Polarity and Point Extensions in Oriented Matroids

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ABSTRACT

A. Bachem and W. Kern have recently extended the notion of polarity (relatively to an $\mathbb{R}$-bilinear form) to oriented matroids [1]. We prove that the usual polarity properties of the face lattices of convex polytopes can be extended to the class of oriented matroids admitting an (oriented) polar. We give also a short proof of the principal result of [1] showing that there is a natural embedding of the poset of signed span of the cocircuits of a polar of an oriented matroid into the extension poset of this matroid. We remark that if M is a matroid admitting a polar, then every hyperplane can be intersected by every line. Oriented matroids satisfying this condition have an important role in oriented-matroid programming.

1. NOTATION

We assume the reader is familiar with the basic results of oriented matroid theory [3, 8, 10, 11]. We specify some of the notation used in this paper.

A *signed set* $X=(X^+,X^-)$ is a finite set $\underline{X}=X^+\cup X^-$ partitioned into two distinguished subsets: the set X^+ of *positive* elements and the set X^- of *negative* elements. The *opposite* $-X$ of a signed set X is defined by $(-X)^+=X^-$ and $(-X)^-=X^+$. For the sake of simplicity we use also the notation $+X=X$, and if A is a subset of $\underline{X}$, $X-A=(X^+-A,X^--A)$ and $X\cap A=(X^+\cap A, X^-\cap A)$, $\overline{}_A X=((X^+-A)\cup(X^-\cap A),\ (X^--A)\cup(X^+\cap A))$. If $\mathscr{X}$ is a family of signed sets, we denote by $\overline{}_A\mathscr{X}$ the set $\{\overline{}_A Y: Y\in\mathscr{X}\}$. If X and Y are two signed sets, we say X is *orthogonal* to Y and we write $X\perp Y$ if $\underline{X}\cap\underline{Y}=\varnothing$ or $(X^+\cap Y^+)\cup(X^-\cap Y^+)\neq\varnothing$ and $(X^+\cap$

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$Y^-)\cup(X^- \cap Y^+) \neq \varnothing$. An *oriented matroid* $M = M(E, \mathcal{O})$ on a finite set E is defined by its collection $\mathcal{O}$ of *signed circuits*, i.e. a set $\mathcal{O}$ of signed subsets of E satisfying

(O1) $X \in \mathcal{O}$ implies $\underline{X} \neq \varnothing$ and $-X \in \mathcal{O}$; if $X_1, X_2 \in \mathcal{O}$ and $\underline{X}_1 \subset \underline{X}_2$ then $X_1 = X_2$ or $X_1 = -X_2$;

(O2) (*Elimination property*) for all $X_1, X_2 \in \mathcal{O}$, $x \in X_1^+ \cap X_2^-$, $y \in X_1^+ - X_2^-$ there is $X_3 \in \mathcal{O}$ such that $y \in \underline{X}_3$, $X_3^+ \subset (X_1^+ \cup X_2^+) - \{x\}$, and $X_3^- \subset (X_1^- \cup X_2^-) - \{x\}$.

As usual, if M is an oriented matroid, we denote by $E(M)$ and $\mathcal{O}(M)$ respectively the underlying set of M and the set of signed circuits of M. By forgetting the orientation of M we obtain a (nonoriented) matroid $\underline{M}$ [16]. The cocircuits of $\underline{M}$ (circuits of the orthogonal matroid $\underline{M}^\perp$) can be oriented in a unique way such that for all signed circuits $X \in \mathcal{O}$ and signed cocircuits $Y \in \mathcal{O}^\perp$ we have $X \perp Y$. A *positive circuit* X is a signed set such that $X^- = \varnothing$. If $M(E, \mathcal{O})$ is an oriented matroid, we denote by $\mathcal{K}(\mathcal{O})$ *the signed span* of $\mathcal{O}$: i.e., if X is a signed set having support contained in E, then $X \in \mathcal{K}(\mathcal{O})$ if and only if there are oriented circuits $X_1, \ldots, X_n$ such that $X^+ = X_1^+ \cup \cdots \cup X_n^+$, $X^- = X_1^- \cup \cdots \cup X_n^-$, and $(X_i^+ \cap X_j^-) = (X_i^- \cap X_j^+) = \varnothing$, $1 \leqslant i < j \leqslant n$. By the definitions, $\mathcal{K}(\mathcal{O})$ is the family of the signed sets X of support contained in E such that $X \perp Y$ for all $Y \in \mathcal{O}^\perp$. In this paper we suppose $\mathcal{K}(\mathcal{O})$ ordered with the relation $X \leqslant Y$ if $X^+ \subset Y^+$ and $X^- \subset Y^-$.

Let E be a finite subset of $\mathbb{R}^n$. The minimal linear dependencies of E over $\mathbb{R}$ constitute the signed circuits of an oriented matroid on E, denoted $\mathbb{L}\mathrm{in}(E)$ and called *the oriented matroid on E determined by linear dependencies over* $\mathbb{R}$. More precisely, if $\underline{C} = \{e_{i_1}, \ldots, e_{i_k}\}$ is a circuit of $\underline{\mathbb{L}\mathrm{in}(E)}$ and $\sum_{j=1}^k \lambda_j e_{i_j} = 0$, then $C = (C^+, C^-)$ with $C^+ = \{e_{i_j} : \lambda_j > 0\}$, $C^- = \{e_{i_j} : \lambda_j < 0\}$ is a signed circuit of $\mathbb{L}\mathrm{in}(E)$ (see [3, Example 3.1]). Similarly the minimal affine dependencies of E over $\mathbb{R}$ constitute the signed circuits of an oriented matroid on E, denoted $\mathbb{A}\mathrm{ff}(E)$ is called *the oriented matroid on E determined by affine dependencies over* $\mathbb{R}$ (see [3, Example 3.5]).

2. SINGLE-ELEMENT EXTENSIONS

A *single-element oriented extension* of an oriented matroid $M(E, \mathcal{O})$ is an oriented matroid $N(E \cup \{p\}, \mathcal{O}')$ of the same rank as M and such that $N \setminus p = M$. We start by looking at the work of Las Vergnas and refer the reader to [10] for details and proofs. However, for convenience, we also use a terminology introduced by A. Mandel [12].

DEFINITION 2.1. Let $M(E, \mathcal{O})$ be an oriented matroid. A pair $(\mathcal{Y}, \mathcal{X})$ of collections of cocircuits of M is said to be a *localization* in M if there is an oriented matroid $N(E \cup \{p\}, \mathcal{O}')$ that is a point extension of M and of the same rank as M such that for every signed cocircuit X of N such that $X - \{p\}$ is also a cocircuit of M,

(2.1.1) $p \notin \underline{X}$ if and only if $X \in \mathcal{X}$;
(2.1.2) $p \in X^+$ if and only if $X - \{p\} \in \mathcal{Y}$.

The theorem quoted below is an easy consequence of a remarkable characterization of localization due to Las Vergnas [10]. It appears in [12] in a similar form. A variety of alternative characterizations is also given in [10].

