Continuous spectral decompositions of Abelian group actions on $C^*$-algebras

Alcides Buss $^a$, Ralf Meyer $^b, *$

$^a$ Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, 48149 Münster, Germany
$^b$ Mathematisches Institut, Georg-August-Universität Göttingen, Bunsenstr. 3–5, 37073 Göttingen, Germany

Received 9 March 2007; accepted 24 April 2007
Available online 8 June 2007
Communicated by Alain Connes

Abstract

Let $G$ be a locally compact Abelian group. Following Ruy Exel, we view Fell bundles over the Pontrjagin dual group of $G$ as continuous spectral decompositions of $G$-actions on $C^*$-algebras. We classify such spectral decompositions using certain dense subspaces related to Marc Rieffel’s theory of square-integrability. There is a unique continuous spectral decomposition if the group acts properly on the primitive ideal space of the $C^*$-algebra. But there are also examples of group actions without or with several inequivalent spectral decompositions.

Keywords: Spectral decomposition; Fell bundle; Square-integrable representation; Proper action; Generalized fixed point algebra

1. Introduction

Let $G$ be a locally compact Abelian group, let $\widehat{G}$ be its Pontrjagin dual, and let $A$ be a $C^*$-algebra with a strongly continuous action $\alpha$ of $G$; we briefly call $A$ or, more precisely, the pair $(A, \alpha)$ a $G$-$C^*$-algebra.
Suppose first that $G$ is compact, and let

$$A_\chi := \{ a \in A \mid \alpha_t(a) = \langle \chi \mid t \rangle \cdot a \text{ for all } t \in G \}$$

for $\chi \in \hat{G}$, where $\langle \chi \mid t \rangle := \chi(t)$. For $\chi = 1$, this yields the fixed-point subalgebra

$$A_1 := \{ a \in A \mid \alpha_t(a) = a \text{ for all } t \in G \}.$$ 

Since $G$ acts by $*-$automorphisms, we have

$$A_\chi \cdot A_\eta \subseteq A_{\chi \eta} \quad \text{and} \quad A_\chi^* = A_{\chi^{-1}} \quad \text{for all } \chi, \eta \in \hat{G}. \quad (2)$$

Hence the spaces $A_\chi$ for $\chi \in \hat{G}$ form a Fell bundle $A$ over $\hat{G}$ (see [3]); there is no continuity condition because $\hat{G}$ is discrete.

The cross-sectional $C^*$-algebra $C^*(A)$ of a Fell bundle $A$ comes with a canonical action of $G$ called the dual action $\hat{\alpha}$. We define it by $\hat{\alpha}_t(a_\chi) := \langle \chi \mid t \rangle \cdot a_\chi$ for $t \in G, \chi \in \hat{G}, a_\chi \in A_\chi$; often the convention $\hat{\alpha}_t(a_\chi) = \langle \chi \mid t \rangle \cdot a_\chi$ is used instead, but this does not fit with (1).

The representation theory of compact groups shows that $\bigoplus_{\chi \in \hat{G}} A_\chi$ is a dense subspace of $A$. Even more, the dense embedding $\bigoplus_{\chi \in \hat{G}} A_\chi \to A$ induces a $G$-equivariant $*$-isomorphism $C^*(A) \cong A$.

Conversely, let $A$ be any Fell bundle over $\hat{G}$ and let $A$ be its cross-sectional $C^*$-algebra, equipped with the dual action of $G$. Then the spectral decomposition (1) of $A$ recovers the original bundle $A$ (up to isomorphism). Thus $A \mapsto C^*(A)$ yields an equivalence of categories between the categories of Fell bundles over $\hat{G}$ and of $G$-$C^*$-algebras.

Now let $G$ be a non-compact Abelian group. A (continuous) spectral decomposition of a $G$-$C^*$-algebra $A$ is a Fell bundle $B$ over $\hat{G}$ with a $G$-equivariant $*$-isomorphism $C^*(B) \cong A$. Since $\hat{G}$ is no longer discrete, a Fell bundle over $\hat{G}$ is a continuous Banach bundle $(B_\chi)_{\chi \in \hat{G}}$ with algebraic operations as in (2); the topology on the bundle is specified most conveniently by a space of continuous sections (see [3]).

For non-compact $G$, spectral decompositions are harder to construct because (1) does not help: the subspaces in (1) are $\{0\}$ if $A$ is the cross-sectional $C^*$-algebra of a Fell bundle. Moreover, a Fell bundle decomposition no longer exists for general $G$-$C^*$-algebras because a $C^*$-algebra of the form $C^*(B)$ for a Fell bundle $B$ over $\hat{G}$ is never unital unless $G$ is compact.

This gives rise to the following questions:

(I) Which $G$-$C^*$-algebras have a continuous spectral decomposition?

(II) How many such decompositions are there?

The following theorem of Ruy Exel [2, Theorem 11.14] answers the first question. Let $\mathcal{M}(A)$ be the multiplier algebra of $A$.

**Theorem 1.** A $G$-$C^*$-algebra $(A, \alpha)$ has a continuous spectral decomposition if and only if there is a dense subspace $\mathcal{W} \subseteq A$ with $\mathcal{W} = \mathcal{W}^*$ and the following properties:

- $A = \bigoplus_{\chi \in \hat{G}} A_\chi$;
- $A_\chi = \{ a \in A \mid \alpha_t(a) = \langle \chi \mid t \rangle \cdot a \text{ for all } t \in G \}$;
- $A_\chi \cdot A_\eta \subseteq A_{\chi \eta}$ and $A_\chi^* = A_{\chi^{-1}}$ for all $\chi, \eta \in \hat{G}$.

To construct such a dense subspace $\mathcal{W}$, we can proceed as follows:

1. **Step 1:** Define $\mathcal{W}$ as the intersection of all closed subspaces $\mathcal{W}' \subseteq A$ that satisfy the following conditions:
   - $\mathcal{W}'$ is dense in $A$;
   - $\mathcal{W}' \cdot \mathcal{W}' \subseteq \mathcal{W}'$;
   - $\mathcal{W}'^* = (\mathcal{W}')^*$.

2. **Step 2:** Show that $\mathcal{W}$ is a $G$-$C^*$-subalgebra of $A$.

3. **Step 3:** Prove that $\mathcal{W}$ is dense in $A$.

4. **Step 4:** Verify that $\mathcal{W}^* = \mathcal{W}$.

5. **Step 5:** Establish the Fell bundle property, i.e., show that $\mathcal{W}$ is invariant under the dual action $\hat{\alpha}$.

6. **Step 6:** Confirm that $A = \bigoplus_{\chi \in \hat{G}} A_\chi$.

Using these steps, one can construct a continuous spectral decomposition of $(A, \alpha)$ as described in the theorem.

**Example:** Consider the $G$-$C^*$-algebra $A = C^*(\mathbb{T})$, where $\mathbb{T}$ is the circle group. We can take $\mathcal{W} = \{ f \in C^*(\mathbb{T}) \mid f(t) = f(-t) \text{ for all } t \in \mathbb{T} \}$.

**Step 1:** $\mathcal{W}$ is dense in $A$ because $C^*(\mathbb{T})$ is a $G$-$C^*$-algebra.

**Step 2:** $\mathcal{W} \cdot \mathcal{W} \subseteq \mathcal{W}$ because $f(t)f(-t) = f(t)f(t)$ for all $f \in \mathcal{W}$.

**Step 3:** $\mathcal{W}^* = \mathcal{W}$ because $C^*(\mathbb{T})$ has the same topological properties as $\mathcal{W}$.

**Step 4:** $A = \mathcal{W}$ because every element of $A$ has a Fourier transform.

**Step 5:** $\mathcal{W}$ is invariant under the dual action $\hat{\alpha}$.

**Step 6:** $A = \bigoplus_{\chi \in \hat{G}} A_\chi$ because $A_\chi = C^*(\mathbb{T})_\chi$.

Thus, $\mathcal{W}$ is a continuous spectral decomposition of $A = C^*(\mathbb{T})$.
(integrability) for all $a \in \mathcal{W}$ and $\chi \in \hat{G}$, the integrals

$$E_\chi(a) := \int_G \alpha_t(a) \cdot \langle \chi | t \rangle \, dt$$

exist unconditionally in the strict topology on $\mathcal{M}(A)$;

(relative continuity) for all $a, b \in \mathcal{W}$, we have

$$\lim_{\zeta \to 1} \| E_{\chi \zeta}(a) \cdot E_\eta(b) - E_\chi(a) \cdot E_{\zeta \eta}(b) \| = 0$$

uniformly in $\chi, \eta \in \hat{G}$.

If $A$ is of the form $C^*(B)$ for some Fell bundle $B$ over $\hat{G}$ and $\mathcal{W}$ is the linear span of products $a \ast a^*$ for compactly supported continuous sections $a$, then these conditions are satisfied by Proposition 10.2 in [2].

It was not clear to Exel whether the second, technical condition in Theorem 1 is really necessary (compare [2, Question 11.16]); its necessity was established by the second author in [6].

The goal of this article is to classify the continuous spectral decompositions of a $G$-$C^*$-algebra $A$; we show that they correspond bijectively to dense subspaces $\mathcal{W}$ with the properties in Theorem 1 that are also “complete” in a suitable sense.

We use this to describe all continuous spectral decompositions of $\mathbb{K}(L^2G)$ with $G$-action induced by the regular representation of $G$; these are closely related to line bundles on open subsets of $\hat{G}$ of full measure (see Section 4.1). We will exhibit non-isomorphic Fell bundles that provide continuous spectral decompositions of $\mathbb{K}(L^2G)$.

Instead of integrable elements as in Theorem 1, we prefer to use square-integrable elements as in [6]. A subspace $\mathcal{R} \subseteq A$ is square-integrable and relatively continuous if and only if

$$\mathcal{W} := \text{span}\{aa^* | a \in \mathcal{R}\}$$

satisfies the two conditions in Theorem 1. We call $\mathcal{R}$ complete if $\mathcal{R}$ is a $G$-invariant right ideal in $A$ and closed with respect to the norm

$$\|\xi\|_{si} := \|\xi\| + \left\|E_1(\xi\xi^*)\right\|^{1/2}.$$  

A continuously square-integrable $G$-$C^*$-algebra is a pair $(A, \mathcal{R})$ consisting of a $G$-$C^*$-algebra $A$ and a dense, relatively continuous, complete subspace of square-integrable elements $\mathcal{R}$ [6].

If $B$ is a Fell bundle over $\hat{G}$, then

$$(A, \mathcal{R}) := \left( C^*(B), \overline{C_c(B)}^{si} \right)$$

is a continuously square-integrable $G$-$C^*$-algebra; here $\overline{C_c(B)}^{si}$ denotes the closure of $C_c(B)$ with respect to the si-norm. Conversely, given a continuously square-integrable $G$-$C^*$-algebra $(A, \mathcal{R})$, we use results of Exel [2] to construct a spectral decomposition $\mathcal{B}(A, \mathcal{R})$ of $A$. Combining techniques from [2] and [6], we show that the isomorphism $A \cong C^*(\mathcal{B}(A, \mathcal{R}))$ maps $\mathcal{R}$ isomorphically onto $\overline{C_c(\mathcal{B}(A, \mathcal{R}))}^{si}$.  

If \((A, \mathcal{R})\) is already of the form \((C^*\mathcal{B}, \overline{C_c(\mathcal{B})}^{\text{si}})\) for some Fell bundle \(\mathcal{B}\), then we construct a canonical isomorphism \(\mathcal{B} \cong B(A, \mathcal{R})\). As a result, the constructions
\[
\mathcal{B} \mapsto (C^*(\mathcal{B}), \overline{C_c(\mathcal{B})}^{\text{si}}), \quad (A, \mathcal{R}) \mapsto B(A, \mathcal{R})
\]
are inverse to each other. They provide an equivalence between the categories of Fell bundles over \(\hat{G}\) and of continuously square-integrable \(G\)-\(C^*\)-algebras (see Theorem 38).

The main object of interest in [6] is the generalized fixed-point algebra of \((A, \mathcal{R})\), which makes sense for non-Abelian groups as well. In our context, it agrees with the unit fiber \(B_1(A, \mathcal{R})\) of the Fell bundle \(B(A, \mathcal{R})\). Thus the information needed to construct the generalized fixed-point algebra already yields the whole spectral decomposition. But we cannot reconstruct \(\mathcal{R}\) from the generalized fixed-point algebra: there are continuous spectral decompositions of \(K(L^2G)\) with non-isomorphic Fell bundles but with the same generalized fixed-point algebra.

A \(G\)-\(C^*\)-algebra is called spectrally proper if the induced action of \(\hat{G}\) on its primitive ideal space is proper. In this case, it is shown in [6] that there is a unique dense, relatively continuous, complete subspace. As a consequence, there is a unique continuous spectral decomposition. We consider this case in Section 4.2 and specialize even further to \(G\)-\(C^*\)-algebras of the form \(C_0(X)\) for a proper \(G\)-space \(X\). We show that this category of \(G\)-\(C^*\)-algebras is equivalent to the category of commutative Fell bundles.

Marc Rieffel introduced square-integrability as a non-commutative generalization of proper actions on spaces [8,9]. The results in Section 4.2 show once more that proper actions have special properties that are not captured by (square-)integrability. As already suggested in [5,6], square-integrability is closer to stability than to properness.

2. Preliminaries

In this section, we review some relevant results of [1,2,5,6,9] and fix our notation and conventions.

Let \(G\) be a locally compact group and let \((B, \beta)\) be a \(G\)-\(C^*\)-algebra. The completion of \(C_c(G, B)\) with respect to the \(B\)-valued inner product
\[
\langle f | g \rangle_B := \int_G f(t)^* \cdot g(t) \, dt
\]
for \(f, g \in C_c(G, B)\) is a \(G\)-equivariant Hilbert \(B\)-module—briefly Hilbert \(B, G\)-module—for the \(G\)-action
\[
(\delta_t f)(s) := \beta_t(f(t^{-1}s)) \quad \forall s, t \in G, f \in C_c(G, B);
\]
we denote this Hilbert \(B, G\)-module by \(L^2(G, B)\). It is the prototype for square-integrable Hilbert \(B, G\)-modules.

2.1. Square-integrability

Let \((\mathcal{E}, \gamma)\) be a Hilbert \(B, G\)-module. The following constructions are taken from [6]. Given \(\xi \in \mathcal{E}\), we define
where \( \cdot \) denotes the right \( B \)-action on \( \mathcal{E} \). We call \( \xi \in \mathcal{E} \) square-integrable if \( \langle \langle \xi | \eta \rangle \rangle \in L^2(G, B) \) for all \( \eta \in \mathcal{E} \). Then \( \langle \langle \xi \rangle \rangle \) becomes an adjointable operator \( \mathcal{E} \to L^2(G, B) \), whose adjoint extends \( \langle \langle \xi \rangle \rangle \) to an adjointable operator \( L^2(G, B) \to \mathcal{E} \); we denote these extensions by \( \langle \langle \xi \rangle \rangle \) as well. We also write \( \langle \langle \xi | \eta \rangle \rangle := \langle \langle \xi | \circ | \eta \rangle \rangle \), where \( \circ \) denotes the composition. Conversely, if \( \langle \langle \xi \rangle \rangle \) extends to an adjointable operator \( L^2(G, B) \to \mathcal{E} \), then \( \xi \) is square-integrable [6]; a bounded extension of \( \langle \langle \xi \rangle \rangle \) does not suffice for this.

**Remark 2.** What does \( \langle \langle \xi | \eta \rangle \rangle \in L^2(G, B) \) mean?

Let \( (w_i)_{i \in I} \) be a net of continuous, compactly supported functions \( G \to [0, 1] \) with \( w_i(t) \to 1 \) uniformly on compact subsets of \( G \). We call \( f \in C_b(G, B) \) square-integrable and write \( f \in L^2(G, B) \) if the net \( (w_i \cdot f)_{i \in I} \) in \( C_c(G, B) \) is a Cauchy net in \( L^2(G, B) \). We identify \( f \) with the limit of this net in \( L^2(G, B) \). It follows from Proposition 12 that this definition does not depend on the choice of the net \( (w_i)_{i \in I} \) and that the continuity of the functions \( w_i \) is irrelevant; it suffices if the functions \( w_i \) are measurable and uniformly bounded.

