{\mathrm{Sz}(X)}$ for every Asplund space. This is a very satisfactory result, but it is not optimal, as we know from [8] that the optimal value $\psi(\omega) = \omega^2$. Further progress in this area depends on the exact knowledge of indices for concrete spaces. The Szlenk index has been precisely calculated for several classes of spaces, most notably for the class of $C([0,\alpha])$, α countable (Samuel [19], see also [8]). We have $\mathrm{Sz}(C([0,\omega^{\omega^\alpha}])) = \omega^{\alpha+1}$, so it follows from the Bessaga–Pełczyński [3] Theorem 1 below, that the value of the Szlenk index characterizes the isomorphism class [10]. Computations of the Szlenk index for other spaces may be found e.g. in [2,1,11]. On the other hand, the precise value of the weak*-dentability index is known only for superreflexive Banach spaces, where $\mathrm{Dz}(X) = \omega$ [13,10], and for spaces with an equivalent UKK* renorming [8]. For a detailed background information on the Szlenk and dentability indices we refer the reader to [10,15,16,18] and references therein.

The main result of our note, Theorem 2, is a precise evaluation of the w^*-dentability index for the class of $C([0,\alpha])$, α countable. These spaces have been classified isomorphically by C. Bessaga and A. Pełczyński [3] in the following way.

Theorem 1 *(Bessaga–Pełczyński). Let $\omega \leqslant \alpha \leqslant \beta < \omega_1$. Then $C([0,\alpha])$ is isomorphic to $C([0,\beta])$ if and only if $\beta < \alpha^\omega$. Moreover, for every countable compact space K there exists a unique $\alpha < \omega_1$ such that $C(K)$ is isomorphic to $C([0,\omega^{\omega^\alpha}])$.*

It is also well known and easy to show that for $\alpha \geqslant \omega$, $C([0,\alpha])$ is isomorphic to $C_0([0,\alpha])$ where $C_0([0,\alpha]) = \{f \in C([0,\alpha]):\ f(\alpha) = 0\}$. The aim of this note is to prove the next theorem. Note, as a particular consequence, that the weak*-dentability index gives a complete isomorphic characterization of a $C(K)$ space, when K is a metrizable compact space (similarly to the case of the Szlenk index).

Theorem 2. *Let $0 \leqslant \alpha < \omega_1$. Then $\mathrm{Dz}(C([0,\omega^{\omega^\alpha}])) = \omega^{1+\alpha+1}$.*

Proof. We start by proving the upper estimate

$$\mathrm{Dz}\big(C\big(\big[0,\omega^{\omega^\alpha}\big]\big)\big) \leqslant \omega^{1+\alpha+1}. \tag{1}$$

The method of the proof is similar to [8], where a short and direct computation of the Szlenk index of the spaces $C([0,\alpha])$ is presented. The next lemma is a variant of Lemma 2.2 from [8]. We omit the proof which requires only minor notational changes.

Lemma 3. *Let X be a Banach space and α an ordinal. Assume that*

$$\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0 \quad d_\varepsilon^\alpha(B_{X^*}) \subset \big(1-\delta(\varepsilon)\big)B_{X^*}.$$

Then

$$\mathrm{Dz}(X) \leqslant \alpha \cdot \omega.$$

We shall also use the following lemma that can be found in [15].

Lemma 4. *Let X be a Banach space and $L_2(X)$ be the Bochner space $L_2([0,1],X)$. Then*

$$\mathrm{Dz}(X) \leqslant \mathrm{Sz}\big(L_2(X)\big).$$

Thus, in order to obtain the desired upper bound we only need to prove the following.