THEOREM 2.2 [10]. *A pair $(\mathcal{Y}, \mathcal{X})$ of collections of cocircuits of M is a localization of M if and only if*

(2.2.1) *$\mathcal{Y} \cup \{-X: X \in \mathcal{Y}\} \cup \mathcal{X}$ is a partition of the cocircuits of M;*
(2.2.2) *$\mathcal{Y} \cup \mathcal{X}$ satisfies the elimination property for all modular pairs of cocircuits.*

REMARK 2.3. Let $\mathbf{A}\mathrm{ff}(E)$ be the oriented matroid of the affine dependencies of a finite set E in $\mathbb{R}^n$ (see [3, Example 3.5]). The hyperplanes spanned by the elements of E divide $\mathbb{R}^n$ into regions bounded by the hyperplanes. Thus when we extend $\mathbf{A}\mathrm{ff}(E)$ by adding another point p of $\mathbb{R}^n$ to the set E, this point p must lie on one of the existing regions of $\mathbb{R}^n$ or on a boundary. If p' is another point of $\mathbb{R}^n$ different from p but lying in the same region (or boundary) as p, then the corresponding point extensions of $\mathbf{A}\mathrm{ff}(E)$ are isomorphic, i.e., the regions and their boundaries determined by the set E are in bijection with the set of nonisomorphic (acyclic) point extensions, $\mathbf{A}\mathrm{ff}(E \cup \{p\})$, of $\mathbf{A}\mathrm{ff}(E)$. Thus the fact that a point extension N of an oriented matroid M depends on a localization in M is an abstraction (and a generalization) of the requirement that the point p of $\mathbf{A}\mathrm{ff}(E \cup \{p\})$ must lie in a region or boundary of $\mathbb{R}^n$ determined by E.

Let $M(E, \mathcal{O})$ be an oriented matroid, and $\mathcal{X}$ be a collection of cocircuits of M such that $\mathcal{X} \cup \{-X: X \in \mathcal{X}\}$ is a partition of $\mathcal{O}^{\perp}$. Suppose X_1, X_2, X_3 are three cocircuits of $\mathcal{X}$ such that $(E - \underline{X}_1) \cap (E - \underline{X}_2) \cap (E - \underline{X}_3)$ is a hyperplane of M (i.e. for all i, j, $1 \leqslant i < j \leqslant 3$, $\underline{X}_i, \underline{X}_j$ is a modular pair of cocircuits). Hence it is easy to derive from the definitions that one and only one of the two following conditions arises (we leave the proof to the reader):

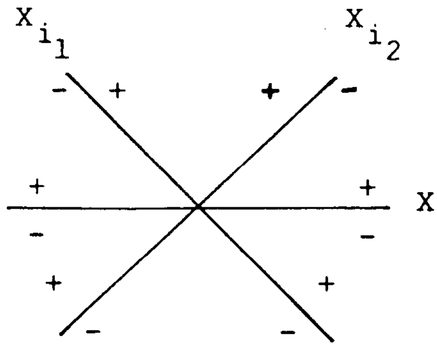

FIG. 1

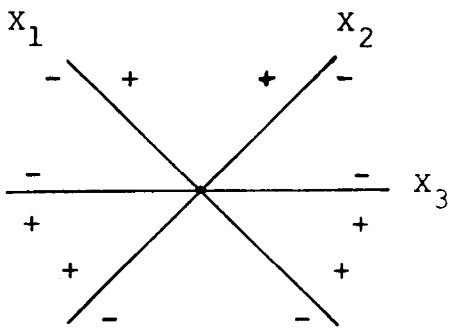

FIG. 2

(2.4.1) there is one (and only one) cocircuit $X \in \{X_1, X_2, X_3\}$ such that if X_{i_1} and X_{i_2} are the remaining cocircuits, then $X^+ \subset X_{i_1}^+ \cup X_{i_2}^+$ and $X_{i_1}^- \cup X_{i_2}^-$ (see Figure 1);

(2.4.2) For every $X \in \{X_1, X_2, X_3\}$, if X_{i_1}, X_{i_2} are the remaining cocircuits, then $X^- \subset X_{i_1}^+ \cup X_{i_2}^+$ and $X^+ \subset X_{i_1}^- \cup X_{i_2}^-$ (see Figure 2).

NOTATION 2.5. Let $\mathscr{X}$ be a collection of cocircuits of an oriented matroid $M(E, \mathscr{O})$ such that $\mathscr{X} \cup \{-X : X \in \mathscr{X}\}$ is a partition of $\mathscr{O}^\perp$. Let $\mathscr{L}(\mathscr{X})$ denote the collection of signed sets on $\mathscr{X}$ satisfying the two conditions below:

(2.5.1) If $Y \in \mathscr{L}(\mathscr{X})$, then Y supports three cocircuits X_1, X_2, X_3 of $\mathscr{X}$ and $(E - \underline{X}_1) \cap (E - \underline{X}_2) \cap (E - \underline{X}_3)$ is a hyperline of M;

(2.5.2) $Y^+ = \{X_{i_1}, X_{i_2}\}$, $Y^- = \{X\}$ if Y satisfies the assumptions of (2.4.1) and X is the distinguished cocircuit of $\underline{Y} = \{X_1, X_2, X_3\}$, and $Y^+ = \{X_1, X_2, X_3\}$ in the opposite case [condition (2.4.2)].

Theorem 2.6 below is a variant of Theorem 2.2, which we have found more useful for our purpose in this work. We leave its proof to the reader.

THEOREM 2.6. *Let $M(E, \mathcal{O})$ be an oriented matroid and $\mathcal{L}(\mathcal{X})$ be a collection of signed sets satisfying the conditions* (2.5.1) *and* (2.5.2). *Let $\mathcal{Y} \cup \{-X : X \in \mathcal{Y}\} \cup \mathcal{X}$ be a partition of the cocircuits of M. Then the pair $(\mathcal{Y}, \mathcal{X})$ is a localization of M if and only if the signed set $\mathcal{A} = (\mathcal{A}^+, \mathcal{A}^-)$, where $\mathcal{A}^+ = \mathcal{Y} \cap \mathcal{X}$ and $\mathcal{A}^- = \{-X : X \in (\mathcal{Y} - \mathcal{X})\}$, is orthogonal to all the signed sets of $\mathcal{L}(\mathcal{X})$.*

As an application of Theorem 2.6, we prove a result that can be useful if we intend to determine all the localizations of an oriented matroid.

For any independent set $I = \{e_1, \ldots, e_n\}$ of $M(E, \mathcal{O})$ and partition $I^+ \cup I^-$ of I, the *localization determined by the partition* $I^+ \cup I^-$ is the localization $(\mathcal{Y}, \mathcal{X})$ of M such that $X \in \mathcal{Y}$ if and only if $\underline{X} \cap I \neq \varnothing$ and if i $(1 \leqslant i \leqslant n)$ is the smallest index such that $e_i \in \underline{X}$, then $e_i \in X^+$ $[e_i \in X^-]$ if $e_i \in I^+$ $[e_i \in I^-]$. These localizations are a slight variation of the localizations corresponding to the *principal extensions* described by Las Vergnas [10].

Let $M(E, \mathcal{O})$ be an oriented matroid, and let $\mathcal{X} \cup \{-X : X \in \mathcal{X}\}$ be a partition of $\mathcal{O}^\perp$. Then it is clear that every localization $(\mathcal{Y}, \mathcal{X})$ of M is determined by the *signed vector v on $\mathcal{X}$* with components indexed by the elements of $\mathcal{X}$ and entries in $\{1, -1, 0\}$ such that the entry of v indexed by X, v_X, is 1, -1, or 0 respectively if $X \in \mathcal{Y}$, $-X \in \mathcal{Y}$, or $X \in \mathcal{X}$.

COROLLARY 2.7. *Let $M(E, \mathcal{O})$ be an oriented matroid and $\mathcal{X} \cup \{-X : X \in \mathcal{X}\}$ be a partition of $\mathcal{O}^\perp$. Then all the localizations of M are determined canonically by two collections*:

(2.7.1) *the set of the signed vectors on $\mathcal{X}$ corresponding to the localizations of M determined by the partitions of the bases of M*;
(2.7.2) *the subset of $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$, $\{\underline{Y} : Y \in \mathcal{L}(\mathcal{X})\}$.*

The following lemma is a necessary tool in the proof of Corollary 2.7.