Let \( \mathcal{E}_{si} \subseteq \mathcal{E} \) be the subspace of square-integrable elements. We call \( \mathcal{E} \) square-integrable if \( \mathcal{E}_{si} \) is dense in \( \mathcal{E} \). We equip \( \mathcal{E}_{si} \) and its subspaces with the norm

\[
\| \xi \|_{si} := \| \xi \| + \| \langle \langle \xi \rangle \rangle \| = \| \xi \| + \| \langle \langle \xi | \rangle \rangle \|^{1/2} + \| \langle \langle \xi | \xi \rangle \rangle \|^{1/2}.
\]

This turns \( \mathcal{E}_{si} \) into a Banach space. The right \( B \)-module structure satisfies \( \| \xi \cdot b \|_{si} \leq \| \xi \|_{si} \cdot \| b \| \) for all \( \xi \in \mathcal{E}_{si} \), \( b \in B \). It is unclear whether \( \mathcal{E}_{si} \) is an essential \( B \)-module, that is, whether \( \mathcal{E}_{si} \cdot B \) is dense in \( \mathcal{E}_{si} \). The group \( G \) acts on \( \mathcal{E}_{si} \) by bounded operators; this action need not be continuous and is uniformly bounded if and only if \( G \) is unimodular. The convolution turns \( C_c(G, B) \) into an algebra. We furnish \( G \)-invariant \( B \)-submodules of \( \mathcal{E} \) such as \( \mathcal{E}_{si} \) with the following \( C_c(G, B) \)-module structure (see [6, Eq. (24)]):

\[
\xi \ast K := \int_G \gamma_t(\xi \cdot K(\gamma_t^{-1})) \, dt \quad \forall \xi \in \mathcal{E}, \ K \in C_c(G, B).
\]

**Definition 3.** A subspace \( \mathcal{R} \subseteq \mathcal{E}_{si} \) is called complete if it is closed (for the si-norm) and a \( C_c(G, B) \)-submodule for the module structure defined in (3). The completion of a subset \( \mathcal{R} \subseteq \mathcal{E}_{si} \) is the smallest complete subset containing \( \mathcal{R} \) or, more explicitly, the si-norm closed linear span of \( \mathcal{R} \cup \mathcal{R} \ast C_c(G, B) \).

It was already noticed by Marc Rieffel [9] that \( \mathcal{E}_{si} \) is too big to construct a generalized fixed-point algebra. We need relatively continuous subsets of \( \mathcal{E}_{si} \). This notion uses the faithful *-representation

\[
\rho : C^*_r(G, B) \to \mathcal{B}(L^2(G, B)), \quad K \mapsto \rho_K,
\]
(ρ_K f)(t) := \int_G \beta_t(K(t^{-1}s)) \cdot f(s) \, ds \quad \forall f, K \in C_c(G, B), \ t \in G \tag{4}

(see [6, Eq. (10)]). Let \mathbb{B}^G(L^2(G, B)) \subseteq \mathbb{B}(L^2(G, B)) be the subalgebra of \textit{G}-equivariant adjointable operators on \textit{L}^2(G, B). The range of \rho is contained in \mathbb{B}^G(L^2(G, B)). We always identify \textit{C}_r^*(G, B) with its image in \mathbb{B}^G(L^2(G, B)).

Definition 4. A subset \mathcal{R} \subseteq \mathcal{E}_si is \textit{relatively continuous} if \langle \xi | \eta \rangle \in \mathcal{C}_r^*(G, B) for all \xi, \eta \in \mathcal{R}.

Proposition 5. Let \mathcal{E} be a \textit{G}-equivariant Hilbert \textit{B}, \textit{G}-module, and let \mathcal{R} \subseteq \mathcal{E}_si be a relatively continuous subspace.

If \mathcal{R} is complete, then the action of \textit{G} on \mathcal{E} restricts to a continuous action on \mathcal{R}, and \mathcal{R} is an essential right \textit{B}-module, that is, \mathcal{R} \cdot \textit{B} = \mathcal{R}.

Conversely, \mathcal{R} is complete if it is si-norm closed, \textit{G}-invariant, and a \textit{B}-submodule.

Proof. The first assertion is Proposition 6.4 in [6]. For the converse assertion, we notice first that the actions of \textit{G} and \textit{B} on \mathcal{R} are continuous because they are continuous on the completion of \mathcal{R} by the first assertion. Hence the function \textit{G} \ni t \mapsto \gamma_t(\xi \cdot K(t^{-1})) \in \mathcal{R} for \xi \in \mathcal{R}, \ K \in C_c(G, B) is a \langle \xi | \eta \rangle-\textit{continuous} function of compact support. Since \mathcal{R} is closed, the integral \textit{ξ} \ast K in (3) belongs to \mathcal{R} as well; thus \mathcal{R} is a closed \textit{C}_c(G, B)-submodule of \mathcal{E}_si as desired. \qed

Corollary 6. Let \mathcal{E} be a Hilbert \textit{B}, \textit{G}-module and let \mathcal{R} \subseteq \mathcal{E}_si be relatively continuous. Suppose that \mathcal{R} is \textit{G}-invariant and \mathcal{R} \cdot \textit{D} \subseteq \overline{\mathcal{R}_si} for a dense subset \textit{D} of \textit{B}. Then the completion of \mathcal{R} is \overline{\mathcal{R}_si}.

Proof. The si-norm closure \overline{\mathcal{R}_si} is closed by construction, \textit{G}-invariant because \mathcal{R} is, and a \textit{B}-submodule as well. Proposition 5 shows that it is the smallest complete subspace containing \mathcal{R}. \qed

Given a relatively continuous subset \mathcal{R} \subseteq \mathcal{E}, we define

\mathcal{F}(\mathcal{E}, \mathcal{R}) := \overline{\text{span}(\mathcal{R})} \circ \textit{C}_r^*(G, B) \subseteq \mathbb{B}^G(L^2(G, B), \mathcal{E}), \tag{5}

where \mathbb{B}^G(L^2(G, B), \mathcal{E}) denotes the space of equivariant, adjointable operators \textit{L}^2(G, B) \rightarrow \mathcal{E}. Proposition 6.1 in [6] asserts that

\mathcal{F}(\mathcal{E}, \mathcal{R}) \circ \textit{C}_r^*(G, B) \subseteq \mathcal{F}(\mathcal{E}, \mathcal{R}), \quad \mathcal{F}(\mathcal{E}, \mathcal{R})^* \circ \mathcal{F}(\mathcal{E}, \mathcal{R}) \subseteq \textit{C}_r^*(G, B),

so that \mathcal{F}(\mathcal{E}, \mathcal{R}) becomes a Hilbert \textit{C}_r^*(G, B)-module in an obvious way [6]. Furthermore, if \mathcal{R} is dense, then \mathcal{F}(\mathcal{E}, \mathcal{R}) is essential in the sense that the linear span of \mathcal{F}(\mathcal{E}, \mathcal{R})(\textit{L}^2(G, B)) is dense in \mathcal{E}.

We use an approximate identity in \textit{C}_r^*(G, B) to show \langle \xi \rangle \in \mathcal{F}(\mathcal{E}, \mathcal{R}) for all \xi \in \mathcal{R}. If \mathcal{R} is complete, then \mathcal{F}(\mathcal{E}, \mathcal{R}) = \overline{\mathcal{R}}. If \mathcal{R} is not complete, we may replace it by its completion.

Definition 7. A \textit{continuously square-integrable} Hilbert \textit{B}, \textit{G}-module is a pair \langle \mathcal{E}, \mathcal{R} \rangle consisting of a Hilbert \textit{B}, \textit{G}-module \mathcal{E} and a dense subspace \mathcal{R} \subseteq \mathcal{E} that is contained in \mathcal{E}_si and is complete and relatively continuous.
The generalized fixed-point algebra \( \text{Fix}(\mathcal{E}, \mathcal{R}) \) is the closed linear span of \( |\mathcal{R}| \langle \langle \mathcal{R} | \) in \( \mathbb{B}^G(\mathcal{E}) \), the \( C^* \)-algebra of \( G \)-equivariant adjointable operators on \( \mathcal{E} \).

There is a canonical isomorphism \( \text{Fix}(\mathcal{E}, \mathcal{R}) \cong \mathbb{K}(\mathcal{F}(\mathcal{E}, \mathcal{R})) \).

**Definition 8.** Let \( (\mathcal{E}, \mathcal{R}) \) and \( (\mathcal{E}', \mathcal{R}') \) be continuously square-integrable Hilbert \( B,G \)-modules. An adjointable operator \( T \in \mathbb{B}(\mathcal{E}, \mathcal{E}') \) is called \( \mathcal{R} \)-continuous if \( T(\mathcal{R}) \subseteq \mathcal{R}' \) and \( T^*(\mathcal{R}') \subseteq \mathcal{R} \).

By definition, the \( \mathcal{R} \)-continuous \( G \)-equivariant operators are the morphisms in the category of continuously square-integrable Hilbert \( B,G \)-modules. It follows from Theorem 6.2 in [6] that \( (\mathcal{E}, \mathcal{R}) \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R}) \) is an equivalence between the categories of continuously square-integrable Hilbert \( B,G \)-modules and of Hilbert \( C^*_r(G, B) \)-modules. Hence it induces a bijection between isomorphism classes of continuously square-integrable Hilbert modules and Hilbert \( C^*_r(G, B) \)-modules, respectively.

We are particularly interested in the following special situation.

**Definition 9.** A continuously square-integrable \( G,C^* \)-algebras (for a locally compact group \( G \)) is a triple \( (A, \alpha, \mathcal{R}) \), where \( (A, \alpha) \) is a \( C^* \)-algebra and \( \mathcal{R} \subseteq A_{si} \) is complete and relatively continuous and dense in \( A \). Here we view \( A \) as a Hilbert \( A,G \)-module as usual.

The continuously square-integrable \( G,C^* \)-algebras form a category: morphisms \( (A_1, \mathcal{R}_1) \to (A_2, \mathcal{R}_2) \) are \( G \)-equivariant *-homomorphisms \( f : A_1 \to A_2 \) with \( f(\mathcal{R}_1) \subseteq \mathcal{R}_2 \).

**Remark 10.** Since continuously square-integrable Hilbert \( B,G \)-modules correspond to Hilbert \( C^*_r(G, B) \)-modules, a continuously square-integrable \( G,C^* \)-algebra is equivalent to a quadruple \( (A, \alpha, \mathcal{F}, \phi) \), where \( (A, \alpha) \) is as above, \( \mathcal{F} \) is a Hilbert \( C^*_r(G, A) \)-module, and \( \phi \) is an isomorphism of Hilbert \( A,G \)-modules

\[
\mathcal{F} \otimes_{C^*_r(G, A)} L^2(G, A) \cong A.
\]

This description is particularly useful if \( G \) is Abelian and \( A = \mathbb{K}(\mathcal{H}) \) for some \( G \)-Hilbert space \( \mathcal{H} \) because then \( C^*_r(G, A) \) is Morita–Rieffel equivalent to \( C^*_r(G, \mathbb{C}) \cong C_0(\hat{G}) \), so that \( C^*_r(G, A) \)-Hilbert modules are easy to classify (see Section 4.1).

### 2.2. Integrable versus square-integrable elements

We mainly work with square-integrable elements, whereas Exel [1,2] uses integrable elements. Since the construction of Fell bundles is rather technical, we do not want to redo too many of Exel’s constructions. Instead, we use the equivalence between the approaches via integrable and square-integrable elements, which goes back to Rieffel [9].

**Definition 11.** (See [1].) A function \( f : G \to A \) is **locally integrable** if it is Bochner integrable over every measurable, relatively compact subset \( K \subseteq G \), that is, \( f|_K \in L^1(K, A) \). It is **unconditionally integrable** if it is locally integrable and the net \( (\int_K f(t) \, dt)_{K \in \mathcal{K}} \) of Bochner integrals converges in the norm topology of \( A \), where \( \mathcal{K} \) denotes the directed set of measurable relatively compact subsets of \( G \) ordered by inclusion. The limit of this net is called the **unconditional integral** and is denoted by \( \int^u_G f(t) \, dt \).
Proposition 12. Let \( f : G \to A \) be a locally integrable function with \( f(t) \geq 0 \) (in \( A \)) for all \( t \in G \). Then the following are equivalent:

- \( f \) is unconditionally integrable;
- \( (\int_G w_i(t) \cdot f(t) \, dt)_{i \in I} \) converges in \( A \) for all nets \( (w_i)_{i \in I} \) as in Remark 2;
- \( (\int_G w_i(t) \cdot f(t) \, dt)_{i \in I} \) converges in \( A \) for some net \( (w_i)_{i \in I} \) of measurable, compactly supported functions \( w_i : G \to [0, 1] \) with \( w_i(t) \to 1 \) uniformly on compact subsets of \( G \).

Furthermore, in this case we have

\[
\lim_{i \in I} \int_G w_i(t) \cdot f(t) \, dt = \int_G f(t) \, dt.
\]

Proof. The first condition implies the other conditions by Proposition 3.4 in [2]. For the converse, it is enough to prove that the third condition implies the first one. Let \( a \in A_+ \) be the limit of the net \( (\int_G w_i(t) \cdot f(t) \, dt)_{i \in I} \). Using the locally uniform convergence \( w_i \to 1 \), we get \( \int_K f(t) \, dt \leq a \) for any measurable relatively compact subset \( K \subseteq G \). Given \( i \in I \), we set \( K_i := \text{supp}(w_i) \). Since \( w_i \leq 1 \), we get

\[
0 \leq \int_G w_i(t) \cdot f(t) \, dt \leq \int_{K_i} f(t) \, dt \leq \int_K f(t) \, dt \leq a
\]

for any measurable relatively compact subset \( K \) of \( G \) that contains \( K_i \). The assertion follows. \( \square \)

Remark 13. The first two conditions in Proposition 12 remain equivalent for any locally integrable function \( G \to A \), that is, no positivity assumption is needed for this. We omit the proof.

Let \((A, \alpha)\) be a \( G\)-\( C^*\)-algebra.

Definition 14. Let \( A_+ \subseteq A \) be the subset of positive elements. We call \( a \in A_+ \) integrable if the function \( G \ni t \mapsto \alpha_t(a) \in A \) is strictly-unconditionally integrable, that is, if the functions

\[
G \to A, \quad t \mapsto \alpha_t(a) \cdot b, \quad t \in G \mapsto b \cdot \alpha_t(a)
\]

are unconditionally integrable for each \( b \) in \( A \). Let \( \int_G^\text{su} \alpha_t(a) \, dt \) be the multiplier \((L, R)\) of \( A \) given by

\[
L(b) = \int_G^u \alpha_t(a) \cdot b \, dt, \quad R(b) = \int_G^u b \cdot \alpha_t(a) \, dt.
\]

Let \( A_i \subseteq A \) be the subspace spanned by the integrable elements in \( A_+ \).

Remark 15. A positive element belongs to \( A_i \) if and only if it is integrable.
Let \((E, \gamma)\) be a Hilbert \(B, G\)-module. Let \(A := \mathbb{K}(E)\) be the \(G\)-C*-algebra of compact operators on \(E\) with \(G\) acting by \(\alpha_t(a) := \gamma_t \circ a \circ \gamma_t^{-1}\) for all \(a \in \mathbb{K}(E)\) and \(t \in G\). For \(\xi, \eta \in E\), we define a compact operator
\[
|\xi\rangle\langle\eta| : E \to E, \quad \zeta \mapsto \xi \cdot \langle \eta | \zeta \rangle \Bigr|
\]

We have \(\alpha_t(|\xi\rangle\langle\eta|) = |\gamma_t\xi\rangle\langle\gamma_t\eta|\) for all \(\xi, \eta \in E\). Let
\[
M : L^\infty(G) \to \mathbb{B}(L^2(G, B)), \quad \phi \mapsto M\phi,
\]
be the representation by pointwise multiplication operators.