Proposition 5. *Let $0 \leqslant \alpha < \omega_1$. Then $\mathrm{Sz}(L_2(C([0,\omega^{\omega^\alpha}]))) \leqslant \omega^{1+\alpha+1}$.*

Proof. For a fixed $\alpha < \omega_1$ and $\gamma < \omega^{\omega^\alpha}$, let us put $Z = L_2(\ell_1([0,\omega^{\omega^\alpha}]))$, together with the weak*-topology induced by $L_2(C_0([0,\omega^{\omega^\alpha}]))$ and $Z_\gamma = L_2(\ell_1([0,\gamma]))$ with the weak*-topology induced by $L_2(C([0,\gamma]))$. We recall that for a Banach space X with separable dual, $L_2(X^*)$ is canonically isometric to $(L_2(X))^*$.

Let P_γ be the canonical projection from $\ell_1([0,\omega^{\omega^\alpha}])$ onto $\ell_1([0,\gamma])$. Then, for $f \in Z$ and $t \in [0,1]$, we define $(\Pi_\gamma f)(t) = P_\gamma(f(t))$. Clearly, Π_γ is a norm one projection from Z onto Z_γ (viewed as a subspace of Z). We also have that for any $f \in Z$, $\|\Pi_\gamma f - f\|$ tends to 0 as γ tends to ω^{ω^α}.

The next lemma is a variant of Lemma 3.3 in [8].

Lemma 6. *Let $\alpha < \omega_1$, $\gamma < \omega^{\omega^\alpha}$, $\beta < \omega_1$ and $\varepsilon > 0$. If $z \in s_{3\varepsilon}^\beta(B_Z)$ and $\|\Pi_\gamma z\|^2 > 1 - \varepsilon^2$, then $\Pi_\gamma z \in s_\varepsilon^\beta(B_{Z_\gamma})$.*

Proof. We will proceed by transfinite induction in β. The cases $\beta = 0$ and β a limit ordinal are clear. Next, we assume that $\beta = \mu + 1$ and the statement has been proved for all ordinals less than or equal to μ. Consider $f \in B_Z$ with $\|\Pi_\gamma f\|^2 > 1 - \varepsilon^2$ and $\Pi_\gamma f \notin s_\varepsilon^\beta(B_{Z_\gamma})$. Assuming $f \notin s_{3\varepsilon}^\mu(B_Z) \supset s_{3\varepsilon}^\beta(B_Z)$ finishes the proof, so we may suppose that $f \in s_{3\varepsilon}^\mu(B_Z)$. By the inductive hypothesis, $\Pi_\gamma f \in s_\varepsilon^\mu(B_{Z_\gamma})$. Thus there exists a weak*-neighborhood V of f such that the diameter of $V \cap s_\varepsilon^\mu(B_{Z_\gamma})$ is less than ε. We may assume that V can be written $V = \bigcap_{i=1}^k H(\varphi_i, a_i)$, where $a_i \in \mathbb{R}$ and $\varphi_i \in L_2(C([0,\gamma]))$. We may also assume, using Hahn–Banach theorem, that $V \cap (1-\varepsilon^2)^{1/2} B_{Z_\gamma} = \emptyset$.

Define $\Phi_i \in L_2(C_0([0, \omega^{\omega^\alpha}]))$ by $\Phi_i(t)(\sigma) = \varphi_i(t)(\sigma)$ if $\sigma \leqslant \gamma$ and $\Phi_i(t)(\sigma) = 0$ otherwise. Then define $W = \bigcap_{i=1}^k H(\Phi_i, a_i)$. Note that for f in Z, $f \in W$ if and only if $\Pi_\gamma f \in V$. In particular W is a weak*-neighborhood of f. Consider now $g, g' \in W \cap s_{3\varepsilon}^\mu(B_Z)$. Then $\Pi_\gamma g$ and $\Pi_\gamma g'$ belong to V and therefore they have norms greater than $(1-\varepsilon^2)^{1/2}$. It follows from the induction hypothesis that $\Pi_\gamma g, \Pi_\gamma g' \in s_\varepsilon^\mu(B_{Z_\gamma})$ thus $\|\Pi_\gamma g - \Pi_\gamma g'\| \leqslant \varepsilon$. Since $\|\Pi_\gamma g\|^2 > 1 - \varepsilon^2$ and $\|g\| \leqslant 1$, we also have $\|g - \Pi_\gamma g\| < \varepsilon$. The same is true for g' and therefore $\|g - g'\| < 3\varepsilon$. This finishes the proof of the lemma. □

