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Howe duality and combinatorial character formula for orthosymplectic Lie superalgebras

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Abstract

We study the Howe dualities involving the reductive dual pairs $(O(d), spo(2m|2n))$ and $(Sp(d), osp(2m|2n))$ on the (super)symmetric tensor of $\mathbb{C}^d \otimes \mathbb{C}^{m|n}$. We obtain complete decompositions of this space with respect to their respective joint actions. We also use these dualities to derive a character formula for these irreducible representations of $spo(2m|2n)$ and $osp(2m|2n)$ that appear in these decompositions.

MSC: 17B10; 81R05

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1. Introduction

Howe duality [13,14] relates the representation theories of a pair of Lie groups/algebras. It enables the study of representations of one Lie group/algebra via the representations of its dual partner, and hence it has become a fundamental tool where representation theory of classical Lie groups/algebras is indispensable. As simple and fundamental a concept it is therefore of no surprise that the Howe duality also applies to generalizations of finite-dimensional Lie groups/algebras. We point

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out here the Howe dualities of finite-dimensional Lie superalgebras in [4,5,22,27,30,31] of infinite-dimensional Lie algebras in [9,12,17,18,32,33] and of infinite-dimensional Lie superalgebras in [6]. In the above-mentioned articles, the main themes revolve around the construction of Howe dualities. In the present article we are also concerned about applications of the Howe dualities that we obtain.

Consider a Lie superalgebra whose representation theory we wish to study. Suppose that on some natural space one has a Howe duality involving this Lie superalgebra with a classical Lie group or Lie algebra as its dual partner. As the representation theory of its classical counterpart is well-understood, one expects that this should enable one to study the representations of the Lie superalgebra in question with the help of the representation theory of its classical dual partner. Of particular interest is a derivation of a character formula for this Lie superalgebra. It appears plausible that knowing the character of the total space and the characters of each of the irreducible representations of the classical group/algebra, one should in principle be able to obtain a character formula for the Lie superalgebra in question. As is well-known, character formulas for Lie superalgebras in general are rather difficult to obtain, and hence such a method could facilitate the computation of characters for certain representations of Lie superalgebras. One of the main purposes of this paper is to demonstrate for the orthosymplectic Lie superalgebra that such an approach to character formulas is indeed viable. The general idea is the following.

Let $\mathfrak{g}_m$ be a classical Lie algebra of rank m and let X be a fixed finite-dimensional classical Lie algebra. Suppose on some space $\mathfrak{F}_m$ the pair $(\mathfrak{g}_m, X)$ forms a dual pair in the sense of Howe. Suppose that this is the case for every m. That is, we have for each m a (multiplicity-free) decomposition with respect to $\mathfrak{g}_m \times X$ of the form

$$\mathfrak{F}_m = \sum_{\lambda} V^{\lambda}_{\mathfrak{g}_m} \otimes V^{\lambda'}_{X},$$

where $V^{\lambda}_{\mathfrak{g}_m}$ and $V^{\lambda'}_{X}$ denote irreducible representations of $\mathfrak{g}_m$ and X, respectively. Here λ is summed over a subset of irreducible representations of $\mathfrak{g}_m$. Since here the correspondence between irreducible representations of $\mathfrak{g}_m$ and X, given by $\lambda \rightarrow \lambda'$, is one-to-one, we will write V^{λ}_{X} for $V^{\lambda'}_{X}$.

Now suppose that $\mathfrak{g}_{m|n}$ is the Lie superalgebraic analogue of $\mathfrak{g}_m$ and we have an action of the dual pair $\mathfrak{g}_{m|n} \times X$ on $\mathfrak{F}_{m|n}$, which is the tensor product of $\mathfrak{F}_m$ with a Grassmann superalgebra depending on n. Thus we have similarly

$$\mathfrak{F}_{m|n} = \sum_{\lambda} V^{\lambda}_{\mathfrak{g}_{m|n}} \otimes V^{\lambda}_{X},$$

where $V^{\lambda}_{\mathfrak{g}_{m|n}}$ denotes an irreducible representation of $\mathfrak{g}_{m|n}$.

Our claim is that if one knows the characters of $V^{\lambda}_{\mathfrak{g}_m}$ for every m, then one, in principle, also knows the characters for $V^{\lambda}_{\mathfrak{g}_{m|n}}$.

Let us now discuss the content of the present article in more detail. Let $X = O$ or $X = Sp$ so that $X(d)$ denotes either the orthogonal or the symplectic group acting on $\mathbb{C}^d$. We have an induced action on $\mathbb{C}^d \otimes \mathbb{C}^m$, thus giving rise to an action on the symmetric tensor $S(\mathbb{C}^d \otimes \mathbb{C}^m)$. Now by classical invariant theory (cf. [13,11]) the invariants of $X(d)$ in the endomorphism ring of $S(\mathbb{C}^d \otimes \mathbb{C}^m)$ is generated by quadratic invariants, which may be identified with the Lie algebra $sp(2m)$ in the case $X = O$ and $so(2m)$ in the case $X = Sp$. This implies that $(O(d), sp(2m))$ and $(Sp(d), so(2m))$ are Howe dual pairs on $S(\mathbb{C}^d \otimes \mathbb{C}^m)$.

Now let $\mathbb{C}^{m|n}$ be the complex superspace of dimension $(m|n)$. The Lie group $X(d)$ acts in a similar fashion on the (super) symmetric tensor $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. Analogously one derives the $(O(d), spo(2m|2n))$- and the $(O(d), osp(2m|2n))$-Howe duality on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. Although these dualities appear already in Howe's classical paper [13], the complete decompositions of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ with respect to these joint actions are unknown to the best of our knowledge. In [27] a partial decomposition is obtained for $X = O$, with a complete answer given in the case of $m = n = 1$ only.

Our first main task is to give the complete decompositions of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ with respect to these Howe dual pairs. This is achieved in the following way. By [13] the decomposition of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ with respect to $X(d)$ and its dual partner is reduced to the decomposition of the subspace of harmonic polynomials H with respect to the dual pair $(X(d), gl(m|n))$. Our task is then reduced to the construction of all $(X(d), gl(m|n))$-highest weight vectors in H. Our analysis of the $(X(d), gl(m|n))$-highest weight vectors in H relies heavily on the $(gl(d), gl(m|n))$-Howe duality in $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ in [5] (see also [30,31]) and the description of their joint highest weight vectors given in [5]. Another important ingredient is the construction of an explicit basis for each irreducible $gl(d) \times gl(m|n)$-component that appears in $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$.

The idea to obtain a character formula for the irreducible representations of $spo(2m|2n)$ or $osp(2m|2n)$ is roughly as follows. In order to simplify notation we take $X = Sp$ in what follows, but note that the same applies to $X = O$ with minor modification. We first consider the classical duality, i.e. the case when $n = 0$. Thus we have an identity of characters of the form

$$ch\, S(\mathbb{C}^d \otimes \mathbb{C}^m) = \sum_{\lambda} ch\, V^{\lambda}_{Sp(d)} \otimes ch\, V^{\lambda}_{so(2m)}.$$

Since now characters are polynomial functions on a Cartan subalgebra, we can write $\chi^{\lambda}_{Sp(d)}(\mathbf{x}) = ch\, V^{\lambda}_{Sp(d)}$ and $\chi^{\lambda}_{so(2m)}(\mathbf{y}) = ch\, V^{\lambda}_{so(2m)}$, where $\mathbf{x}$ and $\mathbf{y}$ denote the linear functions on the respective Cartan subalgebras. The left-hand side is the character of the algebra of polynomials in dm variables, which is a symmetric function in $\mathbf{x}$ and $\mathbf{y}$. Now taking the limit as $m \to \infty$ in an appropriate way one obtains a combinatorial identity involving infinitely many variables $\mathbf{y} = y_1, y_2, \ldots, y_m, \ldots$. Since the right-hand side is symmetric in $\mathbf{y}$, we may apply to this identity the involution ω of symmetric functions that sends the complete symmetric functions to the elementary symmetric functions (see [24]). The ω turns the left-hand side into the character of the tensor product of a polynomial algebra with a Grassmann algebra. Therefore, due to "linear independence" of the $\chi^{\lambda}_{Sp(d)}$, it follows that (modulo some minor manipulation of the variables) the expression $\omega(\lim_{m\to\infty}\chi^{\lambda}_{so(2m)}(\mathbf{y}))$ is essentially the character of the irreducible representation of $osp(2m|2n)$ paired with $V^{\lambda}_{Sp(d)}$. At this point we wish to point out our results imply that the characters of the representations of the Lie superalgebra $osp(2m|2n)$ (respectively $spo(2m|2n)$), for any $m,n\in\mathbb{Z}_+$, that appear under the Howe duality are completely determined by the characters of the representations of the Kac–Moody algebra corresponding to the infinite affine matrix D_∞ (respectively C_∞) (see [16]) that appear under a similar Howe duality.

The next problem is to describe the expression $\chi^{\lambda}_{so(2m)}(\mathbf{y})$. For this we use the beautiful formula of Enright [8,7] for unitarizable irreducible representations associated to a classical Hermitian symmetric pair. The reason for this is that in our case we may express such a character in terms of Schur functions which are carried by ω to the so-called hook Schur functions of Berele and Regev [1]. This allows us to obtain a satisfactory description of the characters.

We now come to the organization of the paper. In Section 2 we recall some basic facts on the orthogonal and symplectic groups and the orthosymplectic superalgebra, where we also take the opportunity to set the notation to be used throughout the paper. In Section 3 we recall the $(gl(d),gl(m|n))$-duality on $S(\mathbb{C}^d\otimes\mathbb{C}^{m|n})$ and construct an explicit basis for each irreducible component that appears in the decomposition of $S(\mathbb{C}^d\otimes\mathbb{C}^{m|n})$. In Sections 4 and 5 we study the $(O(d),spo(2m|2n))$-duality and the $(Sp(d),osp(2m|2n))$-duality and obtain the complete decompositions of $S(\mathbb{C}^d\otimes\mathbb{C}^{m|n})$ with respect to their respective joint actions. In Section 6 we derive a character formula for these representations of $spo(2m|2n)$ and $osp(2m|2n)$ that appear in the decomposition of $S(\mathbb{C}^d\otimes\mathbb{C}^{m|n})$. Here we should mention that in the case of $O(d)\times spo(2m|2n)$ with d even, we are only able to derive the formula for a sum of two irreducible representations in general. We also remark that in [26] a character formula for the oscillator representations is given. This corresponds to our case $O(1)$. In order to obtain a better description of the character formulas we are required to study Enright's formula in more detail. This is done in the Section 7. In Section 8 we study the character formulas in more detail. In Section 9, as another application of our Howe dualities, we give formulas for decomposing tensor products of these irreducible $spo(2m|2n)$- and $osp(2m|2n)$ modules that appear in the decomposition of $S(\mathbb{C}^d\otimes\mathbb{C}^{m|n})$.

Finally all vector spaces, algebras, etc. are over the complex field $\mathbb{C}$ unless otherwise specified. By a *partition* we mean a non-increasing finite sequence of non-negative integers. A *composition* is a finite sequence of either all non-negative integers or all positive half-integers. Furthermore, by a *generalized partition* we will always mean a finite non-increasing sequence of either all integers or all half integers. By a *generalized composition* we will mean a finite sequence of either all integers or all half integers.

2. Parameterization of irreducible representations

In this section we give parameterizations of irreducible representations of the Lie groups and Lie superalgebras that we will be dealing with in this paper. For a more complete treatment of the material on Lie groups the reader is referred to [2].

2.1. Irreducible representations of the general linear Lie superalgebra

Let $\mathbb{C}^{m|n}$ denote the complex $(m|n)$-dimensional superspace. The space of complex linear transformations on $\mathbb{C}^{m|n}$ has a natural structure as a Lie superalgebra, which we will denote by $gl(m|n)$. Choose a homogeneous basis for $\mathbb{C}^{m|n}$ so that we may regard $gl(m|n)$ as $(m+n)\times(m+n)$ matrices. Denote by E_{ij} the elementary matrix with 1 in the ith row and jth column and 0 elsewhere. Then $\mathfrak{h}=\sum_i \mathbb{C}E_{ii}$ is a Cartan subalgebra, while $B=\sum_{i\leqslant j}\mathbb{C}E_{ij}$ is a Borel subalgebra containing $\mathfrak{h}$. Recall that finite-dimensional irreducible $gl(m|n)$-modules are parameterized by $\lambda\in\mathfrak{h}^*$ with $\lambda_i-\lambda_{i+1}\in\mathbb{Z}_+$, for $i=1,\dots,m-1,m+1,\dots,m+n-1$, where $\lambda_i=\lambda(E_{ii})$. We will denote the corresponding finite-dimensional irreducible module by $V^\lambda_{m|n}$. Suppose that λ is a partition (or a Young diagram) with $\lambda_{m+1}\leqslant n$. Then drawing the corresponding diagram λ may be visualized as lying in the $(m|n)$-hook, i.e. from $n+1$th column on the columns of λ all have lengths less than $m+1$. We may interpret λ as a highest weight of $gl(m|n)$ by associating to the diagram λ the labels $\lambda=(\lambda_1,\dots,\lambda_m;\langle\lambda'_1-m\rangle,\dots,\langle\lambda'_n-m\rangle)$, where λ'_i is the length of the ith column of the diagram λ, and $\langle r\rangle$ stands for r, if $r\in\mathbb{N}$, and 0 otherwise. If clear from the context that λ is a Young diagram with $\lambda_{m+1}\leqslant n$, we will mean by $V^\lambda_{m|n}$ the irreducible $gl(m|n)$-module of highest weight λ.

2.2. Irreducible representations of the orthogonal group

Let us denote by $\{e^1,\dots,e^d\}$ the standard basis for $\mathbb{C}^d$. Consider the symmetric non-degenerate bilinear form determined by the $d\times d$ matrix

$$J_d=\begin{pmatrix}0&0&\cdots&0&1\\0&0&\cdots&1&0\\\vdots&\vdots&\vdots&\vdots&\vdots\\0&1&\cdots&0&0\\1&0&\cdots&0&0\end{pmatrix}.$$

The complex orthogonal group $O(d)$ is the subgroup of the complex general linear group $GL(d)$ preserving this form. The Lie algebra of $O(d)$ is $so(d)$, which consists of those $A\in gl(d)$ with $J_dA^tJ_d+A=0$, that is, A is skew-symmetric with respect to the diagonal running from the top right to the bottom left corner.

Consider the case when $d=2k$ is even. We take as a Borel subalgebra $\mathfrak{b}$ the subalgebra of $so(d)$ contained in the subalgebra of upper triangular matrices. Furthermore, we take as a Cartan subalgebra of $\mathfrak{b}$ the subalgebra $\mathfrak{h}$ spanned by the elements $\tilde{E}_{ii}=E_{ii}-E_{d+1-i,d+1-i}$, for $i=1,\dots,k$. Now a finite-dimensional irreducible representation of $so(d)$ is determined by its highest weight $\lambda\in\mathfrak{h}^*$ subject to

$$\lambda(\tilde{E}_{ii}-\tilde{E}_{i+1,i+1})\in\mathbb{Z}_+,$$

$$\lambda(\tilde{E}_{k-1,k-1}+\tilde{E}_{kk})\in\mathbb{Z}_+,$$

for $i=1,\dots,k-1$. Let $\lambda_i=\lambda(\tilde{E}_{ii})$ and identify λ with the sequence of complex numbers $(\lambda_1,\dots,\lambda_k)$. An irreducible representation of $so(2k)$ is finite-dimensional if and only if its highest weight λ satisfies the conditions $\lambda_1\geqslant\lambda_2\cdots\geqslant\lambda_k$ with either $\lambda_i\in\mathbb{Z}$ or else $\lambda_i\in\frac{1}{2}+\mathbb{Z},\ i=1,\dots,k$ and $\lambda_j\geqslant 0,\ j=1,\dots,k-1$. Furthermore such a weight lifts to a representation of $SO(d)$ if and only if $\lambda_i\in\mathbb{Z}_+$.

Let V be a finite-dimensional irreducible $O(d)$-module. When regarded as an $so(d)$-module we have the following possibilities:

(i) V is a direct sum of two irreducible $so(d)$-modules of highest weights $(\lambda_1,\lambda_2,\dots,\lambda_k)$ and $(\lambda_1,\lambda_2,\dots,\lambda_{k-1},-\lambda_k)$, respectively, where $\lambda_k>0$.
(ii) V is an irreducible $so(d)$-module of highest weight $(\lambda_1,\lambda_2,\dots,\lambda_{k-1},0)$.

Here $\lambda_i\in\mathbb{Z}_+$ for all i. In the first case, that is when V is the direct sum of the two irreducible $so(d)$-modules we denote V by $V^\lambda_{O(d)}$, where we let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{k-1},\lambda_k>0)$. In the second case there are two possible choices of V, which we denote by $V^\lambda_{O(d)}$ and $V^\lambda_{O(d)}\otimes\det$, respectively. Recalling that $O(d)$ is a semidirect product of $SO(d)$ and $\mathbb{Z}_2$ these two $O(d)$-modules as $SO(d)$-modules are isomorphic. However as $O(d)$-modules they differ by the determinant representation so that we may distinguish these two modules as follows: consider the element $\tau\in O(d)-SO(d)$ that switches the basis vector e^k with e^{k+1} and leaves all other basis vectors of $\mathbb{C}^d$ invariant. We declare $V^\lambda_{O(d)}$ to be the $O(d)$-module on which τ transforms an $SO(d)$-highest weight vector trivially. Note that τ transforms an $SO(d)$-highest weight vector in the $O(d)$-module $V^\lambda_{O(d)}\otimes\det$ by -1.

We may associate Young diagrams to these $O(d)$-highest weights as follows (cf. [14]). For $\lambda_1\geqslant\lambda_2\cdots\geqslant\lambda_k>0$ we have an obvious Young diagram of length k. When $\lambda_k=0$, we associate to the highest weight of $V^\lambda_{O(d)}$ the usual Young diagram of length less than k. To the highest weight of $V^\lambda_{O(d)}\otimes\det$ we associate the Young diagram obtained from the Young diagram of λ by replacing its first column by a column of length $d-\lambda'_1$. Here and further, for a partition λ, we denote by λ' its conjugate partition. We have thus associated to each finite-dimensional irreducible representation of $O(d)$ a Young diagram λ with $\lambda'_1+\lambda'_2\leqslant d$.

Next consider the case when $d=2k+1$ is odd. We take as a Borel subalgebra $\mathfrak{b}$ the subalgebra of $so(d)$ spanned by upper triangular matrices so that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{b}$ is again spanned by the elements $\tilde{E}_{ii} = E_{ii} - E_{d+1-i,d+1-i}$, for $i = 1, \dots, k$. Now a finite-dimensional irreducible representation of $so(d)$ is determined by its highest weight $\lambda \in \mathfrak{h}^*$ subject to

$$\lambda(\tilde{E}_{ii} - \tilde{E}_{i+1,i+1}) \in \mathbb{Z}_+,$$

$$\lambda(\tilde{E}_{kk}) \in \frac{1}{2}\mathbb{Z}_+,$$

for $i = 1, \dots, k-1$. We set $\lambda_i = \lambda(\tilde{E}_{ii})$ and identify λ with the sequence of complex numbers $(\lambda_1, \lambda_2, \dots, \lambda_k)$. It follows that a highest weight λ of $so(2k+1)$ gives a finite-dimensional irreducible representation if and only $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_k$ and $\lambda_i \in \mathbb{Z}_+$ or else $\lambda_i \in \frac{1}{2} + \mathbb{Z}_+$, for $i = 1, \dots, k$.

Recall that when d is odd $O(d)$ is a direct product of $SO(d)$ and $\mathbb{Z}_2$. Thus any finite-dimensional irreducible representation of $O(d)$, when regarded as an $SO(d)$-module, remains irreducible. Conversely an irreducible representation of $SO(d)$ gives rise to two non-isomorphic $O(d)$-modules that differ from each other by the determinant representation det. We let $V^{\lambda}_{O(d)}$ stand for the irreducible $O(d)$-module corresponding to $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0)$ on which the element $-I$ transforms trivially, so that $\{V^{\lambda}_{O(d)}, V^{\lambda}_{O(d)} \otimes \det\}$ with λ ranging over all partitions as above is a complete set of finite-dimensional non-isomorphic irreducible $O(d)$-modules.

Similarly as before we may associate Young diagrams to these $O(d)$-highest weights. For the highest weight $\lambda = (\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_k \geqslant 0)$ of $V^{\lambda}_{O(d)}$ we have an obvious Young diagram with $l(\lambda) \coloneqq \lambda'_1 \leqslant k$. To the highest weight of $V^{\lambda}_{O(d)} \otimes \det$ we associate the Young diagram obtained from the Young diagram of λ by replacing its first column by a column of length $d - \lambda'_1$.

