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Minimax estimator of regression coefficient in normal distribution under balanced loss function

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ABSTRACT

This article investigates linear minimax estimators of regression coefficient in a linear model with an assumption that the underlying distribution is a normal one with a nonnegative definite covariance matrix under a balanced loss function. Some linear minimax estimators of regression coefficient in the class of all estimators are obtained. The result shows that the linear minimax estimators are unique under some conditions.

1. Introduction

We open this section with some notations: For a matrix A, $\mathscr{M}(A)$, A', A^+, $rk(A)$, $\mathrm{tr}(A)$, $\lambda(A)$ denote range space, transpose, Moore–Penrose inverse, rank, trace, maximum eigenvalue of matrix A, respectively. The $n \times n$ identity matrix is denoted by I_n. For nonnegative definite matrices A and B, $A \geqslant B$ and $A > B$ stand for the nonnegative and positive definiteness of matrix $A - B$, respectively.

Consider the following linear model

$$\begin{cases} y = X\beta + \varepsilon, \\ \varepsilon \sim N_n(0, \sigma^2 V) \end{cases} \tag{1.1}$$

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where $y \in R^n$ is an observable random vector, ε is a random error vector. $X \in R^{n\times p}$ with $rk(X) = p$ is a known matrix. $V \in R^{n\times n}$ is a known nonnegative definite matrix, whereas $\beta \in R^p$ and $\sigma^2 > 0$ are unknown parameters.

For estimating regression coefficient β, we concern ourselves with the minimaxity of linear estimators of β. We denote by £ the class of homogeneous linear estimators of β, i.e.,

$$£ = \{Ly|L \text{ is any } p \times n \text{ real constant matrix}\}.$$

We denote $\mathscr{D}$ by the space of all estimators $d(y)$ of β such that the expected value of the following loss L_θ is finite. To evaluate estimators $d(y)$ of β in general, for every $\beta \in R^p$ and $\sigma^2 > 0$, we define the loss function as

$$L_\theta(\beta, \sigma^2; d(y)) = \frac{\theta(y - Xd(y))'T^+(y - Xd(y)) + (1-\theta)(d(y) - \beta)'X'T^+X(d(y) - \beta)}{\sigma^2 + \beta'X'V^+X\beta}, \tag{1.2}$$

where $\theta \in [0, 1]$, $T = V + XX'$. The numerator of the loss function L_θ which is called balanced loss function was proposed by Hu and Peng [7] using the idea of Zellner's [19] balanced loss and the unified theory of least squares formulated by Rao [12]. We choose the denominator $\sigma^2 + \beta'X'V^+X\beta$ in the loss function (1.2) in order that the maximum risk function of Ly does not rely on parameters σ^2 and β. on the other hand, if we choose the denominator σ^2, then the maximum risk function of Ly is dependent on σ^2 and β. The way of choosing the denominator is similar to the one used by Yu [18].

The balanced loss function takes both precision of estimation and goodness of fit of model into account, so it is a more comprehensive and reasonable standard. It has received considerable attention in the literature under different setups. For more details, the readers are referred to Rodrigues and Zellner [13], Giles et al. [5], Ohtani et al. [9], Ohtani [10,11], Gruber [6], Jozani et al. [8] and Arashi [2].

Moreover, it is well known that the balanced loss function is more sensitive than the quadratic loss function, which means that if an estimator is admissible under the balanced loss function, it is also admissible under the quadratic loss function. Therefore, the study about the admissibility under the balanced loss function are significant. Xu and Wu [16] studied the admissibility of linear estimators under the balanced loss function in a linear model if its covariance matrix is an identity matrix and there is no assumption that the underlying distribution is a normal one. Cao [3] proposed a matrix balanced loss function using Zellner's idea of balanced loss, and obtained Φ admissible estimators for regression coefficient matrix. Hu and Peng [7] extended the result of Xu and Wu [16] to $V \geqslant 0$. However, no systematic work about the minimaxity of linear estimators in the class of all estimators under a balanced loss function has been done.