We are now in position to prove Proposition 5. For that purpose it is enough to show that for all $\alpha < \omega_1$:

$$\forall \gamma < \omega^{\omega^\alpha}\ \forall \varepsilon > 0 \quad s_\varepsilon^{\omega^{1+\alpha}}(B_{Z_\gamma}) = \emptyset. \tag{2}$$

We will prove this by transfinite induction on $\alpha < \omega_1$.

For $\alpha = 0$, γ is finite and the space Z_γ is isomorphic to L_2 and therefore $s_\varepsilon^\omega(B_{Z_\gamma})$ is empty. So (2) is true for $\alpha = 0$.

Assume that (2) holds for $\alpha < \omega_1$. Let $Z = L_2(C_0([0, \omega^{\omega^\alpha}]))$. It follows from Lemma 6 and the fact that for all $f \in Z$ $\|\Pi_\gamma f - f\|$ tends to 0 as γ tends to ω^{ω^α}, that

$$\forall \varepsilon > 0 \quad s_\varepsilon^{\omega^{1+\alpha}}(B_Z) \subset \left(1-\varepsilon^2\right)^{1/2} B_Z.$$

From this and Lemma 3 it follows that

$$\forall \varepsilon > 0 \quad s_\varepsilon^{\omega^{1+\alpha+1}}(B_Z) = \emptyset.$$

By Theorem 1 we know that the spaces $C([0,\gamma])$, $C([0,\omega^{\omega^\alpha}])$, and also $C_0([0,\omega^{\omega^\alpha}])$ are isomorphic, whenever $\omega^{\omega^\alpha} \leqslant \gamma < \omega^{\omega^{\alpha+1}}$. Thus $s_\varepsilon^{\omega^{1+\alpha+1}}(B_{Z_\gamma}) = \emptyset$ for any $\varepsilon > 0$ and $\gamma < \omega^{\omega^{\alpha+1}}$, i.e. (2) holds for $\alpha + 1$.

Finally, the induction is clear for limit ordinals. □

In the rest of the note, we will focus on proving the converse inequality. Note that it suffices to deal with the spaces $C([0,\omega^{\omega^\alpha}])$ where $\alpha < \omega$. Indeed, in case $\alpha \geqslant \omega$, our inequality (1) implies that

$$\mathrm{Dz}\big(C\big(\big[0,\omega^{\omega^\alpha}\big]\big)\big) = \mathrm{Sz}\big(C\big(\big[0,\omega^{\omega^\alpha}\big]\big)\big) = \omega^{\alpha+1}.$$

Proposition 7. *Let X, Z be Banach spaces and let $Y \subset X^*$ be a closed subspace. Let there be $T \in \mathcal{B}(X, Z)$ such that T^* is an isometric isomorphism from Z^* onto Y. Let $\varepsilon > 0$, α be an ordinal such that $B_{X^*} \cap Y \subset d_\varepsilon^\alpha(B_{X^*})$, and $z \in Z^*$. If $z \in d_\varepsilon^\beta(B_{Z^*})$, then $T^*z \in d_\varepsilon^{\alpha+\beta}(B_{X^*})$.*