**Proposition 16.** Let \(E\) be a Hilbert \(B, G\)-module.

(i) If \(\xi, \eta \in E_{\text{si}}\) and \(\phi \in L^\infty(G)\), then \(|\xi\rangle\langle\eta| \in \mathbb{K}(E)_{\text{i}}\) and
\[
\sup_G \int \phi(t) \cdot \alpha_t(|\xi\rangle\langle\eta|) \, dt = |\xi\rangle \circ M\phi \circ \langle\eta| \quad \text{in } \mathbb{B}(E).
\]

(ii) Conversely, if \(|\xi\rangle\langle\xi| \in \mathbb{K}(E)_{\text{i}}\) then \(\xi \in E_{\text{si}}\).

Thus \(\mathcal{R} \subseteq E_{\text{si}}\) if and only if \(|\mathcal{R}\rangle\langle\mathcal{R}| := |\xi\rangle\langle\eta| \mid \xi, \eta \in \mathcal{R}\| \subseteq \mathbb{K}(E)_{\text{i}}\).

**Proof.** Polarization allows us to assume that \(\xi = \eta\) and \(\phi \geq 0\). Let \((w_i)_{i \in I}\) be a net as in Remark 2. Consider the net
\[
I_i := \int_G w_i(t) \cdot \alpha_t(|\xi\rangle\langle\eta|) \, dt
\]
in \(\mathbb{K}(E)\). For \(\zeta \in E\), we compute
\[
I_i(\zeta) = \int_G w_i(t) \cdot \gamma_t(\xi) \langle \gamma_t(\eta) | \zeta \rangle \, dt = \int_G \gamma_t(\xi) \cdot \langle \eta | \gamma_t(\eta) (t) \rangle \, dt
\]
\[
= |\xi\rangle \langle \eta | (w_i \cdot \langle \eta | (\zeta) (t) \rangle = |\xi\rangle \circ M\gamma_t \circ \langle \eta | (\zeta).
\]
Hence \((I_i)_{i \in I}\) is a bounded net with \(\| I_i \| \leq \| |\xi\rangle\| \cdot \| \langle \eta | \| \); it converges pointwise to \(|\xi\rangle\langle\eta|\) because \(|\xi\rangle\langle \eta |(w_i \cdot \langle \eta | (\zeta) (t) \rangle)\) converges to \(|\xi\rangle\langle \eta | (\gamma_t(\eta) (t) \rangle\) by Remark 2. It follows that \((I_i)_{i \in I}\) converges in the strict topology of \(\mathbb{K}(E)\) to \(|\xi\rangle\langle\eta|\) by Remark 2. It follows that \((I_i)_{i \in I}\) converges in the strict topology of \(\mathbb{K}(E)\) to \(|\xi\rangle\langle\eta|\). Since \(\xi = \eta\), we have \(|\xi\rangle\langle\eta| \geq 0\), so that Proposition 12 applies. We get \(|\xi\rangle\langle\eta| \in \mathbb{K}(E)_{\text{i}}\) and
\[
\sup_G \int \alpha_t(|\xi\rangle\langle\eta|) \, dt = |\xi\rangle\langle\eta|\]
as desired. The statement with \(\phi \in L^\infty(G)\) is proved similarly. This finishes the proof of (i). Assertion (ii) is Lemma 8.1(v) in [5]. \(\square\)
Let \((A, \alpha)\) be a \(G\)-\(C^*\)-algebra viewed as a Hilbert module over itself. Then the map \(\langle a | b \rangle \mapsto a \cdot b^*\) induces an isomorphism \(\mathbb{K}(E) \cong A\). Hence Proposition 16 and polarization yield
\[
\mathcal{R} \subseteq A_{si} \iff \{a \cdot a^* | a \in \mathcal{R}\} \subseteq A_i.
\]

2.3. Relative continuity via Fourier coefficients

From now on, we suppose that \(G\) is Abelian, and we let \(\hat{G}\) be its dual group. To emphasize the symmetry between \(G\) and \(\hat{G}\) we write \(\langle \chi | t \rangle := \chi(t)\) for \(\chi \in \hat{G}, t \in G\). The following results are due to Exel [2].

The literature contains several conventions for the dual action and the Fourier transform, which differ by the inversion automorphism \(\chi \mapsto \chi^{-1} = \overline{\chi}\) on \(\hat{G}\). We follow the conventions of [1], which are different from those of [2]. Thus the Fourier transform of a function \(f : G \to \mathbb{C}\) is defined by
\[
\hat{f} : \hat{G} \to \mathbb{C}, \quad \hat{f}(\chi) := \int_G f(t) \cdot \overline{\langle \chi | t \rangle} \, dt. \tag{6}
\]

**Definition 17.** (See [2, Definition 6.1].) Let \(A\) be a \(G\)-\(C^*\)-algebra. For \(a \in A_i\) and \(\chi \in \hat{G}\), we define
\[
\hat{a}(\chi) = E_{\chi}(a) := \sup_{G} \alpha_t(a) \cdot \overline{\langle \chi | t \rangle} \, dt \in \mathcal{M}(A).
\]

The resulting function \(\hat{a} : \hat{G} \to \mathcal{M}(A)\) is bounded and uniformly continuous for the strict topology on \(\mathcal{M}(A)\) by [2, Propositions 6.2, 6.3].

We denote the unique extension of \(\alpha\) to \(\mathcal{M}(A)\) again by \(\alpha\); this extension need not be continuous any more. The \(\chi\)-spectral subspace \(\mathcal{M}_\chi(A)\) of \(\mathcal{M}(A)\) is defined by
\[
\mathcal{M}_\chi(A) := \left\{ m \in \mathcal{M}(A) | \alpha_t(m) = \langle \chi | t \rangle \cdot m \text{ for all } t \in G \right\}
\]
as in (1). It is easy to see that \(E_{\chi}(a) \in \mathcal{M}_\chi(A)\), that is,
\[
\alpha_t(E_{\chi}(a)) = \langle \chi | t \rangle \cdot E_{\chi}(a) \quad \text{for all } t \in G, \ \chi \in \hat{G}, \ a \in A_i. \tag{7}
\]

**Definition 18.** (See [2, Definition 8.1].) Let \(A\) be a \(G\)-\(C^*\)-algebra. A pair of integrable elements \((a, b)\) is relatively continuous if
\[
\lim_{\eta \to 1} \left\| E_{\chi \eta}(a) \cdot E_{\zeta}(b) - E_{\chi}(a) \cdot E_{\eta \zeta}(b) \right\| = 0 \quad \text{uniformly in } \chi, \zeta \in \hat{G}.
\]
We write \(a \sim b\) in this case. A subset \(\mathcal{W} \subseteq A_i\) is called relatively continuous if \(a \sim b\) for all \(a, b \in \mathcal{W}\).

Definitions 4 and 18 define relative continuity for square-integrable and integrable elements in terms of \(G\) and \(\hat{G}\), respectively. We need some preparations to relate these two notions.
Let $B$ be a $G$-$C^*$-algebra and let $\chi \in \hat{G}$. We let $M_\chi$ be the pointwise multiplication operator

$$(M_\chi f)(t) := \langle \chi | t \rangle \cdot f(t) \quad \text{for} \quad f \in L^2(G, B), \ t \in G.$$  

This defines a strongly continuous unitary representation $\chi \mapsto M_\chi$ of $\hat{G}$ on $L^2(G, B)$; this operator is denoted $V_\chi$ in [2].

Recall that the dual action $\hat{\beta}$ of $\hat{G}$ on the crossed product $C^*_r(G, B)$ is defined by continuous extension of the action $(\hat{\beta}_\chi f)(t) := \langle \chi | t \rangle \cdot f(t)$ for $f \in C_c(G, B)$, $t \in G$, $\chi \in \hat{G}$. The representation $\rho$ in (4) relates $\hat{\beta}$ and the operators $M_\chi$:  

$$M_\chi \circ \rho_K \circ M_\chi^* = \rho_{\hat{\beta}_\chi(K)} \quad \forall \chi \in \hat{G}, \ K \in C^*_r(G, B);$$  

(8)

it suffices to check this for $K \in C_c(G, B)$. In particular, we get

$$M_\chi \circ \hat{\beta} \circ M_\chi^* \in C^*_r(G, B) \quad \forall \chi \in \hat{G}.$$  

(9)

Let $E$ be a Hilbert $B, G$-module for a $G$-$C^*$-algebra $B$. We apply the above definitions and results to $A = \mathbb{K}(E)$, so that $M(A) = \mathbb{B}(E)$ with $G$ acting by conjugation: $\alpha_t(T) = \gamma_t \circ T \circ \gamma_t^{-1}$ for all $T \in \mathbb{B}(E), t \in G$. Proposition 16 and Eq. (7) yield

$$E_\chi(|\xi\rangle\langle\xi|) = |\xi\rangle \circ M_\chi \circ \langle\langle\eta|,$$  

(10)

$$\gamma_t \circ E_\chi(|\xi\rangle\langle\eta|) \circ \gamma_t^{-1} = \langle \chi | t \rangle \cdot E_\chi(|\xi\rangle\langle\eta|)$$  

(11)

for all $\chi \in \hat{G}$, $\xi, \eta \in E_{si}$, $t \in G$ (compare [2, Lemma 7.4]). The following proposition relates our two notions of relative continuity in terms of $G$ and $\hat{G}$ and is one of the main technical results in [2].

**Proposition 19.** Let $G$ be Abelian and let $E$ be a Hilbert $B, G$-module for a $G$-$C^*$-algebra $B$. If $\xi, \eta \in E_{si}$, then

$$|\xi\rangle\langle\xi| \xrightarrow{\text{rc}} |\eta\rangle\langle\eta| \quad \iff \quad \langle\langle\xi | \eta\rangle \in C^*_r(G, B).$$

(12)

A set of square-integrable elements $\mathcal{R} \subseteq E_{si}$ is relatively continuous (Definition 4) if and only if the set of integrable elements

$$\mathcal{W}_\mathcal{R} := \{|\xi\rangle\langle\eta| \mid \xi, \eta \in \mathcal{R}\} \subseteq \mathbb{K}(E)_i$$

is relatively continuous (Definition 18).

**Proof.** The first assertion follows as in the proof of [2, Theorem 7.5] with appropriate changes of notation, using (10). It implies the second assertion because the relative continuity of $\mathcal{W}_\mathcal{R}$ is equivalent to the relative continuity of $\{|\xi\rangle\langle\xi| \mid \xi \in \mathcal{R}\}$ by polarization. \qed
2.4. Spectral invariance

As before, the group $G$ is Abelian with dual group $\hat{G}$, and $A = \mathbb{K}(\mathcal{E})$ for a Hilbert $B, G$-module $(\mathcal{E}, \gamma)$ and a $G$-$C^*$-algebra $(B, \beta)$, with $a_t(a) = \gamma_t \circ a \circ \gamma_t^{-1}$ for all $t \in G, a \in A$.

**Lemma 20.** Given $\xi, \eta \in \mathcal{E}_{\text{si}}$ and $\chi \in \mathcal{G}$, let $X := E_X(|\xi\rangle\langle \eta|) \in \mathcal{B}(\mathcal{E})$. Then $X(\mathcal{E}_{\text{si}}) \subseteq \mathcal{E}_{\text{si}}$ and

$$|X(\zeta)| = X \circ |\xi\rangle \circ M^*_X = |\xi\rangle \circ M_X \circ \langle \eta | \zeta\rangle \circ M^*_X$$

(13)

for all $\zeta \in \mathcal{E}_{\text{si}}$.

**Proof.** Equation (11) yields $\gamma_t(X(\zeta)) = \langle \chi | t \rangle \cdot X(\gamma_t(\zeta))$. Now we verify (13) on $f \in C_c(G, B)$:

$$|X(\zeta)| = X \circ |\xi\rangle \circ M^*_X = |\xi\rangle \circ M_X \circ \langle \eta | \zeta\rangle \circ M^*_X = X \circ |\xi\rangle \circ M^*_X(f) = X \circ |\xi\rangle \circ M^*_X(f).$$

Since $\zeta \in \mathcal{E}_{\text{si}}$ and $M_X \in \mathcal{B}(L^2(G, B))$, this extends to an adjointable operator $L^2(G, B) \to \mathcal{E}$. Hence $X(\zeta) \in \mathcal{E}_{\text{si}}$ and the first equality in (13) holds. The second one follows from (10).

**Definition 21.** A subset $\mathcal{W} \subseteq A_1$ is spectrally invariant if $E_X(a) \cdot b$ and $b \cdot E_X(a)$ belong to $\mathcal{W}$ for all $a, b \in \mathcal{W}$, $\chi \in \mathcal{G}$.

**Proposition 22.** Let $\mathcal{R} \subseteq \mathcal{E}_{\text{si}}$ be relatively continuous and complete, and define $\mathcal{W}_{\mathcal{R}}$ as in (12). Let $\xi, \eta \in \mathcal{R}$, $\chi \in \mathcal{G}$, and define $X$ as in Lemma 20. Then $X(\mathcal{R}) \subseteq \mathcal{R}$ and $X^*(\mathcal{R}) \subseteq \mathcal{R}$. Hence $X$ is $\mathcal{R}$-continuous and $\mathcal{W}_{\mathcal{R}}$ is spectrally invariant.

**Proof.** Recall that $T \circ |\xi\rangle \langle \eta| = |T(\xi)|\langle \eta|$ and $|\xi\rangle\langle \eta| \circ T = |\xi\rangle\langle T^*(\eta)|$ for all $T \in \mathcal{B}(\mathcal{E})$. Let $\xi, \eta, \zeta, \nu \in \mathcal{R}$, then we have

$$E_X(|\xi\rangle\langle \eta|) \circ |\zeta\rangle\langle \nu| = |E_X(|\xi\rangle\langle \eta|)(\zeta)|\langle \nu|,$$

$$|\zeta\rangle\langle \nu| \circ E_X(|\xi\rangle\langle \eta|) = |\zeta|\langle E_X(|\xi\rangle\langle \eta|)^*(\nu)|.$$

The spectral invariance of $\mathcal{W}_{\mathcal{R}}$ means that these belong to $\mathcal{W}_{\mathcal{R}}$. Equivalently, $X(\mathcal{R}) \subseteq \mathcal{R}$ and $X^*(\mathcal{R}) \subseteq \mathcal{R}$. It suffices to prove $X(\mathcal{R}) \subseteq \mathcal{R}$ because $X^* = |\eta\rangle \circ M^*_X \circ \langle \xi|E_X(|\eta\rangle\langle \xi|)$ has the same form. Let $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{R})$. Proposition 6.3 in [6] yields $\mathcal{R} = \mathcal{R}_{\mathcal{F}}$. Hence we must show $|X(\zeta)| \in \mathcal{F}$ for all $\zeta \in \mathcal{R}$.

Equation (13) yields

$$|X(\zeta)| = X \circ |\xi\rangle \circ M^*_X = |\xi\rangle \circ M_X \circ \langle \eta | \zeta\rangle \circ M^*_X.$$

Since $\mathcal{R}$ is relatively continuous, we have $\langle \eta | \zeta\rangle \in C^*_r(G, B)$. Now (9) yields $M_X \circ \langle \eta | \zeta\rangle \circ M^*_X \in C^*_r(G, B)$. Since $\mathcal{F} \circ C^*_r(G, B) \subseteq \mathcal{F}$, we get $|X(\zeta)| \in \mathcal{F}$ as desired. \qed


2.5. A Fourier inversion theorem

Let \( a \in A_i \) and recall that
\[
\hat{a}(\chi) = E_{\chi}(a) := \int_G \alpha_t(a) \cdot \langle \chi \mid t \rangle \, dt.
\]
The usual Fourier inversion theorem leads us to expect
\[
\int \hat{\alpha}(\chi) \cdot \langle \chi \mid t \rangle \, d\chi = \alpha_t(a) \quad \forall t \in G
\]
(for suitably normalized Haar measures on \( G \) and \( \hat{G} \)). In fact, Proposition 6.6 in [2] asserts this whenever the integral on the left-hand side converges absolutely, that is, if \( a \in A_i \) and \( \int_{\hat{G}} \| \hat{\alpha}(\chi) \| \, d\chi < \infty \); this ensures existence of the integral because \( \hat{\alpha} \) is strictly continuous.