Let $\varepsilon_i \in \mathfrak{h}^*$ so that $\varepsilon_i(\tilde{E}_{jj}) = \delta_{ij}$. We put $x_i = e^{\varepsilon_i}$ when dealing with characters of $O(d)$.

2.3. *Irreducible representations of the symplectic group*

Let $d = 2k$ and consider the non-degenerate skew-symmetric bilnear form $\langle \cdot | \cdot \rangle$ given by the $d \times d$ matrix

$$\begin{pmatrix} 0 & J_k \\ -J_k & 0 \end{pmatrix}.$$

The symplectic group $Sp(d)$ is the subgroup of $GL(d)$ preserving $\langle \cdot | \cdot \rangle$. We take as a Borel subalgebra $\mathfrak{b}$ the subalgebra of $sp(d)$ that is contained in the subalgebra of upper triangular matrices and a Cartan subalgebra of $\mathfrak{b}$ as the subalgebra $\mathfrak{h}$ spanned by the elements $\tilde{E}_{ii} = E_{ii} - E_{d+1-i,d+1-i}$, for $i = 1, \dots, k$. A finite-dimensional irreducible representation of $sp(d)$ is determined by its highest weight $\lambda \in \mathfrak{h}^*$

subject to

$$\lambda(\tilde{E}_{ii} - \tilde{E}_{i+1,i+1}) \in \mathbb{Z}_+,$$

$$\lambda(\tilde{E}_{k,k}) \in \mathbb{Z}_+,$$

for $i = 1, \dots, k-1$. As before we let $\lambda_i = \lambda(\tilde{E}_{ii})$ and identify λ with the sequence $(\lambda_1, \lambda_2, \dots, \lambda_k)$. A highest weight λ of $sp(2k)$ gives a finite-dimensional irreducible representation if and only if $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_k$ and $\lambda_i \in \mathbb{Z}_+$ for $i = 1, \dots, k$. Furthermore each such representation lifts to a unique irreducible representation of $Sp(d)$ and so we obtain an obvious parameterization of $Sp(d)$-highest weight in terms of Young diagrams λ with $l(\lambda) \leqslant \frac{d}{2}$.

We let $\varepsilon_i \in \mathfrak{h}^*$ so that $\varepsilon_i(\tilde{E}_{jj}) = \delta_{ij}$. We put $y_i = e^{\varepsilon_i}$ when dealing with characters of $Sp(2k)$.

2.4. Irreducible representations of the orthosymplectic Lie superalgebra

Let $\mathbb{C}^{m|n}$ be the $(m|n)$-dimensional complex superspace. Suppose that n is even and $(\cdot|\cdot)$ is a supersymmetric non-degenerate bilinear form, i.e. it is symmetric on the even subspace $\mathbb{C}^{m|0}$ and symplectic on the odd subspace $\mathbb{C}^{0|n}$. The orthosymplectic Lie superalgebra $osp(m|n)$ (cf. [15]) is defined to be the subalgebra of $gl(m|n) = gl(m|n)_{\bar{0}} \oplus gl(m|n)_{\bar{1}}$ consisting of those linear transformations preserving the form $(\cdot|\cdot)$, i.e. $osp(m|n) = osp(m|n)_{\bar{0}} \oplus osp(m|n)_{\bar{1}}$ with

$$osp(m|n)_\varepsilon = \{A \in gl(m|n)_\varepsilon | (Av|w) + (-1)^{\varepsilon \deg v}(v|Aw) = 0\},$$

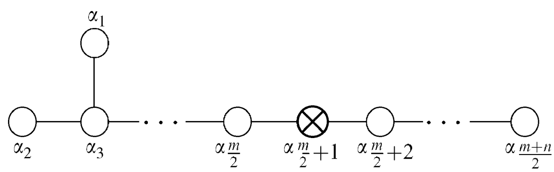
where v and w are any homogeneous vectors of $\mathbb{C}^{m|n}$, $\deg v$ here and further denotes the degree of the homogeneous element v and $\varepsilon \in \mathbb{Z}_2$. We will fix the bilinear form associated to matrix

$$\begin{pmatrix} J_m & 0 & 0 \\ 0 & 0 & J_{n/2} \\ 0 & -J_{n/2} & 0 \end{pmatrix}.$$

We note that $osp(m|n)_{\bar{0}} \cong so(m) \oplus sp(n)$. Let $\mathfrak{b}$ be a Borel subalgebra of $osp(m|n)$ containing the Borel subalgebras of $so(m)$ and $sp(n)$ as chosen above so that a Cartan subalgebra $\mathfrak{h}$ of $osp(m|n)$ can be taken to be the subalgebra spanned by the diagonal matrices $\tilde{E}_{ii} = E_{ii} - E_{m+1-i,m+1-i}$, $i = 1, \dots, [\frac{m}{2}]$, $\tilde{E}_{[\frac{m}{2}]+j,[\frac{m}{2}]+j} = E_{m+j,m+j} - E_{m+n+1-j,m+n+1-j}$, $j = 1, \dots, \frac{n}{2}$. Here and further the symbol $[r]$ stands for the largest integer smaller than or equal to r. As usual, highest weight irreducible representations of $osp(m,n)$ are parameterized by $\lambda \in \mathfrak{h}^*$ and we denote by λ_i the ith label $\lambda(\tilde{E}_{ii})$, for $i = 1, \dots, [\frac{m}{2}] + \frac{n}{2}$. As usual, we will identify λ with $(\lambda_1, \lambda_2, \dots)$.

Suppose that m is an even integer and consider the following $\mathbb{Z}$-gradation of $osp(m|n)$. Let $\mathbb{C}^{m|0} = V \oplus V^*$ be a sum of two isotropic subspaces of $\mathbb{C}^{m|0}$ with respect to the restriction of the form $(\cdot|\cdot)$ on $\mathbb{C}^{m|0}$. Likewise let $\mathbb{C}^{0|n} = W \oplus W^*$ be such an isotropic decomposition of $\mathbb{C}^{0|n}$. We have $osp(m|n)_{\bar{0}} \cong S^2(\mathbb{C}^{0|n}) \oplus \Lambda^2(\mathbb{C}^{m|0})$ and $osp(m|n)_{\bar{1}} \cong \mathbb{C}^{m|0} \otimes \mathbb{C}^{0|n}$. Set $\mathfrak{g}_0 = (V \oplus W) \otimes (V \oplus W)^*$, $\mathfrak{g}_1 = S^2(V) \oplus \Lambda^2(W) \oplus (V \otimes W)$ and $\mathfrak{g}_{-1} = S^2(V^*) \oplus \Lambda^2(W^*) \oplus (V^* \otimes W^*)$. This equips $osp(m|n)$ with a $\mathbb{Z}$-gradation with $\mathfrak{g}_0$ isomorphic to $gl(\frac{m}{2}|\frac{n}{2})$ such that its standard Cartan subalgebra is also $\mathfrak{h}$.

Now take a finite-dimensional irreducible $\mathfrak{g}_0$-module $V^{\lambda}_{\frac{m}{2}|\frac{n}{2}}$ of highest weight $\lambda \in \mathfrak{h}^*$, which we again will identify with a sequence $(\lambda_1, \lambda_2, \ldots)$. We may extend $V^{\lambda}_{\frac{m}{2}|\frac{n}{2}}$ trivially to a module over the parabolic subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. Inducing it to an $osp(m|n)$-module, it is clear that it has a unique irreducible quotient, which we will denote by $V^{\lambda}_{osp(m|n)}$. Of course $V^{\lambda}_{osp(m|n)}$ is not finite-dimensional in general. As such $osp(m|n)$-modules play an important role in the sequel, we will give a more detailed description of their parameterizations. Let $\varepsilon_i \in \mathfrak{h}^*$, $i = 1, \ldots, [\frac{m}{2}] + \frac{n}{2}$, be defined by $\varepsilon_i(\tilde{E}_{jj}) = \delta_{ij}$. We will label the simple roots and coroots of $osp(m|n)$ according to the following diagram.



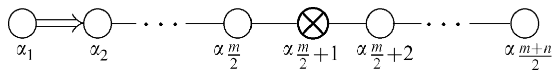
Here $\alpha_1 = -\varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{\frac{m}{2}} = \varepsilon_{\frac{m}{2}-1} - \varepsilon_{\frac{m}{2}}, \ldots, \alpha_{\frac{m+n}{2}} = \varepsilon_{\frac{m+n}{2}-1} - \varepsilon_{\frac{m+n}{2}}$, and, as is customary, $\otimes$ denotes an isotropic root. Thus if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\frac{m+n}{2}})$ is the highest weight of a finite-dimensional irreducible $gl(\frac{m}{2}|\frac{n}{2})$-module $V^{\lambda}_{\frac{m}{2}|\frac{n}{2}}$, then the labels of the irreducible highest weight module $V^{\lambda}_{osp(m|n)}$ with respect to the above Dynkin diagram is given by

$$(-\lambda_1 - \lambda_2, \lambda_1 - \lambda_2, \ldots, \lambda_{\frac{m}{2}-1} - \lambda_{\frac{m}{2}}, \lambda_{\frac{m}{2}} + \lambda_{\frac{m}{2}+1}, \lambda_{\frac{m}{2}+1} - \lambda_{\frac{m}{2}+2}, \ldots). \tag{2.1}$$

When dealing with characters of $osp(m|n)$ we will use the notation $x_j = e^{\varepsilon_j}$, for $j = 1, \ldots, \frac{m}{2}$ and $z_l = e^{\varepsilon_{\frac{m}{2}+l}}$, for $l = 1, \ldots, \frac{n}{2}$.

On the superspace $\mathbb{C}^{m|n}$ with m even we may take a skew-supersymmetric non-degenerate bilinear form $(\cdot|\cdot)$, i.e. it is symplectic on the even subspace $\mathbb{C}^{m|0}$ and symmetric on the odd subspace $\mathbb{C}^{0|n}$. In the same fashion we may define the symplectic-orthogonal Lie superalgebra $spo(m|n)$ to be the subalgebra of $gl(m|n)$ preserving $(\cdot|\cdot)$. We remark that as Lie superalgebras we have $spo(m|n) \cong osp(n|m)$ and hence our discussion of the orthosymplectic Lie superalgebra carries over to

$spo(m|n)$, for n even, with minor modification. We label the simple roots and coroots according to the following diagram.



Here $\alpha_1 = -2\varepsilon_1, \alpha_2 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{\frac{m}{2}} = \varepsilon_{\frac{m}{2}-1} - \varepsilon_{\frac{m}{2}}, \ldots, \alpha_{\frac{m+n}{2}} = \varepsilon_{\frac{m+n}{2}-1} - \varepsilon_{\frac{m+n}{2}}$. Similarly we will denote the irreducible quotient of the induced $gl(m|n)$-module $V^{\lambda}_{\frac{m}{2}|\frac{n}{2}}$ by $V^{\lambda}_{spo(m|n)}$. So if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\frac{m+n}{2}})$ is the $gl(\frac{m}{2}|\frac{n}{2})$-labels of $V^{\lambda}_{\frac{m}{2}|\frac{n}{2}}$, then the $spo(m|n)$-labels of $V^{\lambda}_{spo(m|n)}$ are

$$(-\lambda_1, \lambda_1 - \lambda_2, \ldots, \lambda_{\frac{m}{2}-1} - \lambda_{\frac{m}{2}}, \lambda_{\frac{m}{2}} + \lambda_{\frac{m}{2}+1}, \lambda_{\frac{m}{2}+1} - \lambda_{\frac{m}{2}+2}, \ldots). \tag{2.2}$$

When dealing with characters of $spo(m|n)$ we will use the notation $y_j = e^{\varepsilon_j}$, for $j = 1, \ldots, \frac{m}{2}$ and $z_l = e^{\varepsilon_{\frac{m}{2}+l}}$, for $l = 1, \ldots, \frac{n}{2}$.

3. The $(gl(d), gl(m|n))$-duality

In this section we present some results on $(gl(d), gl(m|n))$-duality that will be used later on. In particular, Theorem 3.4 constructs explicit bases for irreducible $gl(d) \times gl(m|n)$-modules appearing in the decomposition of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$, and we believe the result to be new.

Consider the natural actions of $gl(d|q)$ on $\mathbb{C}^{d|q}$ and $gl(m|n)$ on $\mathbb{C}^{m|n}$. We can form the $gl(d|q) \times gl(m|n)$-module $\mathbb{C}^{d|q} \otimes \mathbb{C}^{m|n}$. We have an induced action on the symmetric tensor $S(\mathbb{C}^{d|q} \otimes \mathbb{C}^{m|n})$. This action is completely reducible and in fact $(gl(d|q), gl(m|n))$ is a dual pair in the sense of Howe [5] (see also [30]). Since in this paper we will only concern ourselves with the case when $q = 0$, we will make this assumption in what follows. In this case we have the following decomposition:

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \cong \sum_{\lambda} V^{\lambda}_d \otimes V^{\lambda}_{m|n}. \tag{3.1}$$

The sum in (3.1) is over all partitions of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ of length $l(\lambda)$ not exceeding d subject to $\lambda_{m+1} \leqslant n$. Since $l(\lambda) \leqslant d$ we may regard λ as a highest weight for an irreducible $gl(d)$-module so that there is no ambiguity in V^{λ}_d. The meaning of $V^{\lambda}_{m|n}$ as a $gl(m|n)$-module was explained in Section 2.1.

In the sequel it is important to have an explicit formula for the joint highest weight vectors of the irreducible component $V^{\lambda}_d \otimes V^{\lambda}_{m|n}$ in (3.1). (See also [25,28] for different descriptions of these vectors.) In order to present them we need to introduce some more notation.

We let $e^1, \dots, e^d$ denote the standard basis for the standard $gl(d)$-module. Similarly we let $e_1, \dots, e_m; f_1, \dots, f_n$ denote the standard homogeneous basis for the standard $gl(m|n)$-module. The weights of e^i, e_l and f_k are denoted by $\tilde{\varepsilon}_i$, ε_l and δ_k, for $1 \leqslant i \leqslant d$, $1 \leqslant l \leqslant m$ and $1 \leqslant k \leqslant n$, respectively. We set

$$x_l^i := e^i \otimes e_l, \quad \eta_k^i := e^i \otimes f_k. \tag{3.2}$$

We will denote by $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ the polynomial superalgebra generated by (3.2). By identifying $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ with the polynomial superalgebra $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ the commuting pair $(gl(d), gl(m|n))$ may be realized as first-order differential operators as follows: $(1 \leqslant i, i' \leqslant d,\ 1 \leqslant s, s' \leqslant m$ and $1 \leqslant k, k' \leqslant n)$:

$$\sum_{j=1}^{m} x_j^i \frac{\partial}{\partial x_j^{i'}} + \sum_{j=1}^{n} \eta_j^i \frac{\partial}{\partial \eta_j^{i'}}, \tag{3.3}$$

$$\sum_{j=1}^{d} x_s^j \frac{\partial}{\partial x_{s'}^j}, \quad \sum_{j=1}^{d} \eta_{k'}^j \frac{\partial}{\partial \eta_k^j}, \quad \sum_{j=1}^{d} x_s^j \frac{\partial}{\partial \eta_k^j}, \quad \sum_{j=1}^{d} \eta_k^j \frac{\partial}{\partial x_s^j}. \tag{3.4}$$

Here (3.3) spans a copy of $gl(d)$, while (3.4) spans a copy of $gl(m|n)$.

The standard Cartan subalgebras of $gl(d)$ and $gl(m|n)$ are spanned, respectively, by

$$\sum_{j=1}^{m} x_j^i \frac{\partial}{\partial x_j^i} + \sum_{j=1}^{n} \eta_j^i \frac{\partial}{\partial \eta_j^i} \quad \text{and} \quad \sum_{j=1}^{d} x_s^j \frac{\partial}{\partial x_s^j}, \ \sum_{j=1}^{d} \eta_k^j \frac{\partial}{\partial \eta_k^j},$$

while the nilpotent radicals are, respectively, generated by the simple root vectors

$$\sum_{j=1}^{m} x_j^{i-1} \frac{\partial}{\partial x_j^i} + \sum_{j=1}^{n} \eta_j^{i-1} \frac{\partial}{\partial \eta_j^i}, \quad 1 < i \leqslant d,$$

and

$$\sum_{j=1}^{d} x_{s-1}^j \frac{\partial}{\partial x_s^j}, \ \sum_{j=1}^{d} \eta_{k-1}^j \frac{\partial}{\partial \eta_k^j}, \ \sum_{j=1}^{d} x_m^j \frac{\partial}{\partial \eta_1^j}, \quad 1 < s \leqslant m, 1 < k \leqslant n.$$

We will consider two separate cases, namely $m \geqslant d$ and $m < d$.

First suppose that $m \geqslant d$. Here the condition $\lambda_{m+1} \leqslant n$ is vacuous. For $1 \leqslant r \leqslant \min(d, m)$ define

$$\Delta_r := \det \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_r^1 \\ x_1^2 & x_2^2 & \cdots & x_r^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^r & x_2^r & \cdots & x_r^r \end{pmatrix}. \tag{3.5}$$

Theorem 3.1 (Cheng and Wang [5]). *In the case when $m \geqslant d$, the $gl(d) \times gl(m|n)$ highest weight vectors in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ associated to the weight λ is given by the product $\Delta_{\lambda'_1}\Delta_{\lambda'_2}\cdots\Delta_{\lambda'_{\lambda_1}}$.*

We now consider the case $d > m$. It is readily checked that the highest weight vectors associated to Young diagrams λ with $\lambda_{m+1} = 0$ can be obtained just as in the previous case so that we may assume that $l(\lambda) > m$. Let $\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1}$ denote its column lengths as usual. We have $d \geqslant \lambda'_1 \geqslant \lambda'_2 \cdots \geqslant \lambda'_{\lambda_1}$ and $m \geqslant \lambda'_{n+1}$. For $m < r \leqslant d$, consider the following determinant of an $r \times r$ matrix:

$$\Delta_{k,r} := \det \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^r \\ x_2^1 & x_2^2 & \cdots & x_2^r \\ \vdots & \vdots & \cdots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^r \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \\ \vdots & \vdots & \cdots & \vdots \\ \eta_k^1 & \eta_k^2 & \cdots & \eta_k^r \end{pmatrix}, \quad k = 1, \ldots, n. \tag{3.6}$$

That is, the first m rows are filled by the vectors $(x_j^1, \ldots, x_j^r)$, for $j = 1, \ldots, m$, in increasing order and the last $r - m$ rows are filled with the same vector $(\eta_k^1, \ldots, \eta_k^r)$. Here the determinant of a matrix

$$A := \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^r \\ a_2^1 & a_2^2 & \cdots & a_2^r \\ \vdots & \vdots & \cdots & \vdots \\ a_r^1 & a_r^2 & \cdots & a_r^r \end{pmatrix},$$

with matrix entries possibly involving Grassmann variables η_k^i, is by definition the expression $\sum_{\sigma \in S_r} (-1)^{l(\sigma)} a_1^{\sigma(1)} a_2^{\sigma(2)} \cdots a_r^{\sigma(r)}$, where $l(\sigma)$ is the length of σ in the symmetric group S_r.

Theorem 3.2 (Cheng and Wang [5]). *In the case when $m < d$, the $gl(d) \times gl(m|n)$ highest weight vectors in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ associated to the weight λ is given by the product*

$$\prod_{k=1}^{v} \Delta_{k,\lambda'_k} \prod_{j=r+1}^{\lambda_1} \Delta_{\lambda'_j}, \tag{3.7}$$

where v is defined by $\lambda'_v > m$ and $\lambda'_{v+1} \leqslant m$.

For application purposes it is useful to construct an explicit basis for the $gl(d) \times gl(m|n)$-modules $V^{\lambda}_{m|n}$ that appear in the decomposition of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. This we will do now.

