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Complete positivity of matrices of special form

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Abstract

Given that $B, C_2, \ldots, C_k$ are positive semidefinite (PSD) n-by-n real matrices and B is entrywise nonnegative, we characterize (fully for $n = 2$ and partially for $n \geqslant 3$) when B may be written as $B = \sum_{i=2}^{k} B_i$ so that B_i and $B_i - C_i$ are PSD and B_i is entrywise nonnegative, $i = 2, \ldots, k$. These characterizations are used to give conditions under which an entrywise nonnegative, PSD matrix A with a special block form can be written as $A = BB^{\mathrm{T}}$, in which B is entrywise nonnegative. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

An n-by-n matrix A is called *completely positive* (CP) if it may be written as $A = BB^{\mathrm{T}}$, in which B is n-by-m and entrywise nonnegative. Equivalently, $A = \sum_{i=1}^{m} b_i b_i^{\mathrm{T}}$, in which each $b_i \in R^n$ is nonnegative.

More generally, A is called *doubly nonnegative* (DN) if A is positive semidefinite *and* entrywise nonnegative. Of course CP matrices are DN, but the containment is

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PII: S 0 0 2 4 - 3 7 9 5 (0 0) 0 0 3 1 5 - 3

proper for $n \geqslant 5$. Double nonnegativity is easily checked, but, thus far, there is no definitive test for a matrix to be CP. The two classes coincide for $n \leqslant 4$ and also for certain sparsity patterns when n is larger.

The graph $G = G(A)$ of the n-by-n symmetric matrix $A = [a_{ij}]$ is the undirected graph on n vertices in which is the edge $\{i, j\}$, $i \neq j$, if and only if $a_{ij} \neq 0$. All doubly nonnegative matrices whose graph is G are completely positive if and only if G contains no odd length cycle of length $\geqslant 5$ [2–4,6,9]. We refer to such graphs as NLOC.

For some further graphs, complete positivity may be checked. For example, if the graph of a doubly nonnegative matrix A is triangle free, then A is CP if and only if the comparison matrix $M(A)$ is an M-matrix (possibly singular) [5] and tests are given for certain other graphs in [1], which overlaps prior work.

Our purpose here is to extend conditions for complete positivity by considering doubly nonnegative matrices of the special block form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1k} \\ A_{21} & A_{22} & & \mathbf{0} & \\ A_{31} & & A_{33} & & \\ \vdots & \mathbf{0} & & \ddots & \\ A_{k1} & & & & A_{kk} \end{bmatrix}, \tag{1}$$

which generalizes the special case (up to permutation similarity)

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}. \tag{2}$$

In this process, we extend the graphs for which complete positivity may be checked, in a simple and unified way. In the case that A_{11} is 2-by-2, a parallel result can be found in [1].

We begin with a general discussion. Let $\succeq$ denote the positive semidefinite partial order of symmetric matrices and let $|X|$ denote the entrywise absolute value of a matrix X. Using Schur complements [7], a symmetric matrix A of form (1) is positive semidefinite (PSD) if and only if A_{ii} is PSD, $i = 2, \ldots, k$, and $A_{11} \succeq \sum_{i=2}^{k} A_{1i} A_{ii}^{-1} A_{i1}$. Here, we assume that A_{ii} is invertible, $i = 2, \ldots, k$, but, if not, A_{ii}^{-1} may be replaced by the Moore–Penrose generalized inverse $A_{ii}^{\dagger}$.

Further, A is CP if and only if A_{11} may be decomposed as $A_{11} = \sum_{i=2}^{k} B_i$ in such a way that

$$A_i = \begin{bmatrix} B_i & A_{1i} \\ A_{i1} & A_{ii} \end{bmatrix} \tag{3}$$

is CP, $i = 2, \ldots, k$. (This may be seen by using the rank 1 decomposition already mentioned, $A = \sum_{i=1}^{m} b_i b_i^{\mathrm{T}}$, noting that each vector b_i can have nonzero entries only in those positions corresponding to the rows of A occupied by A_{11} and some other single block A_{i1}, then considering the sums of the $b_j b_j^{\mathrm{T}}$ corresponding to each A_{i1}.)