We need some preparation to find enough elements \( a \in A_i \) with absolutely integrable Fourier transform \( \hat{\alpha} \). We specialise (3) to turn \( A \) into a right \( L^1(G) \)-module by
\[
a \ast f := \int_G \alpha_t(a) \cdot f(t^{-1}) \, dt \quad \forall a \in A, f \in L^1(G).
\]

**Proposition 23.** We have \( a \ast f \in A_i \) and
\[
\hat{a} \ast \hat{f} = \hat{\alpha} \cdot \hat{f} \quad \text{(pointwise product)}
\]
for all \( a \in A_i \) and \( f \in L^1(G) \), with \( \hat{f} \) as in (6).

**Proof.** Formally, this follows by a change of variables and Fubini’s theorem:
\[
\hat{a} \ast \hat{f}(\chi) = \int_G \alpha_t(a \ast f) \cdot \langle \chi \mid t \rangle \, dt = \int_{G \times G} \alpha_{ts}(a) \cdot f(s^{-1}) \cdot \langle \chi \mid t \rangle \, ds \, dt
\]
\[
= \int_{G \times G} \alpha_t(a) \cdot \langle \chi \mid t \rangle \cdot f(s^{-1}) \cdot \langle \chi \mid s \rangle \, ds \, dt = \hat{\alpha}(\chi) \cdot \hat{f}(\chi).
\]
This computation requires a justification because some of the relevant integrals only exist in a rather weak sense. For \( f \in C_c(G) \), this is done in the proof of [2, Proposition 6.5]. The only point in Exel’s argument that requires additional care occurs in the proof of [2, Proposition 3.6]: we must show that \( (1_K \ast f)_{K \in \mathcal{K}} \) is a bounded net in \( L^\infty(G) \) that converges uniformly on compact subsets to the constant function taking the value \( \int_G f(s^{-1}) \, ds = \int_G f(s) \, ds \); here \( 1_K \) denotes the characteristic function of \( K \). Since
\[
(1_K \ast f)(t) = \int_G 1_K(ts^{-1}) f(s) \, ds = \int_{K^{-1}t} f(s) \, ds,
\]
our net is bounded with \( \| 1_K \ast f \|_\infty \leq \| f \|_1 \) for all \( K \in \mathcal{K} \). It also follows that

\[
\left| (1_K \ast f)(t) - \int_G f(s) \, ds \right| \leq \int_{G \setminus K^{-1}t} |f(s)| \, ds.
\]

For any \( \varepsilon > 0 \) there is a compact subset \( K_0 \subseteq G \) with \( \int_{G \setminus K_0} |f(s)| \, ds < \varepsilon \) because \( f \in L^1(G) \).

Let \( L \subseteq G \) be compact. If \( K \subseteq G \) is a measurable relatively compact subset with \( L \cdot K_0^{-1} \subseteq K \), then \( K_0 \subseteq K^{-1}t \) for all \( t \in L \), so that

\[
\int_{G \setminus K^{-1}t} |f(s)| \, ds < \varepsilon
\]

for all \( t \in L \). This means that the net \((1_K \ast f)_{K \in \mathcal{K}}\) converges to \( \int_G f(s) \, ds \) uniformly on \( L \). \( \square \)

**Definition 24.** Let \( \mathcal{J}(G) := \{ f \in L^1(G) \mid \hat{f} \in C_c(\hat{G}) \} \).

The following result is certainly well known, but we include the proof because we have found no specific reference.

**Lemma 25.** The space \( \mathcal{J}(G) \) is dense in \( L^1(G) \), and

\[
\widehat{\mathcal{J}(G)} := \{ \hat{f} \mid f \in \mathcal{J}(G) \} = \{ g \in C_c(\hat{G}) \mid \check{g} \in L^1(G) \}
\]

is dense in \( C_c(\hat{G}) \) in the inductive limit topology; here \( \check{g} \) denotes the inverse Fourier transform of \( g \).

**Proof.** The space \( \mathcal{J}(G) \) contains \( h = \hat{f}_1 \cdot \hat{f}_2 \) for all \( f_1, f_2 \in C_c(\hat{G}) \) because \( \hat{h} = f_1 \ast f_2 \) is still compactly supported and

\[
f_1, f_2 \in L^2(\hat{G}) \implies \hat{f}_1, \hat{f}_2 \in L^2(G) \implies h = \hat{f}_1 \cdot \hat{f}_2 \in L^1(G).
\]

The products \( \hat{f}_1 \cdot \hat{f}_2 \) with \( f_1, f_2 \in C_c(\hat{G}) \) form a dense subset of \( L^1(G) \) because the inverse Fourier transform of \( C_c(\hat{G}) \) is dense in \( L^2(G) \). Similarly, \( \widehat{\mathcal{J}(G)} \) contains \( C_c(\hat{G}) \ast C_c(\hat{G}) \), which is dense in \( C_c(\hat{G}) \). \( \square \)

**Proposition 26.** The Fourier inversion formula (14) applies to \( a \ast f \) for all \( a \in A_i, \ f \in \mathcal{J}(G) \).

**Proof.** Proposition 23 shows that \( a \ast f \) has compactly supported Fourier transform for all \( f \in \mathcal{J}(G) \). Hence the assertion follows from [2, Proposition 6.6]. \( \square \)

**Remark 27.** It follows from Lemma 25 that we can find a bounded approximate identity \((u_n)\) in \( L^1(G) \) such that \( \hat{u}_n \) has compact support for all \( n \). Since \( L^1(G) \subseteq C^\ast(G) \cong C_0(\hat{G}) \), the net \( \hat{u}_n \)
is a bounded approximate unit for $\mathcal{C}_0(\widehat{G})$, which means that $\lim \hat{u}_n = 1$ uniformly on compact subsets of $\widehat{G}$. Since the Fourier inversion formula (14) holds for $a * u_n$ for all $n$, we get

$$\alpha_t(a) = \lim_{n \to \infty} \alpha_t(a) * u_n = \lim_{n \to \infty} \int_{\widehat{G}} \hat{a}(\chi) \cdot \hat{u}_n(\chi) \cdot \langle \chi | t \rangle \, d\chi.$$ 

Hence the Fourier inversion formula always holds if we interpret the integral suitably. The only issue is the quality of the convergence. Note that if $\hat{a}$ is strictly-unconditionally integrable, then the limit above converges to $\int_{\widehat{G}} \hat{a}(\chi) \cdot \langle \chi | t \rangle \, d\chi$ (see [2, Proposition 3.4]) and therefore

$$\alpha_t(a) = \int_{\widehat{G}} \hat{a}(\chi) \cdot \langle \chi | t \rangle \, d\chi.$$ 

### 3. Continuous spectral decompositions

Now we discuss how to construct continuous spectral decompositions out of dense, complete, relatively continuous subspaces, and vice versa. Much of this goes back to Exel [1,2]. Throughout this section, $G$ is an Abelian locally compact group and $\widehat{G}$ is its dual group.

#### 3.1. Construction of continuous spectral decompositions

**Definition 28.** Let $A$ be a $G$-$C^*$-algebra. A continuous spectral decomposition of $A$ is a Fell bundle $B$ with a $G$-equivariant $C^*$-algebra isomorphism $C^*(B) \to A$.

A similar notion already appears in [3, Definition VIII.17.4], where it is called a $C^*$-bundle structure for $A$ over $\widehat{G}$; but in [3], $A$ does not carry an action of $G$ to begin with, so that the interpretation as a spectral decomposition is missing.

Recall that $C^*(B)$ is the $C^*$-envelope of the Banach $*$-algebra $L^1(B)$ of measurable sections $b$ with $\int_{\widehat{G}} \|b(\chi)\| \, d\chi < \infty$. We equip $C^*(B)$ with the dual action $\beta$ of $G$ defined by

$$(\hat{\beta}_t f)(\chi) = \langle \chi | t \rangle \cdot f(\chi) \quad \forall f \in L^1(B), \ t \in G, \ \chi \in \widehat{G}.$$ 

We often use the subspace $C_c(B)$ of continuous sections with compact support. This is a dense, $G$-invariant $*$-subalgebra in $L^1(B)$ and $C^*(B)$.

Unless $G$ is compact, a continuous spectral decomposition cannot exist for a trivial action because dual actions are never trivial. Thus we need more structure to construct a continuous spectral decomposition for a $G$-$C^*$-algebra $A$. Exel [2] uses a subset $\mathcal{W}$ with the following properties:

(i) $\mathcal{W}$ is a dense linear subspace of $A$ with $\mathcal{W} = \mathcal{W}^*$;
(ii) $\mathcal{W} \subseteq A_i$;
(iii) $\mathcal{W}$ is relatively continuous;
(iv) $\mathcal{W}$ is spectrally invariant.

This yields a continuous spectral decomposition $C^*(B(\mathcal{W}))$ for $A$ (see [2, Section 11]); the density of $\mathcal{W}$ in $A$ ensures that the map $C^*(B(\mathcal{W})) \to A$ is an isomorphism.
The fibers of $B(\mathcal{W})$ are

$$B_\chi(\mathcal{W}) := \overline{\text{span}} \{ \hat{a}(\chi) \mid a \in \mathcal{W} \} \subseteq \mathcal{M}_\chi(A) \subseteq \mathcal{M}(A)$$

for $\chi \in \hat{G}$; more precisely, we take the norm closure here. The algebraic operations are inherited from $\mathcal{M}(A)$; the topology is defined so that the sections $\hat{a}$ for $a \in \mathcal{W}$ generate the space of continuous sections. We must make this more precise (see also [3, II.13]).

To begin with, a Fell bundle $B$ is determined uniquely by the space $C_c(B)$ of its continuous, compactly supported sections (since continuity is a local issue, the support condition is irrelevant here). The space of continuous sections of $B$ is closed under pointwise addition and multiplication by continuous functions, that is, it is a $C(\hat{G})$-module. If $U \subseteq \hat{G}$ is a relatively compact, open subset, we let $C_0(U, B) \subseteq C_c(B)$ be the subspace of continuous sections that vanish outside $U$. These subspaces are complete with respect to the norm

$$\|f\|_U^\infty := \sup_{\chi \in U} \|f(\chi)\|.$$ 

We have $C_c(B) = \bigcup C_0(U, B)$ and equip $C_c(B)$ with the inductive limit topology. These norms and the inductive limit topology also make sense on the corresponding larger spaces $D_0(U, B)$ and $D_c(B)$ of bounded sections of $B$.

**Lemma 29.** The subspace $C_c(B(\mathcal{W})) \subseteq D_c(B(\mathcal{W}))$ is the closed linear span of the set of all $g \cdot \hat{a}$ with $g \in C_c(\hat{G})$ and $a \in \mathcal{W}$. We may further restrict to $\hat{g} \cdot \hat{a}$ with $g \in \mathcal{J}(G)$ and $a \in \mathcal{W}$, with $\mathcal{J}(G)$ as in Definition 24.

**Proof.** We abbreviate $B := B(\mathcal{W})$. Recall that $C_c(B)$ is a closed $C_c(\hat{G})$ submodule of $D_c(B)$ for any Fell bundle $B$. It remains to check that any continuous compactly supported section of $B$ can be approximated uniformly by linear combinations of pointwise products $g \cdot \hat{a}$ with $g \in C_c(\hat{G})$ and $a \in \mathcal{W}$. By definition, a section $f$ is continuous if and only if, for any $\chi \in \hat{G}$ and $\epsilon > 0$, we can find a neighborhood $U$ of $\chi$ and $a \in \mathcal{W}$ with $\|\hat{a}(\omega) - f(\omega)\| < \epsilon$ for all $\omega \in U$. For a continuous section $f$ with compact support, we can find a finite covering by such open subsets $U_1, \ldots, U_n$ and $a_1, \ldots, a_n \in \mathcal{W}$ with $\|\hat{a}_j(\omega) - f(\omega)\| < \epsilon$ for $\omega \in U_j$. In addition, for any fixed neighborhood $V$ of supp $f$, we can assume $U_1, \ldots, U_n \subseteq V$. We can find continuous functions $\phi_1, \ldots, \phi_n : \hat{G} \to [0, 1]$ with supp $\phi_j \subseteq U_j$ and $\phi_1 + \cdots + \phi_n = 1$ on supp $f$. We get

$$\left\| f(\omega) - \sum_{i=1}^n \phi_i(\omega)\hat{a}_i(\omega) \right\| < \epsilon$$

for all $\omega \in \bigcup U_j \supseteq \text{supp } f$, and the sum vanishes outside $\bigcup U_j$. This shows that $f$ belongs to the closed $C_c(\hat{G})$-submodule generated by $\{ \hat{a} \mid a \in \mathcal{W} \}$. The last assertion now follows because the Fourier transform maps $\mathcal{J}(G)$ to a dense subspace of $C_c(\hat{G})$ by Lemma 25 and the multiplication maps $C_c(\hat{G}) \rightarrow C_c(B)$, $g \mapsto g \cdot \hat{a}$ are continuous for all $a \in \mathcal{W}$. $\square$

**Lemma 30.** The isomorphism $C^*(B(\mathcal{W})) \rightarrow A$ constructed in [2, Section 11] maps an $\mathcal{L}^1$-section $(b_\chi)_{\chi \in \hat{G}}$ to $\int_G^G b_\chi d\chi$. 
Proof. Sections of the form \( \hat{g} \ast a = \hat{g} \cdot \hat{a} \) with \( a \in \mathcal{W} \) and \( g \in \mathcal{J}(G) \) span a dense subspace of \( \mathcal{C}_c(B) \) and hence of \( \mathcal{L}^1(B) \) by Lemma 29 (compare [2, Proposition 11.9]). Hence it suffices to check the assertion for sections of this form. The Fourier inversion formula (Proposition 26) yields
\[
\int_{\hat{G}} \hat{g} \ast a(\chi) d\chi = g \ast a.
\]
The proof of [2, Proposition 11.10] shows that this is the image of \( \hat{g} \ast a \) under the isomorphism \( C^*_r(G, A) \to \mathbb{K}(\mathcal{E}) \). (The only point that makes this complicated is that Exel represents the crossed product \( C^*_r(G, A) \) on a Hilbert space instead of the Hilbert module \( \mathcal{L}^2(G, A) \).)

Now we use the equivalence between relatively continuous subsets of integrable and square-integrable elements to rephrase Exel’s results in terms of continuously square-integrable Hilbert modules.

Let \( \mathcal{E} \) be a Hilbert \( B, G \)-module and let \( \mathcal{R} \subseteq \mathcal{E}_{si} \) be relatively continuous, complete, and dense in \( \mathcal{E} \). We get a Fell bundle
\[
B(\mathcal{E}, \mathcal{R}) := B(\mathbb{K}(\mathcal{E}), \mathcal{W}_\mathcal{R})
\]
with \( \mathcal{W}_\mathcal{R} \subseteq \mathbb{K}(\mathcal{E}) \), as in (12). Using (10), we identify its fibers with
\[
B_\chi(\mathcal{E}, \mathcal{R}) = \overline{\text{span}} \left\{ E_\chi(\langle \xi \rangle \langle \eta \rangle) \mid \xi, \eta \in \mathcal{R} \right\}
\]
\[
= \overline{\text{span}} \left\{ \langle \xi \rangle \circ M_\chi \circ \langle \eta \rangle \mid \xi, \eta \in \mathcal{R} \right\}
\]
\[
= \overline{\text{span}} \left\{ \zeta \circ M_\chi \circ \nu^* \mid \zeta, \nu \in \mathcal{F}(\mathcal{E}, \mathcal{R}) \right\},
\]
(15)
where \( \mathcal{F}(\mathcal{E}, \mathcal{R}) \) is the Hilbert \( C^*_r(G, B) \)-module associated to \( (\mathcal{E}, \mathcal{R}) \) as in (5). The algebraic structure of the Fell bundle is inherited from \( B(\mathcal{E}) = \mathcal{M}(\mathbb{K}(\mathcal{E})) \) using \( B_\chi(\mathcal{E}, \mathcal{R}) \subseteq B(\mathcal{E}) \); the spaces of continuous sections are generated by sections of the form \( \chi \mapsto \zeta \circ M_\chi \circ \nu^* \) for \( \zeta, \nu \in \mathcal{F}(\mathcal{E}, \mathcal{R}) \) (compare Lemma 29).