For every $\beta \in R^p$ and $\sigma^2 > 0$, we define the risk function of $d(y)$ as

$$R_{L_\theta}(\beta, \sigma^2; d(y)) = E[L_\theta(\beta, \sigma^2; d(y))].$$

If the element is finite, thus the optimality of an estimator $d_0(Y) \in \mathscr{D}$, such as domination, admissibility, minimaxity and so on, can be evaluated by its risk in the range spaces of the risk function. Because this paper only deals with the linear minimax estimator of β, we only give the concept of minimax estimator.

Definition 1.1. $d^*(y)$ is said to be a minimax estimator, if

$$\sup_{\substack{\beta\in R^p\\ \sigma^2>0}} R_{L_\theta}(\beta, \sigma^2; d^*(y)) = \inf_{d(y)} \sup_{\substack{\beta\in R^p\\ \sigma^2>0}} R_{L_\theta}(\beta, \sigma^2; d(y)).$$

Some results related to linear minimax estimators in linear models have been established for scalar quadratic loss function. For the fixed effects model, Alam [1], Efron and Morris [4] studied the minimax estimators of the mean of a multivariate normal distribution. Xu [15] obtained the linear minimax estimators of estimable function of regression coefficient in the class of linear estimators if $V > 0$. Yu

[17] extended the result to $V \geqslant 0$ and obtained the minimax estimators in the subset of homogeneous linear estimators class. For the stochastic effects linear model, Yu [18] studied the linear minimax estimator of stochastic regression coefficients and parameters under quadratic loss function.

In this paper, we will study the unique linear minimax estimator of β in $\mathscr{D}$ and the linear model (1.1) under the balanced loss function (1.2).

The rest of this paper is organized as follows. In section 2, we give some important preliminaries. In section 3, we demonstrate the main theorems concerning the minimax estimators. Concluding remarks are given in section 4.

2. Some important preliminaries

Suppose $rk(V) = r$ and let $Q = (Q_1, Q_2)$ be an orthogonal matrix such that

$$Q'VQ = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}, \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r) \text{ with } \lambda_i > 0, i = 1, 2, \ldots, r.$$

Obviously, $V = Q_1 \Lambda Q_1'$, $V^+ = Q_1 \Lambda^{-1} Q_1'$, $Q_2 Q_2' = I - VV^+$, $\mathscr{M}(X') = \mathscr{M}(X'Q_1) + \mathscr{M}(X'Q_2)$.

Let $B = X'T^+X$, $S = (1-\theta)B^{-\frac{1}{2}}X'T^+X$. Obviously, $S\beta$ is estimable and

$$S = (1-\theta)B^{-\frac{1}{2}}X'T^+Q_1Q_1'X + (1-\theta)B^{-\frac{1}{2}}X'T^+Q_2Q_2'X \triangleq T_1Q_1'X + T_2Q_2'X, \tag{2.1}$$

where $T_1 = (1-\theta)B^{-\frac{1}{2}}X'T^+Q_1$, $T_2 = (1-\theta)B^{-\frac{1}{2}}X'T^+Q_2$. According to the following lemma, the decomposition of (2.1) is unique if and only if $VXX'(I - VV^+) = 0$.

Lemma 2.1. *$\mathscr{M}(X'Q_1)$ and $\mathscr{M}(X'Q_2)$ are orthogonal subspaces of R^p if and only if $VXX'(I - VV^+) = 0$.*

The proof of this lemma is omitted here, since it can be verified directly. We suppose that the singular value decomposition of matrix $T_1Q_1'X(X'V^+X)^+X'Q_1\Lambda^{-\frac{1}{2}}$ is

$$T_1Q_1'X(X'V^+X)^+X'Q_1\Lambda^{-\frac{1}{2}} = KFR', \tag{2.2}$$

where $F = \mathrm{diag}(f_1, f_2, \ldots, f_t)$ with $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_t > 0$, $t = rk[T_1Q_1'X(X'V^+X)^+X'Q_1]$ and $K'K = R'R = I_t$.