Recall that λ is a partition (or a Young diagram) which lies in the $(m|n)$-hook of length not exceeding d. Let $x_1, \dots, x_m$ and $\eta_1, \dots, \eta_n$ be even and odd indeterminates, respectively. We form a tableau of shape λ by filling the nodes of λ from the set $\{x_1, \dots, x_m, \eta_1, \dots, \eta_n\}$ so that the resulting tableau T is $(m|n)$-semi-standard. This means that we first fill the nodes of a subdiagram $\mu \subseteq \lambda$ with the even indeterminates $\{x_1, \dots, x_m\}$ so that the resulting subtableau is semi-standard. Then we fill the skew-diagram λ/μ with odd indeterminates $\{\eta_1, \dots, \eta_n\}$ so that its transpose is semi-standard. Let us suppose that the ith column of T has length r and is filled from top to bottom by

$$(x_{i_1}, x_{i_2}, \dots, x_{i_s}, \eta_{j_1}, \dots, \eta_{j_t}). \tag{3.8}$$

We associate to (3.8) the following determinant:

$$\Delta_i^T := \det \begin{pmatrix} x_{i_1}^1 & x_{i_1}^2 & \cdots & x_{i_1}^r \\ x_{i_2}^1 & x_{i_2}^2 & \cdots & x_{i_2}^r \\ \vdots & \vdots & \cdots & \vdots \\ x_{i_s}^1 & x_{i_s}^2 & \cdots & x_{i_s}^r \\ \eta_{j_1}^1 & \eta_{j_1}^2 & \cdots & \eta_{j_1}^r \\ \eta_{j_2}^1 & \eta_{j_2}^2 & \cdots & \eta_{j_2}^r \\ \vdots & \vdots & \cdots & \vdots \\ \eta_{j_t}^1 & \eta_{j_t}^2 & \cdots & \eta_{j_t}^r \end{pmatrix}, \tag{3.9}$$

where $r = s + t$. We set $\Delta^T = \prod_{i=1}^{\lambda_1} \Delta_i^T$.

Theorem 3.3. *The set $\{\Delta^T\}$, with T running over all $(m|n)$-semi-standard tableaux of shape λ, is a basis for the space of $gl(d)$-highest weight vectors in $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ of highest weight λ.*

Proof. It is easy to see that every Δ^T is a $gl(d)$-highest weight vector of $gl(d)$-highest weight λ. Now according to [1] the dimension of $V^{\lambda}_{m|n}$ equals the number of $(m|n)$-semi-standard tableaux of shape λ and hence it is enough to show that the set $\{\Delta^T\}$ is a linearly independent set. Now due to weight considerations it is enough to prove that the set of $\{\Delta^T\}$, where T is over all $(m|n)$-semi-standard tableaux with fixed occurrence of $\{x_1, \dots, x_m, \eta_1, \dots, \eta_n\}$, is linearly independent. We proceed by induction on the number of odd indeterminates that occur inside the T's. If that number is zero, then the conclusion of the theorem is know to be true (see e.g. [10]). Thus we may assume that at least one odd indeterminate occurs in all of the T's.

Now let η_i be the odd indeterminate appearing in all Δ^T with i minimal. Let

$$\sum_T \lambda_T \Delta^T = 0. \tag{3.10}$$

We embed $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ into $S(\mathbb{C}^d \otimes \mathbb{C}^{m+1|n})$ so that we may regard (3.10) as a sum in $S(\mathbb{C}^d \otimes \mathbb{C}^{m+1|n})$. We apply to (3.10) the linear map

$$A = \sum_{j=1}^{d} x^j_{m+1} \frac{\partial}{\partial \eta^j_i}.$$

It is clear that the resulting sum is of the form

$$\sum_T \sum_{S \in \Lambda_T} \lambda_S \Delta^S,$$

where Λ_T is the set of all tableaux obtained from T by replacing one of the η_i-nodes by an x_{m+1}-node. We may assume that all S are $(m+1|n)$-semi-standard with one less odd node. Furthermore each λ_S is a non-zero positive integral multiple of λ_T. (Note that $\lambda_S = p\lambda_S$ if and only if η_i appears with multiplicity p in some column and S is obtained from T by replacing the first η_i node of this column by x_{m+1}.)

We claim that all S are distinct $(m+1|n)$-semi-standard tableaux and thus by induction they are linearly independent. This implies $\lambda_S = 0$ and hence $\lambda_T = 0$ and we are done.

In order to prove the claim we consider two cases.

In the first case suppose that S and S' are obtained from the same T. But in this case S and S' are obviously different, since S and S' are obtained from T by replacing η_i by x_{m+1} in different columns.

Now suppose that S and S' are obtained from T and T', respectively, and $T \neq T'$. If the positions of η_i in T and T' are the same, then T and T' differ at some η_s node, $i \neq s$. But then S and S' also differ at this particular η_s-node as well. If on the other hand T and T' differ at some η_i node, then this means that T at a node has η_i, while at the same node T' has some η_s, $i \neq s$, or x_k, $k \neq m+1$. But then in all of S' this particular node is always η_s or x_k, while in all S this particular node is either η_i or x_{m+1}. Thus S and S' must be distinct. $\square$

Let λ be a Young diagram lying in the $(m|n)$-hook of with $l(\lambda) \leqslant d$ and T be an $(m|n)$-semi-standard tableau of shape λ. We may fill the boxes of the Young diagram λ from the set $\{x^1, \ldots, x^d\}$ in a way so that the resulting tableau T' is semi-standard. Let the ith column of T' be filled by $\{x^{k_1}, \ldots, x^{k_r}\}$. Suppose that a joint $gl(m|n) \times gl(d)$-highest weight vector is of the form (3.7). We may replace the upper indices $1, 2, \ldots, r$ of all the entries in Δ_{i,λ'_i} (or $\Delta_{\lambda'_i}$) by $k_1, k_2, \ldots, k_r$. Let us call the resulting determinant $\Delta^{T'}_{i,\lambda'_i}$ (or $\Delta^{T'}_{\lambda'_i}$) and consider the following product of determinant.

$$\prod_{k=1}^{v} \Delta^{T'}_{k,\lambda'_k} \prod_{j=v+1}^{\lambda_1} \Delta^{T'}_{\lambda'_j}.$$

It is clear from symmetry between the upper and lower indices that when T' ranges over all semi-standard tableaux we obtain a basis for the $gl(m|n)$-highest weight vectors of highest weight λ in $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$.

Now let λ be a Young diagram lying in the $(m|n)$-hook of with $l(\lambda) \leqslant d$ and T and T' as before. Let the ith column of T' and T be filled by

$$\{x^{k_1}, \dots, x^{k_r}\}$$

$$(x_{i_1}, x_{i_2}, \dots, x_{i_s}, \eta_{j_1}, \dots, \eta_{j_t}),$$

respectively, from top to the bottom. To the ith column of the bi-tableau (T, T') we associate the following determinant:

$$\Delta_i^{(T,T')} := \det \begin{pmatrix} x_{i_1}^{k_1} & x_{i_1}^{k_2} & \cdots & x_{i_1}^{k_r} \\ x_{i_2}^{k_1} & x_{i_2}^{k_2} & \cdots & x_{i_2}^{k_r} \\ \vdots & \vdots & \cdots & \vdots \\ x_{i_s}^{k_1} & x_{i_s}^{k_2} & \cdots & x_{i_s}^{k_r} \\ \eta_{j_1}^{k_1} & \eta_{j_1}^{k_2} & \cdots & \eta_{j_1}^{k_r} \\ \eta_{j_2}^{k_1} & \eta_{j_2}^{k_2} & \cdots & \eta_{j_2}^{k_r} \\ \vdots & \vdots & \cdots & \vdots \\ \eta_{j_t}^{k_1} & \eta_{j_t}^{k_2} & \cdots & \eta_{j_t}^{k_r} \end{pmatrix}, \tag{3.11}$$

where again $r = s + t$. We set $\Delta^{(T,T')} = \prod_{i=1}^{\lambda_1} \Delta_i^{(T,T')}$. The following theorem gives an explicit basis for each irreducible $gl(d) \times gl(m|n)$-component in $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$.

Theorem 3.4. *The set $\Delta^{(T,T')}$, where T' is semi-standard in $\{x^1, \dots, x^d\}$ and T is $(m|n)$-semi-standard in $\{x_1, \dots, x_m, \eta_1, \dots, \eta_n\}$, is a basis for $V_d^\lambda \otimes V_{m|n}^\lambda$ in $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$.*

Proof. Given $\Delta^{(T,T')}$ with (T, T') fixed. By Theorem 3.3 and the Jacobson density theorem (more precisely by Burnside's theorem) we can find an element $(a \otimes b) \in U(gl(d)) \otimes U(gl(m|n))$ such that $(a \otimes b)\Delta^{(T,T')}$ is the joint $gl(d) \times gl(m|n)$-highest weight vector and $a \otimes b$ annihilates all $\Delta^{(S,S')}$, for $(S, S') \neq (T, T')$. This implies that the set $\{\Delta^{(T,T')}\}$ is linearly independent. But the number of semi-standard tableaux in $\{x^1, \dots, x^d\}$ times the number of $(m|n)$-semi-standard tableaux in $\{x_1, \dots, x_m, \eta_1, \dots, \eta_n\}$ is precisely the dimension of the space $V_d^\lambda \otimes V_{m|n}^\lambda$. $\square$

Remark 3.1. The above theorem is known in the case when $n = 0$ (see e.g. [10]).

4. The (O, spo)- and (Sp, osp)-duality

Let $\mathbb{C}^d$ be the d-dimensional complex vector space with standard basis $\{e^1, e^2, \dots, e^d\}$. Let $O(d)$ be the orthogonal group leaving invariant the symmetric bilinear form $(\cdot|\cdot)$ as in Section 2, and let $\mathbb{C}^{m|n}$ be the superspace of dimension $(m|n)$. The natural action of $O(d)$ on $\mathbb{C}^d$ extends to an action on $\mathbb{C}^d \otimes \mathbb{C}^{m|n}$. This action gives rise to an action of $O(d)$ on the supersymmetric tensor $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$, which, as in Section 3, we identify with $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$, the commutative superalgebra in (3.2). As the action $gl(d)$ under this identification gets identified with certain first-order differential operators as in (3.3), the action of the Lie algebra of $O(d)$ naturally gets identified with certain first-order differential operators as well.

Consider the following first-order differential operators:

$$E^{xx}_{is} = \sum_{j=1}^{d} x^j_i \frac{\partial}{\partial x^j_s} + \frac{d}{2}\delta_{is}, \quad E^{x\eta}_{ik} = \sum_{j=1}^{d} x^j_i \frac{\partial}{\partial \eta^j_k},$$

$$E^{\eta x}_{ki} = \sum_{j=1}^{d} \eta^j_k \frac{\partial}{\partial x^j_i}, \quad E^{\eta\eta}_{tk} = \sum_{j=1}^{d} \eta^j_t \frac{\partial}{\partial \eta^j_k} - \frac{d}{2}\delta_{ik}, \tag{4.1}$$

where $i, s = 1, \dots, m$ and $k, t = 1, \dots, n$. It is evident that they form a basis for the Lie superalgebra $gl(m|n)$ and it is clear that $O(d)$ commutes with $gl(m|n)$.

Next consider another set of operators on $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$.

$${}^O I^{xx}_{is} = \sum_{j=1}^{d} x^j_i x^{d+1-j}_s, \quad {}^O I^{x\eta}_{ik} = \sum_{j=1}^{d} x^j_i \eta^{d+1-j}_k, \quad {}^O I^{\eta\eta}_{kt} = \sum_{j=1}^{d} \eta^j_k \eta^{d+1-j}_t,$$

$${}^O\Delta^{xx}_{is} = \sum_{j=1}^{d} \frac{\partial}{\partial x^j_i}\frac{\partial}{\partial x^{d+1-j}_s}, \quad {}^O\Delta^{x\eta}_{ik} = \sum_{j=1}^{d} \frac{\partial}{\partial x^j_i}\frac{\partial}{\partial \eta^{d+1-j}_k}, \quad {}^O\Delta^{\eta\eta}_{kt} = \sum_{j=1}^{d} \frac{\partial}{\partial \eta^j_k}\frac{\partial}{\partial \eta^{d+1-j}_t},$$

where $1 \leqslant i \leqslant s \leqslant m$ and $1 \leqslant k < t \leqslant n$. We note that these operators also commute with the action of $O(d)$ on $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$. It is not hard to see that these operators together with (4.1) form a basis of the symplectic-orthogonal Lie superalgebra $spo(2m|2n)$. In fact, using the $\mathbb{Z}$-gradation of $spo(2m|2n)$ given in Section 2, we have $\mathfrak{g}_1 = \sum \mathbb{C}\,{}^O\Delta^{xx}_{is} + \sum \mathbb{C}\,{}^O\Delta^{x\eta}_{ik} + \sum \mathbb{C}\,{}^O\Delta^{\eta\eta}_{kl}$ and $\mathfrak{g}_{-1} = \sum \mathbb{C}\,{}^O I^{xx}_{is} + \sum \mathbb{C}\,{}^O I^{x\eta}_{ik} + \sum \mathbb{C}\,{}^O I^{\eta\eta}_{kl}$. Thus on $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ we have an action of $O(d) \times spo(2m|2n)$.

An element $f \in \mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ will be called *${}^O\Delta$-harmonic*, if ${}^O\Delta^{xx}_{is} f = {}^O\Delta^{x\eta}_{ik} f = {}^O\Delta^{\eta\eta}_{kl} f = 0$. The space of ${}^O\Delta$-harmonics will be denoted by ${}^O H$. Note that since $[gl(m|n), \mathfrak{g}_1] \subseteq \mathfrak{g}_1$ the space ${}^O H$ is invariant under the action of $gl(m|n)$. Also ${}^O H$ is clearly invariant under the action of $O(d)$. Hence we have an action of $O(d) \times gl(m|n)$ on ${}^O H$. Let ${}^O I$ be the subalgebra of $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ generated by ${}^O I^{xx}_{is}$, ${}^O I^{x\eta}_{ik}$ and ${}^O I^{\eta\eta}_{kl}$. It is clear that ${}^O I$ is the subalgebra of $O(d)$-invariants in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$. We have the following theorem.

Theorem 4.1 (Howe [13]). *The pairs* $(O(d), spo(2m|2n))$ *and* $(O(d), gl(m|n))$ *form dual reductive Howe pairs on* $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ *and on* ${}^{O}H$, *respectively. Thus we have*

$$\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}] \cong \sum_{\lambda} V^{\lambda}_{O(d)} \otimes V^{\lambda'}_{spo(2m|2n)},$$

$$^{O}H \cong \sum_{\lambda} V^{\lambda}_{O(d)} \otimes V^{\lambda''}_{m|n},$$

where λ *is summed over a set of irreducible* $O(d)$*-highest weights. Here* λ' *and* λ'' *are certain non-isomorphic irreducible* $spo(2m|2n)$*- and* $gl(m|n)$*-highest weights, respectively. Furthermore the map* ${}^{O}I \otimes {}^{O}H \to \mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ *given by multiplication is surjective and we have, for each* λ, $V^{\lambda'}_{spo(2m|2n)} = {}^{O}IV^{\lambda''}_{m|n}$.

Let d be an even integer and consider the d-dimensional complex vector space with the standard basis $e^1, e^2, \ldots, e^d$ and equipped with the non-degenerate skew-symmetric bilinear form $\langle \cdot | \cdot \rangle$ as in Section 2. Let $Sp(d)$ be the corresponding symplectic group. Again we have an action of $Sp(d)$ on $\mathbb{C}^d \otimes \mathbb{C}^{m|n}$, inducing an action of $Sp(d)$ on the supersymmetric tensor $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$, which we again identify with $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$.

Introduce the following operators:

$$^{Sp}I^{xx}_{is} = \sum_{j=1}^{\frac{d}{2}} (x^j_i x^{d+1-j}_s - x^{d+1-j}_i x^j_s), \quad ^{Sp}I^{x\eta}_{ik} = \sum_{j=1}^{\frac{d}{2}} (x^j_i \eta^j_k - x^{d+1-j}_i \eta^{d+1-j}_k),$$

$$^{Sp}I^{\eta\eta}_{kt} = \sum_{j=1}^{\frac{d}{2}} (\eta^j_k \eta^{d+1-j}_t - \eta^{d+1-j}_k \eta^j_t), \quad ^{Sp}\Delta^{xx}_{is} = \sum_{j=1}^{\frac{d}{2}} \left(\frac{\partial}{\partial x^j_i} \frac{\partial}{\partial x^{d+1-j}_s} - \frac{\partial}{\partial x^{d+1-j}_i} \frac{\partial}{\partial x^j_s} \right),$$

$$^{Sp}\Delta^{x\eta}_{ik} = \sum_{j=1}^{\frac{d}{2}} \left(\frac{\partial}{\partial x^j_i} \frac{\partial}{\partial \eta^{d+1-j}_k} - \frac{\partial}{\partial x^{d+1-j}_i} \frac{\partial}{\partial \eta^j_k} \right),$$

$$^{Sp}\Delta^{\eta\eta}_{kt} = \sum_{j=1}^{\frac{d}{2}} \left(\frac{\partial}{\partial \eta^j_k} \frac{\partial}{\partial \eta^{d+1-j}_t} - \frac{\partial}{\partial \eta^{d+1-j}_k} \frac{\partial}{\partial \eta^j_t} \right),$$

where $1 \leqslant i < s \leqslant m$ and $1 \leqslant k \leqslant t \leqslant n$. It is again not hard to see that these operators together with (4.1) form a basis for the Lie superalgebra $osp(2m|2n)$ and their actions and that of $Sp(d)$ on $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ commute.

An element $f \in \mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ will be called ${}^{Sp}\Delta$*-harmonic*, if ${}^{Sp}\Delta^{xx}_{is}f = {}^{Sp}\Delta^{x\eta}_{ik}f = {}^{Sp}\Delta^{\eta\eta}_{kl}f = 0$. The space of ${}^{Sp}\Delta$-harmonics will be denoted by ${}^{Sp}H$. Similarly, we have an action of $Sp(d) \times gl(m|n)$ on ${}^{Sp}H$. Let ${}^{Sp}I$ be the subalgebra of $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ generated by ${}^{Sp}I^{xx}_{is}$, ${}^{Sp}I^{x\eta}_{ik}$ and ${}^{Sp}I^{\eta\eta}_{kl}$ so that ${}^{Sp}I$ is the subalgebra of $Sp(d)$-invariants in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$. In a similar fashion we have the following theorem.

Theorem 4.2 (Howe [13]). *The pairs $(Sp(d), osp(2m|2n))$ and $(Sp(d), gl(m|n))$ form Howe dual reductive pairs on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ and on ${}^{Sp}H$, respectively. Therefore we have a decomposition of modules*

$$\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}] \cong \sum_{\lambda} V^{\lambda}_{Sp(d)} \otimes V^{\lambda'}_{osp(2m|2n)},$$

$${}^{Sp}H \cong \sum_{\lambda} V^{\lambda}_{Sp(d)} \otimes V^{\lambda''}_{m|n},$$

where λ is summed over a set of irreducible $Sp(d)$-highest weights. Here λ' and λ'' are certain non-isomorphic irreducible $osp(2m|2n)$- and $gl(m|n)$-highest weights, respectively. Furthermore the map ${}^{Sp}I \otimes {}^{Sp}H \to \mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ given by multiplication is surjective and we have, for each λ, $V^{\lambda'}_{osp(2m|2n)} = {}^{Sp}IV^{\lambda''}_{m|n}$.

The proofs of Theorems 4.1 and 4.2 are based on the fact that the invariants of the classical group of the corresponding dual pair in the endomorphism ring of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ are generated by quadratic invariants. Although in [13] it is shown that the pairs $(O(d), spo(2m|2n))$ and $(Sp(d), osp(2m|2n))$ are indeed dual pairs on $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$, the explicit decomposition of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ was not given. We will embark on this task in Section 5.

We conclude this section by showing that the representations of $spo(2m|2n)$ and $osp(2m|2n)$ that appear in Theorems 4.1 and 4.2 are unitarizable. We first recall some definitions.

Let A be a superalgebra and † an anti-linear map with $(ab)^\dagger = b^\dagger a^\dagger$, for a, b in A. We call † an anti-linear *anti-involution* if $(a^\dagger)^\dagger = a$. Now let A be a superalgebra equipped with an anti-linear anti-involution † and let V be an A-module. A Hermitian form $(\cdot|\cdot)$ on V is said to be *contravariant* if $(av|w) = (v|a^\dagger w)$, for $a \in A$ and $v, w \in V$. If furthermore $(\cdot|\cdot)$ is positive-definite, then V is said to be a *unitarizable* A-module. We remark here that we have defined the anti-involution and the contravariant form without "super signs". It follows then that any unitarizable module is completely reducible.