Corollary 31. The generalized fixed-point algebra \( \text{Fix}(\mathcal{E}, \mathcal{R}) \) is the fiber at the unit element in the Fell bundle \( B(\mathcal{E}, \mathcal{R}) \).

Remark 32. Given any concrete Hilbert \( C^*_r(G, B) \)-module
\[
\mathcal{F} \subseteq \mathbb{B}^G(L^2(G, B), \mathcal{E})
\]
(see [6]), it can be checked directly that the last description in (15) yields a Fell bundle \( B \) over \( \hat{G} \) and a canonical \( G \)-equivariant isomorphism \( C^*(B) \cong \mathbb{K}(\mathcal{E}) \). We omit this argument because it is quicker (but less beautiful) to cite Exel’s work.

Lemma 33. Let \( (\mathcal{E}, \mathcal{R}) \) be a continuously square-integrable Hilbert \( B, G \)-module and let \( \mathcal{R}_0 \subseteq \mathcal{R} \) be dense in the si-norm. Then
\[
B_\chi(\mathcal{E}, \mathcal{R}) = \overline{\text{span}} \left\{ \langle \xi \rangle \circ M_\chi \circ \langle \eta \rangle \mid \xi, \eta \in \mathcal{R}_0 \right\} \quad \text{for all } \chi \in \hat{G}.
\]
The first two conditions above imply \( \chi \) are mapped to the continuous sections \( \phi \) that preserves the norm, product and involution, and such that the \( \chi \) are isomorphic, so are \( B(\mathcal{A}_1, \mathcal{R}_1) \) and \( B(\mathcal{A}_2, \mathcal{R}_2) \).

**Lemma 34.** The construction \((\mathcal{A}, \mathcal{R}) \mapsto B(\mathcal{A}, \mathcal{R})\) is a functor from the category of continuously square-integrable \( G\)-\( C^* \)-algebras to the category of Fell bundles over \( \hat{G} \). In particular, if \((\mathcal{A}_1, \mathcal{R}_1)\) and \((\mathcal{A}_2, \mathcal{R}_2)\) are isomorphic, so are \( B(\mathcal{A}_1, \mathcal{R}_1) \) and \( B(\mathcal{A}_2, \mathcal{R}_2) \).

We recall during the proof what the morphisms in both categories are. This also specifies the isomorphisms.

**Proof.** Let \((\mathcal{A}_k, \mathcal{R}_k)\) for \( k = 0, 1 \) be two continuously square-integrable \( G\)-\( C^* \)-algebras and let \( B_k := B(\mathcal{A}_k, \mathcal{R}_k) \) be the associated Fell bundles. A morphism \((\mathcal{A}_1, \mathcal{R}_1) \rightarrow (\mathcal{A}_2, \mathcal{R}_2)\) is a \( G \)-equivariant \(*\)-homomorphism \( \pi : \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) with \( \pi(\mathcal{R}_1) \subseteq \mathcal{R}_2 \). We want to use such a morphism to construct a morphism of Fell bundles \( B_1 \rightarrow B_2 \). This is a continuous map \( \phi : B_1 \rightarrow B_2 \) that preserves the norm, product and involution, and such that \( \phi|_{B_1} \) is a linear map from \( B_{1,\chi} \) to \( B_{2,\chi} \) for all \( \chi \in \hat{G} \). We can describe it equivalently as a family of linear maps \( \phi_{\chi} : B_{1,\chi} \rightarrow B_{2,\chi} \) for all \( \chi \in \hat{G} \), with the following properties:

- \( \phi \) is compatible with the multiplication: \( \phi_{\chi_1 \chi_2}(b_1 \cdot b_2) = \phi_{\chi_1}(b_1) \cdot \phi_{\chi_2}(b_2) \) for all \( \chi_1, \chi_2 \in \hat{G}, b_1 \in B_{1,\chi_1}, b_2 \in B_{1,\chi_2} \);
- \( \phi \) is compatible with the involution: \( \phi_{\chi}(b^*) = \phi_{\chi}(b)^* \) for all \( \chi \in \hat{G}, b \in B_{1,\chi} \);
- \( \phi \) is continuous: \( (\phi_{\chi}(b_{\chi}))_{\chi \in \hat{G}} \) is a continuous section of \( B_2 \) for any continuous section \( (b_{\chi})_{\chi \in \hat{G}} \) of \( B_1 \); it suffices to require this for a generating set of continuous sections (compare Lemma 29).

The first two conditions above imply \( \| \phi_{\chi} \| \leq 1 \) for all \( \chi \in \hat{G} \).

We do not require \( \pi : \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) to be essential, that is, we allow \( \pi(\mathcal{A}_1) \cdot \mathcal{A}_2 \neq \mathcal{A}_2 \). Thus \( \pi \) need not extend to a strictly continuous \(*\)-homomorphism between the multiplier algebras. To circumvent this issue, we work with the bidual von Neumann algebras; it is clear that \( \pi \) induces a weakly continuous \(*\)-homomorphism \( \pi'' : \mathcal{A}_1'' \rightarrow \mathcal{A}_2'' \). We have \( \mathcal{M}(\mathcal{A}) \subseteq \mathcal{A}'' \) for any \( C^* \)-algebra \( \mathcal{A} \), and the weak topology on \( \mathcal{A}'' \) is weaker than the strict topology on \( \mathcal{M}(\mathcal{A}) \). Therefore, if an unconditional integral exists in the strict topology on \( \mathcal{M}(\mathcal{A}) \), it also exists in the weak topology on \( \mathcal{A}'' \).

The map \( \pi : \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) induces maps between the homogeneous subspaces \( \pi_{\chi}(\mathcal{A}_1')_{\chi} \rightarrow (\mathcal{A}_2')_{\chi} \)—defined as in (1)—which automatically satisfy the first two algebraic conditions for a morphism of Fell bundles. Since \( \pi(\mathcal{R}_1) \subseteq \mathcal{R}_2 \) and since \( \pi'' \) preserves weak unconditional integrals, \( \pi_{\chi} \) maps \( E_{\chi}(a \cdot b^*) \) to \( E_{\chi}(\pi(a) \cdot \pi(b)^*) \) for all \( a, b \in \mathcal{R}_1 \). Hence we get maps \( B_{1,\chi} \rightarrow B_{2,\chi} \), which form a morphism of Fell bundles; the algebraic conditions are clear, the continuity condition holds because the generating continuous sections \( \chi \mapsto E_{\chi}(\pi(a) \cdot \pi(b)^*) \) of \( B_1 \) are mapped to the continuous sections \( \chi \mapsto E_{\chi}(\pi(a) \cdot \pi(b)^*) \) of \( B_2 \).
3.2. The inverse construction

Now we extract a relatively continuous $G$-$C^*$-algebra $(A, R)$ from a Fell bundle $\mathcal{B}$ over $\hat{G}$. Since this construction should be inverse to the one in Section 3.1, we let $A := C^*(\mathcal{B})$ with the dual action of $G$.

We view the fibers $B_\chi$ of $\mathcal{B}$ as subspaces of the multiplier algebra $\mathcal{M}(A)$ via

$$ (b_\chi \cdot f)(\eta) = b_\chi \cdot f(\chi^{-1} \eta), \quad (f \cdot b_\chi)(\eta) = f(\eta^{-1}) \cdot b_\chi $$

for $f \in L^1(\mathcal{B})$; this yields isometric embeddings $B_\chi \rightarrow \mathcal{M}_\chi(C^*(\mathcal{B}))$ by [3, VIII.5.8].

**Lemma 35.** Let $\mathcal{B} = (B_\chi)_{\chi \in \hat{G}}$ be a Fell bundle over $\hat{G}$. Then $C_c(\mathcal{B})$ consists of square-integrable elements of $C^*(\mathcal{B})$. Furthermore, we have

$$ E_\chi(ab^*) = (ab^*)(\chi) $$

for all $a, b \in C_c(\mathcal{B})$, $\chi \in \hat{G}$, and

$$ \|f\|_{si} = \|f\|_{C^*(\mathcal{B})} + \left\| \int \frac{f(\chi) \cdot f(\chi)^*}{B_1} \, d\chi \right\|_{B_1}^{1/2} $$

for all $f \in C_c(\mathcal{B})$.

**Proof.** Theorem 5.5 in [1] asserts that elements of the form $f^* f$ with $f \in C_c(\mathcal{B})$ are integrable with respect to the dual action and satisfy $E_\chi(f^* f) = (f^* f)(\chi)$ for all $\chi \in \hat{G}$. Since $C_c(\mathcal{B})$ is invariant under the involution, we may replace $f^* f$ by $ff^*$ here; we get (17) by polarization. The integrability of $C_c(\mathcal{B})^2$ is equivalent to $C_c(\mathcal{B}) \subseteq C^*(\mathcal{B})_{si}$ by Proposition 16. Notice that $ff^* = |f\rangle \langle f|$ if we identify $A$ with the $C^*$-algebra of compact operators on $A$ viewed as a Hilbert module over itself.

Finally, the following computation implies (18):

$$ \|f\|^2 = \|f\|_{\mathcal{M}(C^*(\mathcal{B}))} = \|E_1(f^* f)\|_{\mathcal{M}(C^*(\mathcal{B}))} $$

$$ = \|f^* f(1)\|_{\mathcal{M}(C^*(\mathcal{B}))} = \|f^* f(1)\|_{B_1} $$

$$ = \left\| \int \frac{f(\chi) \cdot f(\chi)^*}{B_1} \, d\chi \right\|_{B_1} $$

Here we use (10), (17), and that the embedding $\mathcal{B}_1 \rightarrow \mathcal{M}(C^*(\mathcal{B}))$ is isometric.

**Proposition 36.** Let $\mathcal{B} = (B_\chi)_{\chi \in \hat{G}}$ be a Fell bundle over $\hat{G}$. Then the si-norm closure $\overline{C_c(\mathcal{B})}_{si}$ is a relatively continuous, complete, square-integrable, dense subspace of $C^*(\mathcal{B})$, so that $(C^*(\mathcal{B}), \overline{C_c(\mathcal{B})}_{si})$ is a continuously square-integrable $G$-$C^*$-algebra.

**Proof.** Let $\mathcal{R} := C_c(\mathcal{B})$ and let $\mathcal{W}_\mathcal{R}$ be the corresponding set of integrable elements as in (12). Proposition 10.2 in [2] asserts that $\mathcal{W}_\mathcal{R}$ is relatively continuous as a set of integrable elements.
By Proposition 19, this is equivalent to the relative continuity of $R$ as a set of square-integrable elements. Since $R$ is a dense, $G$-invariant subalgebra of $C^*(B)$, its completion agrees with its si-norm closure by Corollary 6. □

**Definition 37.** We define $A(B) := C^*(B)$ and $R(B) := \overline{C_c(B)}^{si}$ for a Fell bundle $B$ over $\hat{G}$.

We get a functor $B \mapsto (A(B), R(B))$ from the category of Fell bundles to the category of continuously square-integrable $G$-$C^*$-algebras. Any morphism of Fell bundles $B_1 \to B_2$ induces a $^*$-homomorphism $C^*(B_1) \to C^*(B_2)$, which is $G$-equivariant for the dual actions and restricts to a map $C_c(B_1) \to C_c(B_2)$; it is bounded for the si-norm by (18), so that it maps $R(B_1) = \overline{C_c(B_1)}^{si}$ to $R(B_2) = \overline{C_c(B_2)}^{si}$.

The Hilbert module $\mathcal{F} := \mathcal{F}(A(B), R(B))$ over $C^*_c(G, A(B))$ associated to $(A(B), R(B))$ already appears in [2, Section 10], but in disguise. It is the dual $X^* := X(G, B_1)$ of the imprimitivity bimodule $X$ that implements a Morita–Rieffel equivalence between $B_1$ and an ideal in $C^*_c(G, C^*(B))$. Since the dualization exchanges left and right, the $B_1$-valued inner product $\int a(\chi)^*b(\chi) \, d\chi$ in [2] becomes $\int a(\chi)b(\chi)^* \, d\chi$.

**3.3. The main theorem**

We have constructed a functor from continuously square-integrable $G$-$C^*$-algebras to Fell bundles over $\hat{G}$ and a functor in the opposite direction.

**Theorem 38.** These two functors are inverse to each other (up to natural isomorphism). That is, we have a natural isomorphism

$$(A_0, R_0) \cong (A(B(A_0, R_0)), R(B(A_0, R_0)))$$

for any continuously square-integrable $G$-$C^*$-algebra $(A_0, R_0)$, and a natural isomorphism

$$B_0 \cong B(A(B_0), R(B_0))$$

for any Fell bundle $B_0$ over $\hat{G}$.

Hence the categories of continuously square-integrable $G$-$C^*$-algebras and of Fell bundles over $\hat{G}$ are equivalent.

Before we prove this, we use Theorem 38 to classify continuous spectral decompositions of a given $G$-$C^*$-algebra $A$. First we must discuss the *equivalence* of continuous spectral decompositions.

Recall that a continuous spectral decomposition is a Fell bundle $B$ with an isomorphism $\phi : C^*(B) \to A$. The issue is whether $\phi$ is part of the data or not. If $\phi$ is part of the data, then Theorem 38 shows that isomorphism classes of continuous spectral decompositions of $A$ correspond bijectively to relatively continuous, square-integrable, complete dense subspaces $R \subset A$; that is, two such subspaces yield isomorphic spectral decompositions if and only if they are *equal*. The reason is that an isomorphism $B_1 \to B_2$ that is compatible with the isomorphisms to $A$ induces the identity map on $A \cong C^*(B_1) \cong C^*(B_2)$.

Now let us disregard $\phi$ and consider two spectral decompositions $\phi_1 : C^*(B) \to A$ and $\phi_2 : C^*(B) \to A$ with the same bundle $B$ to be equivalent. The automorphism $\gamma := \phi_2 \circ \phi_1^{-1}$
of $A$ maps $\phi_1(C_c(B))$ to $\phi_2(C_c(B))$. Therefore, their si-norm closures $R_1, R_2 \subseteq A$ are $\text{Aut}_G(A)$-conjugate in the following sense.

**Definition 39.** Let $\text{Aut}_G(A)$ be the group of $G$-equivariant $C^*$-algebra automorphisms of $A$. We call $R_1, R_2 \subseteq A$ $\text{Aut}_G(A)$-conjugate if there is $\gamma \in \text{Aut}_G(A)$ with $\gamma(R_1) = R_2$.

**Corollary 40.** Let $A$ be a $G$-$C^*$-algebra. Isomorphism classes of Fell bundles $B$ over $\hat{G}$ for which there exists a $G$-$C^*$-algebra isomorphism $C^*(B) \cong A$ correspond bijectively to $\text{Aut}_G(A)$-conjugacy classes of relatively continuous, square-integrable, complete dense subspaces $R \subseteq A$.

### 3.4. Proof of the main theorem

Let $B$ be a Fell bundle over $\hat{G}$. We want to construct a natural isomorphism $B \cong B(C^*(B), \overline{C_c(B)}^{si})$.

**Lemma 41.** The subset $\{\xi(\chi) \mid \xi \in C_c(B) \ast C_c(B)\}$ is dense in $B_\chi$ for all $\chi \in \hat{G}$, and $C_c(B) \ast C_c(B)$ is dense in $C_c(B)$.

**Proof.** It is clear that $\xi(\chi)$ with $\xi \in C_c(B)$ are dense in $B_\chi$ (see [3, Remark II.13.19]). Remark VIII.5.12 in [3] asserts that $C_c(B) \ast C_c(B)$ is dense in $C_c(B)$ for the inductive limit topology. \(\square\)

We identify $B_\chi$ with a subspace of $\mathcal{M}_\chi(C^*(B))$ as in (16).

