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TRANSITIVE MAPS FROM POSETS TO DYNKIN DIAGRAMS

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Let G be a group with a BN-pair and Γ its Dynkin diagram. Let P be a partially ordered set and $\lambda : P \to 2^\Gamma$ be a map from P into the induced subgraphs of Γ. We say that the map λ is transitive if for any $a, b, c \in P$ with $a < b < c$, each component of $\lambda(b)$ is contained in either $\lambda(a)$ or $\lambda(c)$. Transitive maps characterize the systems of idempotents of certain monoids having G as the group of units. In this paper we construct a universal transitive map, which is then used to describe all transitive maps.

1. Introduction

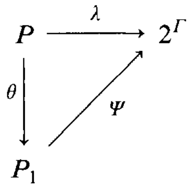
The discrete problem being considered in this paper, has its origins in the classification of the system of idempotents of certain monoids [3, 4]. In [4], it is shown that the system of idempotents of a connected regular monoid M with zero is completely determined by a 'type map' from the finite lattice P of principal ideals of M into the power set of the Dynkin diagram of the group of units of M. In [3], the more general situation of monoids on groups with BN-pairs is considered. Again there is a type map $\lambda : P \to 2^\Gamma$ characterizing the system of idempotents. Moreover, it is shown in [3] that an abstract type map $\lambda : P \to 2^\Gamma$ arises if and only if it is 'transitive'. This means by definition that if $a, b, c \in P$ with $a < b < c$, then each connected component of $\lambda(b)$ is either contained in $\lambda(a)$ or contained in $\lambda(c)$.

In this paper we introduce the concept of an irreducible transitive map and show that any transitive map is derived from an irreducible transitive map. Then we show that for any finite graph Γ, there is a universal irreducible transitive map $u : \mathbf{U} \to 2^\Gamma$ with $|\mathbf{U}| = |\mathbf{U}(\Gamma)| \le 3^{|\Gamma|}$ and any irreducible transitive map $\lambda : P \to 2^\Gamma$ can be embedded in $\mathbf{U}(\Gamma)$. If P is linearly ordered and $\lambda : P \to 2^\Gamma$ is an irreducible transitive map, we show that $|P| \le 2|\Gamma| + 1$. When Γ is one of the connected simply laced Dynkin diagrams, we determine $|\mathbf{U}(\Gamma)|$. Finally we discuss some concrete examples.

2. Main results

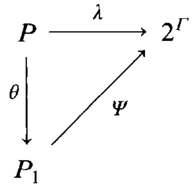
Let Γ be any finite undirected graph. Let $|\Gamma|$ denote the number of points in Γ.

In general, if X is any set, then $|X|$ denotes the cardinality of X. Let 2^{Γ} denote the set of all induced subgraphs of Γ. Let P be a partially ordered set with a maximum element 1. A map $\lambda: P \to 2^{\Gamma}$ with $\lambda(1) = \Gamma$ is *transitive* if $a, b, c \in P$ such that $a < b < c$, then each connected component of $\lambda(b)$ is either contained in $\lambda(a)$ or contained in $\lambda(c)$. Let $\psi: P_1 \to 2^{\Gamma}$ be another transitive map. Then we consider λ and ψ to be the same (or isomorphic) if there is an isomorphism of partially ordered sets $\theta: P \to P_1$ such that the diagram



commutes.

Let $\lambda: P \to 2^{\Gamma}$ be a transitive map. If Q is any partially ordered set with a maximum element **1** and $\theta: Q \to P$ an order preserving map with $\theta(\mathbf{1}) = 1$, then we observe that $\lambda \circ \theta: Q \to 2^{\Gamma}$ is also transitive. Keeping this observation in mind, we define the transitive map $\lambda: P \to 2^{\Gamma}$ to be *irreducible* if whenever we have a commuting diagram



with θ being an order preserving surjection and $\psi: P_1 \to 2^{\Gamma}$ a transitive map, $\theta: P \to P_1$ is an isomorphism of partially ordered sets. It is clear that any transitive map on a finite partially ordered set P comes from an irreducible transitive map. We will show this to be true even for infinite P.