We now denote

$$C_i = \left(\sum_{j=1}^{i}(f_j - f_i)^2 + C^2\right)^{\frac{1}{2}}, i = 1, 2, \ldots, t,$$
$$m = \max_{1 \leqslant i \leqslant t}\{i : C_i \leqslant f_i\}. \tag{2.3}$$

and

$$J_f = \frac{\sum_{i=1}^m f_i^2 + C^2}{\sum_{i=1}^m f_i^2 + \sqrt{(\sum_{i=1}^m f_i)^2 - (m-1)(\sum_{i=1}^m f_i^2 + C^2)}},$$

where $C^2 = \mathrm{tr}(\theta T^+V - \theta^2 T^+XB^{-1}X'T^+V)$. Obviously, we have

$$C_t \geqslant C_{t-1} \geqslant \cdots \geqslant C_1 = |C| > 0 \tag{2.4}$$

and

$$C^2 + \sum_{i=1}^m (J_f - f_i)^2 = J_f^2.$$

Inequality $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_t > 0$ together with inequality (2.4) implies that the number m defined in Eq. (2.3) exists iff $C^2 \leqslant f_1^2$.

Lemma 2.2. *If $C^2 \leqslant f_1^2$, then $f_m \geqslant J_f$. Moreover, if $m < t$, then $f_{m+1} < J_f$.*

Proof. By $C_m \leqslant f_m$, we have

$$(m-1)f_m^2 - 2\left(\sum_{i=1}^m f_i\right) f_m + C^2 + \sum_{i=1}^m f_i^2 \leqslant 0, \tag{2.5}$$

then

$$\left(\sum_{i=1}^m f_i\right)^2 - (m-1)\left(C^2 + \sum_{i=1}^m f_i^2\right) \geqslant 0.$$

This together with Eq. (2.5) will yield

$$\begin{aligned} &\left(C^2 + \sum_{i=1}^m f_i^2 - f_m \sum_{i=1}^m f_i\right)^2 + (m-1)f_m^2\left(C^2 + \sum_{i=1}^m f_i^2\right) - f_m^2\left(\sum_{i=1}^m f_i\right)^2 \\ = &\left(C^2 + \sum_{i=1}^m f_i^2\right)\left[(m-1)f_m^2 - 2\left(\sum_{i=1}^m f_i\right) f_m + C^2 + \sum_{i=1}^m f_i^2\right] \geqslant 0. \end{aligned}$$

By $\sum_{i=1}^m f_i^2 \geqslant f_m \sum_{i=1}^m f_i$, we have

$$f_m\left[\left(\sum_{i=1}^m f_i\right)^2 - (m-1)\left(C^2 + \sum_{i=1}^m f_i^2\right)\right]^{\frac{1}{2}} \geqslant C^2 + \sum_{i=1}^m f_i^2 - f_m \sum_{i=1}^m f_i.$$

Hence, $f_m \geqslant J_f$. Let $m < t$, if $f_{m+1} \geqslant J_f$, then

$$f_i - J_f \geqslant f_i - f_{m+1} \geqslant 0, i = 1, 2, \ldots, m+1,$$

This together with the definition of the number m will yield

$$J_f^2 = C^2 + \sum_{i=1}^m (J_f - f_i)^2 \geqslant C^2 + \sum_{i=1}^m (f_i - f_{m+1})^2 = C^2 + \sum_{i=1}^{m+1} (f_i - f_{m+1})^2 > f_{m+1}^2,$$

which implies $J_f > f_{m+1}$. This is a contradiction to the inequality $f_{m+1} \geqslant J_f$. The proof of this Lemma is completed. □

Lemma 2.3 (Wu [14]). *Assume a model $y = X\beta + e$, $e \sim N_n(0, \sigma^2 I_n)$, where β, σ^2 are same as that in model (1.1), $X \in R^{n\times p}$ is a known matrix. Let L and F be known $t \times n$ matrices. If L satisfies the following conditions:*

(1) $L = LX(X'X)^- X'$,

(2) $LX(X'X)^- X'F'$ *is symmetric and* $LX(X'X)^- X'L' \leqslant LX(X'X)^- X'F'$,

(3) $rk(LX(X'X)^- X'(F-L)') \geqslant rk(L) - 2$.