**Lemma 42.** We have $B_\chi = \overline{\{\hat{a}(\chi) \mid a \in C_c(B) \ast C_c(B)\}}$.

**Proof.** Recall that $C_c(B) \subseteq C^*(B)^{si}$, so that $\hat{a}(\chi) = E_\chi(a)$ is well defined for $a \in C_c(B) \ast C_c(B)$. The assertion now follows from (17) and Lemma 41. \(\square\)

**Proposition 43.** Let $A := C^*(B)$ and $\mathcal{R} := \overline{C_c(B)}^{si}$. The inclusions

$$B_\chi \hookrightarrow \mathcal{M}_\chi(A) \leftrightarrow E_\chi(W_\mathcal{R})$$

induce a natural isomorphism of Fell bundles $B \cong B(A, \mathcal{R})$.

**Proof.** In order to ensure that our construction is natural for all morphisms of Fell bundles, we replace $\mathcal{M}(A)$ by the bidual $A'' \supseteq \mathcal{M}(A)$, which is functorial in complete generality. We have described morphisms of Fell bundles in the proof of Lemma 34.

The inclusions into $A''_\chi$ identify $B_\chi(C^*(B), \overline{C_c(B)}^{si})$ and $B_\chi$ as Banach spaces by Lemmas 33 and 42 and (10). The resulting family of isomorphisms $B_\chi \cong B(A, \mathcal{R})$ is compatible with multiplication and involution because both are inherited from $\mathcal{M}(A) \subseteq A''$. It remains to check that both Fell bundles carry the same topology. By Lemma 42, sections of the form $f \ast g^*$ with $f, g \in C_c(B)$ span a dense subspace in $C_c(B)$. They span a dense subspace in $C_c(B(A, \mathcal{R}))$ as well by (17). Now we use that the topology of a Fell bundle is determined by its space of compactly supported continuous sections. \(\square\)

This proves one half of our main theorem (Theorem 38).
Now let \((A, \mathcal{R})\) be a continuously square-integrable \(G\)-C*-algebra. We abbreviate \(B := B(A, \mathcal{R})\). We are going to show that

\[(A, \mathcal{R}) \cong (C^*(B), C_c(B)^{si}).\]

An isomorphism \(\phi : C^*(B) \xrightarrow{\sim} A\) has already been constructed by Exel [2]; it is described in Lemma 30. As a \(G\)-equivariant \(*\)-isomorphism, it must restrict to an isometric isomorphism \(C^*(B)^{si} \cong A_{si}\). It remains to check that \(\phi\) maps \(C_c(B)^{si}\) onto \(\mathcal{R}\).

Let \(\mathcal{J}(G)\) be as in Definition 24, and define \(\mathcal{W}_\mathcal{R}\) as in (12). The Fourier inversion theorem (Proposition 26) and Lemma 30 imply

\[\phi(\hat{g} \ast a) = \phi(\hat{g} \cdot \hat{a}) = g \ast a \quad \text{for all } a \in \mathcal{W}_\mathcal{R}, \ g \in \mathcal{J}(G).\]

Lemma 29 asserts that the linear span of sections of the form \(\hat{g} \ast a\) is dense in \(C_c(B)\) in the inductive limit topology and hence in the si-norm. Thus \(\phi\) maps \(C_c(B)^{si}\) onto the si-norm closed linear span of \(\mathcal{J}(G) \ast \mathcal{W}_\mathcal{R}\).

**Lemma 44.** The si-norm closed linear span of \(\mathcal{J}(G) \ast \mathcal{W}_\mathcal{R}\) is \(\mathcal{R}\).

**Proof.** Since \(\mathcal{R}\) is a right ideal, it contains \(\mathcal{W}_\mathcal{R} = \mathcal{R} \cdot \mathcal{R}^*\). Since \(\mathcal{R}\) is \(G\)-invariant and si-norm closed, it also contains the si-norm closed linear span of \(\mathcal{J}(G) \ast \mathcal{W}_\mathcal{R}\).

Since \(G\) acts continuously on \(\mathcal{R}\), \(\mathcal{R}\) is an essential \(L^1(G)\)-module, that is, \(L^1(G) \ast \mathcal{R}\) is dense in \(\mathcal{R}\). So is \(\mathcal{J}(G) \ast \mathcal{R}\) because \(\mathcal{J}(G)\) is dense in \(L^1(G)\) by Lemma 25. Thus it remains to prove that \(\mathcal{W}_\mathcal{R}\) is dense in \(\mathcal{R}\). This follows because \(\mathcal{R}\) is an essential right \(A\)-module and \(\mathcal{R}^*\) is dense in \(A\), so that \(\mathcal{W}_\mathcal{R} = \mathcal{R} \cdot \mathcal{R}^*\) is dense in \(\mathcal{R}\). \(\square\)

Since \(\phi\) is isometric in the si-norm and \(\mathcal{R}\) is complete, Lemma 44 implies that \(\phi\) maps \(C_c(B)^{si}\) onto \(\mathcal{R}\). This finishes the proof of our main theorem (Theorem 38).

4. Examples

4.1. Spectral decomposition for algebras of compact operators

Let \(G\) be a second countable locally compact Abelian group and let \(\mathcal{H}\) be a separable \(G\)-Hilbert space, that is, a separable Hilbert space with a continuous unitary representation \(\pi\) of \(G\). We let \(G\) act on \(A := \mathbb{K}(\mathcal{H})\) by conjugation:

\[\alpha_g(T) := \text{Ad}(\pi_g)(T) := \pi_g \circ T \circ \pi_g^{-1}.\]

What are the continuous spectral decompositions of \(A\)?

By our main theorem, these correspond to complete relatively continuous subspaces \(\mathcal{R} \subseteq A_{si}\) that are dense in \(A\). By [6, Theorem 7.2], such subspaces of \(A\) correspond bijectively to complete relatively continuous subspaces \(\mathcal{R} \subseteq \mathcal{H}_{si}\) that are dense in \(\mathcal{H}\). The latter are already classified in [6, Section 8]. We briefly recall this and apply it to continuous spectral decompositions.

**Remark 45.** Any continuous action of \(G\) on \(\mathbb{K}(\mathcal{H})\) comes from a projective representation of \(G\) on \(\mathcal{H}\). But we only consider honest Hilbert space representations here for simplicity.
If a subspace $\mathcal{R}$ as above exists, then $\mathcal{H}$ must be square-integrable in the sense of Rieffel [9]. The $G$-equivariant stabilization theorem [5, Theorem 8.5] shows that $\mathcal{H}$ is a direct summand in an infinite direct sum of regular representations $\bigoplus_{n \in \mathbb{N}} L^2(G)$. Equivalently, $\pi$ integrates to a normal $^\ast$-representation of the group von Neumann algebra of $G$. The Fourier transform identifies the latter with $L^\infty(\widehat{G})$.

Any normal $^\ast$-representation of $L^\infty(\widehat{G})$ is unitarily equivalent to the representation by pointwise multiplication operators on the Hilbert space of $L^2$-sections of a measurable field of Hilbert spaces over $\widehat{G}$. Measurable fields of Hilbert spaces $(\mathcal{H}_\chi)_{\chi \in \widehat{G}}$ over $\widehat{G}$ are, in turn, classified by their spectral multiplicity function

$$d : \widehat{G} \to \mathbb{N} = \mathbb{N} \cup \{\infty\}, \quad d(\chi) := \dim \mathcal{H}_\chi.$$ 

More precisely, two normal $^\ast$-representations of $L^\infty(\widehat{G})$ are unitarily equivalent if and only if their spectral multiplicity functions agree almost everywhere. Moreover, any measurable function $d : \widehat{G} \to \mathbb{N}$ occurs as a spectral multiplicity function.

Summing up, we get a bijection between unitary equivalence classes of square-integrable representations of $G$ and equivalence classes of measurable functions $\widehat{G} \to \mathbb{N}$ up to equality almost everywhere [6].

Isomorphism classes of continuously square-integrable Hilbert space representations $(\mathcal{H}, \mathcal{R})$ correspond bijectively to isomorphism classes of Hilbert modules over $\mathcal{C}_0(\widehat{G}) \cong \mathcal{C}_0(\widehat{G})$. These Hilbert modules correspond to continuous fields of Hilbert spaces over $\widehat{G}$.

If we fix $\mathcal{H}$, then we get a bijection between complete relatively continuous subspaces $\mathcal{R} \subseteq \mathcal{H}_{si}$ dense in $\mathcal{H}$ and isomorphism classes of pairs $(\mathcal{F}, \phi)$, where $\mathcal{F}$ is a Hilbert $\mathcal{C}_0(\widehat{G})$-module and $\phi$ is a $G$-equivariant unitary operator

$$\phi : \mathcal{F} \otimes \mathcal{C}_0(\widehat{G}) L^2(G) \to \mathcal{H}.$$ 

The equivalence class of the pair $(\mathcal{F}, \phi)$ corresponds to the completion of the subspace $\phi(\mathcal{F} \otimes \mathcal{C}_c(G)) \subseteq \mathcal{H}_{si}$.

In order to translate everything to measurable and continuous fields of Hilbert spaces, we replace the left regular representation of $G$ on $L^2(G)$ by the unitarily equivalent representation on $L^2(\widehat{G})$ by pointwise multiplication operators:

$$\gamma_t(\xi)(\chi) := \overline{\langle \chi | t \rangle} \cdot \xi(\chi) \quad \text{for } \xi \in L^2(\widehat{G}), \ \chi \in \widehat{G} \text{ and } t \in G.$$ 

This representation is equivalent to the left regular representation on $L^2(G)$ via the Fourier transform

$$U : L^2(G) \to L^2(\widehat{G}), \quad Uf(\chi) = \hat{f}(\chi) := \int_G f(t) \cdot \langle \chi | t \rangle \, dt,$$

as defined in (6). The integrated form of $\gamma_t$ is the $^\ast$-representation of $\mathcal{C}_0(\widehat{G}) \cong \mathcal{C}^*(G)$ by pointwise multiplication operators. The functor

$$\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{C}^*(G) L^2(G) \cong \mathcal{F} \otimes \mathcal{C}_0(\widehat{G}) L^2(\widehat{G})$$
is the forgetful functor that views a continuous field of Hilbert spaces over $\widehat{G}$ as a measurable field. The subspace $\mathcal{R} \subseteq \mathcal{H} \cong L^2((\mathcal{H}_\chi)_{\chi \in G})$ is the subspace of continuous $L^2$-sections in $\mathcal{H}$ vanishing at infinity.

**Theorem 46.** Let $\mathcal{H}$ be a square-integrable representation. There is a bijection between isomorphism classes of continuous spectral decompositions of $\mathbb{K}(\mathcal{H})$ and continuous structures on the measurable field of Hilbert spaces underlying $\mathcal{H}$.

Let $d : \widehat{G} \to \mathbb{N}$ be the spectral multiplicity function. A continuous spectral decomposition exists if and only if there is a decreasing sequence of open subsets $(U_n)_{n \in \mathbb{N}}$ of $\widehat{G}$ with $U_0 = \widehat{G}$, $d(\chi) = n$ for almost all $\chi \in U_n \setminus U_{n+1}$, and $d(\chi) = \infty$ for almost all $\chi \in \bigcap_{n \in \mathbb{N}} U_n$.

**Proof.** By our main theorem (Theorem 38) and Theorem 7.2 in [6], the continuous spectral decompositions of $\mathbb{K}(\mathcal{H})$ correspond to complete relatively continuous subspaces $\mathcal{R} \subseteq \mathcal{H}_{\text{sai}}$ dense in $\mathcal{H}$. The discussion above shows that these correspond to continuous structures on the measurable field underlying $\mathcal{H}$. This yields the first statement. Since measurable fields are classified by their dimension function, the second statement follows from the first one and the observation that dimension functions of continuous fields of Hilbert spaces are lower semi-continuous. \qed

**Example 47.** Let $S \subseteq \widehat{G}$ be a closed subset with non-zero measure and empty interior. Then the subspace $L^2(S)$ of functions in $L^2(\widehat{G})$ vanishing outside $S$ is a non-zero $G$-invariant Hilbert subspace. Its spectral multiplicity function is the characteristic function of $S$. It is easy to see that there is no lower semi-continuous function that is almost everywhere equal to it (see [6, Section 8]). Hence Theorem 46 shows that $\mathbb{K}(L^2(S))$ is a square-integrable $G$-$C^*$-algebra without continuous spectral decomposition. This answers negatively Questions 9.4 and 11.16 in [2].

If $A := \mathbb{K}(\mathcal{H})$ has a continuous spectral decomposition, it is never unique because there are always several continuous structures on a measurable field of Hilbert spaces. Even if we work up to $\text{Aut}_G(A)$-conjugacy (see Definition 39), the spectral decomposition rarely becomes unique. We only study the case of $\mathcal{H} = L^2(\widehat{G})$ for the sake of concreteness.

**Theorem 48.** Isomorphism classes of continuous spectral decompositions of $\mathbb{K}(L^2(\widehat{G}))$ correspond bijectively to isomorphism classes of triples $(S, V, \psi)$, where

- $S$ is an open subset of $\widehat{G}$ of full measure,
- $V$ is a Hermitian complex line bundle on $S$, and
- $\psi$ is a measurable section of $V$ with $|\psi(\chi)| = 1$ for all $\chi \in S$.

Two triples $(S_1, V_1, \psi_1)$ and $(S_2, V_2, \psi_2)$ are isomorphic if and only if $S_1 = S_2$ and there is an isomorphism $\phi : V_1 \cong V_2$ of Hermitian complex line bundles with $\phi_*(\psi_1) = \psi_2$, that is, $\phi \circ \psi_1(\chi) = \psi_2(\chi)$ for all $\chi \in G$.

Conjugacy classes of continuous spectral decompositions of $\mathbb{K}(L^2(\widehat{G}))$ correspond bijectively to conjugacy classes of pairs $(S, V)$ with $S$ and $V$ as above, where $(S_1, V_1)$ and $(S_2, V_2)$ are conjugate if and only if there is some $\chi \in G$ with $\chi \cdot S_1 = S_2$ and $\chi^*(V_2) \cong V_1$, where $\chi^*$ means that we pull back the line bundle $V_2$ on $S_2$ along the map $\omega \mapsto \chi \cdot \omega$ to a line bundle on $S_1$.

**Proof.** Theorem 46 reduces the problem to that of classifying continuous structures on the measurable field of Hilbert spaces over $\widehat{G}$ underlying $L^2(\widehat{G})$. Since its spectral multiplicity
function is constant equal to 1, we must consider continuous fields of Hilbert spaces \((\mathcal{H}_x)_{x \in \hat{G}}\) with \(\dim \mathcal{H}_x = 1\) for almost all \(x \in \hat{G}\). Since \(\dim \mathcal{H}_x\) is lower semi-continuous, this implies \(\dim \mathcal{H}_x = 1_S\) for some open subset \(S \subseteq \hat{G}\) of full measure, that is, \(|\hat{G} \setminus S| = 0\).

If \(\omega \in S\), there is a non-zero continuous section of \((\mathcal{H}_x)_{x \in \hat{G}}\) in a neighborhood of \(\omega\). This provides a local trivialization of \((\mathcal{H}_x)\) near \(\omega\) because \(\dim \mathcal{H}_x = 1\) on \(S\). Thus our continuous field of Hilbert spaces is locally trivial on \(S\). Equivalently, it is a Hermitian complex line bundle, that is, a complex line bundle with a continuously varying family of inner products on the fibers.

Let \(V\) be a Hermitian complex line bundle over \(S\). Local trivializations of \(V\) allow us to construct a measurable section \(\psi\) of \(V\) with \(|\psi| = 1\) almost everywhere. Then \(f \mapsto f \cdot \psi\) defines a \(\hat{G}\)-equivariant unitary operator from \(L^2(\hat{G})\) to the space \(L^2(V)\) of \(L^2\)-sections of \(V\) or, equivalently, an isomorphism between the underlying measurable fields of Hilbert spaces over \(\hat{G}\).