In order that a symmetric matrix of form (3) be CP, it must be DN, and, for this it is necessary and sufficient that A_i be entrywise nonnegative, A_{ii} be PSD and $B_i \succeq A_{1i}A_{ii}^{-1}A_{i1}$. Again A_{ii}^{-1} may be replaced by $A_{ii}^{\dagger}$. In case the graph of each A_i is NLOC, these conditions become necessary and sufficient for each A_i to be CP and thus for A to be CP. We conclude that the existence of a decomposition of the doubly nonnegative matrix A_{11} into a sum of doubly nonnegative matrices $\sum_{i=2}^{k} B_i$ in such a way that $B_i \succeq A_{1i}A_{ii}^{-1}A_{i1}$, each of which is PSD, is a central question in CP theory. We next formalize and study this question and then apply the results to matrices of form (1).

Suppose that B and $C_2, \ldots, C_k$ are symmetric matrices. We say that B is *decomposable* relative to $C_2, \ldots, C_k$ if B may be written as $B = \sum_{i=2}^{k} B_i$ so that $B_i \succeq C_i, i = 2, \ldots, k$; when $C_2, \ldots, C_k$ are naturally understood from the context, we just say that B is decomposable. It is an easy exercise that B is decomposable if and only if

$$B \succeq \sum_{i=2}^{k} C_i. \tag{4}$$

A more stringent condition that is relevant to our inquiry is the following. We say that B is *DN-decomposable* relative to $C_2, \ldots, C_k$ if B may be written as $B = \sum_{i=2}^{k} B_i$ so that $B_i \succeq C_i$ and B_i is DN, $i = 2, \ldots, k$. Condition (4) remains necessary, but is no longer generally sufficient. Now, B must be DN (by virtue of being a sum of DN matrices), but, even together with (4), this is not sufficient beyond the case $n = 1$ (as we shall see). We are able to characterize DN-decomposability for $n = 2$. For $n \geqslant 3$, we give conditions and characterizations only in certain circumstances.

2. DN-decomposition in the 2-by-2 case

Theorem 1. *Suppose that $B, C_2, \ldots, C_k$ are 2-by-2 PSD matrices with B DN and $B \succeq \sum_{i=2}^{k} C_i$. Then B is DN-decomposable relative to $C_2, \ldots, C_k$ if and only if $B \succeq \sum_{i=2}^{k} |C_i|$.*

Proof. ($\Rightarrow$): We assume that B may be written as $B = \sum_{i=2}^{k} B_i$, in which $B_i \succeq C_i$ and B_i is DN, $i = 2, \ldots, k$. It is easy to verify that, for 2-by-2 matrices, since B_i is DN and C_i is PSD, $B_i \succeq C_i$ implies $B_i \succeq |C_i|$. Thus,

$$B = \sum_{i=2}^{k} B_i \succeq \sum_{i=2}^{k} |C_i|.$$

($\Leftarrow$): We assume that $B \succeq \sum_{i=2}^{k} |C_i|$ and let

$$C_i = \begin{bmatrix} a_i & c_i \\ c_i & b_i \end{bmatrix}, \quad i = 2, \ldots, k.$$

We may assume that the c_i are nonzero since otherwise we could redefine B by subtracting it from each C_i that is diagonal. By diagonal congruence, we may assume that

$$B=\begin{bmatrix}1 & \phi\\ \phi & 1\end{bmatrix} \quad \text{with } \phi \geqslant 0.$$

We consider several cases.