Then the estimator Ly of $FX\beta$ is admissible in $\mathscr{D}$ under the loss function $(d - FX\beta)'(d - FX\beta)$.

3. Main results

The matrices K and R can be written in the partitioned form: $K = (K_1, K_2)$, $R = (R_1, R_2)$, where K_1 and R_1 are the first m columns of K and R, respectively. Denote $\Delta_f = \mathrm{diag}(f_1 - J_f, f_2 - J_f, \ldots, f_m - J_f)$ and $H = T_2 Q_2' X[X'(I - VV^+)X]^+ X'(I - VV^+)$. Then we have the following theorem.

Theorem 3.1. *If $VXX'(I - VV^+) = 0$, then the following statements hold*

(1) If $C^2 \leqslant f_1^2$, then $L_1 y$ is the unique minimax estimator of β in the class of all estimators under the model (1.1) and the loss function (1.2). Moreover, the maximum risk is J_f^2, where $L_1 = B^{-\frac{1}{2}}H + B^{-\frac{1}{2}}K_1\Delta_f R_1'\Lambda^{-\frac{1}{2}}Q_1' + \theta B^{-1}X'T^+$;

(2) If $C^2 > f_1^2$, then $L_2 y$ is the unique minimax estimator of β in the class of all estimators under the model (1.1) and the loss function (1.2). Moreover, the maximum risk is C^2, where $L_2 = B^{-\frac{1}{2}}H + \theta B^{-1}X'T^+$.

Proof. We first prove (1). According to Eq. (2.2), we have

$$T_1Q_1'X = T_1Q_1'X(X'V^+X)^+X'Q_1\Lambda^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}Q_1'X = KFR'\Lambda^{-\frac{1}{2}}Q_1'X. \tag{3.1}$$

By Eqs. (2.2) and (3.1), we have

$$t = rk(R'R) \geqslant rk(R'\Lambda^{-\frac{1}{2}}Q_1'X) \geqslant rk(T_1Q_1'X) \geqslant rk(KFR'),$$

hence, $t = rk(R'\Lambda^{-\frac{1}{2}}Q_1'X) \leqslant rk(Q_1'X)$, and $R'\Lambda^{-\frac{1}{2}}Q_1'X$ is a row full rank matrix. Let $a = rk(Q_1'X)$, it is easy to verify that there exist a $r \times (a - t)$ matrix R_3 and a $r \times (r - a)$ matrix R_4 such that

$$\begin{aligned}
&(R_1, R_2, R_3, R_4)'(R_1, R_2, R_3, R_4) = I_r,\\
&a = rk(R_1, R_2, R_3) = rk[(R_1, R_2, R_3)'\Lambda^{-\frac{1}{2}}Q_1'X],\\
&R_4'\Lambda^{-\frac{1}{2}}Q_1'X = 0.
\end{aligned}$$

Denote

$$\begin{aligned}
&\tilde{y} = (y_1', y_2', y_3', y_4')',\\
&y_i = R_i'\Lambda^{-\frac{1}{2}}Q_1'y, \quad i = 1, 2, 3, 4,\\
&\beta_i = R_i'\Lambda^{-\frac{1}{2}}Q_1'X\beta, \quad i = 1, 2, 3, 4.
\end{aligned}$$

Obviously, $\tilde{y} \sim N_r((\beta_1', \beta_2', \beta_3', 0')', \sigma^2 I_r)$.

Let$F_1 = \mathrm{diag}(f_1, \ldots, f_m)$, $F_2 = \mathrm{diag}(f_{m+1}, \ldots, f_t)$ $(m < t)$. By $Q_2Q_2' = I - VV^+$ and the definition of T_2, we have $HX\beta = T_2Q_2'X\beta$. This together with Eq. (2.2) will yield

$$\begin{aligned}
&(H + K_1\Delta_f R_1'\Lambda^{-\frac{1}{2}}Q_1')X\beta - S\beta\\
&\quad = K_1\Delta_f R_1'\Lambda^{-\frac{1}{2}}Q_1'X\beta - T_1Q_1'X\beta + HX\beta - T_2Q_2'X\beta\\
&\quad = K_1\Delta_f R_1'\Lambda^{-\frac{1}{2}}Q_1'X\beta - (K_1, K_2)F(R_1, R_2)'\Lambda^{-\frac{1}{2}}Q_1'X\beta\\
&\quad = K_1\Delta_f\beta_1 - (K_1, K_2)F(\beta_1', \beta_2')'.
\end{aligned} \tag{3.2}$$