LEMMA 2.8. *Let $M(E, \mathcal{O})$ be an oriented matroid, and $\mathcal{X}$ be a collection of cocircuits of M such that $\mathcal{X} \cup \{-X : X \in \mathcal{X}\}$ is a partition of $\mathcal{O}^\perp$. Suppose $Y \in \mathcal{L}(\mathcal{X})$ and $Y^+ = \{X_1, X_2, X_3\}$. Then for every $i = 1,2,3$ there is a localization $(\mathcal{Y}_i, \varnothing)$ of M such that $-X_i \in \mathcal{Y}_i$ and $Y^+ - \{X_i\} \subset \mathcal{Y}_i$.*

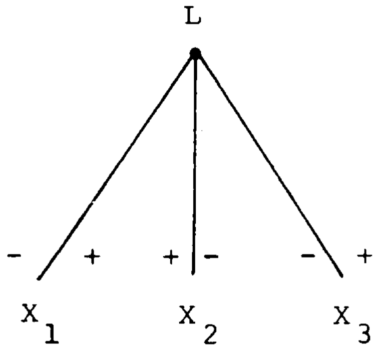

FIG. 3

Proof of Lemma 2.8. Note that if $A \subset E$ and $\underset{A}{-}\mathscr{X} := \{\underset{A}{-}X : X \in \mathscr{X}\}$, then $Y_0 \in \mathscr{L}(\underset{A}{-}\mathscr{X})$ if and only if there is $Y \in \mathscr{L}(\mathscr{X})$ such that $Y_0^+ = \{\underset{A}{-}X : X \in Y^+\}$ and $Y_0^- = \{\underset{A}{-}X : X \in Y^-\}$. Otherwise $(\mathscr{Y}, \mathscr{X})$ is a localization of $M(E, \mathscr{O})$ if and only if $(\underset{A}{-}\mathscr{Y}, \underset{A}{-}\mathscr{X})$ is a localization of $\underset{A}{-}M$. Consequently we can suppose that $M(E, \mathscr{O})$ is acyclic and that $L = (E - \underline{X}_1) \cap (E - \underline{X}_2) \cap (E - \underline{X}_3)$ and $E - \underline{X}_1$ are faces of $M(E, \mathscr{O})$. As Y is a positive signed set, for appropriate reorientations of M, one of two cases of Figures 3, 4 holds.

We treat the case of Figure 3. (The case of Figure 4 is similar.) Let $\{b_1, \ldots, b_{r-1}\}$ be a base of the hyperplane $E - \underline{X}_1$, and b be a point in

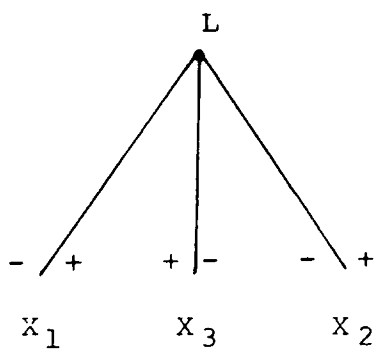

FIG. 4

$(E - \underline{X}_2) - L$. Let $(\mathscr{Y}', \varnothing)$ $[(\mathscr{Y}'', \varnothing)]$ be the localization of M determined by $B^+ = \{b_1, \ldots, b_{r-1}, b_r = b\}$ $[B^+ = \{b\}, B^- = \{b_1, \ldots, b_{r-1}\}]$. It is clear that $X_1, X_2, -X_3 \in \mathscr{Y}'$ $[X_1, X_3, -X_2 \in \mathscr{Y}'']$. By taking X_2 or X_3 instead of X_1 we can obtain the localizations in the conditions of the lemma. ∎

Proof of Corollary 2.7. Suppose $Y \in \mathscr{L}(\mathscr{X})$, and let $Y^- = \{X_1\}$, $Y^+ = \{X_2, X_3\}$. Consider the family $\mathscr{X}' = \{\mathscr{X} - \{X_1\}\} \cup \{-X_1\}$. The positive signed set Y_1, $Y_1^+ = \{-X_1, X_2, X_3\}$, is a signed set of $\mathscr{L}(\mathscr{X}')$. By Lemma 2.8 there are three localizations $(\mathscr{Y}_1, \varnothing)$, $(\mathscr{Y}_2, \varnothing)$, and $(\mathscr{Y}_3, \varnothing)$ of M such that $\{X_1, X_2, X_3\} \subset \mathscr{Y}_1$, $\{-X_1, -X_2, X_3\} \subset \mathscr{Y}_2$, and $\{-X_1, X_2, -X_3\} \subset \mathscr{Y}_3$. But then if Y_0 is a signed set such that $\underline{Y}_0 = \{X_1, X_2, X_3\}$ and if $Y_0 \perp \mathscr{Y}_1$, $Y_0 \perp \mathscr{Y}_2$, and also $Y_0 \perp \mathscr{Y}_3$, then we must have either $Y_0^- = \{X_1\}$, $Y_0^+ = \{X_2, X_3\}$ or $Y_0^+ = \{X_1\}$, $Y_0^- = \{X_2, X_3\}$.

The case where $Y \in \mathscr{L}(\mathscr{X})$ and Y is a positive signed set is similar. Hence Corollary 2.7 is a clear consequence of Theorem 2.6 and Lemma 2.8. ∎

COROLLARY 2.9. *Let $M(E, \mathscr{O})$ be an oriented matroid. Suppose that $(\mathscr{Y}_1, \mathscr{X}_1)$ and $(\mathscr{Y}_2, \mathscr{X}_2)$ are localizations of M. Then $(\mathscr{Y}_1 \cup (\mathscr{Y}_2 \cap \mathscr{X}_1), \mathscr{X}_1 \cap \mathscr{X}_2)$ is also a localization of M.*

Proof. Let $\mathscr{X}$ be a collection of cocircuits of M such that $\mathscr{X} \cup \{-Y : Y \in \mathscr{X}\}$ is a partition of $\mathscr{O}^\perp$. Let $\mathscr{A}_1 = (\mathscr{A}_1^+, \mathscr{A}_1^-)$ $[\mathscr{A}_2 = (\mathscr{A}_2^+, \mathscr{A}_2^-)]$ be such that $\mathscr{A}_1^+ = \mathscr{L}_1 \cap \mathscr{X}$, $\mathscr{A}_1^- = \mathscr{L}_1 - \mathscr{X}$ $[\mathscr{A}_2^+ = \mathscr{L}_2 \cap \mathscr{X}, \mathscr{A}_2^- = \mathscr{L}_2 - \mathscr{X}]$. The signed sets $\mathscr{A}_1$ and $\mathscr{A}_2$ are orthogonal to $\mathscr{L}(\mathscr{X})$ by Theorem 2.6. Then the *composition* of the signed sets $\mathscr{A}_1$ and $\mathscr{A}_2$ [i.e. the signed set $\mathscr{A}_{12}$ such that $\mathscr{A}_{12}^+ = \mathscr{A}_1^+ \cup (\mathscr{A}_2^+ - \underline{\mathscr{A}}_1)$ and $\mathscr{A}_{12}^- = \mathscr{A}_1^- \cup (\mathscr{A}_2^- - \underline{\mathscr{A}}_1)$] is also orthogonal to $\mathscr{L}(\mathscr{X})$, and Corollary 2.9 follows. ∎