Case 1. If $\phi=0$, for each i let $p_i=|c_i|\,/\sum_{j=2}^{k}|c_j|$ and let

$$B_i=\begin{bmatrix}p_i\left(1-\sum_{j=2}^{k}a_j\right)+a_i & 0\\ 0 & p_i\left(1-\sum_{j=2}^{k}b_j\right)+b_i\end{bmatrix}.$$

For each i, $B_i \succeq C_i$ since B_i-C_i has nonnegative diagonal entries and

$$\begin{aligned}\det(B_i-C_i)&=p_i^2\left\{\left(1-\sum_{j=2}^{k}a_j\right)\left(1-\sum_{j=2}^{k}b_j\right)-\left(\sum_{j=2}^{k}|c_j|\right)^2\right\}\\ &=p_i^2\det\left(B-\sum_{j=2}^{k}|C_j|\right)\geqslant 0.\end{aligned}$$

Since $\sum_{j=2}^{k}B_j=B$ and $B_i\succeq 0$, matrix B is DN-decomposable relative to $C_2,\ldots,C_k$.

Case 2. If at least one of the c_i (say c_2) is such that $c_i \geqslant \phi$, then define $\widetilde{B}$ and $\widetilde{C_2}$ by subtracting ϕ from the off-diagonal entries of B and C. From case 1 it follows that $\widetilde{B}$ has a DN-decomposition relative to $\widetilde{C_2},C_3,\ldots,C_k$ and hence that B has a DN-decomposition relative to $C_2,\ldots,C_k$.

Case 3. If $c_i<0$ for each i, let

$$p_i=\left(\frac{\phi}{k-1}-c_i\right)\Bigg/\left(\phi-\sum_{j=2}^{k}c_j\right), \quad i=2,\ldots,k.$$

Then $\sum_{i=2}^{k}p_i=1$ and, for each i, $p_i>0$ and

$$p_i\left(\phi-\sum_{j=2}^{k}c_j\right)+c_i=\frac{\phi}{k-1}\geqslant 0. \tag{5}$$

For each i, let

$$B_i=C_i+p_i\left(B-\sum_{j=2}^{k}C_j\right).$$

Each B_i is entrywise nonnegative, by (5), and $B=\sum_{i=2}^{k}B_i$. Moreover, for each i, $B_i\succeq C_i$ since B_i-C_i has nonnegative diagonal entries and

$$\det(B_i - C_i) = p_i^2 \det\left(B - \sum_{i=2}^{k} C_i\right) \geqslant 0.$$

Thus B_i is DN for each i and hence B is DN-decomposable relative to $C_2, \dots, C_k$.

Case 4. If at least one c_i (say c_2) is such that $0 < c_i < \phi$, then define $\widetilde{B} = B - C_2$, yielding $\widetilde{B} \succeq \sum_{i=3}^{k} C_i$ and $\widetilde{B} \succeq \sum_{i=3}^{k} |C_i|$. If $\widetilde{B}$ has a DN-decomposition relative to $C_3, \dots, C_k$, then B has a DN-decomposition relative to $C_2, \dots, C_k$. Thus, since the theorem is trivially true when $k = 2$, induction on k shows that it is true in this final case. □

Theorem 1 cannot be extended to include 3-by-3 matrices, as shown by the following example.

$$B = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 12 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 4 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Matrix B is DN-decomposable with respect to C_2 and C_3 since $B = B_2 + B_3$ with $B_2 \succeq C_2$ and $B_3 \succeq C_3$, but $B \not\succeq |C_2| + |C_3|$. Thus $B \succeq \sum_{i=2}^{k} |C_i|$ is not a necessary condition for DN-decomposability when $n = 3$. It will be shown below that the condition is sufficient when B is diagonal.

3. DN-decomposition in the *n*-by-*n* case

If $B, C_2, \dots, C_k$ are n-by-n symmetric matrices and $B \succeq \sum_{i=2}^{k} C_i$, it is easy to show that B is decomposable relative to $C_2, \dots, C_k$ since B may be written as $B = \sum_{i=2}^{k} B_i$, in which

$$B_i = C_i + \frac{1}{k-1}\left(B - \sum_{i=2}^{k} C_i\right) \quad \text{for } i = 2, \dots, k.$$

Although DN-decomposability is a more stringent condition, it can be demonstrated under certain circumstances by using the same B_i as above (or a slight generalization).