By direct operation, we have

$$\begin{aligned}
&E[\theta(y - XLy)'T^+(y - XLy) + (1 - \theta)(Ly - \beta)'X'T^+X(Ly - \beta)]\\
&\quad = E[(B^{\frac{1}{2}}Ly - \theta B^{-\frac{1}{2}}X'T^+y) - S\beta]'[(B^{\frac{1}{2}}Ly - \theta B^{-\frac{1}{2}}X'T^+y) - S\beta] + \sigma^2C^2.
\end{aligned} \tag{3.3}$$

By Eqs. (3.2), (3.3) and Lemma 2.2, we have

$$\begin{aligned}
&E[\theta(y - XL_1y)'T^+(y - XL_1y) + (1-\theta)(L_1y - \beta)'X'T^+X(L_1y - \beta)] \\
&= E[(B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y) - S\beta]'[(B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y) - S\beta] + \sigma^2C^2 \\
&= \sigma^2\{C^2 + \mathrm{tr}[(H + K_1\Delta_fR_1'\Lambda^{-\frac{1}{2}}Q_1')'V(H + K_1\Delta_fR_1'\Lambda^{-\frac{1}{2}}Q_1')]\} \\
&\quad + \beta'[(H + K_1\Delta_fR_1'\Lambda^{-\frac{1}{2}}Q_1')X - S]'[(H + K_1\Delta_fR_1'\Lambda^{-\frac{1}{2}}Q_1')X - S]\beta \\
&= \sigma^2[C^2 + \mathrm{tr}(K_1\Delta_fR_1'\Lambda^{-\frac{1}{2}}Q_1'VQ_1\Lambda^{-\frac{1}{2}}R_1\Delta_fK_1')] \\
&\quad + (K_1\Delta_f\beta_1 - (K_1, K_2)F(\beta_1', \beta_2')')'(K_1\Delta_f\beta_1 - (K_1, K_2)F(\beta_1', \beta_2')') \\
&= \sigma^2[C^2 + \mathrm{tr}(\Delta_f^2)] + \left[\begin{pmatrix}\Delta_f\beta_1 \\ 0\end{pmatrix} - \begin{pmatrix}F_1\beta_1 \\ F_2\beta_2\end{pmatrix}\right]'\left[\begin{pmatrix}\Delta_f\beta_1 \\ 0\end{pmatrix} - \begin{pmatrix}F_1\beta_1 \\ F_2\beta_2\end{pmatrix}\right] \\
&= \sigma^2J_f^2 + J_f^2\beta_1'\beta_1 + \beta_2'F_2^2\beta_2 \leqslant J_f^2(\sigma^2 + \beta_1'\beta_1 + \beta_2'\beta_2).
\end{aligned}$$

Note that

$$\begin{aligned}
\sigma^2 + \beta'X'V^+X\beta &= \sigma^2\beta'X'Q_1\Lambda^{-\frac{1}{2}}(R_1, R_2, R_3, R_4)(R_1, R_2, R_3, R_4)'\Lambda^{-\frac{1}{2}}Q_1'X\beta \\
&= \sigma^2 + \beta_1'\beta_1 + \beta_2'\beta_2 + \beta_3'\beta_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
R_{L_\theta}(\beta, \sigma^2; L_1y) &= E[L_\theta(\beta, \sigma^2; L_1y)] \\
&= \frac{E[\theta(y - XL_1y)'T^+(y - XL_1y) + (1-\theta)(L_1y - \beta)'X'T^+X(L_1y - \beta)]}{\sigma^2 + \beta'X'V^+X\beta} \\
&\leqslant \frac{J_f^2(\sigma^2 + \beta_1'\beta_1 + \beta_2'\beta_2)}{\sigma^2 + \beta_1'\beta_1 + \beta_2'\beta_2 + \beta_3'\beta_3} \leqslant J_f^2,
\end{aligned}$$