We wish to begin by constructing a universal irreducible transitive map. For a fixed finite undirected graph Γ let

$$\mathbf{U} = \mathbf{U}(\Gamma) = \{(A, B) \mid A, B \in 2^{\Gamma},\ A \cap B = \emptyset \text{ and each connected component of } A \cup B \text{ is either contained in } A \text{ or contained in } B\}.$$

For (A, B), $(A', B') \in \mathbf{U}$ we define $(A, B) \leq (A', B')$ if $A \subseteq A'$ and $B' \subseteq B$, i.e. A is an induced subgraph of A' and B' is an induced subgraph of B. Define the map $u: \mathbf{U} \to 2^{\Gamma}$ as

$$u(A, B) = A \cup B \quad \textit{for all } (A, B) \in \mathbf{U}.$$

Note that a finite linearly ordered set P is a *chain* with length $|P| - 1$. When P is a chain we say the map $\lambda: P \to 2^{\Gamma}$ is linear.

Theorem 2.1. *For a fixed finite undirected graph Γ, the poset $\mathbf{U}=\mathbf{U}(\Gamma)$ is a finite distributive lattice with maximum element $(\Gamma,\emptyset)\in\mathbf{U}$ and maximal chains of length $2|\Gamma|$. Furthermore, the map $u:\mathbf{U}\to 2^\Gamma$ is an irreducible transitive map.*

Proof. Let (A,B), (A',B'), $(A'',B'')\in\mathbf{U}$. It is easy to check that

$$(A,B)\wedge(A',B')=(A\cap A',B\cup B'),$$

and

$$(A,B)\vee(A',B')=(A\cup A',B\cap B').$$

So

$$(A,B)\wedge((A',B')\vee(A'',B''))=((A,B)\wedge(A',B'))\vee((A,B)\wedge(A'',B'')),$$

and

$$(A,B)\vee((A',B')\wedge(A'',B''))=((A,B)\vee(A',B'))\wedge((A,B)\vee(A'',B'')).$$

Hence $\mathbf{U}$ is a distributive lattice. Clearly $(\Gamma,\emptyset)\in\mathbf{U}$ is the maximum element.

It follows from [1, Corollary 7.14] that every maximal chain in $\mathbf{U}$ has the same length. Let $\{s_1,s_2,\dots,s_n\}$ be the set points in Γ. Then clearly $\mathbf{U}$ contains the following maximal chain of length $2|\Gamma|$:

$$\begin{array}{c}(\Gamma,\emptyset)\\ |\\ (\{s_1,\dots,s_{n-1}\},\emptyset)\\ |\\ \vdots\\ (\{s_1\},\emptyset)\\ |\\ (\emptyset,\emptyset)\\ |\\ (\emptyset,\{s_1\})\\ |\\ \vdots\\ (\emptyset,\{s_1,\dots,s_{n-1}\})\\ |\\ (\emptyset,\Gamma)\end{array}$$

(Here $\{s_1,\dots,s_i\}$ denotes the corresponding induced subgraph of Γ.) Therefore, every maximal chain in $\mathbf{U}$ must have length $2|\Gamma|$.

Let $(A',B')<(A,B)<(A'',B'')$. Then $A'\subseteq A\subseteq A''$ and $B''\subseteq B\subseteq B'$. Let C be any connected component of $u(A,B)=A\cup B$. Then by definition either

$$C\subseteq A\subseteq A''\subseteq A''\cup B''=u(A'',B'')\quad\text{or}\quad C\subseteq B\subseteq B'\subseteq A'\cup B'=u(A',B').$$