Theorem 2. *Suppose that $B, C_2, \dots, C_k$ are n-by-n PSD matrices with B DN and $B \succeq \sum_{i=2}^{k} C_i$. If there exist nonnegative numbers $\alpha_2, \dots, \alpha_k$ such that $\sum_{i=2}^{k} \alpha_i = 1$ and $B_i = C_i + \alpha_i(B - \sum_{i=2}^{k} C_i)$ is entrywise nonnegative for $i = 2, \dots, k$, then $B = \sum_{i=2}^{k} B_i$ is a DN-decomposition of B relative to $C_2, \dots, C_k$.*

Proof. Since both C_i and $B - \sum_{i=2}^{k} C_i$ are PSD for each i, so is B_i and hence B_i is DN for each i. Clearly $B = \sum_{i=2}^{k} B_i$ and $B_i \succeq C_i$ for each i. □

Theorem 2 also follows from a more general observation: a DN matrix B is DN-decomposable relative to the PSD matrices $C_2, \ldots, C_k$ if and only if $B - \sum_{i=2}^{k} C_i$ can be partitioned into PSD matrices $X_2, \ldots, X_k$ such that $C_i + X_i$ is DN for $i = 2, \ldots, k$. Moreover, because each B_i is nonnegative, such a partition $X_2, \ldots, X_k$ has the property that if any C_i has a negative entry $(C_i)_{pq}$, then $(X_i)_{pq} \geqslant |(C_i)_{pq}|$.

In the following theorem, we denote by Z the set of n-by-n matrices whose off-diagonal entries are all $\leqslant 0$.

Theorem 3. *Suppose that $B, C_2, \ldots, C_k$ are n-by-n DN matrices with $B \succeq \sum_{i=2}^{k} C_i$ and $B - \sum_{i=2}^{k} C_i \in Z$. Then B is DN-decomposable relative to $C_2, \ldots, C_k$.*

Proof. We may assume that $B - \sum_{i=2}^{k} C_i$ is irreducible since otherwise DN-decompositions for each of its irreducible diagonal blocks could be combined to form a DN-decomposition for B, forming each matrix B_i from C_i by replacing each diagonal block of C_i by a larger ($\succeq$) DN matrix, generated from a DN-decomposition of the corresponding diagonal block of B relative to the corresponding diagonal blocks of $C_2, \ldots, C_k$.

We also may assume that $B - \sum_{i=2}^{k} C_i$ is singular since otherwise it could be made singular by decreasing the diagonal elements of B. If this new B, which satisfies the hypotheses of the theorem, is DN-decomposable relative to $C_2, \ldots, C_k$, then it follows easily that the original B is also.

Since $B - \sum_{i=2}^{k} C_i \in Z$ is PSD, it is an (irreducible and singular) M-matrix [8]. Hence there is a vector $x = (x_1, \ldots, x_n)^{\mathrm{T}} > 0$ such that

$$\left(B - \sum_{i=2}^{k} C_i\right) x = 0. \tag{6}$$

We now construct matrices $B_2, \ldots, B_k$ that satisfy $(B_i - C_i)x = 0$ and $B_i - C_i \in Z$, implying that $B_i - C_i$ is a singular M-matrix and thus is PSD. Let $B = [b_{ij}]$ and let the off-diagonal entry in row s and column t of each B_i be given by

$$(B_i)_{st} = b_{st} \frac{(C_i)_{st}}{\sum_{j=2}^{k} (C_j)_{st}} \quad \text{for } s, t = 1, \ldots, n.$$

In this way, the ratio $(B_i)_{st}/(C_i)_{st}$ remains fixed (and $\leqslant 1$) as i varies and s, t remain fixed. On the diagonal of each B_i, let

$$(B_i)_{ss} = (C_i)_{ss} + \sum_{j=1, j \neq s}^{n} \frac{x_j}{x_s} \{(C_i)_{sj} - (B_i)_{sj}\}, \quad s = 1, \ldots, n$$

so that $(B_i - C_i)x = 0$ for $i = 2, \ldots, k$. Straightforward calculation shows that $B_i - C_i \in Z$ for $i = 2, \ldots, k$. Thus $B_i - C_i$ is a singular M-matrix and hence is PSD.