In particular, if $\beta_2 = 0$ and $\beta_3 = 0$, the equality of the above expression holds. Hence

$$\sup_{\substack{\beta \in R^p \\ \sigma^2 > 0}} R_{L_\theta}(\beta, \sigma^2; L_1y) = J_f^2.$$

We next prove that L_1y is the unique minimax estimator of β in the class of all estimators. Suppose, to the contrary, that L_1y is not the unique minimax estimator of β in the class of all estimators. Then there exists an estimator δ such that

$$\sup_{\substack{\beta \in R^p \\ \sigma^2 > 0}} R_{L_\theta}(\beta, \sigma^2; \delta) \leqslant \sup_{\substack{\beta \in R^p \\ \sigma^2 > 0}} R_{L_\theta}(\beta, \sigma^2; L_1y),$$

and

$$P(\delta = L_1y) < 1 \tag{3.4}$$

for every $\beta \in R^p$ and $\sigma^2 > 0$, where $P(\cdot)$ denotes the probability of random event. Therefore, if $\beta_2 = 0$ and $\beta_3 = 0$, we have

$$R_{L_\theta}(\beta, \sigma^2; \delta) \leqslant R_{L_\theta}(\beta, \sigma^2; L_1y) = J_f^2,$$

which becomes by the definition of $R_{L_\theta}(\beta, \sigma^2; d(y))$

$$E[\theta(y - X\delta)'T^+(y - X\delta) + (1-\theta)(\delta - \beta)'X'T^+X(\delta - \beta)]$$
$$\leqslant E[\theta(y - XL_1y)'T^+(y - XL_1y) + (1-\theta)(L_1y - \beta)'X'T^+X(L_1y - \beta)]. \quad (3.5)$$

If $\beta_2 = 0$ and $\beta_3 = 0$, it follows from Eqs. (3.4) and (3.5) that

$$E\left[\theta\left(y - X\frac{\delta + L_1y}{2}\right)'T^+\left(y - X\frac{\delta + L_1y}{2}\right) + (1-\theta)\right.$$
$$\left.\left(\frac{\delta + L_1y}{2} - \beta\right)'X'T^+X\left(\frac{\delta + L_1y}{2} - \beta\right)\right]$$
$$= \frac{1}{2}E[\theta(y - X\delta)'T^+(y - X\delta) + (1-\theta)(\delta - \beta)'X'T^+X(\delta - \beta)]$$
$$+ \frac{1}{2}E[\theta(y - XL_1y)'T^+(y - XL_1y) + (1-\theta)(L_1y - \beta)'X'T^+X(L_1y - \beta)]$$
$$- \frac{1}{4}E[(B^{\frac{1}{2}}\delta - B^{\frac{1}{2}}L_1y)'(B^{\frac{1}{2}}\delta - B^{\frac{1}{2}}L_1y)]$$
$$< E[\theta(y - XL_1y)'T^+(y - XL_1y) + (1-\theta)(L_1y - \beta)'X'T^+X(L_1y - \beta)] \quad (3.6)$$

By Eq. (3.3), we have

$$E\left[\theta\left(y - X\frac{\delta + L_1y}{2}\right)'T^+\left(y - X\frac{\delta + L_1y}{2}\right) + (1-\theta)\right.$$
$$\left.\left(\frac{\delta + L_1y}{2} - \beta\right)'X'T^+X\left(\frac{\delta + L_1y}{2} - \beta\right)\right]$$
$$= E\left[\left(B^{\frac{1}{2}}\frac{\delta + L_1y}{2} - \theta B^{-\frac{1}{2}}X'T^+y - S\beta\right)'\right.$$
$$\left.\left(B^{\frac{1}{2}}\frac{\delta + L_1y}{2} - \theta B^{-\frac{1}{2}}X'T^+y - S\beta\right)\right] + \sigma^2C^2. \quad (3.7)$$

and

$$E[\theta(y - XL_1y)'T^+(y - XL_1y) + (1-\theta)(L_1y - \beta)'X'T^+X(L_1y - \beta)]$$
$$= E[(B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)'(B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)] + \sigma^2C^2 \quad (3.8)$$