REMARK 2.10. Corollary 2.9 has an interesting geometrical interpretation when the oriented point extensions $M_1(E \cup \{p_1\}, \mathscr{O}_1)$, $M_2(E \cup \{p_2\}, \mathscr{O}_2)$ determined respectively by the two different localizations $(\mathscr{Y}_1, \mathscr{X}_1)$ and $(\mathscr{Y}_2, \mathscr{X}_2)$ are compatible: i.e., when there is an oriented matroid $M_{12}(E \cup \{p_1, p_2\}, \mathscr{O}_{12})$ such that $M_{12} \setminus p_1 = M_2$ and $M_{12} \setminus p_2 = M_1$. Let $I = \{p_1, p_2\}$, and let $(\mathscr{Y}, \mathscr{X})$ be the localization of M_{12} determined by the partition $I = I^+ \cup I^-$, $I^+ = I$. Let $M'(E \cup \{p_1, p_2, p_3\}, \mathscr{O}')$ be the single-element extension of M_{12} determined by the localization $(\mathscr{Y}, \mathscr{X})$: i.e., the new point p_3 is placed in the segment $[p_1, p_2]$ in the neighborhood of p_1. The restriction of M_{12}' to the set $E \cup \{p_3\}$ is a point extension of M. The localization $(\mathscr{Y}_1 \cup (\mathscr{Y}_2 \cap \mathscr{X}_1), \mathscr{X}_1 \cap \mathscr{X}_2)$ is exactly the localization determined by this point extension.

3. POLARITY AND ORIENTED MATROIDS

The notion of adjoint of a geometric lattice presented in Definition 3.1 above is only apparently more general then that of Crapo [7]. Indeed, the function φ here defined can be extended to a one-to-one order-reversing function betwen L and $\tilde{L}$ (see [1, Proposition 2.1] or [2, Lemma 5.1]). For additional information concerning the notion of adjoint, see [4] and [14, Proposition 3.5].

DEFINITION 3.1. Let L and $\tilde{L}$ be two geometric lattices of same rank. We say $\tilde{L}$ is an *adjoint* of L if there is an injective function φ mapping the points (atoms) of L on the copoints of $\tilde{L}$ and the copoints of L onto the points of $\tilde{L}$ in such a way that if p is a point and H a copoint of L, then $p \leqslant H$ if and only if $\varphi(H) \leqslant \varphi(p)$. Similarly, given two matroids M and $\tilde{M}$ of the same rank, we call $\tilde{M}$ an *adjoint* of M if the lattice of flats of $\tilde{M}$ is an adjoint of the lattice of flats of M.

EXAMPLE 3.2. Let V be a real vector space of finite dimension, E a finite set of V, and $\mathbb{L}\text{in}(E)$ the (oriented) matroid of linear dependencies of E over $\mathbb{R}$. We suppose that E spans V. Let $\Phi: V \times V \to \mathbb{R}$ be a nondegenerate $\mathbb{R}$-bilinear form. For every hyperplane H_i $(1 \leqslant i \leqslant m)$ of V spanned by elements of E, let h_i be a nonnull vector such that for every $x \in H_i$, $\Phi(h_i, x) = 0$. Then the matroid $\underline{\mathbb{L}\text{in}}(\{h_1, \ldots, h_m\})$ is an adjoint of $\underline{\mathbb{L}\text{in}}(E)$.

First note that the matroids $\mathbb{L}\text{in}(E)$ and $M = \mathbb{L}\text{in}(\{h_1, \ldots, h_m\})$ have the same rank. Indeed, let $B = \{e_1, \ldots, e_r\}$ be a base of $\mathbb{L}\text{in}(E)$, and $B' = \{h'_1, \ldots, h'_r\}$ be the vectors of $\{h_1, \ldots, h_m\}$ such that for every $1 \leqslant i, j \leqslant r$, $\Phi(e_i, h'_j) \neq 0$ if and only if $i = j$. Then it is clear that B' is an independent set, and as E spans V, B' is necessarily a base of M. On the other hand, for every $e \in E$, $e^{\perp} = \{h_i : 1 \leqslant i \leqslant m,\ \Phi(e, h_i) = 0\}$ is a flat of M because Φ is an $\mathbb{R}$-bilinear form. We can suppose always that $e \in B$. But then there are $r - 1$ vectors of B' in the flat $e^{\perp}$, and $e^{\perp}$ is a hyperplane of M. This proves what we wanted.

The following extension of the notion of adjoint to oriented matroids is equivalent to the one introduced recently in [1]. These authors use the term "adjoint of an oriented matroid." But definition 3.3 below is a clear generalization of the notion of "polar relative to a positive definite $\mathbb{R}$-bilinear form." For this reason we prefer to use the term "polar" instead of "adjoint."

Let $\Phi: E \times \tilde{E} \to \{0, 1, -1\}$, and for every $\tilde{e} \in \tilde{E}$ $[e \in E]$ let $\Phi_{\tilde{e}}$ $[\Phi_e]$ be the function $\Phi_{\tilde{e}}: E \to \{0, 1, -1\}$, $e \rightsquigarrow \Phi(e, \tilde{e})$ $[\Phi_e: \tilde{E} \to \{0, 1, -1\}$, $\tilde{e} \rightsquigarrow$

$\Phi(e, \tilde{e})]$. In the sequel we identify the support of $\Phi_{\tilde{e}}$ $[\Phi_e]$, denoted supp$(\Phi_{\tilde{e}})$ [supp(Φ_e)], with the signed set

$$\left(\{e: e \in E,\ \Phi_{\tilde{e}}(e) = 1\}^+, \{e: e \in E,\ \Phi_{\tilde{e}}(e) = -1\}^-\right)$$

$$\left[\left(\{\tilde{e}: \tilde{e} \in \tilde{E},\ \Phi_e(\tilde{e}) = 1\}^+, \{\tilde{e}: \tilde{e} \in \tilde{E},\ \Phi_e(\tilde{e}) = -1\}^-\right)\right].$$

Definition 3.3. Let $M(E, \mathcal{O})$ and $\tilde{M}(\tilde{E}, \tilde{\mathcal{O}})$ be two simple oriented matroids of the same rank. Suppose there is a function $\Phi: E \times \tilde{E} \to \{0, 1, -1\}$ satisfying the following two conditions:

(3.3.1) for every $\tilde{e} \in \tilde{E}$ the signed set supp$(\Phi_{\tilde{e}})$ is a signed cocircuit of M and for every signed cocircuit Y of M there is one and only one element $\tilde{e} \in \tilde{E}$ such that supp$(\Phi_{\tilde{e}}) = Y$ or supp$(\Phi_{\tilde{e}}) = -Y$;

(3.3.2) for every $e \in E$ the signed set supp(Φ_e) is a signed cocircuit of $\tilde{M}$ and for every signed cocircuit $\tilde{Y}$ of $\tilde{M}$ there is at most one element $e \in E$ such that supp$(\Phi_e) = \tilde{Y}$ or supp$(\Phi_e) = -\tilde{Y}$.

We say that $(\tilde{M}, \Phi)$, or briefly $\tilde{M}$, is *a polar of* M (*relative to the function* Φ).

Suppose that $(\tilde{M}, \Phi)$ is a polar of M. Let φ be the function mapping set of points and hyperplanes of M on the set of points and hyperplanes of $\tilde{M}$ such that if p $[H]$ is a point of M [hyperplane] then $\varphi(p) = p^{\perp} = \{\tilde{e}: \tilde{e} \in \tilde{E}, \Phi(p, \tilde{e}) = 0\}$ $[\varphi(H) = H^{\perp} = \{\tilde{e}: \tilde{e} \in \tilde{E};$ for every $e \in H, \Phi(e, \tilde{e}) = 0\}]$. It is clear that the function φ verifies the conditions of Definition 3.1 and hence $\underline{\tilde{M}}$ is an adjoint of $\underline{M}$. We observe that the function $\bar{\varphi}: L(\underline{M}) \to L(\underline{\tilde{M}})$, such that for every flat F of $\underline{M}$, $\bar{\varphi}(F) = F^{\perp} = \{\tilde{e}: \tilde{e} \in \tilde{E},\ \Phi(\tilde{e}, e) = 0$ for every $e \in F\}$, is a one-to-one order-reversing function. The Vámos matroid is orientable (see [3, Example 3.10]), but it has no adjoint [4]. It follows that there are oriented matroids which do not admit a polar.