Since each B_i is entrywise nonnegative and each C_i is PSD, each B_i is DN. The fact that $B = \sum_{i=2}^{k} B_i$ follows from Eq. (6). $\square$

4. DN-decomposition in the diagonal case

In the case that B is a diagonal matrix and $C_2, \ldots, C_k$ are nonnegative symmetric matrices, Theorem 3 provides a characterization of exactly when B is DN-decomposable relative to $C_2, \ldots, C_k$.

Corollary 4. *Suppose $B, C_2, \ldots, C_k$ are n-by-n entrywise nonnegative symmetric matrices, with B diagonal. Then $B \succeq \sum_{i=2}^{k} C_i$ if and only if there exist diagonal matrices B_i such that $B = \sum_{i=2}^{k} B_i$ and $B_i \succeq C_i$, for $i = 2, \ldots, k$.*

Proof. ($\Leftarrow$): $B = \sum_{i=2}^{k} B_i \succeq \sum_{i=2}^{k} C_i$.

($\Rightarrow$): As in the proof of Theorem 3, since $B - \sum_{i=2}^{k} C_i$ is PSD and in Z, B can be written as $\sum_{i=2}^{k} B_i$, in which B_i is entrywise nonnegative and diagonal, with $B_i \succeq C_i$, for $i = 2, \ldots, k$. $\square$

If the hypothesis in Corollary 4 is relaxed so that the symmetric matrices $C_2, \ldots, C_k$ are not required to be entrywise nonnegative, a sufficient condition can be found for DN-decomposability, but first the following lemma is required.

Lemma 5. *Suppose D is an n-by-n nonnegative diagonal matrix and C is an n-by-n real symmetric matrix. If $D \succeq |C|$, then $D \succeq C$.*

Proof. We may assume D is a positive diagonal matrix since any zero diagonal entry would imply that the corresponding row and column of C have all zero entries. Since $D - |C|$ is PSD, so is $D^{-1/2}(D - |C|)D^{-1/2}$ and thus $I_n \succeq D^{-1/2}|C|D^{-1/2}$, which equals $|D^{-1/2}CD^{-1/2}|$. Since

$$\rho\left(D^{-1/2}CD^{-1/2}\right) \leqslant \rho\left(\left|D^{-1/2}CD^{-1/2}\right|\right) \leqslant \rho(I_n) = 1,$$

the eigenvalues of $D^{-1/2}CD^{-1/2}$ lie on the interval $[-1, 1]$, implying that the eigenvalues of $I - D^{-1/2}CD^{-1/2}$ lie on $[0, 2]$, as do the eigenvalues of $D^{-1/2}(I_n - D^{-1/2}CD^{-1/2})D^{1/2} = D - C$. $\square$

Theorem 6. *Suppose B is an n-by-n nonnegative diagonal matrix and matrices $C_2, \ldots, C_k$ are n-by-n symmetric. If $B \succeq \sum_{i=2}^{k} |C_i|$, then there exist nonnegative diagonal matrices B_i such that $B = \sum_{i=2}^{k} B_i$ and $B_i \succeq C_i$, for $i = 2, \ldots, k$.*

Proof. By Corollary 4, since $B, |C_2|, \ldots, |C_k|$ are entrywise nonnegative symmetric matrices, with B diagonal, there are nonnegative diagonal matrices B_i such that $B = \sum_{i=2}^{k} B_i$ and $B_i - |C_i|$ is PSD, for each i. Applying Lemma 5, $B_i \succeq C_i$ for $i = 2, \ldots, k$. □

Theorem 6 gives a sufficient condition for a diagonal matrix B to be DN-decomposable relative to the symmetric matrices $C_2, \ldots, C_k$. However, this condition is not necessary, as shown by the example given after Theorem 1.