By Eqs. (3.6)–(3.8), we have

$$E\left[\left(B^{\frac{1}{2}}\frac{\delta + L_1y}{2} - \theta B^{-\frac{1}{2}}X'T^+y - S\beta\right)'\left(B^{\frac{1}{2}}\frac{\delta + L_1y}{2} - \theta B^{-\frac{1}{2}}X'T^+y - S\beta\right)\right]$$
$$< E[(B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)'(B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)]. \quad (3.9)$$

We next prove that $B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y$ is an admissible estimator of $S\beta$ in the class of all estimators under the loss $(d - S\beta)'(d - S\beta)$ if $\beta_2 = 0$ and $\beta_3 = 0$. In fact, if $\beta_2 = 0$ and $\beta_3 = 0$, it follows from Eqs. (2.1), (3.1) and the definition of H that

$$E[(B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)'(B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)]$$
$$= E[(K_1\Delta_fR_1'\Lambda^{-\frac{1}{2}}Q_1'y - T_1Q_1'X\beta)'(K_1\Delta_fR_1'\Lambda^{-\frac{1}{2}}Q_1'y - T_1Q_1'X\beta)]$$
$$= E(K_1\Delta_fy_1 - K_1F_1\beta_1)'(K_1\Delta_fy_1 - K_1F_1\beta_1).$$

To prove that $B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^{+}y$ is an admissible estimator of $S\beta$ in the class of all estimators under the loss $(d - S\beta)'(d - S\beta)$, we only need to prove that under the model $y_1 \sim N_m(\beta_1, \sigma^2 I_m)$ and the loss function $(d_1 - K_1F_1\beta_1)'(d_1 - K_1F_1\beta_1)$, $K_1\Delta_f y_1$ is an admissible estimator of $K_1F_1\beta_1$ in the class of all estimators. If $\Delta_f \geqslant 0, J_f > 0$, we have $\Delta_f < F_1$ by Lemma 2.2. This together with Lemma 2.3 will yield $K_1\Delta_f y_1$ is an admissible estimator of $K_1F_1\beta_1$ in the class of all estimators. In other words, $B^{\frac{1}{2}}L_1y - \theta B^{-\frac{1}{2}}X'T^{+}y$ is an admissible estimator of $S\beta$ in the class of all estimators. This is a contradiction to the inequality (3.9). Therefore, L_1y is the unique minimax estimator of β in the class of all estimators. Moreover, the maximum risk is J_f^2.

We next prove (2). If $C^2 > f_1^2$, then $K = K_2, R = R_2, \beta_2 = R'\Lambda^{-\frac{1}{2}}Q_1'X\beta$, and β_1 does not exist. Furthermore,

$$|C| > f_1 \geqslant f_2 \geqslant \cdots \geqslant f_t > 0. \tag{3.10}$$

By the same way used to prove (1), we have

$$HX\beta - S\beta = -KF\beta_2. \tag{3.11}$$

By Eqs. (3.3), (3.10) and (3.11), we have

$$\begin{aligned}
&E[\theta(y - XL_2y)'T^{+}(y - XL_2y) + (1-\theta)(L_2y - \beta)'X'T^{+}X(L_2y - \beta)]\\
&= E[(B^{\frac{1}{2}}L_2y - \theta B^{-\frac{1}{2}}X'T^{+}y) - S\beta]'[(B^{\frac{1}{2}}L_2y - \theta B^{-\frac{1}{2}}X'T^{+}y) - S\beta] + \sigma^2C^2\\
&= \sigma^2[C^2 + \mathrm{tr}(HVH)] + (HX\beta - S\beta)'(HX\beta - S\beta)\\
&= \sigma^2C^2 + (KF\beta_2)'(KF\beta_2) = \sigma^2C^2 + \beta_2'F^2\beta_2 \leqslant C^2(\sigma^2 + \beta_2'\beta_2).
\end{aligned} \tag{3.12}$$