Proposition 4.1. *The representations $V^{\lambda'}_{spo(2m|2n)}$ and $V^{\lambda'}_{osp(2m|2n)}$ that occur in the decompositions of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ are unitarizable.*

Proof. We need to construct a contravariant positive-definite Hermitian form on $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$. We proceed as follows. First note that the space $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ is an irreducible representation of the direct sum of a Heisenberg algebra and a Clifford superalgebra with generators mapped to x_i^j, η_k^j, $\frac{\partial}{\partial x_i^j}$, $\frac{\partial}{\partial \eta_k^j}$, for $i = 1, \dots, m$, $k = 1, \dots, n$ and $j = 1, \dots, d$, and 1. Identifying this superalgebra with its image we have an anti-linear anti-involution given by

$$(x_i^j)^\dagger = \frac{\partial}{\partial x_i^j}, \quad \left(\frac{\partial}{\partial x_i^j}\right)^\dagger = x_i^j, \quad (\eta_k^j)^\dagger = \frac{\partial}{\partial \eta_k^j}, \quad \left(\frac{\partial}{\partial \eta_k^j}\right)^\dagger = \eta_k^j, \quad 1^\dagger = 1.$$

This gives rise to a unique contravariant Hermitian form $(\cdot|\cdot)$ on $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ with $(1|1) = 1$. Furthermore for any non-zero monomial $f \in \mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ we have $(f|f) > 0$, and hence $(\cdot|\cdot)$ is positive-definite. Therefore $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$, as a representation of the Clifford superalgebra, is unitarizable.

Now it is easy to see, using (4.1) along with the formulas for ${}^{Sp}I$, ${}^{Sp}\Delta$ and ${}^{O}I$, ${}^{O}\Delta$ in this section, that $osp(2m|2n)$ and $spo(2m|2n)$ are invariant under the anti-involution †. This implies that the representations of $osp(2m|2n)$ and $spo(2m|2n)$ on $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ are unitarizable. $\square$

5. Joint highest weight vectors

In this section we will describe the explicit decomposition of the space $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ under the joint actions of the relevant dual pairs. We will do so by explicitly finding a joint highest weight vector for each irreducible component.

5.1. *The case of* (O, spo)*-duality*

Consider the $(O(d), spo(2m|2n))$-duality on the space $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. Using the notation from Section 4 we make the identification of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ with the polynomial superalgebra $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ so that the Lie algebra $so(d)$ and $spo(2m|2n)$ are identified with differential operators.

By Theorem 4.1 we only need to find the decomposition of the space of harmonic polynomials ${}^{O}H$ into irreducible $O(d) \times gl(m|n)$-modules. By Theorem 3.2 $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ as a $gl(d) \times gl(m|n)$-module decomposes into $\sum_\lambda V_d^\lambda \otimes V_{m|n}^\lambda$, where the summation is over all partitions λ with $l(\lambda) \leqslant d$ and $\lambda_{m+1} \leqslant n$.

Consider first the case when $m \geqslant \frac{d}{2}$. Take a diagram λ with $\lambda_1' + \lambda_2' \leqslant d$ and let v_λ be the corresponding joint $gl(d) \times gl(m|n)$-highest weight vector in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ of the form in Theorems 3.1 or 3.2. Note that in this case it is automatic that $\lambda_{m+1} \leqslant n$, as long as $n \geqslant 1$.

Here and further we use $\mathbf{\frac{1}{2}}$ to denote the $(m+n)$-tuple $(\frac{1}{2}, \dots, \frac{1}{2}; -\frac{1}{2}, \dots, -\frac{1}{2})$. That is, the first m entries are $\frac{1}{2}$, while the last n entries are $-\frac{1}{2}$.

Proposition 5.1. *Suppose that $n \geqslant 1$ and $m \geqslant \frac{d}{2}$. The vector v_λ is ${}^O\Delta$-harmonic of $O(d)$-weight corresponding to the diagram λ. Therefore*

$$ {}^O H \cong \sum_\lambda V^\lambda_{O(d)} \otimes V^{\lambda + d\mathbf{\frac{1}{2}}}_{m|n}, $$

where λ ranges over all diagrams with $\lambda'_1 + \lambda'_2 \leqslant d$. Here the weight $\lambda + d\mathbf{\frac{1}{2}}$ denotes the sum of the $gl(m|n)$-weight corresponding to the Young diagram λ with the $(m+n)$-tuple $d\mathbf{\frac{1}{2}}$.

Proof. Note that by our choice of the Borel subalgebra of $so(d)$, it is automatic that v_λ is an $O(d) \times gl(m|n)$-highest weight vector. (In fact this is true for any λ.) Thus in order to show that v_λ is ${}^O\Delta$-harmonic it is enough to show that it is annihilated by ${}^O\Delta^{xx}_{11} = \sum_{j=1}^d \frac{\partial}{\partial x^j_1} \frac{\partial}{\partial x^{d+1-j}_1}$. This is because v_λ is already annihilated by the nilpotent radical of the Borel subalgebra of $gl(m|n)$, which together with ${}^O\Delta^{xx}_{11}$ generates the nilpotent radical of the Borel subalgebra of $spo(2m|2n)$. Also note that if $\lambda'_1 \leqslant m$, then the joint highest weight vector is the usual joint highest weight vector in the classical $O(d) \times sp(2m)$-duality and hence is killed by ${}^O\Delta^{xx}_{11}$ [14]. So we may assume that $\lambda'_1 > m$. In this case in order to show that ${}^O\Delta^{xx}_{11} v_\lambda = 0$, we consider the classical $O(d) \times sp(2m+2n)$-duality. Here the joint highest weight vector w_λ of $O(d) \times sp(2m+2n)$ is a product of determinants of the form $\Delta_{\lambda'_1}\Delta_{\lambda'_2} \cdots \Delta_{\lambda'_{\lambda_1}}$, with only λ'_1 exceeding m. That is, only in $\Delta_{\lambda'_1}$ can we possibly have variables of the form x^j_{m+i} with $i = 1, \dots, n$. From the duality in the classical case we know that w_λ is harmonic [14] and hence in particular ${}^O\Delta^{xx}_{11} w_\lambda = 0$. Consider the first-order differential operators $\Gamma_i = \sum_{j=1}^d \eta^j_1 \frac{\partial}{\partial x^j_{m+i}}$, for $i = 1, \dots, n$. We see that ${}^O\Delta^{xx}_{11}$ commutes with all Γ_i and hence

$$ 0 = \Gamma_1 \cdots \Gamma_{\lambda'_1 - m} {}^O\Delta^{xx}_{11} w_\lambda = {}^O\Delta^{xx}_{11} \Gamma_1 \cdots \Gamma_{\lambda'_1 - m} w_\lambda. $$

But $\Gamma_1 \cdots \Gamma_{\lambda'_1 - m} w_\lambda = (-1)^{\lambda'_1 - m - 1} (\lambda'_1 - m)! v_\lambda$ and hence ${}^O\Delta^{xx}_{11} v_\lambda = 0$.

Finally the addition of $d\mathbf{\frac{1}{2}}$ to the $gl(m|n)$-highest weight λ is of course due to (4.1). $\square$

As in [14] one shows that v_λ indeed has $O(d)$-weight corresponding to the Young diagram λ. But as λ ranges over all partitions with $\lambda'_1 + \lambda'_2 \leqslant d$ we conclude that the v_λ's generate the complete set of all finite-dimensional irreducible $O(d)$-modules. Due to $O(d) \times gl(m|n)$-duality in ${}^O H$ we see that

$$ {}^O H \cong \sum_\lambda V^\lambda_{O(d)} \otimes V^{\lambda + d\mathbf{\frac{1}{2}}}_{m|n}, $$

where λ ranges over all Young diagrams with $\lambda'_1 + \lambda'_2 \leqslant d$.

Proposition 5.2. *Suppose that $n \geqslant 1$ and $m \geqslant \frac{d}{2}$. Then as an $O(d) \times spo(2m|2n)$-module we have the following decomposition.*

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \cong \sum_{\lambda} V^{\lambda}_{O(d)} \otimes V^{\lambda + d\mathbf{\frac{1}{2}}}_{spo(2m|2n)},$$

where λ ranges over all Young diagrams with $\lambda'_1 + \lambda'_2 \leqslant d$.

Now consider the case when $m < \frac{d}{2}$. For this case we introduce new even variables so that the total number of even variables is at least $\frac{d}{2}$. Since the case when d is odd is analogous we assume for simplicity that d is even and we add new variables $x^j_{m+1}, \dots, x^j_{\frac{d}{2}}$, $j = 1, \dots, d$, to the polynomial superalgebra $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ and denote the resulting superalgebra by $\mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$. That is, we are considering the embedding $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \subseteq S(\mathbb{C}^d \otimes \mathbb{C}^{\frac{d}{2}|n})$. Without further mentioning we adopt the convention of adding a $'$ to operators, vectors, etc., when we are regarding them as over $S(\mathbb{C}^d \otimes \mathbb{C}^{\frac{d}{2}|n})$. So for example we denote the corresponding Laplacian of $\mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$ by ${}^O\Delta'^{xx}_{ij}$, $1 \leqslant i, j \leqslant \frac{d}{2}$, etc. and call ${}^O\Delta'$-harmonic an element $f \in \mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$ that is annihilated by all these Laplacians. We note that ${}^O\Delta^{xx}_{is} = {}^O\Delta'^{xx}_{is}$, for $1 \leqslant i, s \leqslant m$, etc. Furthermore ${}^O\Delta'^{xx}_{ij}$, with either i or j not in $\{1, \dots, m\}$, is a sum of second-order differential operators, each of them involving differentiation with respect to some of the new variables $x^j_{m+1}, \dots, x^j_{\frac{d}{2}}$ that we have introduced. It follows that if $f \in \mathbb{C}[\mathbf{x}, \boldsymbol{\eta}] \subseteq \mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$, then f is ${}^O\Delta'$-harmonic if and only if f is ${}^O\Delta$-harmonic. Thus ${}^OH \subseteq {}^OH'$.

Now in $\mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$ we know that the subspace of ${}^O\Delta'$-harmonics is

$${}^OH' = \sum_{\lambda} V^{\lambda}_{O(d)} \otimes V^{\lambda + d\mathbf{\frac{1}{2}}}_{\frac{d}{2}|n},$$

where λ ranges over all Young diagrams with $\lambda'_1 + \lambda'_2 \leqslant d$ by Proposition 5.1. Let v'_λ be a joint highest weight vector in ${}^OH'$ for the component $V^{\lambda}_{O(d)} \otimes V^{\lambda + d\mathbf{\frac{1}{2}}}_{\frac{d}{2}|n}$. Then if the first column exceeds $\frac{d}{2}$ we have up to a scalar multiple

$$v'_\lambda = \Delta_{1\lambda'_1} \Delta_{\lambda'_2} \cdots \Delta_{\lambda'_{\lambda_1}}.$$

Otherwise we have up to a scalar multiple

$$v'_\lambda = \Delta_{\lambda'_1} \Delta_{\lambda'_2} \cdots \Delta_{\lambda'_{\lambda_1}}.$$

Suppose λ is such a diagram with $\lambda_{m+1} > n$. In this case the nth column of λ exceeds m and hence $\Delta_{\lambda'_n}$ contains at least one row with entries consisting entirely of newly introduced variables. Now by Theorems 3.3 and 4.1 all the $O(d)$-highest weight vectors of highest weight λ in ${}^OH'$ are, up to a scalar multiple, of the form Δ^T, where T runs over all $(\frac{d}{2}|n)$-semi-standard tableaux. But then it is not hard to see that one of the rows in some Δ_i^T must consist entirely of newly introduced variables so that Δ^T reduces to zero when setting $x^j_{m+1} = \cdots = x^j_{\frac{d}{2}} = 0$. Since ${}^OH \subseteq {}^OH'$, this implies that there are no $O(d)$-highest weight vectors of highest weight λ with $\lambda_{m+1} > n$ in ${}^OH$.

On the other hand if $\lambda'_1 + \lambda'_2 \leqslant d$ and $\lambda_{m+1} \leqslant n$, it is quite easy to see, using Theorem 3.2, that v_λ (that is the $O(d) \times gl(m|n)$-joint highest weight vector in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$) is annihilated by ${}^O\Delta^{xx}_{11} = \sum_{j=1}^d \frac{\partial}{\partial x_1^j} \frac{\partial}{\partial x_1^{d+1-j}}$, and hence v_λ is indeed ${}^O\Delta$-harmonic. Combining the results of this section we have proved the following.

Theorem 5.1. *We have the following decomposition of ${}^OH$ as an $O(d) \times gl(m|n)$-module*:

$$ {}^OH \cong \sum_\lambda V^\lambda_{O(d)} \otimes V^{\lambda + d\mathbf{\frac{1}{2}}}_{m|n}, $$

where λ ranges over all diagrams with $\lambda'_1 + \lambda'_2 \leqslant d$ and $\lambda_{m+1} \leqslant n$. Thus as an $O(d) \times spo(2m|2n)$-module we have

$$ S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \cong \sum_\lambda V^\lambda_{O(d)} \otimes V^{\lambda + d\mathbf{\frac{1}{2}}}_{spo(2m|2n)}, $$

where λ ranges over all Young diagrams with $\lambda'_1 + \lambda'_2 \leqslant d$ and $\lambda_{m+1} \leqslant n$. Here the labels of the $spo(2m|2n)$-highest weight of $V^{\lambda + d\mathbf{\frac{1}{2}}}_{spo(2m|2n)}$ with respect to the Dynkin diagram of Section 2.4 *is obtained by applying* (2.2) *to the $gl(m|n)$-weight $\lambda + d\mathbf{\frac{1}{2}}$.*

Proof. The preceding discussion already shows that the theorem hold when $n \geqslant 1$ and $m < \frac{d}{2}$. Since in the case when $m \geqslant \frac{d}{2}$ and $n \geqslant 1$ the condition $\lambda_{m+1} \leqslant n$ is vacuous, the theorem is true in this case due to Proposition 5.2. But of course the case $n = 0$ is the well-known classical case for which the conclusion of the theorem hold as well. $\square$

Remark 5.1. Partial results on the decomposition of the space $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ with respect to the joint action of $O(d) \times spo(2m|2n)$ were obtained earlier by Nishiyama in [27] by constructing certain $O(d) \times spo(2m|2n)$-joint highest weight vectors in $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. However, the full set of such joint highest weight vectors (and hence the complete decomposition) was not obtained in there.

5.2. The case of (Sp, osp)*-duality*

Now consider the action of the dual pair $(Sp(d), osp(2m|2n))$ on the space $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. The procedure is similar to that of Section 5.1.

In view of Theorem 4.2 we again only need to find the decomposition of the space ${}^{Sp}H$ with respect to the joint action of $Sp(d) \times gl(m|n)$. According to Theorem 3.2 $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ as a $gl(d) \times gl(m|n)$-module decomposes into $\sum_\lambda V_d^\lambda \otimes V_{m|n}^\lambda$, where the summation is over all partitions λ with $l(\lambda) \leqslant d$ and $\lambda_{m+1} \leqslant n$.

Let us first consider the case when $m \geqslant \frac{d}{2}$. We take a Young diagram λ with $l(\lambda) \leqslant \frac{d}{2}$ so that the condition $\lambda_{m+1} \leqslant n$ here is automatic. We recall from Section 2 that the finite-dimensional irreducible representations of $Sp(d)$ are parameterized by diagrams with length not exceeding $\frac{d}{2}$.

Proposition 5.3. *Suppose that $m \geqslant \frac{d}{2}$ and let λ be a diagram with $l(\lambda) \leqslant \frac{d}{2}$. Let $v_\lambda \in V_d^\lambda \otimes V_{m|n}^\lambda$ be a $gl(d) \times gl(m|n)$-joint highest weight vector in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$. Then v_λ is ${}^{Sp}\Delta$-harmonic of $Sp(d)$-weight corresponding to the diagram λ. Therefore*

$$ {}^{Sp}H \cong \sum_\lambda V_{Sp(d)}^\lambda \otimes V_{m|n}^{\lambda + d\mathbf{\frac{1}{2}}}, $$

where λ ranges over all diagrams with $l(\lambda) \leqslant \frac{d}{2}$. Here $\lambda + d\mathbf{\frac{1}{2}}$ denotes the sum of the $gl(m|n)$-weight corresponding to λ and the $(m+n)$-tuple $d\mathbf{\frac{1}{2}}$.

Proof. Since $m \geqslant \frac{d}{2}$ and $l(\lambda) \leqslant \frac{d}{2}$, the $gl(d) \times gl(m|n)$-joint highest weight vector is of the form $v_\lambda = \prod_{i=1}^{\lambda_1} \Delta_{\lambda_i'}$, that is, only the $\mathbf{x}$ variables are involved. Since the Borel subalgebra of $sp(d)$ is contained in the standard Borel subalgebra of $gl(d)$, v_λ is an $sp(d) \times gl(m|n)$-highest weight vector. We need to show that it is ${}^{Sp}\Delta$-harmonic. For this it is again sufficient to show that v_λ is annihilated by ${}^{Sp}\Delta_{12}^{xx} = \sum_{j=1}^{\frac{d}{2}} \left(\frac{\partial}{\partial x_1^j} \frac{\partial}{\partial x_2^{d+1-j}} - \frac{\partial}{\partial x_1^{d+1-j}} \frac{\partial}{\partial x_2^j}\right)$. But this is clear by the classical $Sp(d) \times so(2m)$-duality [14], because the formulas for the joint highest weight vector v_λ and for the Laplacian ${}^{Sp}\Delta_{12}^{xx}$ in the classical case are identical with our formulas here. Now the proposition follows from Theorem 4.2 together with the fact that we have constructed an $Sp(d)$-highest weight vector corresponding to every finite-dimensional irreducible $Sp(d)$-module. □

We now consider the case $m < \frac{d}{2}$. In this case the condition $\lambda_{m+1} \leqslant n$ is not an empty condition. Here we can apply the idea of Section 4.1 by inserting enough new even variables $x_{m+1}^j, \ldots, x_{\frac{d}{2}}^j$ and consider the $Sp(d) \times osp(d|2n)$-duality on the space $S(\mathbb{C}^d \otimes \mathbb{C}^{\frac{d}{2}|n})$. We identify $S(\mathbb{C}^d \otimes \mathbb{C}^{\frac{d}{2}|n})$ with $\mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$ as before and regard $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}] \subset \mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$. Again we will use $'$ to distinguish elements in $\mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$ from elements in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$. As in Section 4.1 it is easy to see that an element $f \in \mathbb{C}[x, \boldsymbol{\eta}] \subset \mathbb{C}[\mathbf{x}', \boldsymbol{\eta}]$ is ${}^{Sp}\Delta$-harmonic if and only if it is ${}^{Sp}\Delta'$-harmonic and therefore ${}^{Sp}H \subset {}^{Sp}H'$.

Now by Proposition 5.3 we have

$$^{Sp}H' = \sum_{\lambda} V^{\lambda}_{Sp(d)} \otimes V^{\lambda+d\mathbf{\frac{1}{2}}}_{\frac{d}{2}|n},$$

where λ is summed over all partitions of length $l(\lambda) \leqslant \frac{d}{2}$. A joint highest weight vector v_λ is given by $\prod_{i=1}^{\lambda_1} \Delta_{\lambda'_i}$ and hence by Theorem 3.3, the set of Δ^T's, where T runs over all $(\frac{d}{2}|n)$-semi-standard tableaux of shape λ, is a basis for the space of $Sp(d)$-highest weight vectors of highest weight λ in $^{Sp}H'$.

Now suppose that $\lambda_{m+1} > n$. Let T be a $(\frac{d}{2}|n)$-semi-standard tableau and $\Delta^T = \prod_i \Delta_i^T$. It is clear that in this case one of the Δ_i^T's must contain a row consisting entirely of newly introduced variables. But then this means that, by setting these newly introduced variables equal to zero, Δ^T is zero. This implies that in ^{Sp}H there are no $Sp(d)$-highest weight vectors of highest weight λ, and hence no $Sp(d)$-module of the form $V^{\lambda}_{Sp(d)}$ can occur in the decomposition of ^{Sp}H with respect to the action of $Sp(d)$.

The above argument combined with Proposition 5.3 gives the complete description of the $Sp(d) \times osp(2m|2n)$-duality on the space $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$, which we summarize in the following theorem.

Theorem 5.2. *We have the following decomposition of ^{Sp}H as an $Sp(d) \times gl(m|n)$-module*:

$$^{Sp}H \cong \sum_{\lambda} V^{\lambda}_{Sp(d)} \otimes V^{\lambda+d\mathbf{\frac{1}{2}}}_{m|n},$$

where λ ranges over all diagrams with $l(\lambda) \leqslant \frac{d}{2}$ and $\lambda_{m+1} \leqslant n$. Thus as an $Sp(d) \times osp(2m|2n)$-module we have

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \cong \sum_{\lambda} V^{\lambda}_{Sp(d)} \otimes V^{\lambda+d\mathbf{\frac{1}{2}}}_{osp(2m|2n)},$$

where λ ranges over all Young diagrams with $l(\lambda) \leqslant \frac{d}{2}$ and $\lambda_{m+1} \leqslant n$. Here the labels of the $osp(2m|2n)$-highest weight of $V^{\lambda+d\mathbf{\frac{1}{2}}}_{osp(2m|2n)}$ with respect to the Dynkin diagram of Section 2.4 *is obtained by applying* (2.1) *to the $gl(m|n)$-weight $\lambda + d\mathbf{\frac{1}{2}}$.*

Proof. Let λ be a diagram with $l(\lambda) \leqslant \frac{d}{2}$ and $\lambda_{m+1} \leqslant n$. In view of the discussion above and Proposition 5.3 it remains to prove that in the case when $m < \frac{d}{2}$, the $gl(d) \times gl(m|n)$-joint highest weight vector in $\mathbb{C}[\mathbf{x}, \boldsymbol{\eta}]$ is indeed $^{Sp}\Delta$-harmonic. For this it is enough to show that it is annihilated by $^{Sp}\Delta_{12}^{xx} = \sum_{j=1}^{\frac{d}{2}} (\frac{\partial}{\partial x_1^j} \frac{\partial}{\partial x_2^{d+1-j}} - \frac{\partial}{\partial x_1^{d+1-j}} \frac{\partial}{\partial x_2^j})$.