A necessary condition for a diagonal matrix to be DN-decomposable follows, in which a *signature matrix* is a diagonal matrix with each diagonal entry either $+1$ or -1.

Theorem 7. *Suppose the diagonal matrix B is DN-decomposable relative to the symmetric matrices $C_2, \ldots, C_k$. Then $B \succeq \sum_{i=2}^{k} S_i C_i S_i$ for any signature matrices $S_2, \ldots, S_k$.*

Proof. There are diagonal matrices $B_2, \ldots, B_k$ such that $B = \sum_{i=2}^{k} B_i$ and $B_i \succeq C_i$ for each i. Since $B_i = S_i B_i S_i \succeq S_i C_i S_i$ for each i, then $B \succeq \sum_{i=2}^{k} S_i C_i S_i$. □

Note that if $n = 2$, B is diagonal and $C_2, \ldots, C_k$ are PSD, then the necessary condition of Theorem 7 is also a sufficient condition for DN-decomposability, by Theorem 1. However, for 3-by-3 diagonal matrices the condition is not sufficient for DN-decomposability, as will be shown immediately after the following required (and well-known) lemma is introduced.

Lemma 8. *If A_1 and A_2 are n-by-n PSD matrices, then* nullspace$(A_1 + A_2) \subset$ nullspace(A_1).

Consider the following PSD matrices:

$$C_2 = \begin{bmatrix} 2 & -1.4 & -1.1 \\ -1.4 & 3.1 & -1 \\ -1.1 & -1 & 2.1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 3.2 & -0.8 & 1 \\ -0.8 & 1.7 & 1.2 \\ 1 & 1.2 & 1.8 \end{bmatrix}.$$

Let r denote the largest eigenvalue of $C_1 + C_2$, with corresponding eigenvector x, and let $B = rI_3$. It is easy to verify that $B \succeq S_2C_2S_2 + S_3C_3S_3$ for any signature matrices S_2 and S_3. We now show that B is not DN-decomposable relative to C_2 and C_3. Suppose there are nonnegative diagonal matrices B_2 and B_3 such that $B = B_2 + B_3$ with $B_2 \succeq C_2$ and $B_3 \succeq C_3$. Since $(B_1 + B_2)x = rx = (C_2 + C_3)x$, $x \in$ nullspace$[(B_2 - C_2) + (B_3 - C_3)]$ and hence, by Lemma 8, $x \in$ nullspace$(B_2 - C_2)$ and $x \in$ nullspace$(B_3 - C_3)$. The diagonal matrices B_2 and B_3 are uniquely determined by the equations $B_2x = C_2x$ and $B_3x = C_3x$.

$$B_2 \approx \begin{bmatrix} 3.38 & 0 & 0 \\ 0 & 4.54 & 0 \\ 0 & 0 & 4.27 \end{bmatrix}, \quad B_3 \approx \begin{bmatrix} 3.85 & 0 & 0 \\ 0 & 2.69 & 0 \\ 0 & 0 & 2.95 \end{bmatrix}.$$

However, $B_3 \not\succeq C_3$ since the eigenvalues of $B_3 - C_3$ are (approximately) 0.00, -0.19 and 2.27. Thus, in the 3-by-3 case, the necessary condition of Theorem 7 is not sufficient.