Equation $\sigma^2 + \beta'X'V^{+}X\beta = \sigma^2 + \beta_2'\beta_2 + \beta_3'\beta_3$ together with Eq. (3.12) will yield

$$\begin{aligned}
R_{L_\theta}(\beta, \sigma^2; L_2y) &= E[L_\theta(\beta, \sigma^2; L_2y)]\\
&= \frac{E[\theta(y - XL_2y)'T^{+}(y - XL_2y) + (1-\theta)(L_2y - \beta)'X'T^{+}X(L_2y - \beta)]}{\sigma^2 + \beta'X'V^{+}X\beta}\\
&\leqslant \frac{C^2(\sigma^2 + \beta_2'\beta_2)}{\sigma^2 + \beta_2'\beta_2 + \beta_3'\beta_3} \leqslant C^2,
\end{aligned}$$

Especially, if $\beta_2 = 0$ and $\beta_3 = 0$, the equality of the above expression holds. Hence,

$$\sup_{\substack{\beta \in R^p\\ \sigma^2 > 0}} R_{L_\theta}(\beta, \sigma^2; L_2y) = C^2.$$

We next prove that L_2y is the unique minimax estimator of β in the class of all estimators. Suppose, to the contrary, that L_2y is not the unique minimax estimator of β in the class of all estimators. Then there exists an estimator δ such that

$$\sup_{\substack{\beta \in R^p\\ \sigma^2 > 0}} R_{L_\theta}(\beta, \sigma^2; \delta) \leqslant \sup_{\substack{\beta \in R^p\\ \sigma^2 > 0}} R_{L_\theta}(\beta, \sigma^2; L_2y),$$

and

$$P(\delta = L_2y) < 1 \tag{3.13}$$

for every $\beta \in R^p$ and $\sigma^2 > 0$. Therefore, if $\beta_2 = 0$ and $\beta_3 = 0$, we have

$$R_{L_\theta}(\beta, \sigma^2; \delta) \leqslant R_{L_\theta}(\beta, \sigma^2; L_2y) = C^2,$$

which becomes by the definition of $R_{L_\theta}(\beta, \sigma^2; d(y))$

$$E[\theta(y - X\delta)'T^+(y - X\delta) + (1-\theta)(\delta - \beta)'X'T^+X(\delta - \beta)] \leqslant E[\theta(y - XL_2y)'T^+(y - XL_2y) + (1-\theta)(L_2y - \beta)'X'T^+X(L_2y - \beta)]. \quad (3.14)$$

Furthermore, if $\beta_2 = 0$ and $\beta_3 = 0$, it follows from Eqs.(2.7), (3.13), (3.14) and the same way used to prove (1) that

$$E\left[\left(B^{\frac{1}{2}}\frac{\delta + L_2y}{2} - \theta B^{-\frac{1}{2}}X'T^+y - S\beta\right)'\left(B^{\frac{1}{2}}\frac{\delta + L_2y}{2} - \theta B^{-\frac{1}{2}}X'T^+y - S\beta\right)\right] < E[(B^{\frac{1}{2}}L_2y - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)'(B^{\frac{1}{2}}L_2y - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)] = 0.$$

This is a contradiction to the inequality $E[(B^{\frac{1}{2}}\frac{\delta+L_2y}{2} - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)'(B^{\frac{1}{2}}\frac{\delta+L_2y}{2} - \theta B^{-\frac{1}{2}}X'T^+y - S\beta)] \geqslant 0$. Hence, L_2y is the unique minimax estimator of β in the class of all estimators. Moreover, the maximum risk is C^2. This completes the proof of this theorem. □

4. Concluding remarks

In the model (1.1), the unique linear minimax estimator of regression coefficient under the balanced loss function (1.2) is obtained in the class of all estimators by the admissibility theory. Furthermore, we may discuss the admissibility of the linear minimax estimator in the class of all estimators in the future.

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