Hence $u:\mathbf{U}\to 2^\Gamma$ is a transitive map. Now consider the commuting diagram

$$\begin{array}{ccc} U & \xrightarrow{u} & 2^{\Gamma} \\ {\scriptstyle\theta}\downarrow & \nearrow {\scriptstyle\psi} & \\ P & & \end{array}$$

where P is a poset, θ is an order preserving surjection and ψ is a transitive map. Let (A,B), $(A_1,B_1)\in \mathbf{U}$, with $\theta(A,B)\le\theta(A_1,B_1)$. Since $(A_1,B_1)\le(A_1,\emptyset)$, $(\emptyset,B)\le(A,B)$ and θ is order preserving, so

$$\theta(\emptyset,B)\le\theta(A,B)\le\theta(A_1,B_1)\le\theta(A_1,\emptyset).$$

Since $A\cap B=\emptyset$ (resp. $A_1\cap B_1=\emptyset$) and any connected component of $A\cup B$ (resp. $A_1\cup B_1$) is either contained in A (resp. A_1) or in B (resp. B_1), and since $\psi\circ\theta=u$ is transitive it follows that $A\subseteq A_1$ and $B_1\subseteq B$. Hence $(A,B)\le(A_1,B_1)$ in $\mathbf{U}$. Therefore, θ is an isomorphism, which proves the theorem. □

Theorem 2.2. *Let Γ be a fixed finite undirected graph, P a partially ordered set and $\lambda:P\to 2^{\Gamma}$ a transitive map. For $a\in P$, let $\alpha(a)$ denote the union of all connected components of $\lambda(a)$ contained in $\lambda(b)$ for all $b\in P$ with $b\ge a$. Let $\beta(a)=\lambda(a)\setminus\alpha(a)$. Define $\theta:P\to\mathbf{U}(\Gamma)$ as $\theta(a)=(\alpha(a),\beta(a))$. Then θ is an order preserving map and the diagram*

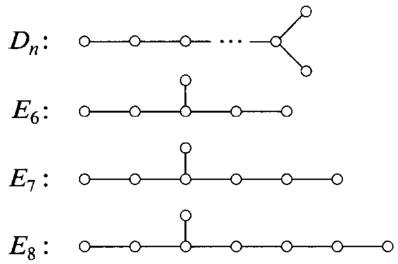
$$\begin{array}{ccc} P & \xrightarrow{\lambda} & 2^{\Gamma} \\ {\scriptstyle\theta}\downarrow & \nearrow {\scriptstyle u} & \\ U(\Gamma) & & \end{array}$$

commutes. In particular, if λ is irreducible, then θ is an isomorphism onto $\theta(P)$.

Proof. It suffices to show that θ is order preserving. Let $a_1,a_2\in P$, and $a_1\le a_2$. Let c be any connected component of $\lambda(a_1)$ such that $c\subseteq\lambda(b)$ for all $b\ge a_1$. In particular, $c\subseteq\lambda(a_2)$ and $c\subseteq\lambda(b)$ for all $b\ge a_2$. Hence $\alpha(a_1)\subseteq\alpha(a_2)$. Now let c be any connected component of $\lambda(a_2)$ such that $c\not\subseteq\alpha(a_2)$. Then $c\not\subseteq\lambda(a_3)$ for some $a_3\ge a_2$. Since $a_3\ge a_2\ge a_1$ by transitivity of λ, it follows that $c\subseteq\lambda(a_1)$. But $c\not\subseteq\alpha(a_1)$ since $a_3\ge a_1$ and $c\not\subseteq\lambda(a_3)$. Hence $c\subseteq\beta(a_1)$. Thus $\beta(a_2)\subseteq\beta(a_1)$. So $(\alpha(a_1),\beta(a_1))\le(\alpha(a_2),\beta(a_2))$ in $\mathbf{U}$. □

For a finite undirected graph Γ we denote the cardinality of $\mathbf{U}=\mathbf{U}(\Gamma)$ by $m(\Gamma)$. Our main interest is in the Dynkin diagrams Γ. For our purposes we will restrict ourselves to the simply laced Dynkin diagrams. The connected simply laced Dynkin diagrams are:

A_n: 



Theorem 2.3. *Let Γ be a finite undirected graph with $|\Gamma|=n$.*

(1) *If $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ are the connected components of Γ, then $m(\Gamma)=m(\Gamma_1)\ldots m(\Gamma_t)$.*

(2) *$m(\Gamma)\leq 3^n$, with equality occurring when Γ is totally disconnected.*