5. Applications to complete positivity

We now apply the above DN-decomposition theorems to the question of complete positivity for DN matrices of form (1). Let

$$M_i = \begin{bmatrix} A_{11} & A_{1i} \\ A_{i1} & A_{ii} \end{bmatrix} \quad \text{for } i = 2, \dots, k.$$

Theorem 9. *Suppose A is a DN matrix of form* (1), *with* A_{11} 2-*by*-2, *and* $G(M_i)$ *is NLOC for* $i = 2, \dots, k$. *Then A is CP if and only if*

$$A_{11} \succeq \sum_{i=2}^{k} \left| A_{1i} A_{ii}^{\dagger} A_{i1} \right|.$$

Proof. By Theorem 1, A_{11} has a DN-decomposition $\sum_{i=2}^{k} B_i$ relative to $A_{12}A_{22}^{\dagger}A_{21}, \dots, A_{1k}A_{kk}^{\dagger}A_{k1}$ if and only if $A_{11} \succeq \sum_{i=2}^{k} \left| A_{1i}A_{ii}^{\dagger}A_{i1} \right|$. If A_{11} has such a decomposition, then, with A_i defined as in (3), since $G(A_i)$ is NLOC and A_i is DN, A_i is CP for each i and hence A is CP. If A_{11} has no such decomposition, A cannot be CP. □

Theorem 10. *Suppose A is a DN matrix of form* (1) *and* $G(M_i)$ *is NLOC for* $i = 2, \dots, k$. *If there exist nonnegative numbers* $\alpha_2, \dots, \alpha_k$ *such that* $\sum_{i=2}^{k} \alpha_i = 1$ *and* $A_{1i}A_{ii}^{\dagger}A_{i1} + \alpha_i(A_{11} - \sum_{i=2}^{k} A_{1i}A_{ii}^{\dagger}A_{i1})$ *is entrywise nonnegative for* $i = 2, \dots, k$, *then A is CP.*

Proof. A_{11} has a DN-decomposition relative to $A_{12}A_{22}^{\dagger}A_{21}, \dots, A_{1k}A_{kk}^{\dagger}A_{k1}$ by Theorem 2. For each i, A_i, as defined in (3), is CP since $G(A_i)$ is NLOC and A_i is DN. Hence A is CP. □

In the same way that Theorem 10 follows directly from Theorem 2, Theorems 11 and 13 below follow directly from Theorems 3 and 6, respectively.

Theorem 11. *Suppose A is a DN matrix of form* (1) *and* $G(M_i)$ *is NLOC for* $i = 2, \ldots, k$. *If the matrices* $A_{12}A_{22}^{\dagger}A_{21}, \ldots, A_{1k}A_{kk}^{\dagger}A_{k1}$ *are entrywise nonnegative and* $A_{11} - \sum_{i=2}^{k} A_{1i}A_{ii}^{\dagger}A_{i1} \in Z$, *then A is CP.*

If A_{11} is a diagonal matrix, the condition involving Z in the hypothesis of Theorem 11 is automatically satisfied.

Corollary 12. *Suppose A is a DN matrix of form* (1), *with* A_{11} *diagonal*, *and* $G(M_i)$ *is NLOC for* $i = 2, \ldots, k$. *If the matrices* $A_{12}A_{22}^{\dagger}A_{21}, \ldots, A_{1k}A_{kk}^{\dagger}A_{k1}$ *are entrywise nonnegative*, *then A is CP.*

Theorem 13. *Suppose A is a DN matrix of form* (1), *with* A_{11} *diagonal*, *and* $G(M_i)$ *is NLOC for* $i = 2, \ldots, k$. *If* $A_{11} \succeq \sum_{i=2}^{k} |A_{1i}A_{ii}^{\dagger}A_{i1}|$, *then A is CP.*

We note that Corollary 12 follows directly from Theorem 13, as well as from Theorem 11.

Theorem 14. *Suppose A is a DN matrix of form* (1), *with* A_{11} *diagonal. If there are signature matrices* $S_2, \ldots, S_k$ *such that* $A_{11} \not\succeq \sum_{i=2}^{k} S_i A_{1i}A_{ii}^{\dagger}A_{i1}S_i$, *then A is not CP.*

Proof. If A were CP, then A_{11} would be DN-decomposable relative to $A_{12}A_{22}^{\dagger}A_{21}, \ldots, A_{1k}A_{kk}^{\dagger}A_{k1}$ but, by Theorem 7, such a decomposition cannot exist. □

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