Example 3.4. Under the conditions of Example 3.2, suppose that Φ is a positive definite $\mathbb{R}$-bilinear form. Then $\mathbb{L}\text{in}(\{h_1, \ldots, h_m\})$ is a polar of $\mathbb{L}\text{in}(E)$ relative to the function $\Phi': E \times \{h_1, \ldots, h_m\} \to \{0, 1, -1\}$, where $\Phi'(e, h_i) = 0$ if $\Phi(e, h_i) = 0$ and $\Phi'(e, h_i) = 1$ $[\Phi'(e, h_i) = -1]$ if $\Phi(e, h_i) > 0$ $[\Phi(e, h_i) < 0]$. To prove this, note that the vector space V endowed with the form Φ is a Euclidean space. Hence by [3, Proposition 3.1], for every h_i, $1 \leqslant i \leqslant m$, the signed set $X = (X^+, X^-)$, where $X^+ = \{e: e \in E, \Phi(e, h_i) > 0\}$ and $X^- = \{e: e \in E,\ \Phi(e, h_i) < 0\}$, is equal to one of the two opposite cocircuits Y, $-Y$ of $\mathbb{L}\text{in}(E)$ such that $\underline{Y} = E - \{e : e \in E,\ \Phi(e, h_i) = 0\}$. Thus the condition (3.3.1) is true. The proof of the condition (3.3.2) is similar.

EXAMPLE 3.5. Every (simple) rank-three oriented matroid has a polar. This result has been proved, by means of an equivalent language, in [8, Théorème 3.6].

The transformations most naturally associated with polar sets (in $\mathbb{R}^n$) are projective (see [15, §2.2, Theorem 14]). As we have proved in [6], the projective transformations correspond in the oriented-matroid theory to the sign-reversal operations. A consequence of this remark is:

PROPOSITION 3.6. *Let $M(\tilde{E}, \tilde{\mathscr{A}})$ be a polar of the oriented matroid $M(E, \mathscr{O})$. Then for every $A \subset \tilde{E}$ and $B \subset E$ the oriented matroid $\underset{A}{-}\tilde{M}$ is a polar of $\underset{B}{-}M$.*

Proof. Suppose that $\tilde{M}$ is a polar of M relative to the function Φ. Then $\underset{A}{-}\tilde{M}$ is a polar of $\underset{B}{-}M$ relative to the function

$$\underset{B \times A}{\overline{}}\Phi : E \times \tilde{E} \to \{0, 1, -1\},$$

where

$$\underset{B \times A}{\overline{}}\Phi(e, \tilde{e}) = \Phi(e, \tilde{e})\varepsilon(e, \tilde{e}),$$

and $\varepsilon(e, \tilde{e})$ is equal to -1 if $\tilde{e} \in A$ or $e \in B$ and equal to 1 if $\tilde{e} \notin A$ and $e \notin B$. ∎

The following proposition was suggested to the author by A. Mandel [13]. It is implicit in [1].

PROPOSITION 3.7. *Let $\tilde{M}$ be a polar of the oriented matroid M. Suppose N is the simplification of a minor of M. Then there is a polar $\tilde{N}$ of N such that $\tilde{N}$ is a restriction of $\tilde{M}$.*

The proposition is an immediate consequence of the two lemmas below.

LEMMA 3.8. *Suppose that $(M(\tilde{E}, \tilde{\mathscr{O}}), \Phi)$ is a polar of $M(E, \mathscr{O})$, and let e be a point of M. Define $e^{\perp} = \{\tilde{e} : \tilde{e} \in \tilde{E},\ \Phi(e, \tilde{e}) = 0\}$. Then the restriction of*

M to the set $e^{\perp}$ is a polar of the simple oriented matroid $\overline{M/e}$ obtained from M/e by the identification of parallel elements.

Proof. For every point $\bar{p}$ of M/e (i.e. element of $\overline{M/e}$), let p be an element of $E-\{e\}$ such that $\bar{p}^{M'}=\bar{p}$, where $M'=M/e$. Let $\Phi'\colon \overline{E}(\overline{M/e})\times e^{\perp} \to \{0,1,-1\}$ be the map such that for every $(\bar{p},\tilde{e})\in \overline{E}\times e^{\perp}$, $\Phi'(\bar{p},\tilde{e})=\Phi(p,\tilde{e})$. We prove that $(\tilde{M}(e^{\perp}),\Phi')$ is a polar of $\overline{M/e}$. First note that the map Φ' is well defined. Indeed, suppose that the elements e_1 and e_2 are parallel elements of M/e. Then there is a circuit X of M such that $\underline{X}=\{e,e_1,e_2\}$. For every $\tilde{e}\in e^{\perp}$, if Y is the signed cocircuit $\operatorname{supp}(\Phi_{\tilde{e}})$, then $\underline{Y}\cap\underline{X}=\{e_1,e_2\}$, and by the orthogonality property $\operatorname{sg}_Y(e_1)\operatorname{sg}_Y(e_2)=-\operatorname{sg}_X(e_1)\operatorname{sg}_X(e_2)$, i.e., there is $\varepsilon\in\{1,-1\}$ such that $\Phi(e_1,\tilde{e})=\varepsilon\Phi(e_2,\tilde{e})$ for every $\tilde{e}\in e^{\perp}$.

We prove (3.3.1). By the definitions, for every $\tilde{e}\in e^{\perp}$, $\operatorname{supp}(\Phi_{\tilde{e}})$ is a signed cocircuit of M/e [i.e., is a cocircuit of M such that $e\notin \operatorname{supp}(\Phi_{\tilde{e}})$], and hence $\operatorname{supp}(\Phi'_{\tilde{e}})$ is a signed cocircuit of $\overline{M/e}$. Conversely, let Y be a cocircuit of M such that $e\notin\underline{Y}$. Then as $(\tilde{M},\Phi)$ is a polar of M, there are $\tilde{e}\in\tilde{E}$ and $\varepsilon\in\{+,-\}$ such that $\operatorname{supp}(\Phi_{\tilde{e}})=\varepsilon Y$ and $\Phi(e,\tilde{e})=0$ (i.e. $\tilde{e}\in e^{\perp}$). But in this case, if $\overline{Y}$ is the corresponding signed cocircuit of $\overline{M/e}$, we have also $\operatorname{supp}(\Phi'_{\tilde{e}})=\varepsilon\overline{Y}$.