(3) *If $\Gamma=A_n$, then $m(\Gamma)=\sum_{k=0}^{[(n+1)/2]} 2^k\binom{n+1}{2k}$.*

(4) *If $\Gamma=D_n$, then $m(\Gamma)=\sum_{k=0}^{[n/2]} 2^{k+1}\binom{n}{2k}+3\sum_{k=0}^{[(n-2)/2]} 2^k\binom{n-2}{2k}$.*

(5) *If $\Gamma=E_6$, then $m(\Gamma)=209$.*

(6) *If $\Gamma=E_7$, then $m(\Gamma)=499$.*

(7) *If $\Gamma=E_8$, then $m(\Gamma)=1339$.*

Proof. (1) Follows from the definitions. (2) Follows from (1) by observing that if $|\Gamma|=1$, then $m(\Gamma)=3$. The rest of the theorem follows by observing from the definition of $\mathbf{U}(\Gamma)$ that

$$m(\Gamma)=\sum_{k=0}^{n} 2^k f(k),$$

where $f(k)$ is the number of k-tuples of disjoint connected subgraphs of Γ such that the union of subgraphs in any k-tuple has exactly k connected components. $\square$

Finally we consider the case of a linearly ordered set P.

Theorem 2.4. *Let Γ be a fixed finite undirected graph, $P=\{1=e_0>e_1>\cdots>e_k\}$ be a linearly ordered set and $\lambda: P\to 2^\Gamma$ be a transitive map.*

(1) *The map λ is irreducible if and only if $\lambda(e_i)\neq\lambda(e_{i+1})$, $i=0,1,\ldots,k-1$.*

(2) *If λ is irreducible, then λ extends to an irreducible transitive map $\bar{\lambda}: P_1\to 2^\Gamma$, where P_1 is a linearly ordered set containing P and $|P_1|=2|\Gamma|+1$.*

(3) *If λ is irreducible and $\lambda(e_i)=\emptyset$ for some $i=1,2,\ldots,k$, then $\lambda(e_j)\neq\emptyset$ for $j\neq i$ and also*

$$\lambda(e_i)\subsetneq\lambda(e_{i+1})\subsetneq\cdots\subsetneq\lambda(e_k) \quad \textit{and} \quad \lambda(e_i)\subsetneq\lambda(e_{i-1})\subsetneq\cdots\subsetneq\lambda(e_0).$$

(4) *If $|P|=2|\Gamma|+1$, then there are $(|\Gamma|!)^2$ number of irreducible transitive maps $\lambda: P\to 2^\Gamma$ such that $\lambda(e_i)=\emptyset$ for some $i\in\{1,2,\ldots,k=2|\Gamma|\}$.*

Proof. (1) Suppose $\lambda(e_i)=\lambda(e_{i+1})$ for some $i=0,1,\ldots,k-1$. Take

$$P_1 = \{1 = e_0 > e_1 > \cdots > e_{i-1} > e_{i+1} > \cdots > e_k\}$$

and $\theta : P \to P_1$ as $\theta(e_i) = e_{i+1}$ and $\theta(e_j) = e_j$ for $j \neq i$. Take $\psi : P_1 \to 2^\Gamma$ as $\psi = \lambda|_{P_1}$. Then θ is an order preserving surjection, but not one-to-one and $\lambda = \psi \circ \theta$. Hence λ is not irreducible. Conversely, suppose $\lambda(e_i) \neq \lambda(e_{i+1})$, $i = 0, 1, \ldots, k-1$. Let $\theta : P \to P_1$ be an order preserving surjection and $\psi : P_1 \to 2^\Gamma$ a transitive map such that $\lambda = \psi \circ \theta$. Suppose $\theta(e_i) = \theta(e_j)$. Then since θ is order preserving we have $\theta(e_i) = \theta(e_{i+1}) = \cdots = \theta(e_{j-1}) = \theta(e_j)$. This implies that $e_i = e_j$. Hence θ is one-to-one. So λ is irreducible.