Proof. By induction with respect to β. The cases when $\beta = 0$ or β is a limit ordinal are clear. Let $\beta = \mu + 1$ and suppose that $T^*z \notin d_\varepsilon^{\alpha+\beta}(B_{X^*})$. If $z \notin d_\varepsilon^\mu(B_{Z^*})$, then the proof is finished. So we proceed assuming that $z \in d_\varepsilon^\mu(B_{Z^*})$, which by the inductive hypothesis implies that $T^*z \in d_\varepsilon^{\alpha+\mu}(B_{X^*})$. There exist $x \in X$, $t > 0$, such that $T^*z \in H(x,t) \cap d_\varepsilon^{\alpha+\mu}(B_{X^*}) = S$ and $\operatorname{diam} S < \varepsilon$. Consider the slice $S' = H(Tx, t) \cap d_\varepsilon^\mu(B_{Z^*})$. We have $\langle Tx, z\rangle = \langle x, T^*z\rangle$, so $z \in S'$. Also, $\operatorname{diam} S' \leqslant \operatorname{diam} S < \varepsilon$ as T^* is an isometry. We conclude that $z \notin d_\varepsilon^\beta(B_{Z^*})$, which finishes the argument. □

Let us introduce a shift operator $\tau_m : \ell_1([0,\omega]) \to \ell_1([0,\omega])$, $m \in \mathbb{N}$, by letting $\tau_m h(n) = h(n-m)$ for $n \geqslant m$, $\tau_m h(n) = 0$ for $n < m$ and $\tau_m h(\omega) = h(\omega)$.

Corollary 8. *Let $h \in d_\varepsilon^\alpha(B_{\ell_1([0,\omega])})$. Then $\tau_m h \in d_\varepsilon^\alpha(B_{\ell_1([0,\omega])})$ for every $m \in \mathbb{N}$.*

Proof. Indeed, consider the mapping $T : C([0,\omega]) \to C([0,\omega])$ defined as $T((x(0), x(1), \ldots, x(\omega))) = (x(1), x(2), \ldots, x(\omega))$. Clearly, $T^* = \tau_1$ and the assertion for $m = 1$ follows by the previous proposition. For $m > 1$ one may use induction. □

Definition 9. Let α be an ordinal and $\varepsilon > 0$. We will say that a subset M of X^* is an *ε-α-obstacle* for $f \in B_{X^*}$ if

(i) $\operatorname{dist}(f, M) \geqslant \varepsilon$,
(ii) for every $\beta < \alpha$ and every w^*-slice S of $d_\varepsilon^\beta(B_{X^*})$ with $f \in S$ we have $S \cap M \neq \emptyset$.

It follows by transfinite induction that if f has an ε-α-obstacle, then $f \in d_\varepsilon^\alpha(B_{X^*})$.

An (n,ε)-*tree* in a Banach space X is a finite sequence $(x_i)_{i=0}^{2^{n+1}-1} \subset X$ such that

$$x_i = \frac{x_{2i}+x_{2i+1}}{2} \quad \text{and} \quad \|x_{2i}-x_{2i+1}\| \geqslant \varepsilon$$

for $i=0,\dots,2^n-1$. The element x_0 is called the *root* of the tree $(x_i)_{i=0}^{2^{n+1}-1}$. Note that if $(h_i)_{i=0}^{2^{n+1}-1} \subset B_{X^*}$ is an (n,ε)-tree in X^*, then $h_0 \in d_\varepsilon^n(B_{X^*})$.

Define $f_\beta \in \ell_1([0,\alpha])$, for $\alpha \geqslant \beta$, by $f_\beta(\xi)=1$ if $\xi=\beta$ and $f_\beta(\xi)=0$ otherwise.

Lemma 10.

$$f_\omega \in d_{1/2}^\omega(B_{\ell_1([0,\omega])}).$$

Proof. In [7, Exercise 9.20] a sequence is constructed of $(n,1)$-trees in $B_{\ell_1([0,\omega])}$ with roots

$$r_n = \Big(\underbrace{\frac{1}{2^n},\dots,\frac{1}{2^n}}_{2^n\text{-times}},0,\dots\Big)$$