Now we prove (3.3.2). For every $p\in E$, $p\neq e$, $\operatorname{supp}(\Phi_p)\cap e^{\perp}$ is a signed cocircuit of $\tilde{M}(e^{\perp})$. Indeed, by the definitions, $p^{\perp}$ and $e^{\perp}$ are hyperplanes of $\underline{\tilde{M}}$, and as $\underline{\tilde{M}}$ is an adjoint of $\underline{M}$, $p^{\perp}\cap e^{\perp}$ is a hyperline of $\underline{\tilde{M}}$. Hence for every point $\bar{p}$ of $\overline{M/e}$, $\operatorname{supp}(\Phi'_{\bar{p}})$ is a cocircuit of $M(\tilde{e}^{\perp})$. Conversely, let $\tilde{Y}$ be a signed cocircuit of $\tilde{M}(e^{\perp})$, and suppose there are two elements e_1 and e_2 of E such that $\operatorname{supp}(\Phi_{e_1})\cap e^{\perp}=\varepsilon Y$ and $\operatorname{supp}(\Phi_{e_2})\cap e^{\perp}=\varepsilon' Y$, where $\varepsilon,\varepsilon'\in\{+,-\}$. Then $e_1^{\perp}\cap e^{\perp}=e_2^{\perp}\cap e^{\perp}$ is a hyperline of $\underline{\tilde{M}}$. But in this case $\{e_1,e_2,e\}$ is a circuit of $\underline{M}$, e_1,e_2 are parallel elements of M/e, and the lemma follows. ∎

LEMMA 3.9. *Suppose that $(M(\tilde{E},\mathcal{O}),\Phi)$ is a polar of $M(E,\mathcal{O})$, and let e be a point of M. Suppose that e is not an isthmus of M. Let $\mathscr{H}$ be the set of hyperplanes of M which do not vanish on deleting e. Define $\tilde{e}^{\perp}=\{e'\colon e'\in E,\ \Phi(e',\tilde{e})=0\}$. Then the restriction of M to the set $\overline{E}=\{\tilde{e}\colon \tilde{e}\in\tilde{E},\ \tilde{e}^{\perp}\in\mathscr{H}\}$ is a polar of the oriented matroid $M\setminus e$.*

Proof. First note that the restriction $\tilde{M}(\overline{E})$ has the same rank as $\tilde{M}$. Indeed, let $\bar{\varphi}\colon L(\underline{M})\to\tilde{L}(\underline{\tilde{M}})$ be the antiisomorphism of lattices determined by Φ. Suppose that M has rank r. As e is not an isthmus, there are r different hyperplanes $H_1,H_2,\ldots,H_r\in\mathscr{H}$ such that $\varnothing=\bigcap_{i=1}^{r}H_i<\bigcap_{i=1}^{r-1}H_i$

$< \cdots < H_1$. Let $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_r\}$ be the subset of $\bar{E}$ such that $\tilde{e}_i^{\perp} = H_i$ for $i = 1, 2, \ldots, r$. As $\bar{\varphi}$ is an antiisomorphism of the lattices $L(\underline{M})$ and $L(\underline{\tilde{M}})$, $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_r\}$ is a base of the matroid $\tilde{M}$ and hence $\text{rank}(\tilde{M}(\bar{E})) = r$. On the other hand, let e_1 be a point of $M \setminus e$. As $\text{rank}(M \setminus e) = \text{rank}(M)$, then there are $r - 1$ hyperplanes $H_1, H_2, \ldots, H_{r-1}$ of $M \setminus e$ such that $e_1 = \bigcap_{i=1}^{r-1} H_i < \bigcap_{i=1}^{r-2} H_i < \cdots < H_1$. Let $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{r-1}\}$ be the subset of $\bar{E}$ such that $\tilde{e}_i^{\perp} = H_i$ for $i = 1, 2, \ldots, r - 1$. Then $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{r-1}\}$ is a base of the hyperplane $e_1^{\perp}$, and $e_1^{\perp} \cap \bar{E}$ is a hyperplane of $\tilde{M}(\bar{E})$. Hence it is clear that $\tilde{M}(\bar{E})$ is a polar of $M \setminus e$ relative to the restriction of the function Φ to the domain $(E \setminus e) \times \bar{E}$. ∎

In the following, if $(\tilde{M}, \Phi)$ is a polar of the oriented matroid M, we suppose that for every positive cocircuit $X = (X^+, X^- = \varnothing)$ of M there is an element $\tilde{e} \in \tilde{E}(\tilde{M})$ such that $\text{supp}(\Phi_{\tilde{e}}) = X$, i.e., $X^+ = \{e : e \in E, \Phi(e, \tilde{e}) = 1\}$. (By Proposition 3.6, this is always possible without loss of generality.)

A. L. Cheung has proved that the adjoint of a geometric lattice is embeddable in its lattice of extensions (see [4]). The next important theorem, due to A. Bachem and W. Kern [1], makes this result more transparent. We give a short proof of Theorem 3.10 using the results established in Section 2 of this paper. If $M(E, \mathcal{O})$ is an oriented matroid, we denote by $\mathscr{E}(M)$ the poset (i.e. the partial ordered set) of all localizations of M with the relation $(\mathscr{Y}, \mathscr{X}) \leqslant (\mathscr{Y}', \mathscr{X}')$ if and only if $\mathscr{Y} \subset \mathscr{Y}'$. In the following we denote the localization $(\mathscr{Y}, \mathscr{X})$ by $\mathscr{Y}$ for short.

THEOREM 3.10 [1]. *Let $M(\tilde{E}, \tilde{\mathcal{O}})$ be a polar of $M(E, \mathcal{O})$ relative to the function $\Phi : E \times \tilde{E} \to \{0, 1, -1\}$. Then the map $\psi : \mathscr{K}(\tilde{\mathcal{O}}^{\perp}) \to \mathscr{E}(M)$, $X \rightsquigarrow \{\pm \text{supp}(\Phi_{\tilde{e}}) : \tilde{e} \in X^{\pm}\}$, defines an embedding of the poset $\mathscr{K}(\tilde{\mathcal{O}}^{\perp})$ into the poset $\mathscr{E}(M)$.*

The following proposition completes the information given by Theorem 3.10.

PROPOSITION 3.10′. *In the conditions of Theorem 3.10, the localizations determined by a partition of an ordered independent set of M (in particular the principal extensions) can be identified with elements of $\psi(\mathscr{K}(\tilde{\mathcal{O}}^{\perp}))$.*

Proof. Let $\mathscr{Y}$ be the localization determined by a partition $I^+ \cup I^-$ of an ordered independent set $I = \{e_1, e_2, \ldots, e_n\}$ of M. Let $\varepsilon_i = +$ $[\varepsilon_i = -]$ if $e_i \in I^+$ $[e_i \in I^-]$, $1 \leqslant i \leqslant n$. We claim that

$$(3.10'.1) \quad \psi(\varepsilon_1 \text{supp}(\Phi_{e_1}) \circ \varepsilon_2 \text{supp}(\Phi_{e_2}) \circ \cdots \circ \varepsilon_n \text{supp}(\Phi_{e_n})) = \mathscr{Y},$$

where $X \circ Y$ denotes the composite of the signed sets X, Y: i.e. $(X \circ Y)^{\pm} = X^{\pm} \cup (Y^{\pm} - (X^{+} \cup X^{-}))$. We remark that, from the definitions, we have

$$\left(\varepsilon_1 \operatorname{supp}(\Phi_{e_1}) \circ \cdots \circ \varepsilon_n \operatorname{supp}(\Phi_{e_n})\right) \in \mathscr{K}(\tilde{\mathscr{O}}^{\perp}).$$

Also from the definitions we have $\psi(\varepsilon_1 \operatorname{supp}(\Phi_{e_1}) \circ \cdots \circ \varepsilon_n \operatorname{supp}(\Phi_{e_n})) = \{\alpha_j \operatorname{supp}(\Phi_{\tilde{e}})$: there is some $e_i \in I$ such that $\Phi(\tilde{e}, e_i) \neq 0$, and if j is the smallest index such that $\Phi(\tilde{e}, e_j) \neq 0$ then $\alpha_j \Phi(\tilde{e}, e_j) = \varepsilon_{j'}(\alpha_j = \pm)\} = \mathscr{Y}$, and the proposition follows. ■

LEMMA 3.11. *Under the conditions of Theorem* 3.7, *for every* $Y \in \mathscr{L}(\{\operatorname{supp}(\Phi_{\tilde{e}}) : \tilde{e} \in \tilde{E}\})$ *the signed set*

$$\left(\{\tilde{e} : \operatorname{supp}(\Phi_{\tilde{e}}) \in Y^{+}\}^{+}, \{\tilde{e} : \operatorname{supp}(\Phi_{\tilde{e}}) \in Y^{-}\}^{-}\right)$$

is a signed circuit of $\tilde{M}$.