But this is easy to see using the formula for such a joint highest weight vector given in Theorem 3.2. $\square$

6. Character formulas for irreducible unitarizable $spo(2m|2n)$- and $osp(2m|2n)$-modules

In this section we give combinatorial character formulas for the $spo(2m|2n)$- and $osp(2m|2n)$-representations that appear in the decomposition of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ of Section 5. We shall need a result of Enright [8] which we shall recall. Before this we need some preparatory material.

Consider a Hermitian symmetric pair (G, K), where G is a real classical simple Lie group. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the corresponding complexified Lie algebras. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ so that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ so that $\mathfrak{q} = \mathfrak{k} + \mathfrak{b}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ with abelian radical $\mathfrak{u}$. Hence as a vector space we have $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{u}$. Denote by Δ and $\Delta(\mathfrak{k})$ the root systems of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{h})$, respectively, and let Δ_+ be the set of positive roots determined by $\mathfrak{b}$. Furthermore set $\Delta(\mathfrak{k})_+ = \Delta_+ \cap \Delta(\mathfrak{k})$ and let ρ and $\rho_{\mathfrak{k}}$ denote the respective half sums of positive roots. Also let $\Delta(\mathfrak{u}) = \{\alpha \in \Delta | \mathfrak{g}_\alpha \subseteq \mathfrak{u}\}$ and put $\rho_{\mathfrak{u}} = \frac{1}{2}\sum_{\alpha \in \mathfrak{u}} \alpha$. Let W and $W(\mathfrak{k})$ denote the Weyl groups of $\mathfrak{g}$ and $\mathfrak{k}$, respectively.

Now to each $\lambda \in \mathfrak{h}^*$ one can associate a subgroup W_λ of W. Since we will need to explicitly compute W_λ later on, we will give a detailed description of it now. The group W_λ is the subgroup of W generated by the reflections s_α, where $\alpha \in \Delta(\mathfrak{u})$ satisfying the following three conditions [8,7]:

(i) $\langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{N}$.
(ii) If for some $\beta \in \Delta$ we have $(\lambda + \rho | \beta) = 0$, then $(\alpha | \beta) = 0$.
(iii) If for some long root $\beta \in \Delta$ we have $(\lambda + \rho | \beta) = 0$, then α is a short root.

Associated to W_λ one may define a root system Δ_λ consisting of the roots $\gamma \in \Delta$ such that s_γ lies in W_λ. Now we set $\Delta_\lambda(\mathfrak{k}) = \Delta_\lambda \cap \Delta(\mathfrak{k})$, $\Delta_{\lambda+} = \Delta_+ \cap \Delta_\lambda$ and $\Delta_\lambda(\mathfrak{k})_+ = \Delta_\lambda(\mathfrak{k}) \cap \Delta_{\lambda+}$. The group $W_\lambda(\mathfrak{k})$ is defined to be the subgroup of W_λ generated by reflection along the roots lying in $\Delta_\lambda(\mathfrak{k})_+$. We have a decomposition of the group $W_\lambda \cong W_\lambda(\mathfrak{k}) \times W_\lambda^{\mathfrak{k}}$, where

$$W_\lambda^{\mathfrak{k}} = \{w \in W_\lambda | \langle w\rho, \check{\alpha} \rangle \in \mathbb{Z}_+, \ \forall \alpha \in \Delta_\lambda(\mathfrak{k})_+\}. \tag{6.1}$$

Remark 6.1. Note that our definition of W_λ is actually the definition of $W_{\lambda+\rho}$ in [7,8].

For $\mu \in \mathfrak{h}^*$ being a $\Delta(\mathfrak{k})_+$-dominant integral weight we denote the finite-dimensional irreducible $\mathfrak{k}$-module of highest weight μ by $V_{\mathfrak{k}}^{\mu}$, as usual.

Now let $\lambda \in \mathfrak{h}^*$ be a $\Delta(\mathfrak{k})_+$-dominant integral weight. We may extend $V_{\mathfrak{k}}^{\lambda}$ to a $\mathfrak{q}$-module in the trivial way and consider the induced representation $M_{\mathfrak{g}}^{\lambda}$ of $\mathfrak{g}$. It is clear that $M_{\mathfrak{g}}^{\lambda}$ contains a unique maximal submodule and hence has a unique irreducible quotient, which is isomorphic to the highest weight irreducible $\mathfrak{g}$-module of highest weight λ. We will denote this $\mathfrak{g}$-module by $V_{\mathfrak{g}}^{\lambda}$.

For $\xi \in \mathfrak{h}^*$ with $\langle \xi, \check{\alpha} \rangle \in \mathbb{R}$ for all $\alpha \in \Delta(\mathfrak{k})$, we denote the unique $\Delta(\mathfrak{k})_+$-dominant element in the $W(\mathfrak{k})$-orbit of ξ by $\bar{\xi}$.

We have the following character formula for an irreducible unitarizable representation $V_{\mathfrak{g}}^{\lambda}$.

Theorem 6.1 (Davidson et al. [7], Enright [8]). *We have*

$$ch\, V_{\mathfrak{g}}^{\lambda} = \frac{e^{-\rho_{\mathfrak{u}}} \sum_{w \in W_{\lambda}^{\mathfrak{k}}} (-1)^{l(w)} ch\, V_{\mathfrak{k}}^{\overline{w(\lambda+\rho)}-\rho_{\mathfrak{k}}}}{\prod_{\alpha \in \Delta(\mathfrak{u})} (1-e^{-\alpha})},$$

where $l(w)$ is the length of w in W_{λ}.

6.1. Character formula for $spo(2m|2n)$-modules

It follows from Theorem 4.1 in the case when $n=0$ that we have the following identities of characters, for d even and odd, respectively.

$$(y_1 \cdots y_m)^{\frac{d}{2}} \prod_{i=1}^{\frac{d}{2}} \prod_{j=1}^{m} \frac{1}{(1-x_i y_j)(1-x_i^{-1} y_j)} = \sum_{\lambda} ch\, V_{O(d)}^{\lambda} ch\, V_{sp(2m)}^{\lambda + d\mathbf{\frac{1}{2}}}, \quad d \text{ even}, \tag{6.2}$$

$$\begin{aligned}(y_1 \cdots y_m)^{\frac{d}{2}} \prod_{i=1}^{\frac{d-1}{2}} \prod_{j=1}^{m} & \frac{1}{(1-x_i y_j)(1-x_i^{-1} y_j)(1-y_j)} \\ &= \sum_{\lambda} ch\, V_{O(d)}^{\lambda} ch\, V_{sp(2m)}^{\lambda + d\mathbf{\frac{1}{2}}}, \quad d \text{ odd}.\end{aligned} \tag{6.3}$$

Here λ is summed over all partitions with $\lambda_1' + \lambda_2' \leqslant d$ such that $l(\lambda) \leqslant m$ and $\mathbf{\frac{1}{2}}$ stands for the m-tuple $(\frac{1}{2}, \ldots, \frac{1}{2})$. Let us write $\chi_{O(d)}^{\lambda}(\mathbf{x})$ for the character of $V_{O(d)}^{\lambda}$ to stress its dependence on the variables $x_1, \ldots, x_{[\frac{d}{2}]}$. We will now apply Theorem 6.1 to the Hermitian symmetric pair $(Sp(2m), U(m))$, so that $\mathfrak{g} = sp(2m)$ and $\mathfrak{k} = gl(m)$. We may now rewrite $ch\, V_{sp(2m)}^{\lambda + d\mathbf{\frac{1}{2}}}$ in terms of Schur functions as follows. Since $\rho_{\mathfrak{u}} + d\mathbf{\frac{1}{2}}$ is $W(\mathfrak{k})$-invariant, we have by Theorem 6.1

$$ch\, V_{sp(2m)}^{\lambda + d\mathbf{\frac{1}{2}}} = (y_1 \cdots y_m)^{\frac{d}{2}} \frac{\sum_{w \in W_{\lambda + d\mathbf{\frac{1}{2}}}^{\mathfrak{k}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(\mathbf{y})}{\prod_{1 \leqslant i \leqslant j \leqslant m} (1-y_i y_j)},$$

where here and further $\rho_d = \rho + d\mathbf{\frac{1}{2}}$. Here $W^{\mathfrak{k}}_{\lambda+d\mathbf{\frac{1}{2}}}$ is the subset of the Weyl group of $sp(2m)$ defined by (6.1).

Remark 6.2. As we now need to deal with $W^{\mathfrak{k}}_{\lambda+d\mathbf{\frac{1}{2}}}$, $W_{\lambda+d\mathbf{\frac{1}{2}}}$ and $W_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})$ for different m at the same time, we introduce a superscript m in order to distinguish them. So for example $W^{\mathfrak{k},m}_{\lambda+d\mathbf{\frac{1}{2}}}$ is the subset $W^{\mathfrak{k}}_{\lambda+d\mathbf{\frac{1}{2}}}$ of the Weyl group of $sp(2m)$.

Combining this with (6.2) and (6.3), respectively, we have for even and odd d respectively

$$\prod_{i=1}^{\frac{d}{2}} \prod_{j=1}^{m} \frac{1}{(1-x_i y_j)(1-x_i^{-1} y_j)} = \sum_{\lambda} \chi^{\lambda}_{O(d)}(\mathbf{x}) \frac{\sum_{w\in W^{\mathfrak{k},m}_{\lambda+d\mathbf{\frac{1}{2}}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(\mathbf{y})}{\prod_{1\leqslant i\leqslant j\leqslant m}(1-y_i y_j)}, \tag{6.4}$$

$$\prod_{i=1}^{\frac{d-1}{2}} \prod_{j=1}^{m} \frac{1}{(1-x_i y_j)(1-x_i^{-1} y_j)(1-y_j)} = \sum_{\lambda} \chi^{\lambda}_{O(d)}(\mathbf{x}) \frac{\sum_{w\in W^{\mathfrak{k},m}_{\lambda+d\mathbf{\frac{1}{2}}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(\mathbf{y})}{\prod_{1\leqslant i\leqslant j\leqslant m}(1-y_i y_j)}. \tag{6.5}$$

Here $\chi^{\lambda}_{O(d)}(\mathbf{x}) = \chi^{\bar{\lambda}}_{O(d)}(\mathbf{x})$ if and only if $\bar{\lambda}$ is obtained from λ by replacing the first column of λ by a column of length $d-\lambda'_1$. That is, the corresponding representation of the Lie algebra $so(d)$ are isomorphic. Here and further we denote by $\bar{\lambda}$ the Young diagram obtained from λ by replacing its first column by $d-\lambda'_1$ boxes.

In order to distinguish such representations at the level of characters in the case when d is odd let us take $-I \in O(d)\backslash SO(d)$ and let ε denote the eigenvalue of $-I$ so that we have $\varepsilon^2 = 1$. We may then rewrite (6.5) as

$$\prod_{i=1}^{\frac{d-1}{2}} \prod_{j=1}^{m} \frac{1}{(1-\varepsilon x_i y_j)(1-\varepsilon x_i^{-1} y_j)(1-\varepsilon y_j)} = \sum_{\lambda} \chi^{\lambda}_{O(d)}(\varepsilon,\mathbf{x}) \frac{\sum_{w\in W^{\mathfrak{k},m}_{\lambda+d\mathbf{\frac{1}{2}}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(\mathbf{y})}{\prod_{1\leqslant i\leqslant j\leqslant m}(1-y_i y_j)}, \tag{6.6}$$

where now $\chi^{\lambda}_{O(d)}(\varepsilon,\mathbf{x})$ is a polynomial in $\mathbf{x}$ and ε such that when setting $\varepsilon = 1$, we obtain $\chi^{\lambda}_{O(d)}(\mathbf{x})$. Now it is easy to see that if λ is a Young diagram and $\chi^{\lambda}_{SO(d)}(\mathbf{x})$ is the corresponding $SO(d)$-character of $\chi^{\lambda}_{O(d)}(\varepsilon, \mathbf{x})$, then $\chi^{\lambda}_{O(d)}(\varepsilon, \mathbf{x}) = \varepsilon^{|\lambda|}\chi^{\lambda}_{SO(d)}(\mathbf{x})$, where $|\lambda|$ is the size of λ. Hence we have $\chi^{\bar{\lambda}}_{O(d)}(\varepsilon, \mathbf{x}) = \varepsilon\chi^{\lambda}_{O(d)}(\varepsilon, \mathbf{x})$.

Identities (6.4) and (6.6) will be our starting point for a character formula for unitary $spo(2m|2n)$-modules. We need the following lemma.

Lemma 6.1. *Suppose that $f^{\lambda}(\mathbf{y})$ and $g^{\lambda}(\mathbf{y})$ are power series in the variables $\mathbf{y}$.*

(i) *Suppose that d is odd and*

$$\sum_{\lambda} f^{\lambda}(\mathbf{y})\chi^{\lambda}_{O(d)}(\varepsilon, \mathbf{x}) = \sum_{\lambda} g^{\lambda}(\mathbf{y})\chi^{\lambda}_{O(d)}(\varepsilon, \mathbf{x}), \tag{6.7}$$

where the summation is over the full set of irreducible finite-dimensional characters of $O(d)$. Then $f^{\lambda}(\mathbf{y}) = g^{\lambda}(\mathbf{y})$, for all λ.

(ii) *Suppose that d is even and*

$$\sum_{\lambda} f^{\lambda}(\mathbf{y})\chi^{\lambda}_{O(d)}(\mathbf{x}) = \sum_{\lambda} g^{\lambda}(\mathbf{y})\chi^{\lambda}_{O(d)}(\mathbf{x}),$$

where the summation is over the full set of irreducible finite-dimensional characters of $O(d)$. Then $f^{\lambda}(\mathbf{y}) + f^{\bar{\lambda}}(\mathbf{y}) = g^{\lambda}(\mathbf{y}) + g^{\bar{\lambda}}(\mathbf{y})$.

Proof. We shall only show (i), i.e. for d odd, as the case of d even is analogous (in fact easier). The argument is similar to the one given in [3].

We multiply identity (6.7) by the Weyl denominator D of the Lie group $SO(d)$ and using the Weyl character formula for $\chi^{\lambda}_{SO(d)}(\mathbf{x}) = \frac{\sum_{w\in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{D}$ we obtain

$$\sum_{\lambda} f^{\lambda}(\mathbf{y})\varepsilon^{|\lambda|} \sum_{w\in W} (-1)^{l(w)} e^{w(\lambda+\rho)} = \sum_{\lambda} g^{\lambda}(\mathbf{y})\varepsilon^{|\lambda|} \sum_{w\in W} (-1)^{l(w)} e^{w(\lambda+\rho)}. \tag{6.8}$$

Now as λ ranges over all integral dominant weights, $\lambda + \rho$ ranges over all regular integral dominant weights of $SO(d)$. Hence if $\lambda \neq \mu$ as $SO(d)$-dominant weights, then the set of weights $\{w(\lambda + \rho), w(\mu + \rho) | w \in W\}$ are all distinct. Clearly two weights λ and μ are equal as $SO(d)$-dominant weights if and only if $\mu = \bar{\lambda}$. Thus looking at the coefficient of $e^{\lambda+\rho}$ in (6.8) we obtain

$$\varepsilon^{|\lambda|} f_{\lambda}(\mathbf{y})e^{\lambda+\rho} + \varepsilon^{|\bar{\lambda}|} f_{\bar{\lambda}}(\mathbf{y})e^{\bar{\lambda}+\rho} = \varepsilon^{|\lambda|} g_{\lambda}(\mathbf{y})e^{\lambda+\rho} + \varepsilon^{|\bar{\lambda}|} g_{\bar{\lambda}}(\mathbf{y})e^{\bar{\lambda}+\rho}.$$

Since $\varepsilon^{|\lambda|}\varepsilon^{|\bar{\lambda}|} = \varepsilon$, we conclude that $f^{\lambda}(\mathbf{y})e^{\lambda+\rho} = g(\mathbf{y})e^{\lambda+\rho}$ and hence $f^{\lambda}(\mathbf{y}) = g^{\lambda}(\mathbf{y})$. $\square$

From identities (6.6) and (6.4) by using Lemma 6.1 we obtain the following results for every $m \in \mathbb{N}$:

In the case when d is odd:

$$\sum_{w\in W^{\mathfrak{k},m+1}_{\lambda+d\frac{1}{2}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(y_1,\dots,y_m,0) = \sum_{w\in W^{\mathfrak{k},m}_{\lambda+d\frac{1}{2}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(y_1,\dots,y_m).$$

In the case when d is even:

$$\sum_{w\in W^{\mathfrak{k},m+1}_{\lambda+d\frac{1}{2}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(y_1,\dots,y_m,0) + \sum_{w\in W^{\mathfrak{k},m+1}_{\bar{\lambda}+d\frac{1}{2}}} (-1)^{l(w)} s_{\overline{w(\bar{\lambda}+\rho_d)}-\rho_d}(y_1,\dots,y_m,0)$$
$$= \sum_{w\in W^{\mathfrak{k},m}_{\lambda+d\frac{1}{2}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(y_1,\dots,y_m) + \sum_{w\in W^{\mathfrak{k},m}_{\bar{\lambda}+d\frac{1}{2}}} (-1)^{l(w)} s_{\overline{w(\bar{\lambda}+\rho_d)}-\rho_d}(y_1,\dots,y_m).$$

This allows us to define, in the case when d is odd, an element $S^{\lambda}_{sp}(y_1,y_2,\dots)$ in the inverse limit of symmetric polynomials, that is uniquely determined by the property that

$$S^{\lambda}_{sp}(y_1,y_2,\dots,y_m,0,0,\dots) = \sum_{w\in W^{\mathfrak{k},m}_{\lambda+d\frac{1}{2}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(y_1,\dots,y_m).$$

Similarly we may define an element $S^{\lambda}_{sp}(y_1,y_2,\dots)+S^{\bar{\lambda}}_{sp}(y_1,y_2,\dots)$ in the case when d is even.

Remark 6.3. The elements $S^{\lambda}_{sp}(y_1,y_2,\dots)$ and $S^{\lambda}_{sp}(y_1,y_2,\dots)+S^{\bar{\lambda}}_{sp}(y_1,y_2,\dots)$ are in general infinite sums of symmetric functions and hence are strictly speaking not symmetric functions. However, in these infinite sums there are only finitely many summands for any fixed degree.

We now take the limit as $m\to\infty$ in (6.4) and (6.6) and obtain the following identities, respectively

$$\prod_{i=1}^{\frac{d}{2}}\prod_{j=1}^{\infty}\frac{1}{(1-x_iy_j)(1-x_i^{-1}y_j)} = \sum_{\lambda}\chi^{\lambda}_{O(d)}(\mathbf{x})\frac{S^{\lambda}_{sp}(\mathbf{y})}{\prod_{1\leqslant i\leqslant j}(1-y_iy_j)}, \tag{6.9}$$

$$\prod_{i=1}^{\frac{d-1}{2}} \prod_{j=1}^{\infty} \frac{1}{(1-\varepsilon x_i y_j)(1-\varepsilon x_i^{-1} y_j)(1-\varepsilon y_j)} = \sum_{\lambda} \chi^{\lambda}_{O(d)}(\varepsilon, \mathbf{x}) \frac{S^{\lambda}_{sp}(\mathbf{y})}{\prod_{1 \leqslant i \leqslant j} (1-y_i y_j)}, \tag{6.10}$$

where λ is summed over all $O(d)$-highest weights and $\mathbf{y} = (y_1, y_2, \ldots)$.

Identities (6.9) and (6.10) follow from the fact that setting $y_{m+1} = y_{m+2} = \cdots = 0$, they reduce to identities (6.4) and (6.6), respectively. Thus the left- and the right-hand sides of (6.9) and (6.10) give rise to the same elements in the ring of the symmetric functions, respectively.