(2) By Theorems 2.1 and 2.2 it suffices to show that for any maximal chain Λ in $\mathbf{U}$ the map u restricted to Λ is irreducible. Let $(A, B), (A', B') \in \Lambda$, $(A, B) \gneqq (A', B')$ such that (A, B) covers (A', B'). Then $A' \subseteq A$ and $B \subseteq B'$. So $(A', B) \in \mathbf{U}$ and $(A, B) \geq (A', B) \geq (A', B')$. Since Λ is a maximal chain in $\mathbf{U}$, $A = A'$ or $B = B'$. Since $A \cap B = \emptyset$, $A' \cap B' = \emptyset$ (by definition) and $(A, B) \neq (A', B')$, so $u(A, B) = A \cup B \neq A' \cup B' = u(A', B')$. Hence by (1), u restricted to Λ is irreducible.

(3) Suppose $\lambda(e_i) = \emptyset$ for some $i = 1, 2, \ldots, k$. Since λ is irreducible, by (1) $\lambda(e_{i+1}) \neq \emptyset$ and $\lambda(e_{i-1}) \neq \emptyset$. Hence since λ is transitive $\lambda(e_{i+1}) \subseteq \lambda(e_{i+2}) \subseteq \cdots \subseteq \lambda(e_k)$ and $\lambda(e_{i-1}) \subseteq \lambda(e_{i-2}) \subseteq \cdots \subseteq \lambda(e_1)$. So $\lambda(e_j) \neq \emptyset$ for $j \neq i$.

(4) This follows from (1) and (3) by observing that since λ is irreducible, $\lambda(e_i) \neq \emptyset$ unless $i = |\Gamma|$. □

3. Examples

In this section we will restrict to the situation where Γ is one of the following:

○—○, or ○—○—○.

These correspond to monoids on GL(3, F) and GL(4, F), respectively [3]. One monoid on GL(4, F) is $M_4(F)$, the monoid of all 4×4 matrices over the field F. In this situation

$$\Gamma = \left\{ \theta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\quad \theta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad \theta_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

and θ_i and θ_j are joined by an edge if and only if $\theta_i \theta_j \neq \theta_j \theta_i$.

The standard idempotent representatives for matrices in $M_4(F)$ of different ranks are given by the linearly ordered set

$$P = \left\{ I,\ e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{O} \right\}.$$

The corresponding transitive map $\lambda : P \to 2^{\Gamma}$ is given by

$$\lambda(e) = \{\theta \in \Gamma \mid e\theta = \theta e\}, \quad \text{for all } e \in P.$$

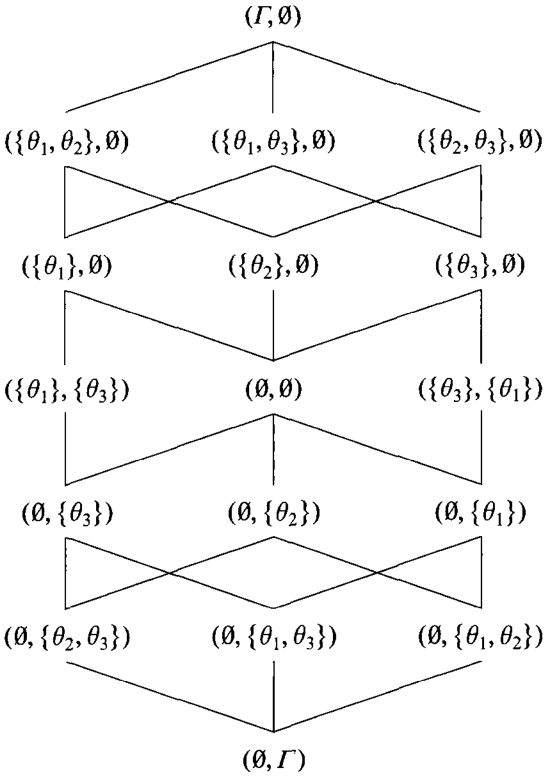
Thus

$$\lambda(I) = \Gamma, \qquad \lambda(e_1) = \{\theta_2, \theta_3\}, \qquad \lambda(e_2) = \{\theta_1, \theta_3\},$$
$$\lambda(e_3) = \{\theta_1, \theta_2\} \quad \text{and} \quad \lambda(0) = \Gamma. \tag{3.1}$$