whose elements belong to $\mathcal{P} = \{h \in B_{\ell_1([0,\omega])}\colon\ \|h\|_1 = 1,\ h(n)\geqslant 0,\ h(\omega)=0\}$. We have $r_n \in d_{1/2}^{2n}(B_{\ell_1([0,\omega])})$, and $\operatorname{dist}(f_\omega,\mathcal{P})=2$. Finally, for every $h\in\mathcal{P}$, every $x\in C([0,\omega])$ and every $t\in\mathbb{R}$ such that $f_\omega \in H(x,t)$, there exists $m\in\mathbb{N}$ such that $\tau_m h \in H(x,t)$. Therefore the set $\{\tau_m r_n\colon (m,n)\in\mathbb{N}^2\}$ is a $\frac12$-ω-obstacle for f_ω. Thus $f_\omega \in d_{1/2}^\omega(B_{\ell_1([0,\omega])})$. $\square$

Proposition 11. *For every $\alpha<\omega$,*

$$f_{\omega^{\omega^\alpha}} \in d_{1/2}^{\omega^{1+\alpha}}(B_{\ell_1([0,\omega^{\omega^\alpha}])}). \tag{3}$$

Proof. The case $\alpha=0$ is contained in Lemma 10. Let us suppose that we have proved the assertion (3) for all ordinals (natural numbers, in fact) less than or equal to α. It is enough to show, for every $n\in\mathbb{N}$, that

$$f_{(\omega^{\omega^\alpha})^n} \in d_{1/2}^{\omega^{1+\alpha}n}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])}). \tag{4}$$

Indeed, (4) implies

$$f_{(\omega^{\omega^\alpha})^n} \in d_{1/2}^{\omega^{1+\alpha}n}(B_{\ell_1([0,\omega^{\omega^{\alpha+1}}])}).$$

Since $f_{(\omega^{\omega^\alpha})^n} \xrightarrow{w^*} f_{\omega^{\omega^{\alpha+1}}}$ and $\|f_{(\omega^{\omega^\alpha})^n} - f_{\omega^{\omega^{\alpha+1}}}\|=2$, we see that $\{f_{(\omega^{\omega^\alpha})^n}\colon n\in\mathbb{N}\}$ is a $\frac12$-$\omega^{1+\alpha+1}$-obstacle for $f_{\omega^{\omega^{\alpha+1}}}$. That implies (3) for $\alpha+1$.

In order to prove (4) we will proceed by induction. The case $n=1$ follows from the inductive hypothesis as indicated above, so let us suppose that $n=m+1$ and (4) holds for m.

Define the mapping $T: C([0,(\omega^{\omega^\alpha})^n]) \to C([0,\omega^{\omega^\alpha}])$ by

$$(Tx)(\gamma) = x\big(\big(\omega^{\omega^\alpha}\big)^m(1+\gamma)\big), \quad \gamma \leqslant \omega^{\omega^\alpha}.$$

A simple computation shows that the dual map T^* is given by

$$\big(T^*g\big)(\gamma) = \begin{cases} g(\xi) & \text{if } \gamma = (\omega^{\omega^\alpha})^m(1+\xi),\ \xi\leqslant\omega^{\omega^\alpha}, \\ 0 & \text{otherwise.}\end{cases}$$

Clearly, T^* is an isometric isomorphism of $\ell_1([0,\omega^{\omega^\alpha}])$ onto $\operatorname{rng} T^*$. We claim that

$$B_{\ell_1([0,(\omega^{\omega^\alpha})^n])} \cap \operatorname{rng} T^* \subset d_{1/2}^{\omega^{1+\alpha}m}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])}). \tag{5}$$

Note that the set of extremal points of $B_{\ell_1([0,(\omega^{\omega^\alpha})^n])} \cap \operatorname{rng} T^*$ satisfies

$$\operatorname{ext}\big(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])} \cap \operatorname{rng} T^*\big) \subset \big\{f_\gamma, -f_\gamma\colon\ \gamma = \big(\omega^{\omega^\alpha}\big)^m(1+\xi),\ \xi\leqslant\omega^{\omega^\alpha}\big\}.$$