Proof of Lemma 3.11. As $\underline{\tilde{M}}$ is an adjoint of $\underline{M}$, if F is a flat of $\overline{M}$ then $F^{\perp} = \{\tilde{e} : \tilde{e} \in \tilde{E}(\underline{\tilde{M}}), \Phi(e, \tilde{e}) = 0 \text{ for every } e \in F\}$ is a flat of $\tilde{M}$ of rank equal to rank($\underline{M}$)-rank(F).

Suppose that $Y \in \mathscr{L}(\{\operatorname{supp}(\Phi_{\tilde{e}}) : \tilde{e} \in \tilde{E}\})$ and $\underline{Y} = \{X_1, X_2, X_3\}$. Let $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ be the elements of $\tilde{E}$ such that $\operatorname{supp}(\Phi_{\tilde{e}_i}) = X_i$, $1 \leqslant i \leqslant 3$. By hypothesis $L = (E - \underline{X}_1) \cap (E - \underline{X}_2) \cap (E - \underline{X}_3)$ is a hyperline of M. Then $L^{\perp}$ is a line of $\tilde{M}$, and $\tilde{e}_i \in L$ for $i = 1,2,3$, i.e., $\underline{\tilde{Y}} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is a circuit of $\underline{\tilde{M}}$. Suppose that Y is positive. (The proof of the other possible case is similar.) Let X be a cocircuit of $\underline{Y} = \{X_1, X_2, X_3\}$ and let X_{i_1}, X_{i_2} be the remaining cocircuits. From condition (2.5.2) we have $X_{i_1} \cap X_{i_2} - \underline{X} \subset (X_{i_1}^{+} \cap X_{i_2}^{-}) \cup (X_{i_1}^{-} \cap X_{i_2}^{+})$ (see Figure 2). In this case, if $e \in \underline{X}_{i_1} \cap \underline{X}_{i_2} - \underline{X}$ then $\operatorname{sg}_{X_{i_1}}(e) = -\operatorname{sg}_{X_{i_2}}(e)$. By the definitions, if $\operatorname{supp}(\Phi_{\tilde{e}_{i_3}}) = X$, $\operatorname{supp}(\Phi_{\tilde{e}_{i_1}}) = X_{i_1}$, and $\operatorname{supp}(\Phi_{\tilde{e}_{i_2}}) = X_{i_2}$, then $\Phi(e, \tilde{e}_{i_3}) = 0$, $\Phi(e, \tilde{e}_{i_1}) = -\Phi(e, \tilde{e}_{i_2})$, i.e., $\tilde{e}_{i_1}$ and $\tilde{e}_{i_2}$ have opposite signs in the cocircuit $\operatorname{supp}(\Phi_e)$, and $\tilde{e}_{i_3} \notin \operatorname{supp}(\Phi_e)$. If Z is a signed circuit of $\tilde{M}$, then $Z \perp \operatorname{supp}(\Phi_e)$ and $\operatorname{sg}_Z(\tilde{e}_{i_1}) = \operatorname{sg}_Z(\tilde{e}_{i_2})$. As this equality is true for all $\tilde{e}_{i_1}, \tilde{e}_{i_2} \in \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$, the signed set $(\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}^{+}, \varnothing)$ is a positive circuit of M and the lemma follows. ■

Proof of Theorem 3.10. It is well known that $X \in \mathscr{K}(\tilde{\mathscr{O}}^{\perp})$ if and only if X is orthogonal to the circuits of $\tilde{M}$. Then the result follows from Lemma 3.11 and Theorem 2.6. ■

REMARK 3.12. Let $M(E, \mathcal{O})$ be an oriented matroid, and assume the existence of a polar of M. Then it results from Theorem 3.10 that for every hyperplane H and every line L of M there is a single-element extension $M'(E \cup \{p\}, \mathcal{O}')$ of M such that $\overline{H}^{M'} \cap \overline{L}^{M'} = \overline{p}^{M'}$. Oriented matroids satisfying this condition have an important role in oriented-matroid programming (see [9]).

Let M be a matroid polytope (i.e. an acyclic oriented matroid). We shall prove that if we assume the existence of a polar $\tilde{M}$ of M, then the construction of the polar cone C^*, of an n-dimensional cone C in $\mathbb{R}^n$, can be generalized in an analogous fashion for the matroid polytope M. We call a matroid polytope N a *polar reciprocal of* M if the lattice of the faces of M is antiisomorphic to the lattice of faces of N.

THEOREM 3.13. *Let M be an oriented matroid, and suppose there is a polar $\tilde{M}$ of M. Then, for every acyclic reorientation M' of M, there is a polar reciprocal of M' canonically determined by M.*

LEMMA 3.14. *Let M be an acyclic oriented matroid and $(\tilde{M}, \Phi)$ be a polar of M. Let $\tilde{E}' = \{\tilde{e} : \tilde{e} \in \tilde{E}(\tilde{M})$, $\mathrm{supp}(\Phi_{\tilde{e}})$ is a positive cocircuit of $M\}$. Then the restriction $\tilde{M}'$ of M to the set $\tilde{E}'$ is an acyclic matroid, and the lattice of faces of $\tilde{M}'$ is antiisomorphic to the lattice of faces of M.*

Proof of Lemma 3.14. We prove that $X = (X^+, \varnothing)$ is a positive cocircuit of $\tilde{M}'$ if and only if there is one extreme point e of M such that $\mathrm{supp}(\Phi_e) \cap \tilde{E}' = X$. Let e be an extreme point of M. It is well known that there are $r-1$ facets of M, $H_1, \ldots, H_{r-1}$, such that $\{e\} = \bigcap_{i=1}^{r-1} H_i$ and for every j, $1 \leqslant j \leqslant r-2$, $\bigcap_{i=1}^{j} H_i \not\subset H_{j+1}$ (see [11]). From condition (3.3.1) there is a subset $\{\tilde{e}_1, \ldots, \tilde{e}_{r-1}\}$ of $\tilde{E}'$ such that, for every i, $1 \leqslant i \leqslant r-1$, $(\tilde{e}_i)^{\perp} = \{e : e \in E(M), \Phi(e, \tilde{e}_i) = 0\} = H_i$. Otherwise, for every flat F of $\underline{M}$ the set $F^{\perp} = \{\tilde{e} : \tilde{e} \in \tilde{E}', \Phi(e, \tilde{e}) = 0$ for every $e \in F\}$ would be a flat of $\underline{\tilde{M}}'$. Then $H_1^{\perp} = \{\tilde{e}_1\} \subsetneq (H_1 \cap H_2)^{\perp} \subsetneq \cdots \subsetneq (\bigcap_{i=1}^{r-1} H_i)^{\perp} = e^{\perp}$. This proves that $e^{\perp}$ is a hyperplane (of rank $r-1$) of $\tilde{M}'$, and hence $\mathrm{supp}(\Phi_e) \cap \tilde{E}'$ is a positive cocircuit of $\tilde{M}'$. Conversely let $X = (X^+, \varnothing)$ be a positive cocircuit of $\tilde{M}'$. From the definition of restriction, there is a cocircuit Y of $\tilde{M}$ such that $X = Y \cap \tilde{E}'$. We prove that there is one extreme point e of M such that $Y = \mathrm{supp}(\Phi_e)$. From Theorem 3.10 we know that $\{\mathrm{supp}(\Phi_{\tilde{e}}) : \tilde{e} \in Y^+\} \cup \{-\mathrm{supp}(\Phi_{\tilde{e}}) : \tilde{e} \in Y^-\} = \mathcal{Y}$ is a localization of M. Since if $e \in \bigcap\{Z^+ : Z \in \mathcal{Y}\}$ then $Y^{\pm} \subset \mathrm{supp}(\Phi_e)^{\pm}$, we have to consider two possible cases:

(i) $\bigcap\{Z^+ : Z \in \mathcal{Y}\} = \{e\}$;
(ii) $\bigcap\{Z^+ : Z \in \mathcal{Y}\} = \varnothing$.