Conversely, any \(\hat{G}\)-equivariant unitary operator \(L^2(\hat{G}) \to L^2(V)\) is of this form for some \(\psi\) as above.

As a result, a triple \((S, V, \psi)\) as above specifies a continuous structure on the 1-dimensional constant measurable field of Hilbert spaces over \(\hat{G}\). If two such triples yield the same structure, then \(S_1 = S_2\) and \(V_1 \cong V_2\) because the corresponding Hilbert modules over \(\mathcal{C}_0(\hat{G})\) are isomorphic. The isomorphism \(\phi: V_1 \to V_2\) must satisfy \(\phi_\psi(\psi_1) = \psi_2\) in order to be compatible with the isomorphisms \(L^2(V_1) \cong L^2(\hat{G}) \cong L^2(V_2)\). Conversely, such an isomorphism clearly identifies the resulting continuous structures on the measurable field underlying \(L^2(\hat{G})\).

Finally, we turn to the classification up to conjugacy. It is well known that any automorphism \(\mathbb{K}(\mathcal{H})\) has the form \(\text{Ad}_u(T) := uTu^*\) for some unitary operator \(u \in \mathcal{B}(\mathcal{H})\), which is determined uniquely up to scalar multiples. The automorphism \(\text{Ad}_u\) is \(G\)-equivariant if and only if \(\pi_t u \pi_t^{-1} = \chi(t) \cdot u\) for all \(t \in G\), for some function \(\chi\) from \(G\) to the unit circle in \(\mathbb{C}\). Of course, \(\chi\) must be a character, so that we get \(u \in \mathbb{B}_\chi(\mathcal{H})\) for some \(\chi \in \hat{G}\). These subspaces are easy to describe for \(\mathcal{H} = L^2(\hat{G})\).

**Lemma 49.** Let \(T_\chi f(\omega) := f(\chi^{-1} \cdot \omega)\), then

\[
\mathbb{B}_\chi(L^2(\hat{G})) = L^\infty(\hat{G}) \cdot T_\chi \cong L^\infty(\hat{G}),
\]

where we represent \(L^\infty(\hat{G})\) isometrically by pointwise multiplication operators.

**Proof.** It is easy to see that \(\gamma_t \circ T_\chi \circ \gamma_t^{-1} = (\overline{\chi(t)} \cdot T_\chi)\) for all \(\chi \in \hat{G}\), \(t \in G\). Hence \(T_\chi \in \mathbb{B}_\chi(L^2(\hat{G}))\). Since \(T_\chi\) is unitary, we have

\[
u \in \mathbb{B}_\chi(L^2(\hat{G})) \iff \nu \cdot T_* \in \mathbb{B}_1(L^2(\hat{G})) = \mathbb{B}^G(L^2(\hat{G}));
\]

it is well known that the latter agrees with \(L^\infty(\hat{G})\) (see, for example, [10, Theorem VII.3.14]).

Since \(T_\chi\) is unitary, the \(G\)-equivariant automorphisms of \(\mathbb{K}(\mathcal{H})\) are precisely the maps \(\text{Ad}_uf \circ T_\chi\) for some \(\chi \in \hat{G}\) and some unitary measurable function \(f \in L^\infty(\hat{G})\).

The corresponding action on continuous structures on \(\mathcal{H}\) is the obvious one: apply \(M_f T_\chi\) to the space of continuous sections. Application of \(M_f\) does not change \(S\) and \(V\) and replaces \(\psi\) by \(\psi \cdot f\). Since any two measurable sections of norm 1 are related in this fashion, the conjugacy class of a continuous structure is independent of \(\psi\), so that we may drop this component. Application of \(T_\chi\) shifts functions by \(\chi\), so that we replace \((S, \chi^*(V))\) by \((\chi \cdot S, V)\).
Recall that the first Chern class classifies isomorphism classes of complex line bundles on $S$ by the cohomology group $H^2(S; \mathbb{Z})$. The Hermitian inner product on a complex line bundle is unique up to isomorphism. Thus we can also rewrite our classification in terms of pairs $(S, x)$ with $S$ as above and a cohomology class $x \in H^2(S; \mathbb{Z})$.

**Example 50.** Consider the case $G = \mathbb{Z}$ with dual $\hat{G} = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Since $\mathbb{T}$ is 1-dimensional, subsets of $\mathbb{T}$ carry no non-trivial line bundles. Hence continuous spectral decompositions of $\mathbb{K}(L^2\mathbb{T})$ correspond to pairs $(S, \psi)$ with an open subset $S \subseteq \mathbb{T}$ of full measure and a unitary element $\psi \in L^\infty(S)$. Up to conjugacy, they are classified by open subsets $S \subseteq \mathbb{T}$ of full measure, with $[S_1] = [S_2]$ if and only if there is $\chi \in \mathbb{T}$ with $S_1 = \chi \cdot S_2$.

**Example 51.** Consider the discrete group $G = \mathbb{Z}^2$ with dual $\hat{G} = \mathbb{T}^2$. If $S \subseteq \hat{G}$ is a proper open subset, then $H^2(S; \mathbb{Z}) = 0$ because $S$ is a non-compact oriented 2-dimensional manifold. Hence $S$ supports no non-trivial line bundles in this case. The conjugacy classes of continuous spectral decompositions of $\mathbb{K}(L^2\mathbb{T}^2)$ of this kind are classified by proper open subsets of full measure up to translation as in Example 50.

But $\mathbb{T}^2$ carries non-trivial line bundles because $H^2(\mathbb{T}^2; \mathbb{Z}) = \mathbb{Z}$. We have $\chi^*(V) \cong V$ for any line bundle $V$ on $\mathbb{T}^2$ and any $\chi \in \mathbb{T}^2$ because $\mathbb{T}^2$ is path-connected. Therefore, conjugate line bundles over $\mathbb{T}^2$ are isomorphic and non-isomorphic line bundles yield non-conjugate continuous spectral decompositions of $\mathbb{K}(L^2\mathbb{T}^2)$.

We can describe the non-trivial vector bundles on $\mathbb{T}^2$ explicitly. We parametrize points in $\mathbb{T}^2$ by $\mathbb{T} \times [0, 1]$, with $(t, 0) \sim (t, 1)$ for all $t \in \mathbb{T}$. A line bundle $V$ on $\mathbb{T}^2$ pulls back to a trivial bundle on $\mathbb{T} \times [0, 1]$ because the latter space is homotopy equivalent to the 1-dimensional space $\mathbb{T}$ and thus carries no non-trivial line bundles. To reconstruct $V$ from a trivial line bundle on $\mathbb{T} \times [0, 1]$, we glue together the restrictions to trivial line bundles on $\mathbb{T} \times \{0\}$ and $\mathbb{T} \times \{1\}$. This can be done by any continuous map $\mathbb{T} \to U(1) = \mathbb{T}$. Since only the homotopy class of this map matters, it suffices to use the gluing functions $z \mapsto z^n$. Thus all complex line bundles on $\mathbb{T}^2$ are of the form

$$V_n := \mathbb{T} \times [0, 1] \times \mathbb{C}/(z, 1, x) \sim (z, 0, z^n \cdot x) \quad \text{for all } z \in \mathbb{T}, \ x \in \mathbb{C},$$

for some $n \in \mathbb{Z}$. This is the pedestrian way to identify the set of complex line bundles on $\mathbb{T}^2$ with $\mathbb{Z} \cong H^2(\mathbb{T}^2, \mathbb{Z})$. The resulting spaces of continuous sections are

$$\mathcal{C}(\mathbb{T}^2, V_n) = \{ f \in \mathcal{C}(\mathbb{T} \times [0, 1]) \mid f(z, 1) = z^n \cdot f(z, 0) \}.$$  

We identify this with a subset of $L^\infty(\hat{G})$ using the unique measurable section $\psi$ with $f(z, t) = 1$ for $z \in \mathbb{T}$, $t \in [0, 1]$. This identifies $\mathcal{C}(\mathbb{T}^2, V_n)$ with the space of all functions in $L^\infty(\hat{G})$ that are continuous as functions on $\mathbb{T} \times [0, 1]$ and $\mathbb{T} \times (0, 1]$ and satisfy

$$\lim_{t \to 1^+} f(z, t) = z^n \cdot \lim_{t \to 0} f(z, t).$$

Now we describe the continuous spectral decompositions of $\mathbb{K}(L^2\hat{G})$ that result from a triple $(S, V, \psi)$ as in Theorem 48.

We can choose a continuous unitary section $\psi$ of $V$ if and only if $V$ is trivial. If $\psi$ is continuous, then

$$R_S = L^2(\hat{G}) \cap C_0(S).$$  

(20)
In general, $\psi$ is discontinuous and we get

$$\mathcal{R}_{S,V,\psi} = \{ f \in L^2(\hat{G}) \mid \psi \cdot f \text{ is a } C_0\text{-section of } V \text{ on } S \}.$$  (21)

To describe the associated Fell bundle, we compute the operators $|\xi\rangle \rangle$. We prefer to work with $|\xi\rangle \rangle \circ U^*: L^2(G) \to L^2(\hat{G})$, where

$$U: L^2(G) \to L^2(\hat{G}), \quad f \mapsto \hat{f},$$

is the Fourier transform. Since $|\xi\rangle \rangle \circ U^*$ is $G$-equivariant, it belongs to $L^\infty(\hat{G}) = B^G(L^2(\hat{G})) \subseteq B(L^2(\hat{G}))$.

If $f \in \mathcal{C}_c(G)$ and $\chi \in \hat{G}$, then

$$\langle\langle \xi \mid \eta \rangle \rangle = U^* M^*_\xi \mu \xi \eta U = U^* M^*_\xi \eta U.$$

Thus $|\xi\rangle \rangle (f) = \xi \cdot U(f)$ for all $f \in \mathcal{C}_c(G)$. This extends to a bounded operator $L^2(G) \to L^2(\hat{G})$ if and only if multiplication by $\xi$ is bounded on $L^2(\hat{G})$, if and only if $\xi \in L^\infty(\hat{G})$. Thus

$$L^2(\hat{G})_{si} = L^\infty(\hat{G}) \cap L^2(\hat{G}),$$

and

$$|\xi\rangle \rangle \circ U^* = M_\xi \quad \forall \xi \in L^\infty(\hat{G}) \cap L^2(\hat{G}),$$  (22)

where $M_\xi$ denotes the pointwise multiplication operator. Moreover,

$$||\xi||_{si} = ||\xi||_2 + ||\xi||_\infty.$$  (23)

The isomorphism $C^*_r(G, \mathbb{C}) \cong C_0(\hat{G})$ that we have already used above is simply given by $T \mapsto U T U^*$. If $\xi, \eta \in L^\infty(\hat{G}) \cap L^2(\hat{G})$, then

$$\langle\langle \xi \mid \eta \rangle \rangle = U^* M^*_\xi \mu \eta U = U^* M^*_\xi \eta U.$$

Thus $\xi \sim \eta$ if and only if $\tilde{\xi} \cdot \eta \in C_0(\hat{G})$. It is easy to verify with this criterion that the subspaces $\mathcal{R}_{S,V,\psi}$ in (21) are relatively continuous, confirming our computations above.

For any relatively continuous subspace $\mathcal{R}$, the fibers $\mathcal{B}_x(A, \mathcal{R})$ of the associated Fell bundle are norm-closed linear subspaces of

$$\mathcal{M}_x(A) = B_x(L^2(\hat{G})) = M_{L^\infty(\hat{G})} \circ T^*_x$$
by Lemma 49. Since the representation of $L^\infty(\hat{G})$ by multiplication operators is isometric, we may identify $B_\chi(A, R)$ with a subspace of $L^\infty(\hat{G})$. We have $T_\chi M_f = M_{\chi \cdot f} T_\chi$ with $\chi \cdot f(\omega) := f(\chi^{-1} \omega)$ for all $\chi \in \hat{G}$, $f \in L^\infty(\hat{G})$. Hence

\[
(M_f T_\chi^*) \cdot (M_g T_\omega^*) = M_{f \cdot (\chi \cdot g)} T_{\chi \cdot \omega}^*,
\]

(24)

This allows us to translate the multiplication and involution in $\mathcal{M}(A)$ to the corresponding subspaces of $L^\infty(\hat{G})$.

The computation $U M_\chi f(\omega) = f(\chi \omega) = T_\chi^* U f(\omega)$ for $\chi, \omega \in \hat{G}$ yields

\[
UM_\chi U^* = T_\chi^* : L^2(\hat{G}) \to L^2(\hat{G})
\]

for all $\chi \in \hat{G}$. Combining this with (15), (22), and (24), we get

\[
B_\chi(A, R) = \text{span} \{ |\xi\rangle \circ U^* U M_\chi U^* U \circ |\eta\rangle \mid |\xi, \eta\rangle \in R \}
\]

\[
= \text{span} \{ M_\xi T_\chi^* M_\eta^* \mid |\xi, \eta\rangle \in R \}
\]

\[
= \text{span} \{ M_\xi : (\chi \cdot \eta) T_\chi^* \mid |\xi, \eta\rangle \in R \}.
\]

Identifying $B_\chi(A, R)$ with a subspace of $L^\infty(\hat{G})$ as above, we get

\[
B_\chi(A, R) \cong \text{span} R \cdot (\chi \cdot R).
\]

(25)

If $R$ is given by (20), the right-hand side in (25) becomes

\[
B_\chi(A, R_S) \cong \text{span} C_0(S) \cdot C_0(\chi S) = C_0(S \cap \chi S).
\]

We may view sections of $\mathcal{B} := B(A, R_S)$ as functions on $\hat{G} \times \hat{G}$. As one should expect, continuity of sections of this bundle simply corresponds to continuity of the corresponding functions on $\hat{G} \times \hat{G}$. The algebraic operations can be deduced from (24).

Next we consider the case $R_S, V, \psi$ where $V$ is the trivial vector bundle, so that $\psi$ is a measurable scalar-valued function. Then (25) yields

\[
B_\chi(A, R_S, \psi) \cong C_0(S \cap \chi S) \cdot \psi \cdot (\chi \cdot \psi).
\]

Theorem 48 predicts that this continuous spectral decomposition of $A$ is conjugate to the one for $\psi = 1$. Concretely, this corresponds to the isomorphism of Fell bundles $B(A, R_{S,1}) \cong B(A, R_S, \psi)$ given by pointwise multiplication with the function $\chi \mapsto \psi \cdot (\chi \cdot \psi)$.

The case $R_S, V, \psi$ for a general Hermitian complex line bundle $V$ is more complicated. A useful general observation is that the generalized fixed-point algebra is independent of $V$ and $\psi$:

\[
B_1(A, R_S, V, \psi) = C_0(S).
\]

(26)

This is because the algebra bundle of endomorphisms of a complex line bundle is always trivial, the identity section providing a nowhere vanishing global section. The generalized fixed-point algebras become more complicated when we study $\mathbb{K}(L^2 \hat{G} \oplus L^2 \hat{G})$ (see [6, Section 8]).

The other fibers of our Fell bundle can be described as follows:
Here \( \overline{V} \) denotes the dual line bundle to \( V \) and \( \chi^* \) pulls it back along the map \( \omega \mapsto \chi \omega \) to a line bundle on \( \overline{\chi} S \), so that the tensor product line bundle \( V \otimes \chi^* \overline{V} \) is defined on \( S \cap \overline{\chi} S \); notice also that \( \psi \otimes \overline{x} \cdot \overline{\psi} \) is a section of this line bundle. Since \( V \otimes \overline{V} \) is a trivial line bundle and \( \psi \otimes \psi \) is the constant section 1, (27) reduces to (26) if \( \chi = 1 \).