Recall that ω, the involution of the ring of symmetric functions which sends the complete symmetric functions to the elementary symmetric functions, is defined by $\omega(\prod_{j \in \mathbb{N}} \frac{1}{1-w_j}) = \prod_{j \in \mathbb{N}} (1 + w_j)$ (see for example [24]). We can now apply ω partially to the variables $y_{m+1}, y_{m+2}, \ldots$. After that we set the variables $y_{m+n+1} = y_{m+n+2} = \cdots = 0$ and we obtain the following identities ($z_l = y_{m+l}$, for $l = 1, \ldots, n$).

$$\begin{aligned} &\prod_{i=1}^{\frac{d}{2}} \prod_{j=1}^{m} \prod_{l=1}^{n} \frac{(1+x_i z_l)(1+x_i^{-1} z_l)}{(1-x_i y_j)(1-x_i^{-1} y_j)} \\ &\quad = \sum_{\lambda} \chi^{\lambda}_{O(d)}(\mathbf{x}) \frac{HS^{\lambda}_{sp}(\mathbf{y}, \mathbf{z}) \prod_{i,l} (1+y_i z_l)}{\prod_{1 \leqslant i \leqslant j \leqslant m} (1-y_i y_j) \prod_{1 \leqslant l < k \leqslant n} (1-z_l z_k)}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} &\prod_{i=1}^{\frac{d-1}{2}} \prod_{j=1}^{m} \prod_{l=1}^{n} \frac{(1+\varepsilon x_i z_l)(1+\varepsilon x_i^{-1} z_l)(1+\varepsilon z_l)}{(1-\varepsilon x_i y_j)(1-\varepsilon x_i^{-1} y_j)(1-\varepsilon y_j)} \\ &\quad = \sum_{\lambda} \chi^{\lambda}_{O(d)}(\varepsilon, \mathbf{x}) \frac{HS^{\lambda}_{sp}(\mathbf{y}, \mathbf{z}) \prod_{i,l} (1+y_i z_l)}{\prod_{1 \leqslant i \leqslant j \leqslant m} (1-y_i y_j) \prod_{1 \leqslant l < k \leqslant n} (1-z_l z_k)}. \end{aligned} \tag{6.12}$$

Remark 6.4. We note that $\omega(\prod_{1 \leqslant l \leqslant k} \frac{1}{1-z_l z_k}) = \prod_{1 \leqslant l < k} \frac{1}{1-z_l z_k}$. This follows from the following identities:

$$\prod_{1 \leqslant l \leqslant k} \frac{1}{1-z_l z_k} = \sum_{\lambda} s_{\lambda}(z_1, z_2, \ldots),$$

$$\prod_{1 \leqslant l < k} \frac{1}{1-z_l z_k} = \sum_{\mu} s_{\mu}(z_1, z_2, \ldots),$$

where λ is summed over all partitions with even row lengths, and μ is summed over all partitions with even column lengths.

Let us now explain the term $HS^{\lambda}_{sp}(\mathbf{y}, \mathbf{z})$. Since setting the variables $y_{m+n+1} = y_{m+n+2} = \cdots = 0$ the expression $S^{\lambda}_{sp}(\mathbf{y})$ reduces to a finite sum whose summands are

Schur polynomials with coefficients ± 1, it follows that applying the involution ω to it, we obtain a sum whose summands consists of hook Schur functions with coefficients ± 1. In fact if $S_{sp}^{\lambda}(y_1, y_2, \ldots) = \sum_{\mu} \varepsilon_\mu s_\mu(y_1, y_2, \ldots)$, where $\varepsilon_\mu = \pm 1$, then (cf. [3])

$$\omega(S_{sp}^{\lambda}(y_1, y_2, \ldots)) = \sum_{\mu} \varepsilon_\mu HS_\mu(y_1, \ldots, y_m; z_1, z_2, \ldots),$$

where $HS_\mu(y_1, \ldots, y_m; z_1, z_2, \ldots)$ is the hook Schur function of [1] in the variables $y_1, \ldots, y_m$ and $z_1, z_2, \ldots$ corresponding to the partition μ. Next setting the variables $z_{n+1} = z_{n+2} = \cdots = 0$ we get the hook Schur polynomial associated to μ, which we denote by $HS_\mu(y_1, \ldots, y_m; z_1, \ldots, z_n)$. One property of hook Schur polynomials is that $HS_\mu(y_1, \ldots, y_m; z_1, \ldots, z_n)$ is non-zero if and only if μ lies in the $(m|n)$-hook, i.e. $\mu_{m+1} \leqslant n$. So if $S_{sp}^{\lambda}(\mathbf{y}) = \sum_{\mu} \varepsilon_\mu s_\mu(\mathbf{y})$, then by $HS_{sp}^{\lambda}(\mathbf{y}, \mathbf{z})$ we mean the expression

$$HS_{sp}^{\lambda}(\mathbf{y}, \mathbf{z}) = \sum_{\mu} \varepsilon_\mu HS_\mu(y_1, \ldots, y_m; z_1, \ldots, z_n).$$

Therefore λ in (6.11) and (6.12) is summed over all $O(d)$-highest weights λ such that $\lambda_{m+1} \leqslant n$.

From Theorem 5.1, Lemma 6.1, and identities (6.11) and (6.12) we obtain the following theorem.

Theorem 6.2. *Let λ be a diagram of Theorem* 5.1 *and let $V_{spo(2m|2n)}^{\lambda + d\mathbf{\frac{1}{2}}}$ be the irreducible $spo(2m|2n)$-module corresponding to $V_{O(d)}^{\lambda}$ under the Howe duality. Here $\mathbf{\frac{1}{2}}$ is the $(m+n)$-tuple $(\frac{1}{2}, \ldots, \frac{1}{2}; -\frac{1}{2}, \ldots, -\frac{1}{2})$.*

(i) *If d is odd, then*

$$ch\, V_{spo(2m|2n)}^{\lambda + d\mathbf{\frac{1}{2}}} = (\mathbf{y}\mathbf{z}^{-1})^{\frac{d}{2}} \frac{HS_{sp}^{\lambda}(\mathbf{y}, \mathbf{z}) \prod_{i,l}(1 + y_i z_l)}{\prod_{1 \leqslant i \leqslant j \leqslant m} (1 - y_i y_j) \prod_{1 \leqslant l < k \leqslant n} (1 - z_l z_k)}.$$

(ii) *If d is even, then*

$$\begin{aligned} &ch\, V_{spo(2m|2n)}^{\lambda + d\mathbf{\frac{1}{2}}} + ch\, V_{spo(2m|2n)}^{\bar{\lambda} + d\mathbf{\frac{1}{2}}} \\ &\quad = (\mathbf{y}\mathbf{z}^{-1})^{\frac{d}{2}} \frac{(HS_{sp}^{\lambda}(\mathbf{y}, \mathbf{z}) + HS_{sp}^{\bar{\lambda}}(\mathbf{y}, \mathbf{z})) \prod_{i,l} (1 + y_i z_l)}{\prod_{1 \leqslant i \leqslant j \leqslant m} (1 - y_i y_j) \prod_{1 \leqslant l < k \leqslant n} (1 - z_l z_k)}. \end{aligned}$$

Here $\mathbf{y}\mathbf{z}^{-1}$ stands for the product $y_1 \cdots y_m z_1^{-1} \cdots z_n^{-1}$.

Remark 6.5. The expression $HS_{sp}^{\lambda}(\mathbf{y}, \mathbf{z})$ in general involves an infinite number of hook Schur functions, so the computation of these characters is a highly non-trivial task. In order to have a method to compute them, it is necessary to have an explicit description of $W^{\mathfrak{k},m}_{\lambda+d\mathbf{\frac{1}{2}}}$. We will do this in Section 7. From this we will then show in Section 8 that the coefficients of the monomials in a character of a fixed degree can be computed by computing a finite number of hook Schur functions.

6.2. *Character formula for $osp(2m|2n)$-modules*

As the arguments in this case are very similar to the one given in the previous section, we will only sketch them here.

It follows from Theorem 5.2 in the case when $n = 0$ that we have the following identity of characters.

$$(x_1 \cdots x_m)^{\frac{d}{2}} \prod_{i=1}^{\frac{d}{2}} \prod_{j=1}^{m} \frac{1}{(1 - x_j y_i)(1 - x_j y_i^{-1})} = \sum_{\lambda} ch\, V^{\lambda}_{Sp(d)} ch\, V^{\lambda+d\mathbf{\frac{1}{2}}}_{so(2m)}. \tag{6.13}$$

Here λ is summed over all partitions with $l(\lambda) \leqslant \min(\frac{d}{2}, m)$ and $\mathbf{\frac{1}{2}}$ stands for the m-tuple $(\frac{1}{2}, \ldots, \frac{1}{2})$. Let us write $\chi^{\lambda}_{Sp(d)}(\mathbf{y})$ for the character of $V^{\lambda}_{Sp(d)}$ to stress its dependence on the variables $y_1, \ldots, y_{\frac{d}{2}}$. We now apply Theorem 6.1 to the Hermitian symmetric pair $(SO^*(2m), U(m))$, so that we have $\mathfrak{g} = so(2m)$ and $\mathfrak{k} = gl(m)$. By Theorem 6.1 we can then write $ch\, V^{\lambda+d\mathbf{\frac{1}{2}}}_{so(2m)}$ in terms of Schur functions as

$$ch\, V^{\lambda+d\mathbf{\frac{1}{2}}}_{so(2m)} = (x_1 \cdots x_m)^{\frac{d}{2}} \frac{\sum_{w \in W^{\mathfrak{k},m}_{\lambda+d\mathbf{\frac{1}{2}}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(\mathbf{x})}{\prod_{i<j}(1 - x_i x_j)}.$$

Here $W^{\mathfrak{k},m}_{\lambda+d\mathbf{\frac{1}{2}}}$ is a subset of the Weyl group of $so(2m)$. Thus we have the following identity.

$$\prod_{i=1}^{\frac{d}{2}} \prod_{j=1}^{m} \frac{1}{(1 - x_j y_i)(1 - x_j y_i^{-1})} = \sum_{\lambda} \chi^{\lambda}_{Sp(d)}(\mathbf{y}) \frac{\sum_{w \in W^{\mathfrak{k},m}_{\lambda+d\mathbf{\frac{1}{2}}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(\mathbf{x})}{\prod_{i<j}(1 - x_i x_j)}. \tag{6.14}$$

Analogous to the proof of Lemma 6.1 one proves the following lemma.

Lemma 6.2. *Suppose that $f^{\lambda}(\mathbf{x})$ and $g^{\lambda}(\mathbf{x})$ are power series in the variables $\mathbf{x}$ and suppose that*

$$\sum_{\lambda} f^{\lambda}(\mathbf{x}) \chi^{\lambda}_{Sp(d)}(\mathbf{y}) = \sum_{\lambda} g^{\lambda}(\mathbf{y}) \chi^{\lambda}_{Sp(d)}(\mathbf{y}), \tag{6.15}$$

where the summation is over the full set of irreducible finite-dimensional characters of $Sp(d)$. *Then* $f^{\lambda}(\mathbf{x}) = g^{\lambda}(\mathbf{x})$, *for all* λ.

From Lemma 6.2 and identity (6.14) it follows that

$$\sum_{w \in W^{\mathfrak{k},m+1}_{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho)}-\rho}(x_1, \dots, x_m, 0) = \sum_{w \in W^{\mathfrak{k},m}_{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(x_1, \dots, x_m),$$

which then allows us to define an element $S^{\lambda}_{so}(x_1, x_2, \dots)$ in the inverse limit of symmetric polynomials, uniquely determined by the property that

$$S^{\lambda}_{so}(x_1, x_2, \dots, x_m, 0, 0, \dots) = \sum_{w \in W^{\mathfrak{k},m}_{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}} (-1)^{l(w)} s_{\overline{w(\lambda+\rho_d)}-\rho_d}(x_1, \dots, x_m).$$

Taking the limit as $m \to \infty$ (6.14) and Lemma 6.2 imply the following identity.

$$\prod_{i=1}^{\frac{d}{2}} \prod_{j=1}^{\infty} \frac{1}{(1 - x_j y_i)(1 - x_j y_i^{-1})} = \sum_{\lambda} \chi^{\lambda}_{Sp(d)}(\mathbf{y}) \frac{S^{\lambda}_{so}(x_1, x_2, \dots)}{\prod_{i<j}(1 - x_i x_j)}. \tag{6.16}$$

We apply to (6.16) the involution of symmetric functions ω partially to the variables $x_{m+1}, x_{m+2}, \dots,$ then set the variables $z_{n+1} = z_{n+2} = \cdots = 0$. We arrive at the following identity ($z_l = x_{m+l}$, for $l = 1, 2, \dots$).

$$\prod_{i=1}^{\frac{d}{2}} \prod_{j=1}^{m} \prod_{l=1}^{n} \frac{(1 + y_i z_l)(1 + y_i^{-1} z_l)}{(1 - y_i x_j)(1 - y_i^{-1} x_j)} = \sum_{\lambda} \chi^{\lambda}_{Sp(d)}(\mathbf{y}) \frac{HS^{\lambda}_{so}(\mathbf{x}; \mathbf{z}) \prod_{i,l} (1 + x_i z_l)}{\prod_{1 \leqslant i<j \leqslant m} (1 - x_i x_j) \prod_{1 \leqslant l \leqslant k \leqslant n} (1 - z_l z_k)}, \tag{6.17}$$

where $HS^{\lambda}_{so}(x_1, \dots, x_m; z_1, \dots, z_n)$ is obtained by applying the involution ω to S^{λ}_{so} and setting the variables $z_{n+1} = z_{n+2} = \cdots = 0$. As before it is also a sum whose summands consist of hook Schur polynomials with coefficients ± 1. By Theorem 5.2 and Lemma 6.2 we then obtain the following theorem.

Theorem 6.3. *Let* λ *be a diagram of Theorem* 5.2 *and let* $V^{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}_{osp(2m|2n)}$ *be the irreducible* $osp(2m|2n)$*-module corresponding to* $V^{\lambda}_{Sp(d)}$ *under the Howe duality. Here* $\frac{\mathbf{1}}{\mathbf{2}}$ *is the*

$(m+n)$-tuple $(\frac{1}{2},\ldots,\frac{1}{2};-\frac{1}{2},\ldots,-\frac{1}{2})$. Then

$$ch\, V^{\lambda+d\mathbf{\frac{1}{2}}}_{osp(2m|2n)} = (\mathbf{x}\mathbf{z}^{-1})^{\frac{d}{2}} \frac{HS^{\lambda}_{so}(\mathbf{x};\mathbf{z}) \prod_{i,l}(1+y_i z_l)}{\prod_{1\leqslant i<j\leqslant m}(1-x_i x_j)\prod_{1\leqslant l\leqslant k\leqslant n}(1-z_l z_k)},$$

where $\mathbf{x}\mathbf{z}^{-1}$ denotes the product $x_1 x_2\cdots x_m z_1^{-1} z_2^{-1}\cdots z_n^{-1}$.

Remark 6.6. We actually have Howe dualities of the dual pairs $(O(d),\mathfrak{g}(C_\infty))$ and $(Sp(d),\mathfrak{g}(D_\infty))$ on the space $S(\mathbb{C}^d\otimes\mathbb{C}^\infty)$. Here the infinite-dimensional Lie algebras $\mathfrak{g}(C_\infty)$ and $\mathfrak{g}(D_\infty)$ are Kac–Moody algebras corresponding to the infinite affine matrices C_∞ and D_∞, respectively [16]. From these dualities one can show that, using similar arguments as we have given here, the corresponding characters of those irreducible representations of $\mathfrak{g}(C_\infty)$- and $\mathfrak{g}(D_\infty)$-modules are given by certain infinite sums of symmetric functions. Applying the involution ω to these characters one obtains the characters for our $spo(2m|2n)$- and $osp(2m|2n)$-modules. Thus the characters of the representations of $\mathfrak{g}(C_\infty)$ (respectively $\mathfrak{g}(D_\infty)$) that appear in these dualities determine the characters of the representations $spo(2m|2n)$ (respectively $osp(2m|2n)$).

7. The group $W_{\lambda+d\mathbf{\frac{1}{2}}}$

Throughout this section $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_s)$ is a partition of non-negative integers of length $s\leqslant d$. We shall describe the groups $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ and $W^m_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})$ for the Hermitian symmetric pairs $(Sp(2m),U(m))$ and $(SO^*(2m),U(m))$.

Recall that the group $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is defined as the subgroup of the Weyl group of $sp(2m)$ or $so(2m)$ generated by reflections corresponding to $\alpha\in\Delta(\mathfrak{u})$ satisfying conditions (i)–(iii) given in Section 6. We will simply refer to them as conditions (i)–(iii) in what follows.

7.1. *The case of $O(d)\times sp(2m)$-duality for d even*

In the case when d is even $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is the subgroup of the Weyl group of $sp(2m)$, which is isomorphic to the sign permutation group $S_m\ltimes\mathbb{Z}_2^m$. The positive roots Δ_+ of $sp(2m)$ are generated by the simple roots $-2\varepsilon_1,\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_3,\ldots,\varepsilon_{m-1}-\varepsilon_m$. We have $\rho=-\varepsilon_1-2\varepsilon_2-\cdots-m\varepsilon_m$, which we write as

$$\rho=(-1,-2,\ldots,-m).$$

We have the condition that $\lambda_1'+\lambda_2'\leqslant d$. Now $\Delta_+(\mathfrak{k})$ is generated by the simple roots $\varepsilon_1-\varepsilon_2,\ldots,\varepsilon_{m-1}-\varepsilon_m$, while $\Delta(\mathfrak{u})$ consists of roots of the form $-\varepsilon_i-\varepsilon_j$, $1\leqslant i\leqslant j\leqslant m$.

Let us first consider the case $s = \frac{d}{2}$. In this case

$$\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho = \left(\lambda_1 + \frac{d}{2} - 1, \lambda_2 + \frac{d}{2} - 2, \ldots, \lambda_{\frac{d}{2}-1} + 1, \lambda_{\frac{d}{2}}, -1, -2, \ldots, -m + \frac{d}{2}\right).$$

We see that $\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho$ has no zero coefficient, and hence condition (iii) is vacuous. It follows that for each $i = 1, \ldots, \frac{d}{2}$ with $m \geqslant \lambda_i + d - i$ we have

$$\left(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho, -\varepsilon_i - \varepsilon_{\lambda_i + d - i}\right) = 0. \tag{7.1}$$

On the other hand if $m < \lambda_i + d - i$, $i = 1 \ldots, \frac{d}{2}$, we have for all $t = 1, \ldots, m$

$$\left(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho, -\varepsilon_i - \varepsilon_t\right) < 0. \tag{7.2}$$

This implies by condition (ii) that if $\alpha = -\varepsilon_k - \varepsilon_l$ is such that $s_\alpha \in W^m_{\lambda + d\frac{\mathbf{1}}{\mathbf{2}}}$, then neither k nor l can be in the index set $J = \{1, \ldots, \frac{d}{2}, \lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_{\frac{d}{2}} + \frac{d}{2}\}$. Let $I^0 = \{1, \ldots, m\} \backslash J$. Let $\alpha = -\varepsilon_k - \varepsilon_l$ with $k, l \in I^0$. Clearly we have

$$\left\langle \lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho, \check{\alpha} \right\rangle \in \mathbb{N},$$

and hence condition (i) is satisfied for such an α. This implies that $W^m_{\lambda + d\frac{\mathbf{1}}{\mathbf{2}}}$ is generated by the reflections s_α with $\alpha = -\varepsilon_k - \varepsilon_l$, $k, l \in I^0$. Hence $W^m_{\lambda + d\frac{\mathbf{1}}{\mathbf{2}}}$ is the sign permutation group on the index set I^0. Therefore $W^m_{\lambda + d\frac{\mathbf{1}}{\mathbf{2}}}(\mathfrak{k})$ is equal to the permutation group of the index set I^0 and hence $\Delta_{\lambda + d\frac{\mathbf{1}}{\mathbf{2}}}(\mathfrak{k})_+$ consists of $\varepsilon_k - \varepsilon_l$ with $k < l$ and $k, l \in I^0$.