Case (i): From the definitions $Y^{\pm} \subset (\mathrm{supp}(\Phi_e))^{\pm}$, i.e., $Y = \mathrm{supp}(\Phi_e)$. If e is an interior point of M, then for every $\tilde{e} \in \tilde{E}'$, $\Phi(e, \tilde{e}) = 1$, i.e. $\tilde{E}' \subset Y^+$, and $Y \cap \tilde{E}'$ cannot be a cocircuit of $\tilde{M}'$. Then there is a facet H of M such that $e \in H$. Let $F = \bigcap\{H: H \text{ facet of } M,\ e \in H\}$. Let e' be an extreme point of the face F, and suppose $e \neq e'$. Assuming that X' is the positive cocircuit of $\tilde{M}'$ such that $(X')^+ = \{\tilde{e}: \tilde{e} \in \tilde{E}',\ \Phi(e', \tilde{e}) = 1\}$, we obtain the contradiction $X^+ \subsetneq (X')^+$. Hence $F = e$.

Case (ii): We shall prove that this case is not possible. Let $M'(E \cup \{p\}, \mathcal{O}')$ be the point extension of M determined by the localization $\mathcal{Y}$. As M is acyclic, one at least of the two matroids M' or $\underset{p}{-}\mathrm{M}'$ is also an acyclic oriented matroid (see [11, Lemma 3.1.1]). By reasoning similar to that in case (i) we may conclude that if M' $[\underset{p}{-} M']$ is acyclic, then necessarily p is an extreme point of M' $[\underset{p}{-} M']$. Assume that p is an extreme point of M' $[\underset{p}{-} M']$. Hence there are two cocircuits Y_1 and Y_2 such that $Y_1^- = \{p\}$, $Y_2^- = \varnothing$, and $p \in Y_2^+$ (see [11]). But in this case $Y_1 - \{p\}$ and $Y_2 - \{p\}$ are two positive cocircuits of M, and hence there are $\tilde{e}_1, \tilde{e}_2 \in \tilde{E}'$ such that $\mathrm{supp}(\Phi_{\tilde{e}_1}) = Y_1 - \{p\}$ and $\mathrm{supp}(\Phi_{\tilde{e}_2}) = Y_2 - \{p\}$. As by hypothesis $X = Y \cap \tilde{E}'$ is a positive cocircuit, it turns out by the definition of the localization $\mathcal{Y}$ that $\mathrm{supp}(\Phi_{\tilde{e}_1}), \mathrm{supp}(\Phi_{\tilde{e}_2}) \in \mathcal{Y}$, which is a contradiction with our hypothesis that M' is a point extension determined by this localization.

We prove that $\tilde{M}'$ is acyclic. $\tilde{M}'$ is acyclic if and only if $(\tilde{M}')^{\perp}$ is totally cyclic. Hence it results from the preceding that $\tilde{M}'$ is acyclic if for every element $\tilde{e} \in \tilde{E}'$ there is an extreme point e of M such that $\tilde{e}$ is in the cocircuit $\mathrm{supp}(\Phi_{\tilde{e}}) \cap \tilde{E}'$, i.e. $\Phi(e, \tilde{e}) = 1$. But for every $\tilde{e} \in \tilde{E}'$, $\mathrm{supp}(\Phi_{\tilde{e}})$ is a positive cocircuit of M and hence there is an extreme point e of M such that $e \in (\mathrm{supp}(\Phi_{\tilde{e}}))^+$, i.e. $\Phi(e, \tilde{e}) = 1$, as expected.

Let $\mathcal{F}(M)$ $[\mathcal{F}(\tilde{M}')]$ be the lattice of faces of M $[\tilde{M}']$. We prove that $\mathcal{F}(M)$ is antiisomorphic to $\mathcal{F}(\tilde{M}')$. From the definitions, for all $\tilde{e}_i, \tilde{e}_j \in \tilde{E}'$, there is an extreme point e of M in the set $(\tilde{e}_j)^{\perp} - (\tilde{e}_i)^{\perp}$, i.e., $\Phi(e, \tilde{e}_i) = 1$ but $\Phi(e, \tilde{e}_j) = 0$. In this case $e^{\perp} = \{\tilde{e}: \tilde{e} \in \tilde{E}',\ \Phi(e, \tilde{e}) = 0\}$ is a facet of $\tilde{M}'$ such that $\tilde{e}_j \in e^{\perp}$ but $\tilde{e}_i \notin e^{\perp}$. Thus $\tilde{E}'$ is the set of points of $\tilde{M}'$. Let φ be the function such that for every point e of the lattice $\mathcal{F}(M)$, $\varphi(e)$ is the copoint $\{\tilde{e}: \tilde{e} \in \tilde{E}',\ \Phi(e, \tilde{e}) = 0\}$ of the lattice $\mathcal{F}(\tilde{M}')$, and for every copoint $\tilde{H}$ of $\mathcal{F}(\tilde{M}')$, $\varphi(\tilde{H})$ is the (unique) point of $\mathcal{F}(M)$ such that $\Phi(e, \tilde{e}) = 0$ for every $\tilde{e} \in \tilde{H}$. From the definitions we conclude that φ is an injective function mapping the points of $\mathcal{F}(M)$ onto the copoints of $\mathcal{F}(\tilde{M}')$ and the copoints of $\mathcal{F}(\tilde{M}')$ onto the points of $\mathcal{F}(M)$ in such a way that if e is a point and H a copoint of $\mathcal{F}(M)$, then $e \leqslant H$ if and only if $\varphi(H) \leqslant \varphi(e)$. Otherwise the lattices $\mathcal{F}(M)$ and $\mathcal{F}(\tilde{M}')$ are point lattices, and every element different from the top element is the meeting of copoints (see [11]). In this case it is known that φ can be extended to an antiisomorphism between $\mathscr{F}(M)$ and $\mathscr{F}(\tilde{M}')$ (for a proof see for example [2, Lemma 5.1]). ∎

Proof of Theorem 3.13. Let $\tilde{M}$ be the polar of M relative to a function Φ. Assume that $\underset{A}{-} M = M'$ is an acyclic reorientation of M. From Proposition 3.6, $\tilde{M}$ is the polar of M' relative to the function $\Phi' = \underset{A \times \varnothing}{\overline{}}\ \Phi$. Then Theorem 3.13 is a clear consequence of Lemma 3.14. ∎

To conclude we present an open question. In [5] we have proved that any (simple) rank-three matroid has a polar. More precisely, it is not too hard to derive from [5, Théorème 3.6] that if M is a rank-three oriented matroid and $\mathscr{Y}$ is a localization of M, then there is a polar $(\tilde{M}, \Phi)$ of M and a signed set $X \in \mathscr{K}((\tilde{\mathscr{O}}(\tilde{M}))^{\perp})$ such that $\mathscr{Y} = \{\pm \operatorname{supp}(\Phi_{\tilde{e}}) : \tilde{e} \in X^{\pm}\}$. In general, if M is an oriented matroid admitting a polar, is a similar result true?

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