Note that by Theorem 2.4, λ is irreducible. The graph structure for Γ is

$$\theta_1 \text{ —— } \theta_2 \text{ —— } \theta_3.$$

The universal partially ordered set $\mathbf{U} = \mathbf{U}(\Gamma)$ is



By Theorem 2.4(4) there are 36 maximal linear irreducible transitive maps λ with some element mapping to the empty subgraph. The remaining ones are given by the following:

$$\lambda_1: \quad 1 \longmapsto \Gamma \qquad\qquad \lambda_2: \quad 1 \longmapsto \Gamma$$
$$\quad | \qquad\qquad\qquad\qquad |$$
$$e_1 \longmapsto \{\theta_2, \theta_3\} \qquad\qquad e_1 \longmapsto \{\theta_2, \theta_3\}$$
$$\quad | \qquad\qquad\qquad\qquad |$$

$$\begin{array}{c@{\ }l} e_2 & \longmapsto \{\theta_3\} \\ | & \\ e_3 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_4 & \longmapsto \{\theta_1\} \\ | & \\ e_5 & \longmapsto \{\theta_1, \theta_2\} \\ | & \\ e_6 & \longmapsto \Gamma \end{array}$$

$$\begin{array}{c@{\ }l} e_2 & \longmapsto \{\theta_3\} \\ | & \\ e_3 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_4 & \longmapsto \{\theta_1\} \\ | & \\ e_5 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_6 & \longmapsto \Gamma \end{array}$$

$$\lambda_3: \begin{array}{c@{\ }l} 1 & \longmapsto \Gamma \\ | & \\ e_1 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_2 & \longmapsto \{\theta_3\} \\ | & \\ e_3 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_4 & \longmapsto \{\theta_1\} \\ | & \\ e_5 & \longmapsto \{\theta_1, \theta_2\} \\ | & \\ e_6 & \longmapsto \Gamma \end{array}$$

$$\lambda_4: \begin{array}{c@{\ }l} 1 & \longmapsto \Gamma \\ | & \\ e_1 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_2 & \longmapsto \{\theta_3\} \\ | & \\ e_3 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_4 & \longmapsto \{\theta_1\} \\ | & \\ e_5 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_6 & \longmapsto \Gamma \end{array}$$

$$\lambda_5: \begin{array}{c@{\ }l} 1 & \longmapsto \Gamma \\ | & \\ e_1 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_2 & \longmapsto \{\theta_1\} \\ | & \\ e_3 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_4 & \longmapsto \{\theta_3\} \\ | & \\ e_5 & \longmapsto \{\theta_2, \theta_3\} \\ | & \\ e_6 & \longmapsto \Gamma \end{array}$$

$$\lambda_6: \begin{array}{c@{\ }l} 1 & \longmapsto \Gamma \\ | & \\ e_1 & \longmapsto \{\theta_1, \theta_2\} \\ | & \\ e_2 & \longmapsto \{\theta_1\} \\ | & \\ e_3 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_4 & \longmapsto \{\theta_3\} \\ | & \\ e_5 & \longmapsto \{\theta_1, \theta_3\} \\ | & \\ e_6 & \longmapsto \Gamma \end{array}$$

$$\lambda_7: \begin{array}{c@{\ }l} 1 & \longmapsto \Gamma \\ | & \\ e_1 & \longmapsto \{\theta_1, \theta_2\} \\ | & \end{array}$$

$$\lambda_8: \begin{array}{c@{\ }l} 1 & \longmapsto \Gamma \\ | & \\ e_1 & \longmapsto \{\theta_1, \theta_3\} \\ | & \end{array}$$