Next consider the case $s < \frac{d}{2}$. In this case we have

$$\lambda + d\tfrac{\mathbf{1}}{\mathbf{2}} + \rho = \left(\lambda_1 + \frac{d}{2} - 1, \ldots, \lambda_s + \frac{d}{2} - s, \frac{d}{2} - s - 1, \ldots, \underbrace{0}_{\frac{d}{2}}, -1, -2, \ldots, -m + \frac{d}{2}\right).$$

Since $(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho, 2\varepsilon_{\frac{d}{2}}) = 0$, condition (iii) implies that if $\alpha \in \Delta(\mathfrak{u})$ is such that $s_\alpha \in W^m_{\lambda + d\frac{\mathbf{1}}{\mathbf{2}}}$, then α is a short root. As in the previous case (7.1) and (7.2) hold in this case as well with $i = 1, \ldots, s$. In addition we have for $j = s + 1, \ldots, \frac{d}{2}$

$$\left(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho, -\varepsilon_j - \varepsilon_{d-j}\right) = 0. \tag{7.3}$$

Let $J = \{1, \dots, d-s-1, \lambda_1 + d - 1, \dots, \lambda_s + d - s\}$ and $I^- = \{1, \dots, m\} \backslash J$. Similarly as in the previous case conditions (i) and (ii) now tell us that $\alpha \in \Delta(\mathfrak{u})$ is such that $s_\alpha \in W^m_{\lambda + d\mathbf{\frac{1}{2}}}$ if and only if $\alpha = -\varepsilon_k - \varepsilon_l$ with $k, l \in I^-$ and $k \neq l$. Clearly $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is equal to the even sign permutation group (i.e. permutations with an even number of sign changes) in the index set I^-. Therefore $W^m_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})$ is the permutation group on the index set I^- and hence $\Delta_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})_+$ consists of $\varepsilon_k - \varepsilon_l$ with $k < l$ and $k, l \in I^-$.

Finally consider the case when $s > \frac{d}{2}$. In this case we have

$$\lambda + d\mathbf{\frac{1}{2}} + \rho = \Bigg(\lambda_1 + \frac{d}{2} - 1, \dots, \underbrace{\lambda_{d-s} - \frac{d}{2} + s}_{d-s}, s - \frac{d}{2}, \dots, \underbrace{1}_{\frac{d}{2}}, 0, \dots, \underbrace{1 + \frac{d}{2} - s}_{s}, \underbrace{-1 + \frac{d}{2} - s}_{s+1}, \dots, -m + \frac{d}{2} \Bigg).$$

Then (7.1) and (7.2) hold for $i = 1, \dots, d-s$ and we have in addition

$$\left(\lambda + d\mathbf{\frac{1}{2}} + \rho, -\varepsilon_j - \varepsilon_{d-j+2}\right) = 0, \quad j = d-s+2, \dots, \frac{d}{2},$$
$$\left(\lambda + d\mathbf{\frac{1}{2}} + \rho, -2\varepsilon_{\frac{d}{2}+1}\right) = 0. \tag{7.4}$$

Let $I^+ = \{d-s+1, s+1, s+2, \dots, m\} \backslash \{\lambda_1 + d - 1, \dots, \lambda_{d-s} + d - (d-s)\}$. Then $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is generated by s_α, where $\alpha = -\varepsilon_k - \varepsilon_l$ with $k, l \in I^+$ and $k \neq l$. This implies that $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is the even sign permutation group on the index set I^+ and hence $W^m_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})$ is the permutation group on the index set I^+ and hence $\Delta_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})_+$ consists of $\varepsilon_k - \varepsilon_l$ with $k < l$ and $k, l \in I^+$.

7.2. *The case of $O(d) \times sp(2m)$-duality for d odd*

Suppose that $s = \frac{d+1}{2}$. We have

$$\lambda + d\mathbf{\frac{1}{2}} + \rho = \left(\lambda_1 + \frac{d}{2} - 1, \dots, \lambda_{\frac{d-3}{2}} - \frac{3}{2}, \frac{3}{2}, \underbrace{\frac{1}{2}}_{\frac{d+1}{2}}, -\frac{3}{2}, \dots, -m + \frac{d}{2} \right).$$

Therefore (7.1) and (7.2) hold for $i = 1, \dots, \frac{d-3}{2}$ and also

$$\left(\lambda + d\mathbf{\frac{1}{2}} + \rho, -\varepsilon_{\frac{d-1}{2}} - \varepsilon_{\frac{d+3}{2}}\right) = 0.$$

Note that the coefficients of $\lambda + d\mathbf{\frac{1}{2}} + \rho$ are all half integers. Hence if $\beta \in \Delta(\mathfrak{u})$ is a long root, then

$$\left\langle \lambda + d\mathbf{\frac{1}{2}} + \rho, \check{\beta} \right\rangle \in \frac{1}{2} + \mathbb{Z}.$$

Thus the long roots are eliminated from the consideration of $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ by condition (i). Now let $J = \{1, \dots, \frac{d-1}{2}, \frac{d+3}{2}, \lambda_1 + d - 1, \dots, \lambda_{d-s} + d - (d-s)\}$ and let $I^0 = \{1, \dots, m\} \backslash J$. Then $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is the even sign permutation group on the index set I^0 so that $W^m_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})$ is the permutation group on I^0.

Now suppose that $s < \frac{d+1}{2}$ so that

$$\lambda + d\mathbf{\frac{1}{2}} + \rho = \left(\lambda_1 + \frac{d}{2} - 1, \dots, \lambda_s + \frac{d}{2} - s, \frac{d}{2} - s - 1, \dots, \frac{1}{2}, \underbrace{-\frac{1}{2}}_{\frac{d+1}{2}}, -\frac{3}{2}, \dots, -m + \frac{d}{2} \right).$$

Thus (7.1) and (7.2) hold for $i = 1, \dots, s$. Furthermore (7.3) holds for $j = s+1, \dots, \frac{d-1}{2}$. As before the long roots in $\Delta(\mathfrak{u})$ are eliminated from considerations of $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$. Set $J = \{1, \dots, d-s-1, \lambda_1 + d - 1, \dots, \lambda_s + d - s\}$ and let $I^- = \{1, \dots, m\} \backslash J$. Then $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is the even sign permutation group on the index set I^- so that $W^m_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})$ is the permutation group on I^-.

Finally consider the case when $s > \dfrac{d+1}{2}$ so that we have

$$\lambda + d\mathbf{\frac{1}{2}} + \rho = \Bigg(\lambda_1 + \frac{d}{2} - 1, \dots, \underbrace{\lambda_{d-s} - \frac{d}{2} + s}_{d-s}, s - \frac{d}{2},$$
$$\dots, \underbrace{\tfrac{1}{2}}_{\frac{d+1}{2}}, \underbrace{-\frac{1}{2}}_{\frac{d+3}{2}}, \dots, \underbrace{1 + \frac{d}{2} - s}_{s}, \underbrace{-1 + \frac{d}{2} - s}_{s+1}, \dots, -m + \frac{d}{2} \Bigg).$$

Thus (7.1) and (7.2) still hold with $i = 1, \dots, (d-s)$ and also (7.4) holds with $j = 1, \dots, \frac{d+1}{2}$. Again the long roots are eliminated from the consideration of $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$. Let $J = \{1, \dots, (d-s), (d-s)+2, \dots, \frac{d+1}{2}, \lambda_1 + d - 1, \dots, \lambda_{d-s} + d - (d-s)\}$ and let

$I^+ = \{1, \dots, m\} \backslash J$. Then $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is the even sign permutation group on the index set I^+ so that $W^m_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})$ is the permutation group on I^+.

7.3. The case of $Sp(d) \times so(2m)$-duality

In this case $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is a subgroup of the Weyl group of $so(2m)$, which is isomorphic to the even sign permutation group $S_m \ltimes \mathbb{Z}_2^{m-1}$. The positive roots Δ_+ is generated by the simple root $-\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{m-1} - \varepsilon_m$, and hence $\rho = -\varepsilon_2 - 2\varepsilon_3 \cdots - (m-1)\varepsilon_m$, which we write as

$$\rho = (0, -1, -2, \dots, -m+1).$$

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\frac{d}{2}})$ be a partition so that

$$\lambda + d\mathbf{\frac{1}{2}} + \rho = \left(\lambda_1 + \frac{d}{2}, \lambda_2 + \frac{d}{2} - 1, \dots, \lambda_{\frac{d}{2}} + 1, \underbrace{0}_{\frac{d}{2}+1}, -1, -2, \dots, -m + \frac{d}{2} + 1\right).$$

The set $\Delta(\mathfrak{u})$ consists of roots of the form $-\varepsilon_k - \varepsilon_l$, with $k \neq l$.

We have in the case $m \geqslant \lambda_i + d - i + 2$

$$\left(\lambda + d\mathbf{\frac{1}{2}} + \rho, -\varepsilon_i - \varepsilon_{\lambda_i+d-i+2}\right) = 0, \quad i = 1, \dots, \frac{d}{2}. \tag{7.5}$$

On the other hand if $m < \lambda_i + d - i + 2$, then for every $t = 1, \dots, m$ we have

$$\left(\lambda + d\mathbf{\frac{1}{2}} + \rho, -\varepsilon_i - \varepsilon_t\right) < 0. \tag{7.6}$$

This implies by condition (ii) that if $\alpha = -\varepsilon_k - \varepsilon_l$ with $s_\alpha \in W^m_{\lambda+d\mathbf{\frac{1}{2}}}$, then k, l cannot be one of the indices in (7.5) and (7.6). On the other hand set $J = \{1, \dots, \frac{d}{2}, \lambda_1 + d + 1, \lambda_2 + d, \dots, \lambda_{\frac{d}{2}} + \frac{d}{2} + 2\}$ and let $I = \{1, 2, \dots, m\} \backslash J$. Clearly if $\alpha = -\varepsilon_k - \varepsilon_l$ with $k, l \in I$, then

$$\left(\lambda + d\mathbf{\frac{1}{2}} + \rho, -\varepsilon_k - \varepsilon_l\right) \in \mathbb{N},$$

and so condition (i) is satisfied. Of course here (iii) is irrelevant, as Δ is simply-laced. Thus $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ is equal to the even sign permutation group on the index set I and hence $W^m_{\lambda+d\mathbf{\frac{1}{2}}}(\mathfrak{k})$ is the permutation group on I.

From our explicit description of $W^m_{\lambda+d\mathbf{\frac{1}{2}}}$ we have the following.

Proposition 7.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ and let $W^m_{\lambda+d\frac{1}{2}}$ be the corresponding group of either sign or even sign permutation group on the index set $I \subseteq \{1, \ldots, m\}$. Write $\mu = \lambda + d\frac{1}{2} + \rho = (\mu_1, \mu_2, \ldots, \mu_m)$ for the generalized partition with either all integral or half-integral row lengths.*

(i) *We have $|\mu_i| \neq |\mu_j|$, for $i, j \in I$ with $i \neq j$.*
(ii) *For $w \in W^m_{\lambda+d\frac{1}{2}}$ the rows of the generalized composition $w(\lambda + d\frac{1}{2} + \rho)$ are all of different length.*
(iii) *In the case $s \leqslant \frac{d}{2}$ we have $\mu_i \leqslant 0$, for $i \in I$.*
(iv) *In the case $s > \frac{d}{2}$ we have $\mu_i > 0$ and $i \in I$ if and only if $i = (d - s + 1)$ and $\mu_{d-s+1} = s - \frac{d}{2}$.*
(v) *For all $m \in \mathbb{N}$ we have $W^m_{\lambda+d\frac{1}{2}} \subseteq W^{m+1}_{\lambda+d\frac{1}{2}}$.*

8. Consequences for the character formula

In this section we will use the result of Section 7 to study the character formulas of Section 6. In Section 7 we gave a description of $W^k_{\lambda+d\frac{1}{2}}$. However, in the character formula we actually need to have a description of $W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$.

Let $w \in W^k_{\lambda+d\frac{1}{2}}$. From Section 7 we know that w is either a sign permutation or an even sign permutation on an index set I.

Recall the decomposition $W^k_{\lambda+d\frac{1}{2}} = W^k_{\lambda+d\frac{1}{2}}(\mathfrak{k}) \times W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$. As $W^k_{\lambda+d\frac{1}{2}}$ is the sign or the even sign permutation group on I and $W^k_{\lambda+d\frac{1}{2}}(\mathfrak{k})$ is the permutation group on I, it follows that the elements of $W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$ are in one-to-one correspondence with either the sign or the even sign changes of the index set I. This correspondence can be made explicit as follows. Let $I = \{j_1 < j_2 < \cdots < j_t\}$ and set $\rho = (\rho_1, \rho_2, \ldots, \rho_k)$ and $\rho_I = (\rho_{j_1}, \rho_{j_2}, \ldots, \rho_{j_t})$. Let τ be either a sign change or an even sign change of I. Let σ be the unique permutation on I which permutes the rows of the generalized composition $\tau(\rho_I)$ so that $\sigma\tau(\rho_I)$ is a generalized partition. Set $w_\tau = \sigma\tau$, then $w_\tau \in W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$ is the element corresponding to τ under the above-mentioned one-to-one correspondence. More explicitly, if τ changes the signs of ρ_I at the rows $i_1 < i_2 < \cdots < i_l$, then σ is the permutation that moves i_l to j_1, i_{l-1} to $j_2, \ldots, i_1$ to j_l. After that the remaining indices $j_{l+1}, \ldots, j_t$ are then assigned from the indices $I \backslash \{i_1, \ldots, i_l\}$ in increasing order.

We are now in a position to describe $\overline{w(\lambda + d\frac{1}{2} + \rho)}$ with $w \in W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$. For this it is convenient to identify $W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$ with either a sign change or an even sign change of the index set $I = \{j_1 < j_2 < \cdots < j_t\}$. Set

$$\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho = \mu = (\mu_1, \mu_2, \ldots, \mu_k).$$

Let $w \in W^{\mathfrak{k},k}_{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}+\rho}$ and let τ_w be the corresponding sign change. Let us suppose that τ_w corresponds to sign changes of the subset $I_w \subseteq I$. Suppose that $I_w = i_1, i_2, \ldots, i_l$. Then

$$\tau_w\left(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho\right) = (\mu_1, \mu_2, \ldots, -\mu_{i_1}, \ldots, -\mu_{i_2}, \ldots, -\mu_{i_l}, \ldots).$$

That is, $\tau_w(\mu)$ is obtained from μ by replacing all the rows indexed by I_w with its negative. Set σ_w equal to the unique permutation on $\{1, 2, \ldots, k\}$ that permutes the rows of generalized composition $\tau_w(\mu)$ so that the resulting is a generalized partition. We denote by $\Lambda_w(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho)$ the partition $\sigma_w \tau_w(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho) - \rho - d\frac{\mathbf{1}}{\mathbf{2}}$.

The following proposition is an easy consequence of the correspondence between sign changes of the index set I and $W^{\mathfrak{k},k}_{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}$.

Proposition 8.1. *With the notation introduced above we have*

$$\Lambda_w\left(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho\right) = \overline{w\left(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho\right)} - \rho - d\frac{\mathbf{1}}{\mathbf{2}}.$$

Using Proposition 8.1 we can now prove the following corollary for the characters of $V^{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}_{spo(2m|2n)}$ and $V^{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}_{osp(2m|2n)}$. Recall the character formulas given in Theorems 6.2 and 6.3. In these formulas the expression $HS_{\overline{w(\lambda+d\frac{\mathbf{1}}{\mathbf{2}}+\rho)}-\rho-d\frac{\mathbf{1}}{\mathbf{2}}}$ is the hook Schur function associated to the partition $\overline{w(\lambda + d\frac{\mathbf{1}}{\mathbf{2}} + \rho)} - \rho - d\frac{\mathbf{1}}{\mathbf{2}}$. Due to Proposition 8.1 we will from now on write $HS_{\Lambda_w(\lambda+d\frac{\mathbf{1}}{\mathbf{2}}+\rho)}$ for $HS_{\overline{w(\lambda+d\frac{\mathbf{1}}{\mathbf{2}}+\rho)}-\rho-d\frac{\mathbf{1}}{\mathbf{2}}}$.

The next corollary shows that in general the character formulas involve an infinite sum of hook Schur functions.

Corollary 8.1. *Fix a diagram λ corresponding to $V^{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}_{spo(2m|2n)}$ or $V^{\lambda+d\frac{\mathbf{1}}{\mathbf{2}}}_{osp(2m|2n)}$ of length $l(\lambda) = s$.*

(i) *Suppose that $m > s + 2$ and $n > 1$. Then HS^{λ}_{so} and HS^{λ}_{sp} are infinite sums of non-zero hook Schur polynomials.*
(ii) *Suppose that $s = m$ and $\lambda_s > n$. Then HS^{λ}_{so} and HS^{λ}_{sp} are finite sums of non-zero hook Schur polynomials.*

Proof. Take any $k > \lambda_1 + d$ so that both k and $k-1$ lie in the index set I associated to $W^k_{\lambda+d\frac{1}{2}}$ of $sp(2k)$ or $so(2k)$. Let $w \in W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$ correspond to τ_w, the even sign permutation that permutes the indices $k-1$ and k and changes the signs of them. It is then easy to see that $\Lambda_w(\lambda + d\frac{1}{2} + \rho)$ is a partition with 2 from the $(s+3)$th row on. Since $n > 1$, it follows that the partition associated to $\Lambda_w(\lambda + d\frac{1}{2} + \rho)$ lies in the $(m|n)$-hook and thus its corresponding hook Schur polynomial is non-zero. This proves (i).

To prove (ii) let $k > \lambda_1 + d$ and consider any $w \in W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$ corresponding to a sign change involving k. Then $\Lambda_w(\lambda + d\frac{1}{2} + \rho)$ is a partition with the first $m+1$ rows exceeding n. But then the corresponding hook Schur polynomial is zero. $\square$

Let $\mathbb{C}[[\mathbf{y}, \mathbf{z}]]$ denote the ring of power series in the variables $\mathbf{y}$ and $\mathbf{z}$. We have a natural filtration of ideals determined by the leading term.

$$\mathbb{C}[[\mathbf{y}, \mathbf{z}]] = \mathfrak{F}_0 \supset \mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \cdots \supset \mathfrak{F}_l \supset \cdots.$$

The formulas of Theorems 6.2 and 6.3 involve in general an infinite number of hook Schur polynomials. However, for a fixed monomial that appears in the character formula we can use a finite number of hook Schur polynomials to compute its coefficient. This follows from the following proposition.

Proposition 8.2. *Let λ be a partition with $l(\lambda) = s$ and $\lambda'_1 + \lambda'_2 \leqslant d$. Let $k > d$.*

(i) *If $s > \frac{d}{2}$ we let $l = 2k + |\lambda| - 2s - 1$. If $s \leqslant \frac{d}{2}$ we let $l = 2k + |\lambda| - d$. Then we have*

$$HS^{\lambda}_{sp}(\mathbf{y}, \mathbf{z}) \equiv \sum_{w \in W^{\mathfrak{k},k-1}_{\lambda+d\frac{1}{2}}} (-1)^{l(w)} HS_{\Lambda_w(\lambda+d\frac{1}{2}+\rho)}(\mathbf{y}; \mathbf{z}) \pmod{\mathfrak{F}_l}.$$

(ii) *Let $l = 2k + |\lambda| - d - 1$. Then*

$$HS^{\lambda}_{so}(\mathbf{x}, \mathbf{z}) \equiv \sum_{w \in W^{\mathfrak{k},k-1}_{\lambda+d\frac{1}{2}}} (-1)^{l(w)} HS_{\Lambda_w(\lambda+d\frac{1}{2}+\rho)}(\mathbf{x}; \mathbf{z}) \pmod{\mathfrak{F}_l}.$$

Proof. The theorem follows rather easily from Proposition 8.1. We will only prove (ii), as (i) is quite similar. We may assume without loss of generality that $k \in I$. Consider $w \in W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$ such that $w \notin W^{\mathfrak{k},k-1}_{\lambda+d\frac{1}{2}}$. This means that τ_w changes the sign of k. We consider the partition $\Lambda_w(\lambda + d\frac{1}{2} + \rho)$. It is not hard to see that the size of this diagram is at least $2k - d + \sum_{i=1}^{s} \lambda_i - 1$. But this means that the hook Schur polynomial determined by $\Lambda_w(\lambda + d\frac{1}{2} + \rho)$ contains only monomials of degree $l = 2k - d + \sum_{i=1}^{s} \lambda_i - 1$. Thus the hook Schur polynomials associated to $\Lambda_w(\lambda + d\frac{1}{2} + \rho)$ with $w \in W^{\mathfrak{k},k-1}_{\lambda+d\frac{1}{2}}$ contain all monomials of HS^{λ}_{so} of degree less than or equal to $l-1$. $\square$

We now compute the functions HS^{λ}_{so} and HS^{λ}_{sp} explicitly in the case of $\lambda = (0, 0, \ldots, 0)$, the trivial partition.