By the inductive assumption and by symmetry, $f_{(\omega^{\omega^\alpha})^m}$ and $-f_{(\omega^{\omega^\alpha})^m}$ belong to $d_{1/2}^{\omega^{1+\alpha}m}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])})$. It is easy to see that more generally, f_γ and $-f_\gamma$ belong to $d_{1/2}^{\omega^{1+\alpha}m}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])})$, whenever $\gamma=(\omega^{\omega^\alpha})^m(1+\xi)$, $\xi\leqslant\omega^{\omega^\alpha}$. Thus we have verified that

$$\operatorname{ext}\big(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])} \cap \operatorname{rng} T^*\big) \subset d_{1/2}^{\omega^{1+\alpha}m}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])}),$$

and the claim (5) follows using the Krein–Milman theorem.

This together with the inductive assumption (3) allows us to apply Proposition 7 (with $\ell_1([0,(\omega^{\omega^\alpha})^n])$ as X^*, $C([0,\omega^{\omega^\alpha}])$ as Z, and $\operatorname{rng} T^*$ as Y) to get

$$f_{(\omega^{\omega^\alpha})^n} = T^* f_{\omega^{\omega^\alpha}} \in d_{1/2}^{\omega^{1+\alpha}n}(B_{\ell_1([0,(\omega^{\omega^\alpha})^n])}). \qquad \square$$

To finish the proof of Theorem 2, we use that for every Asplund space X, $\mathrm{Dz}(X) = \omega^\xi$ for some ordinal ξ (see [15, Proposition 3.3], [10]). Combining Proposition 11 with (1) we obtain

$$\mathrm{Dz}\big(C\big(\big[0,\omega^{\omega^\alpha}\big]\big)\big) = \omega^{1+\alpha+1}$$

for $\alpha < \omega$. For $\omega \leqslant \alpha < \omega_1$, we use that $\omega^{1+\alpha+1} = \omega^{\alpha+1} = \mathrm{Sz}(C([0,\omega^{\omega^\alpha}])) = \mathrm{Dz}(C([0,\omega^{\omega^\alpha}]))$, which finishes the proof. $\square$

Our next proposition is a direct consequence of Theorem 2, Lemma 4 and Proposition 5.

Proposition 12. *Let* $0 \leqslant \alpha < \omega_1$. *Then* $\mathrm{Sz}(L_2(C([0,\omega^{\omega^\alpha}]))) = \omega^{1+\alpha+1}$.

Our main result can be extended to the non-separable case as follows.

Theorem 13. *Let* $0 \leqslant \alpha < \omega_1$. *Let* K *be a compact space whose Cantor derived sets satisfy* $K^{\omega^\alpha} \neq \emptyset$ *and* $K^{\omega^{\alpha+1}} = \emptyset$. *Then* $\mathrm{Dz}(C(K)) = \omega^{1+\alpha+1}$.

Proof. The upper estimate follows from the separable determination of the weak*-dentability index when it is countable and from Theorem 2 (the argument is identical to the one given for the computation of $\mathrm{Sz}(C(K))$ in [14]).

On the other hand, since $K^{\omega^\alpha} \neq \emptyset$, we have that $\mathrm{Sz}(C(K)) \geqslant \omega^{\alpha+1}$ (see [14] or [15, Proposition 7]). Therefore there is a separable subspace X of $C(K)$ such that $\mathrm{Sz}(X) \geqslant \omega^{\alpha+1}$. By considering the closed subalgebra of $C(K)$ generated by X, we may as well assume that X is isometric to $C(L)$, where L is a compact metrizable space. Since $\mathrm{Sz}(C(L)) \geqslant \omega^{\alpha+1}$, it follows from Theorem 2 that $\mathrm{Dz}(C(L)) \geqslant \omega^{1+\alpha+1}$ and finally that $\mathrm{Dz}(C(K)) \geqslant \omega^{1+\alpha+1}$. $\square$

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