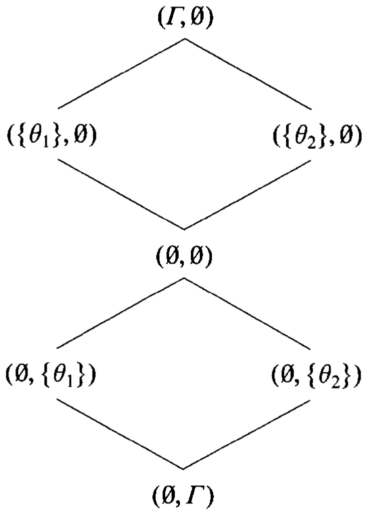
$$\begin{array}{ccc} e_2 & \longmapsto & \{\theta_1\} \\ | & & \\ e_3 & \longmapsto & \{\theta_1,\theta_3\} \\ | & & \\ e_4 & \longmapsto & \{\theta_3\} \\ | & & \\ e_5 & \longmapsto & \{\theta_2,\theta_3\} \\ | & & \\ e_6 & \longmapsto & \Gamma \end{array} \qquad \begin{array}{ccc} e_2 & \longmapsto & \{\theta_1\} \\ | & & \\ e_3 & \longmapsto & \{\theta_1,\theta_3\} \\ | & & \\ e_4 & \longmapsto & \{\theta_3\} \\ | & & \\ e_5 & \longmapsto & \{\theta_1,\theta_3\} \\ | & & \\ e_6 & \longmapsto & \Gamma \end{array}$$

Note that the linear irreducible transitive map λ in (3.1) is not maximal. However, it comes from the maximal linear irreducible transitive map λ_1.

Next we consider the graph $\Gamma=\{\theta_1,\theta_2\}$ corresponding to $\mathrm{GL}(3,F)$. The graph structure is given by

$$\theta_1 \;——\; \theta_2.$$

The universal partially ordered set $\mathbf{U}=\mathbf{U}(\Gamma)$ is given by



The linear irreducible transitive maps of length ≥ 3 are:

$$\lambda_1\colon \begin{array}{ccc} 1 & \longmapsto & \Gamma \\ | & & \\ e_1 & \longmapsto & \{\theta_1\} \\ | & & \\ e_2 & \longmapsto & \emptyset \\ | & & \end{array} \qquad \lambda_2\colon \begin{array}{ccc} 1 & \longmapsto & \Gamma \\ | & & \\ e_1 & \longmapsto & \{\theta_1\} \\ | & & \\ e_2 & \longmapsto & \emptyset \\ | & & \end{array}$$

$$\begin{array}{l} e_3 \longmapsto \{\theta_1\} \\ \mid \\ e_4 \longmapsto \Gamma \end{array} \qquad \begin{array}{l} e_3 \longmapsto \{\theta_2\} \\ \mid \\ e_4 \longmapsto \Gamma \end{array}$$

$$\lambda_3:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \{\theta_2\} \\ \mid \\ e_2 \longmapsto \emptyset \\ \mid \\ e_3 \longmapsto \{\theta_1\} \\ \mid \\ e_4 \longmapsto \Gamma \end{array} \qquad \lambda_4:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \{\theta_2\} \\ \mid \\ e_2 \longmapsto \emptyset \\ \mid \\ e_3 \longmapsto \{\theta_2\} \\ \mid \\ e_4 \longmapsto \Gamma \end{array}$$

$$\lambda_5:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \{\theta_1\} \\ \mid \\ e_2 \longmapsto \emptyset \\ \mid \\ e_3 \longmapsto \Gamma \end{array} \qquad \lambda_6:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \{\theta_1\} \\ \mid \\ e_2 \longmapsto \emptyset \\ \mid \\ e_3 \longmapsto \{\theta_1\} \end{array}$$

$$\lambda_7:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \emptyset \\ \mid \\ e_2 \longmapsto \{\theta_1\} \\ \mid \\ e_3 \longmapsto \Gamma \end{array} \qquad \lambda_8:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \{\theta_1\} \\ \mid \\ e_2 \longmapsto \emptyset \\ \mid \\ e_3 \longmapsto \{\theta_2\} \end{array}$$

$$\lambda_9:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \emptyset \\ \mid \\ e_2 \longmapsto \{\theta_2\} \\ \mid \\ e_3 \longmapsto \Gamma \end{array} \qquad \lambda_{10}:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \{\theta_1\} \\ \mid \\ e_2 \longmapsto \{\theta_2\} \\ \mid \\ e_3 \longmapsto \Gamma \end{array}$$