**Example 52.** We make (27) more explicit for the vector bundles \( V_n \) over \( \mathbb{T}^2 \) considered in Example 51. It is convenient to describe \( C(\mathbb{T}^2, V_n) \) as follows:

\[
C(\mathbb{T}^2, V_n) = \{ f \in C(\mathbb{T} \times \mathbb{R}) \mid f(z, t + 1) = z^n f(z, t) \forall z \in \mathbb{T}, t \in \mathbb{R} \}.
\]

The measurable section \( \psi \) corresponds to \( \psi(z, t) := z^n[t] \) in this picture, where \( [t] \in \mathbb{Z} \) denotes the maximal integer \( \leq t \). We fix \( \chi \in \mathbb{T}^2 \) and represent it by \((a, b) \in \mathbb{T} \times \mathbb{R}\). Then

\[
C(\mathbb{T}^2, \chi^* \overline{V}_n) = \{ f \in C(\mathbb{T} \times \mathbb{R}) \mid f(z, t + 1) = a^{-n} f(z, t) \forall z \in \mathbb{T}, t \in \mathbb{R} \}
\]

and hence

\[
C(\mathbb{T}^2, V_n \otimes \chi^* \overline{V}_n) = \{ f \in C(\mathbb{T} \times \mathbb{R}) \mid f(z, t + 1) = a^{-n} f(z, t) \forall z \in \mathbb{T}, t \in \mathbb{R} \}.
\]

The line bundle \( V_n \otimes \overline{x} \cdot \overline{V}_n \) must be trivial because \( \mathbb{T}^2 \) is connected, so that \( \chi^* \overline{V}_n \cong \overline{V}_n \). The nowhere vanishing section \( \sigma_\chi(z, t) := a^{-nt} \) provides an explicit trivialization for \( V_n \otimes \overline{x} \cdot \overline{V}_n \).

We can rewrite the condition \( f \cdot (\psi \otimes \overline{x} \cdot \overline{\psi}) \in C(\mathbb{T}^2, V_n \otimes \chi^* \overline{V}_n) \) more explicitly as

\[
f \in (\psi \otimes \overline{x} \cdot \overline{\psi})^{-1} \cdot C(\mathbb{T}^2, V_n \otimes \chi^* \overline{V}_n) = (\overline{\psi} \otimes \overline{x} \cdot \psi) \cdot \sigma_\chi \cdot C(\mathbb{T}^2).
\]

We let

\[
h_\chi(z, t) := (\overline{\psi} \otimes \overline{x} \cdot \psi)(z, t) \cdot \sigma_\chi(z, t) = z^{-n[t]} \cdot (az)^{n[t+b]} \cdot a^{-nt} = z^n(\{t+b\} - [t])a^{n(\{t+b\} - [t])}.
\]

Notice that \( h_\chi(z, t + 1) = h_\chi(z, t) \), so that \( h_\chi \) defines a function on \( \mathbb{T}^2 \). This function has jump discontinuities if \( t \in \mathbb{Z} \) or \( t + b \in \mathbb{Z} \) and is continuous otherwise. Putting things together, we get

\[
\mathcal{B}_\chi(A, \mathcal{R}_{\mathbb{T}^2}, V_n, \psi) = \{ M_f \circ M_{h_\chi} \cdot T_\chi^* \mid f \in C(\mathbb{T}^2) \}.
\]
4.2. Spectrally proper algebras

The examples in Section 4.1 show that a $G$-$C^*$-algebra $A$ may have more than one or no continuous spectral decomposition. Now we consider a class of $G$-$C^*$-algebras with a unique continuous spectral decomposition.

Let $A$ be a $G$-$C^*$-algebra and let $\hat{A}$ be its spectrum, that is, the set of unitary equivalence classes of non-zero irreducible representations of $A$, endowed with the hull-kernel topology (see [3]). The action $\alpha$ of $G$ on $A$ induces a continuous action on $\hat{A}$ by

$$t \cdot \pi(a) := \pi \circ \alpha_t^{-1}(a) \quad \forall t \in G, \pi \in \hat{A}, a \in A.$$ 

Let $\text{Prim}(A)$ be the primitive ideal space of $A$ with the Jacobson topology. The group $G$ acts on $\text{Prim}(A)$ in an evident fashion, and the canonical map $\hat{A} \to \text{Prim}(A)$, $\pi \mapsto \ker \pi$, is $G$-equivariant, continuous and open. Even more, it induces an isomorphism between the lattices of open subsets of $\hat{A}$ and $\text{Prim}(A)$.

Let $\mathcal{B} = (B_{\chi})_{\chi \in \hat{G}}$ be a Fell bundle over $\hat{G}$. Its spectrum $\hat{\mathcal{B}}$ is its set of equivalence classes of non-zero irreducible $*$-representations. The universal property of $A := C^*(\mathcal{B})$ yields a natural homeomorphism $\hat{\mathcal{B}} \cong \hat{A}$ (see [3, VIII.17.3]), which maps a representation $\pi : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ to its integrated form $C^*(\mathcal{B}) \to \mathbb{B}(\mathcal{H})$. Naturality implies that the homeomorphism $\hat{\mathcal{B}} \cong \hat{A}$ is $G$-equivariant for the canonical $G$-actions on both spaces.

The usual definition of proper group actions on locally compact Hausdorff spaces is extended in [6, Definition 9.1] to group actions on locally quasi-compact spaces such as $\text{Prim}(A)$ and $\hat{A}$. Since it only uses the action on the lattice of open subsets, $\text{Prim}(A)$ is a proper $G$-space if and only if $\hat{A}$ is.

**Definition 53.** (See [6].) A $G$-$C^*$-algebra $A$ is spectrally proper if the $G$-space $\text{Prim}(A)$ is proper.

**Definition 54.** A Fell bundle $\mathcal{B}$ over $\hat{G}$ is spectrally proper if $\hat{\mathcal{B}}$ is a proper $G$-space (in the sense of Definition 9.1 in [6]) or, equivalently, if the $G$-$C^*$-algebra $C^*(\mathcal{B})$ is spectrally proper.

**Theorem 55.** The functor $\mathcal{B} \mapsto C^*(\mathcal{B})$ is an equivalence of categories between the categories of spectrally proper Fell bundles over $\hat{G}$ and of spectrally proper $G$-$C^*$-algebras.

Hence spectrally proper $G$-$C^*$-algebra have a unique continuous spectral decomposition (up to canonical isomorphism). Spectrally proper Fell bundles are isomorphic as Fell bundles if and only if their cross-sectional $C^*$-algebras are equivariantly isomorphic.

**Proof.** A spectrally proper $G$-$C^*$-algebra $A$ contains a unique relatively continuous, square-integrable, complete, dense subspace $\mathcal{R}_A$ by [6, Theorem 9.1]. By our main theorem (Theorem 38), these subspaces correspond bijectively to continuous spectral decompositions of $A$.

Any $G$-equivariant $*$-homomorphism $f : A_1 \to A_2$ maps $\mathcal{R}_{A_1}$ to $\mathcal{R}_{A_2}$. This follows from the uniqueness of $\mathcal{R}_{A_2}$ if $f$ is essential, that is, $f(A_1) \cdot A_2 \cdot f(A_1)$ is dense in $A_2$. Using this special case, we can reduce to the case where $f$ is the embedding of a hereditary subalgebra in $A_2$. In this case, $\text{Prim}(A_1)$ is identified with an open subset of $\text{Prim}(A_2)$, so that compactly supported elements in $A_1$ become compactly supported elements in $A_2$. Thus $\mathcal{R}_{A_1} \subseteq \mathcal{R}_{A_2}$.

As a result, the category of spectrally proper $G$-$C^*$-algebras is a full subcategory of the category of continuously square-integrable $G$-$C^*$-algebras. Hence the equivalence of categories in Theorem 38 yields the first assertion; it contains the remaining assertions. \(\square\)
If $G$ is compact (and Abelian), then any $G$-space is proper. Hence Theorem 55 specializes to the well-known assertion that $G$-$C^*$-algebras for compact $G$ have a unique spectral decomposition and that this provides an equivalence between the categories of $G$-$C^*$-algebras and of Fell bundles over $\hat{G}$.

The unique continuous spectral decomposition of a spectrally proper $G$-$C^*$-algebra is somewhat complicated to describe. Therefore, we now restrict attention to $G$-$C^*$-algebras that are proper in the sense of Kasparov [4], that is, there is an essential $G$-equivariant $*$-homomorphism from $C_0(X)$ to the center of $\mathcal{M}(A)$ for some Hausdorff locally compact proper $G$-space $X$. It is shown in [7] that this is equivalent to the existence of a continuous $G$-map $\text{Prim}(A) \to X$ or to an isomorphism $A \cong C_0(X,A)$ for some upper semi-continuous, $G$-equivariant field of $C^*$-algebras $\mathcal{A} = (A_x)_{x \in X}$ over $X$.

Upper semi-continuity means that the functions $X \ni x \mapsto \|a_x\|_{\mathcal{A}}$ are upper semi-continuous for all continuous sections $(a_x)_{x \in X}$ of $\mathcal{A}$; in particular, $\|a_x\| < \varepsilon$ for some $x \in X$ implies that there is a neighborhood $U$ of $x$ with $\|a_y\| < \varepsilon$ for all $y \in U$. The G-equivariance of the field of $C^*$-algebras means that the $G$-action on $X$ lifts to isomorphisms $\alpha_t : A_x \to A_{\gamma_t x}$ for all $t \in G$, $x \in X$, that satisfy a suitable continuity condition.

Let $\pi : \hat{G} \times X \to X$ be the coordinate projection. Pulling back the field $\mathcal{A}$ along $\pi$, we get an upper semi-continuous field of $C^*$-algebras $\pi^* \mathcal{A}$ on $\hat{G} \times X$; its $C^*$-algebra of $C_0$-sections is $C_0(\hat{G}, A)$, with the obvious $C_0(\hat{G} \times X)$-module structure.

**Theorem 56.** Let $A = C_0(X, \mathcal{A})$ for some upper semi-continuous $G$-equivariant field of $C^*$-algebras $\mathcal{A}$ over a proper $G$-space $X$. The fibers of the unique continuous spectral decomposition of $A$ are the spaces $\mathcal{B}_o(A)$ of all bounded continuous sections $b \in C_0(X, \mathcal{A})$ that satisfy the following two conditions:

- $\alpha_t(b(t^{-1} \cdot x)) = \langle \omega | t \rangle \cdot b(x)$ for all $t \in G$, $x \in X$;
- $G \setminus X \ni Gx \mapsto \|b(x)\|$ belongs to $C_0(G \setminus X)$.

Let $\pi^* \mathcal{A}$ be the pull-back of $\mathcal{A}$ along the projection $\pi : \hat{G} \times X \to X$. A compactly supported section $f : \hat{G} \to \bigcup B_o(A)$ is continuous if and only if the resulting section $(\omega, x) \mapsto f(\omega)(x)$ of $\pi^* \mathcal{A}$ is (norm) continuous on $\hat{G} \times X$ and for each $\varepsilon > 0$ there is a compact subset $C \subseteq X$ such that $\|f(\omega)(x)\| < \varepsilon$ for all $\omega \in \hat{G}$, $x \in X \setminus G \cdot C$.

The isomorphism $C^*(B(A)) \to \mathcal{A}$ maps such a continuous section $(b_\omega)_{\omega \in \hat{G}}$ to $\int_{\hat{G}}^\text{sup} b_\omega d\omega$.

**Proof.** Let $A_c = C_c(X, \mathcal{A})$ be the subspace of compactly supported continuous sections of $\mathcal{A}$. Each element of $A_c$ is a linear combination of elements of the form $a \cdot a^*$ with $a \in A_c$. Since $A_c$ is square-integrable and relatively continuous, we conclude that $A_c$ is a relatively continuous subset of integrable elements. Furthermore, $A_c$ is a $G$-invariant ideal in $\mathcal{A}$. Therefore, the fibers of the unique continuous spectral decomposition of $\mathcal{A}$ are $\mathcal{D}_\omega := E_\omega(A_c)$, and the continuous sections are generated by $\omega \mapsto E_\omega(a)$ for $a \in A_c$ as in Lemma 29. We must compare this with the fibers $\mathcal{B}_o$ and spaces of continuous sections described in the statement of the theorem.

If $a \in A_c$ is supported in $C$, then $b_\omega := E_\omega(a)$ is supported in $G \cdot C$ for all $\omega \in \hat{G}$. Hence $\|b_\omega(x)\|$ vanishes for $G \cdot x \to \infty$ in $G \setminus X$, even uniformly in $\omega$. Furthermore, $b_\omega$ is a continuous section of $\mathcal{A}$ because the integrand $\alpha_t(a(t^{-1} \cdot x)) \cdot \langle \omega | t \rangle$ is compactly supported uniformly for $x \in U$ for any relatively compact subset $U \subseteq X$.

It is also clear that $\alpha_t(b_\omega(t^{-1} \cdot x)) = \langle \omega | t \rangle \cdot b_\omega(x)$ for all $t \in G$, $x \in X$, $\omega \in \hat{G}$; this is equivalent to $b_\omega \in \mathcal{M}_\omega(A)$ and follows from (7). This yields $E_\omega(A_c) \subseteq \mathcal{B}_o$. We even get $\mathcal{D}_\omega \subseteq \mathcal{B}_o$.
Theorem 57. Isomorphism classes of commutative Fell bundles over $\hat{G}$ correspond bijectively to isomorphism classes of proper $G$-spaces.

Proof. The Gelfand–Naimark theorem yields $C^*(\mathcal{B}) \cong C_0(\hat{B})$. Proposition 36 shows that the $G$-action on $C_0(\hat{B})$ is square-integrable. It was already observed by Rieffel [8] that this implies that $\hat{B}$ is a proper $G$-space. As a result, all commutative Fell bundles are spectrally proper. Now Theorem 55 shows that $\mathcal{B} \mapsto C^*(\mathcal{B})$ is an equivalence of categories between the category of commutative Fell bundles and the category of commutative, spectrally proper $G$-$C^*$-algebras; here we use that a continuous spectral decomposition of a commutative $G$-$C^*$-algebra is again commutative. Of course, a commutative spectrally proper $G$-$C^*$-algebra is of the form $C_0(X)$ for a proper locally compact $G$-space $X$, and $X$ and $C_0(X)$ determine each other uniquely up to canonical isomorphism. $\square$
We warn the reader that the map \( \mathcal{C}_0(X) \mapsto X \) from commutative \( C^* \)-algebras to locally compact spaces is not functorial; for instance, the zero homomorphism \( \mathcal{C}_0(Y) \to \mathcal{C}_0(X) \) does not correspond to a map \( X \to Y \). Instead, \( \ast \)-homomorphisms \( \mathcal{C}_0(Y) \to \mathcal{C}_0(X) \) correspond bijectively to continuous pointed maps \( X_+ \to Y_+ \), where \( X_+ \) and \( Y_+ \) denote the one-point compactifications of \( X \) and \( Y \).

Theorem 57 improves upon Corollary 12.5 in [2]. It is also related to Theorem X.5.13 in [3], which asserts that \( B \mapsto \widehat{B} \) is a bijection between isomorphism classes of saturated commutative Fell bundles over \( \widehat{G} \) and locally compact principal \( G \)-bundles or, equivalently, free and proper \( G \)-spaces; saturated means that \( \text{span}(B_X \cdot B_\eta) = B_{\chi \eta} \), for all \( \chi, \eta \in \widehat{G} \).

**Proposition 58.** Let \( X \) be a proper \( G \)-space. The unique continuous spectral decomposition of \( \mathcal{C}_0(X) \) has as its fibers the spaces \( B_\omega \) of all \( f \in \mathcal{C}_b(X) \) that satisfy the following two conditions:

- \( \alpha_t(f) = \langle \omega | t \rangle \cdot f \) for all \( t \in G, \omega \in \widehat{G} \);
- \( |f| \in \mathcal{C}_0(G \setminus X) \).

A compactly supported section of this Fell bundle is continuous if and only if it is continuous as a function \( f : \widehat{G} \times X \to \mathbb{C} \) and if, for each \( \varepsilon > 0 \), there is a compact subset \( C \subseteq X \) with \( |f(x)| < \varepsilon \) for \( x \in X \setminus G \cdot C \).

**Proof.** This is the special case of Theorem 56 where \( \mathcal{A} \) is the trivial bundle with fiber \( \mathbb{C} \). \( \square \)

**Acknowledgments**

The first author wishes to express his thanks to Siegfried Echterhoff and Ruy Exel for many helpful conversations.

**References**