Let us first consider the case of HS^{λ}_{sp} with λ being the trivial partition. We will write in this case simply HS_{sp} for HS^{λ}_{sp}. In this case $W^{k}_{\lambda+d\mathbf{\frac{1}{2}}}$ is the group of the even sign permutations in the indices $d, \ldots, k$. Let $w \in W^{\mathfrak{k},k}_{\lambda+d\mathbf{\frac{1}{2}}}$ and let τ_w be the corresponding sign changes. Let us suppose that τ_w changes signs at the following l rows: $i_1 < i_2 < \cdots < i_{l-1} < i_l$. Here $i_1 \geqslant d$ and l is an even non-negative integer. Then it is not hard to see that $\Lambda_w(d\mathbf{\tfrac{1}{2}} + \rho)$ is the following partition:

$$\Bigg(i_l - d + 1, i_{l-1} - d + 2, \ldots, i_1 - d + l, \underbrace{l, \ldots, l}_{i_1 - 1},$$
$$\underbrace{l-1, \ldots, l-1}_{i_2 - i_1 - 1}, \ldots, \underbrace{1, \ldots, 1}_{i_l - i_{l-1} - 1}, 0, \ldots \Bigg). \tag{8.1}$$

That is, the first l entries are $i_l - d + 1, i_{l-1} - d + 2, \ldots, i_1 - d + l$, followed by $i_1 - 1$ entries of l, etc. The length of $\Lambda_w(d\mathbf{\tfrac{1}{2}} + \rho)$ is i_l. For a sequence of positive integers $I = \{i_1 < i_2 < \cdots < i_l\}$ with $i_1 \geqslant d$ denote by μ_I partition (8.1). Furthermore we let $|I|$ denote $l + \sum_{j=1}^{l} i_j$.

Proposition 8.3. *We have*

$$HS_{sp}(y_1, \ldots, y_m; z_1, \ldots, z_n) = \sum_{I} (-1)^{|I|} HS_{\mu_I}(y_1, \ldots, y_m; z_1, \ldots, z_n),$$

where the summation is over all tuples $I = (i_1, i_2, \ldots, i_l)$ with l even and $d \leqslant i_1 < i_2 < \cdots < i_l$ satisfying one of the following conditions.

(i) *In the case when $n \geqslant m$ we have $l \leqslant n$ and at most m of the i_j's exceed $d + n - m - 1$.*
(ii) *In the case when $m > n$ we have $l < m$. If in addition we have $l + i_t - t + 1 \leqslant m + 1 \leqslant l + i_{t+1} - t - 1$, for some $t = 0, 1, \ldots, l-1$, then $l - t \leqslant n$.* (*Here by definition $i_0 = 0$.*)

Proof. First we note that if $w \in W^{\mathfrak{k},k}_{\lambda+d\mathbf{\frac{1}{2}}}$ and τ_w its corresponding sign changes at the rows $i_1 < i_2 < \cdots < i_{l-1} < i_l$, then $(-1)^{l(w)} = (-1)^{|I|}$.

(i) Obviously if $l > n$, then the corresponding partition μ_I cannot lie inside the (m, n)-hook. Thus the corresponding hook Schur polynomial is zero. Also clearly if

$l \leqslant m$, then μ_I lies in the (m,n)-hook. Now suppose that $m < l \leqslant n$ and we have

$$i_1 < i_2 < \cdots < i_{l-m} < i_{l-m+1} < \cdots < i_l.$$

Then μ_I lies in the (m,n)-hook if and only if $i_{l-m} - d + (m+1) \leqslant n$, which happens if and only if $i_{l-m} \leqslant d - m + n - 1$.

(ii) Clearly, if $l \geqslant m$, then μ_I does not lie in the (m,n)-hook. Now if $l < m$ and $i_l < m$, then it is easy to see that μ_I lies in the (m,n)-hook. On the other hand if $m \leqslant i_l$, we let $t = 0, 1, \dots, l-1$ be such that

$$l + i_t - t < m + 1 \leqslant l + i_{t+1} - t - 1.$$

It follows from (8.1) that the $(m+1)$th row of μ_I is $l - t$. Thus μ_I lies in the (m,n)-hook if and only if $l - t \leqslant n$. $\square$

Consider now the case of HS^{λ}_{so}, where λ is the trivial partition. We will again write in this case simply HS_{so} for HS^{λ}_{so}. Here $W^k_{\lambda+d\frac{1}{2}}$ is the group of the even sign permutations in the indices $d/2+1, d+2, d+3, \dots, k$. Let $w \in W^{\mathfrak{k},k}_{\lambda+d\frac{1}{2}}$ and let τ_w be the corresponding sign changes, which changes signs at the following l rows: $i_1 < i_2 < \cdots < i_{l-1} < i_l$.

First suppose that $i_1 \neq d/2+1$. In this case $\Lambda_w(d\frac{1}{2}+\rho)$ is the partition

$$\Bigg(i_l - d - 1, i_{l-1} - d, \dots, i_1 - d + l - 2, \underbrace{l, \dots, l}_{i_1 - 1}, \underbrace{l-1, \dots, l-1}_{i_2 - i_1 - 1}, \dots, \underbrace{1, \dots, 1}_{i_l - i_{l-1} - 1}, 0, \dots \Bigg). \tag{8.2}$$

Now if $i_1 = \dfrac{d}{2} + 1$, then $\Lambda_w(d\frac{1}{2}+\rho)$ is

$$\Bigg(i_l - d - 1, i_{l-1} - d, \dots, i_2 - d + l - 3, \underbrace{l-1, \dots, l-1}_{i_2 - 1}, \underbrace{l-2, \dots, l-2}_{i_3 - i_2 - 1}, \dots, \underbrace{1, \dots, 1}_{i_l - i_{l-1} - 1}, 0, \dots \Bigg). \tag{8.3}$$

Note that (8.3) is just (8.2) corresponding to the sequence $i_2 < i_3 < \cdots < i_l$, with $d + 2 \leqslant i_2$.

For a sequence of positive integers $J = \{i_1 < i_2 < \cdots < i_l\}$ with $i_1 \geqslant d+2$ we let ν_J be the partition (8.2). Let $|J| = l + \sum_{j=1}^{l} i_j$.

Proposition 8.4. *We have*

$$HS_{so}(y_1, \dots, y_m; z_1, \dots, z_n) = \sum_J (-1)^{|J|} HS_{\nu_J}(y_1, \dots, y_m; z_1, \dots, z_n),$$

where the summation is over all tuples $J = (i_1, i_2, \dots, i_l)$ with $d+2 \leqslant i_1 < i_2 < \cdots < i_l$ satisfying the following conditions.

(i) *In the case when $n \geqslant m$ we have $l \leqslant n$ and at most m of the i_j's exceed $d+n-m+1$.*
(ii) *In the case when $m > n$ we have $l < m$. If in addition we have $l + i_t - t + 1 \leqslant m+1 \leqslant l + i_{t+1} - t - 1$, for some $t = 1, \dots, l-1$, then $l - t \leqslant n$. (Here again $i_0 = 0$.)*

Proof. As the proof is analogous to that of Proposition 8.3, we omit it. $\square$

The module $V^{\lambda}_{spo(2m|2n)}$ (respectively $V^{\lambda}_{osp(2m|2n)}$) with λ being the trivial partition is the $O(d)$-invariants (respectively $Sp(d)$-invariants) inside $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$. Thus our computations of HS_{sp} and HS_{so} give character formulas of these invariants. On the other hand we can describe the invariants, denoted by $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{O(d)}$ and $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{Sp(d)}$, in the following different way. Since $gl(m|n)$ commutes with $O(d)$ and $Sp(d)$, $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{O(d)}$ and $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{Sp(d)}$ are modules over $gl(m|n)$. We have the following analogue of classical invariant theory.

Proposition 8.5. *We have the following isomorphisms of $gl(m|n)$-modules*

(i) $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{O(d)} \cong \sum_\lambda V^{\lambda}_{m|n}$, *where the summation is over all partitions λ with even row lengths, $l(\lambda) \leqslant d$ and $\lambda_{m+1} \leqslant n$.*
(ii) $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{Sp(d)} \cong \sum_\mu V^{\mu}_{m|n}$, *where the summation is over all partitions μ with even column lengths, $l(\mu) \leqslant d$ and $\mu_{m+1} \leqslant n$.*

Proof. The proof is in the same spirit as the one in the classical case given in [14]. The $(gl(d), gl(m|n))$-duality gives (3.1) and hence taking the $O(d)$-invariants on both sides of (3.1) gives

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{O(d)} = \sum_\lambda (V^{\lambda}_d)^{O(d)} \otimes V^{\lambda}_{m|n}.$$

But it is known that V^{λ}_d has only $O(d)$-invariants if and only if λ is an even partition, i.e. all rows have even length. Furthermore in this case the dimension of $O(d)$-invariants in V^{λ}_d equals 1. This proves (i).

For (ii) we note that V_d^λ has $Sp(d)$-invariants if and only if λ has even columns, in which case the dimension of the invariants is again 1. $\square$

As the character of the $gl(m|n)$-module $V_{m|n}^\lambda$ is given by the hook Schur function associated to λ we obtain the following corollary.

Corollary 8.2. *As $gl(m|n)$-modules we have*

$$ch\, S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{O(d)} = \sum_\lambda HS_\lambda(y_1, \dots, y_m; z_1, \dots, z_n),$$
$$ch\, S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})^{Sp(d)} = \sum_\mu HS_\mu(x_1, \dots, x_m; z_1, \dots, z_n),$$

where the summations over λ and μ are as in Proposition 8.5.

From these two descriptions of the $O(d)$-invariants inside $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$, in the case when d is odd, we have the combinatorial identity

$$\sum_\lambda HS_\lambda(y_1, \dots, y_m; z_1, \dots, z_n)$$
$$= \sum_I (-1)^{|I|} HS_{\mu_I}(y_1, \dots, y_m; z_1, \dots, z_n)$$
$$\times \left(\frac{\prod_{1 \leqslant i \leqslant m, 1 \leqslant l \leqslant n} (1 + y_i z_l)}{\prod_{1 \leqslant i \leqslant j \leqslant m, 1 \leqslant l < k \leqslant n} (1 - y_i y_j)(1 - z_l z_k)} \right),$$

where λ is summed over all partitions with even row lengths, $l(\lambda) \leqslant d$ and $\lambda_{m+1} \leqslant n$, and I is summed over all I as in Proposition 8.3 with μ_I as in (8.1).

Similarly from the descriptions of the $Sp(d)$-invariants we have

$$\sum_\mu HS_\mu(x_1, \dots, x_m; z_1, \dots, z_n)$$
$$= \sum_J (-1)^{|J|} HS_{\nu_J}(x_1, \dots, x_m; z_1, \dots, z_n)$$
$$\times \left(\frac{\prod_{1 \leqslant i \leqslant m, 1 \leqslant l \leqslant n} (1 + x_i z_l)}{\prod_{1 \leqslant i < j \leqslant m, 1 \leqslant l \leqslant k \leqslant n} (1 - x_i x_j)(1 - z_l z_k)} \right),$$

where μ is summed over all partitions with even column lengths, $l(\mu) \leqslant d$ and $\mu_{m+1} \leqslant n$, and J is summed over all J as in Proposition 8.4 with ν_I as in (8.2).

9. Tensor product decomposition

As another application of Theorems 5.1 and 5.2 we derive in this section formulas for the decomposition of tensor products of two representations of either $spo(2m|2n)$ or $osp(2m|2n)$ that appear in the decomposition of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$.

We first recall two Howe dualities involving the dual pairs $(O(d), so(2k))$ and $(Sp(d), sp(2k))$ on the space $\Lambda(\mathbb{C}^d \otimes \mathbb{C}^k)$.

Theorem 9.1 (Howe [14]). *The pairs $(O(d), so(2k))$ and $(Sp(d), sp(2k))$ form dual pairs on the space $\Lambda(\mathbb{C}^d \otimes \mathbb{C}^k)$. Furthermore with respect to their joint actions we have the following decompositions*:

$$\Lambda(\mathbb{C}^d \otimes \mathbb{C}^k) \cong \sum_{\lambda} V^{\lambda}_{O(d)} \otimes V^{\lambda' - d\mathbf{\frac{1}{2}}}_{so(2k)}, \tag{9.1}$$

$$\Lambda(\mathbb{C}^d \otimes \mathbb{C}^k) \cong \sum_{\mu} V^{\mu}_{Sp(d)} \otimes V^{\mu' - d\mathbf{\frac{1}{2}}}_{sp(2k)}, \tag{9.2}$$

where in the first sum λ is summed over all diagrams with $l(\lambda) \leqslant d$, $\lambda'_1 + \lambda'_2 \leqslant d$ and $\lambda_1 \leqslant k$, while in the second sum μ is summed over all diagrams with $l(\mu) \leqslant d/2$ and $\mu_1 \leqslant k$.

Remark 9.1. We regard $so(2k) \cong osp(2k|0)$ and $sp(2k) \cong spo(2k|0)$ and hence the labellings of their highest weights are as in Section 2.4.

Consider for positive integers d and r the decompositions $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \cong \sum_\mu V^{\mu}_{O(d)} \otimes V^{\mu + d\mathbf{\frac{1}{2}}}_{spo(2m|2n)}$ and $S(\mathbb{C}^r \otimes \mathbb{C}^{m|n}) \cong \sum_\gamma V^{\gamma}_{O(r)} \otimes V^{\gamma + r\mathbf{\frac{1}{2}}}_{spo(2m|2n)}$. We have

$$\begin{aligned} & S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \otimes S(\mathbb{C}^r \otimes \mathbb{C}^{m|n}) \\ & \quad \cong \sum_{\mu} V^{\mu}_{O(d)} \otimes V^{\mu + d\mathbf{\frac{1}{2}}}_{spo(2m|2n)} \otimes \sum_{\gamma} V^{\gamma}_{O(r)} \otimes V^{\gamma + r\mathbf{\frac{1}{2}}}_{spo(2m|2n)} \\ & \quad \cong \sum_{\mu,\gamma} (V^{\mu}_{O(d)} \otimes V^{\gamma}_{O(r)}) \otimes (V^{\mu + d\mathbf{\frac{1}{2}}}_{spo(2m|2n)} \otimes V^{\gamma + r\mathbf{\frac{1}{2}}}_{spo(2m|2n)}). \end{aligned}$$

Now writing $V^{\mu + d\mathbf{\frac{1}{2}}}_{spo(2m|2n)} \otimes V^{\gamma + r\mathbf{\frac{1}{2}}}_{spo(2m|2n)} \cong \sum_\lambda c^{\mu\gamma}_{\lambda} V^{\mu + (d+r)\mathbf{\frac{1}{2}}}_{spo(2m|2n)}$ we have therefore

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \otimes S(\mathbb{C}^r \otimes \mathbb{C}^{m|n}) \cong \sum_{\lambda,\mu,\gamma} c^{\mu\gamma}_{\lambda} (V^{\mu}_{O(d)} \otimes V^{\gamma}_{O(r)}) \otimes V^{\mu + (d+r)\mathbf{\frac{1}{2}}}_{spo(2m|2n)}. \tag{9.3}$$

On the other hand we have

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \otimes S(\mathbb{C}^r \otimes \mathbb{C}^{m|n}) \cong S(\mathbb{C}^{d+r} \otimes \mathbb{C}^{m|n})$$

$$\cong \sum_{\lambda} V^{\lambda}_{O(d+r)} \otimes V^{\lambda+(d+r)\mathbf{\frac{1}{2}}}_{spo(2m|2n)}.$$

If we let $V^{\lambda}_{O(d+r)} = \sum_{\mu,\gamma} b^{\lambda}_{\mu\gamma} V^{\mu}_{O(d)} \otimes V^{\gamma}_{O(r)}$, that is, we regard $V^{\lambda}_{O(d+r)}$ as an $O(d) \times O(r)$-module in the obvious way, then we have

$$S(\mathbb{C}^d \otimes \mathbb{C}^{m|n}) \otimes S(\mathbb{C}^r \otimes \mathbb{C}^{m|n}) \cong \sum_{\lambda,\gamma,\mu} b^{\lambda}_{\mu\gamma} (V^{\mu}_{O(d)} \otimes V^{\gamma}_{O(r)}) \otimes V^{\lambda+(d+r)\mathbf{\frac{1}{2}}}_{spo(2m|2n)}. \tag{9.4}$$

Combining (9.3) and (9.4) we see that $c^{\mu\gamma}_{\lambda} = b^{\lambda}_{\mu\gamma}$.

This connection between the branching coefficients and the tensor product coefficients, which may be regarded as a special case of Kudla's seesaw pairs [21], is of course known [14].

Now the same argument applied to the first dual pair of Theorem 9.1 tells us that $b^{\lambda}_{\mu\gamma} = a^{\mu\gamma}_{\lambda}$, where

$$V^{\mu'-d\mathbf{\frac{1}{2}}}_{so(2k)} \otimes V^{\gamma'-r\mathbf{\frac{1}{2}}}_{so(2k)} \cong \sum_{\lambda} a^{\mu\gamma}_{\lambda} V^{\lambda'-(d+r)\mathbf{\frac{1}{2}}}_{so(2k)}.$$

Taking account the fact that the $O(d)$-, $O(r)$- and $O(2d)$-modules that appear in the various decompositions may not be identical we have proved the following theorem.

Theorem 9.2. *Let μ and γ be diagrams lying in the $(m|n)$-hook and satisfying the conditions $\mu'_1 + \mu'_2 \leqslant d$ and $\gamma'_1 + \gamma'_2 \leqslant r$. Let $V^{\mu+d\mathbf{\frac{1}{2}}}_{spo(2m|2n)} \otimes V^{\gamma+r\mathbf{\frac{1}{2}}}_{spo(2m|2n)} \cong \sum_{\lambda} c^{\mu\gamma}_{\lambda} V^{\lambda+(d+r)\mathbf{\frac{1}{2}}}_{spo(2m|2n)}$. Let $k \geqslant \max(\mu_1, \gamma_1)$ and $V^{\mu'-d\mathbf{\frac{1}{2}}}_{so(2k)} \otimes V^{\gamma'-r\mathbf{\frac{1}{2}}}_{so(2k)} \cong \sum_{\lambda} a^{\mu\gamma}_{\lambda} V^{\lambda'-(d+r)\mathbf{\frac{1}{2}}}_{so(2k)}$. Then for λ lying in the $(m|n)$-hook with $\lambda'_1 + \lambda'_2 \leqslant d + r$ we have $c^{\mu\gamma}_{\lambda} = a^{\mu\gamma}_{\lambda}$. Otherwise $c^{\mu\gamma}_{\lambda} = 0$.*

We can derive the following theorem for $osp(2m|2n)$-modules in a completely analogous fashion.

Theorem 9.3. *For d and r even let μ and γ be diagrams lying in the $(m|n)$-hook with $l(\mu) \leqslant d/2$ and $l(\gamma) \leqslant r/2$. Let $V^{\mu+d\mathbf{\frac{1}{2}}}_{osp(2m|2n)} \otimes V^{\gamma+r\mathbf{\frac{1}{2}}}_{osp(2m|2n)} \cong \sum_{\lambda} c^{\mu\gamma}_{\lambda} V^{\lambda+(d+r)\mathbf{\frac{1}{2}}}_{osp(2m|2n)}$. Let $k \geqslant \max(\mu_1, \gamma_1)$ and $V^{\mu'-d\mathbf{\frac{1}{2}}}_{sp(2k)} \otimes V^{\gamma'-r\mathbf{\frac{1}{2}}}_{sp(2k)} \cong \sum_{\lambda} a^{\mu\gamma}_{\lambda} V^{\lambda'-(d+r)\mathbf{\frac{1}{2}}}_{sp(2k)}$. Then for λ lying in the $(m|n)$-hook with $l(\lambda) \leqslant (d+r)/2$ we have $c^{\mu\gamma}_{\lambda} = a^{\mu\gamma}_{\lambda}$. Otherwise $c^{\mu\gamma}_{\lambda} = 0$.*

Remark 9.2. Of course the computation of the coefficients $a^{\mu\gamma}_{\lambda}$ are in general rather difficult. There are combinatorial algorithms that in principle can be used to compute them. See for example [19,23] and references therein. It turns out that the coefficients can be computed once the usual Littlewood–Richardson coefficients (for the general linear group) are known. The precise formulas are given in [20].

Remark 9.3. The tensor product decompositions of the $spo(2m|2n)$-modules and the $osp(2m|2n)$-modules that appear in the decomposition of $S(\mathbb{C}^d \otimes \mathbb{C}^{m|n})$ are stable in the following sense. The coefficients $c_\lambda^{\mu\gamma}$ are independent of m and n for $n \geqslant 1$ and $m \geqslant d/2$. This follows from a minor modification of our argument above.

Remark 9.4. The above method for computing the tensor product decomposition using Howe duality appears to be quite general and could have further applications. For example, using the $gl(d) \times gl(m|n)$-Howe duality of Section 3 one can derive rather easily the fact that the multiplication rule of the Hook Schur functions is the same as that of ordinary Schur functions. This was derived earlier in [29] using purely combinatorial methods.

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