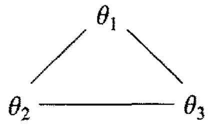
$$\lambda_{11}:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \{\theta_2\} \\ \mid \end{array} \qquad \lambda_{12}:\ \begin{array}{l} 1 \longmapsto \Gamma \\ \mid \\ e_1 \longmapsto \{\theta_2\} \\ \mid \end{array}$$

$$\begin{array}{llll} & e_2 \longmapsto \emptyset & & e_2 \longmapsto \emptyset \\ & \quad | & & \quad | \\ & e_3 \longmapsto \Gamma & & e_3 \longmapsto \{\theta_1\} \\ \lambda_{13}: & 1 \longmapsto \Gamma & \lambda_{14}: & 1 \longmapsto \Gamma \\ & \quad | & & \quad | \\ & e_1 \longmapsto \{\theta_2\} & & e_1 \longmapsto \{\theta_2\} \\ & \quad | & & \quad | \\ & e_2 \longmapsto \{\theta_1\} & & e_2 \longmapsto \emptyset \\ & \quad | & & \quad | \\ & e_3 \longmapsto \Gamma & & e_3 \longmapsto \{\theta_2\} \end{array}$$

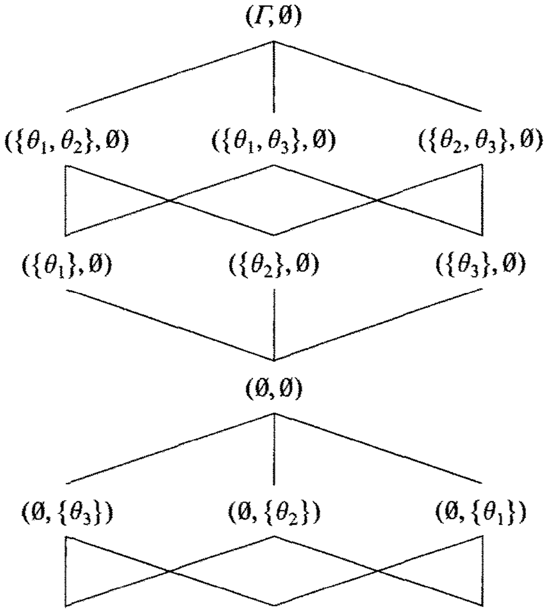
Note that the first 4 maps ($\lambda_1, \ldots, \lambda_4$) are of length 4 and the last 10 maps ($\lambda_5, \ldots, \lambda_{14}$) are of length 3 and are contained in the maximal maps $\lambda_1, \ldots, \lambda_4$. Also observe that any nonmaximal irreducible transitive map may be contained in more than one maximal irreducible transitive map. For example, the map λ_5 is contained in λ_1 and λ_2.

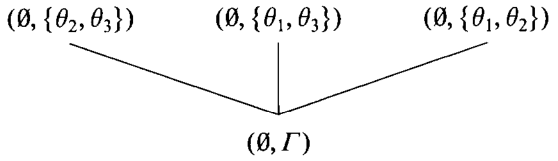
Finally consider the extended Dynkin diagram (see [2])

Γ:



The universal partially ordered set $\mathbf{U} = \mathbf{U}(\Gamma)$ in this case is given by:





Note that in this case there are exactly 36 maximal linear irreducible transitive maps.

References

[1] G. Gratzer, Lattice Theory (Freeman, New York, 1971).
[2] S. Helgason, Differential Geometry, Lie groups, and Symmetric Spaces (Academic Press, New York, 1978).
[3] M.S. Putcha, Monoids on groups with *BN*-pairs, J. Algebra, 120 (1989) 139–169.
[4] M.S. Putcha and L.E. Renner, The system of idempotents and the lattice of *J*-classes of reductive algebraic monoids, J. Algebra 116 (1988) 385–399.