# Collective additive tree spanners for circle graphs and polygonal graphs* 

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## ARTICLE INFO

## Article history:

Received 2 June 2010
Received in revised form 12 March 2012
Accepted 27 March 2012
Available online 25 April 2012

## Keywords:

Graph algorithms
Collective additive tree spanners
Circle graphs
Polygonal graphs
Balanced separators


#### Abstract

A graph $G=(V, E)$ is said to admit a system of $\mu$ collective additive tree $r$-spanners if there is a system $\mathcal{T}(G)$ of at most $\mu$ spanning trees of $G$ such that for any two vertices $u, v$ of $G$ a spanning tree $T \in \mathcal{T}(G)$ exists such that the distance in $T$ between $u$ and $v$ is at most $r$ plus their distance in $G$. In this paper, we examine the problem of finding "small" systems of collective additive tree $r$-spanners for small values of $r$ on circle graphs and on polygonal graphs. Among other results, we show that every $n$-vertex circle graph admits a system of at most $2 \log _{\frac{3}{2}} n$ collective additive tree 2 -spanners and every $n$-vertex $k$-polygonal graph admits a system of at most $2 \log _{\frac{3}{2}} k+7$ collective additive tree 2 -spanners. Moreover, we show that every $n$-vertex $k$-polygonal graph admits an additive $(k+6)$-spanner with at most $6 n-6$ edges and every $n$-vertex 3 -polygonal graph admits a system of at most three collective additive tree 2 -spanners and an additive tree 6 -spanner. All our collective tree spanners as well as all sparse spanners are constructible in polynomial time.


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## 1. Introduction

A spanning subgraph $H$ of $G$ is called a spanner of $G$ if $H$ provides a "good" approximation of the distances in $G$. More formally, for $r \geq 0, H$ is called an additive $r$-spanner of $G$ if for any pair of vertices $u$ and $v$ their distance in $H$ is at most $r$ plus their distance in $G[19,24,25]$. If $H$ is a tree then it is called an additive tree $r$-spanner of $G$ [24]. (A similar definition can be given for multiplicative $t$-spanners [ $8,22,23$ ] and for multiplicative tree $t$-spanners [5].) In this paper, we continue the approach taken in $[10,12-14,18]$ of studying collective tree spanners. We say that a graph $G=(V, E)$ admits a system of $\mu$ collective additive tree $r$-spanners if there is a system $\mathcal{T}(G)$ of at most $\mu$ spanning trees of $G$ such that for any two vertices $u, v$ of $G$ a spanning tree $T \in \mathcal{T}(G)$ exists such that the distance in $T$ between $u$ and $v$ is at most $r$ plus their distance in $G$ (see [14]). We say that system $\mathcal{T}(G)$ collectively $+r$-spans the graph $G$ and that $G$ is (collectively) $+r$-spanned by $\mathcal{T}(G)$. Clearly, if $G$ admits a system of $\mu$ collective additive tree $r$-spanners, then $G$ admits an additive $r$-spanner with at most $\mu \times(n-1)$ edges (take the union of all those trees), and if $\mu=1$ then $G$ admits an additive tree $r$-spanner.

Collective tree spanners were investigated for a number of particular graph classes, including planar graphs, bounded chordality graphs, bounded genus graphs, bounded treewidth graphs, AT-free graphs and others (see [10,12-14,18]). Some families of graphs admit a constant number and some admit a logarithmic number of collective additive tree $r$-spanners, for small values of $r$.

[^0]One of the motivations to introduce this concept stems from the problems of designing compact and efficient distance and routing labeling schemes in graphs. A distance labeling scheme for trees is described in [21] that assigns each vertex of an $n$-vertex tree an $O\left(\log ^{2} n\right)$-bit label such that, given the labels of two vertices $x$ and $y$, it is possible to compute in constant time, based solely on these two labels, the distance in the tree between $x$ and $y$. A shortest path routing labeling scheme for trees is described in [29] that assigns each vertex of an $n$-vertex tree an $O\left(\log ^{2} n / \log \log n\right)$-bit label such that, given the label of a source vertex and the label of a destination, it is possible to compute in constant time, based solely on these two labels, the neighbor of the source that heads in the direction of the destination. Hence, if an $n$-vertex graph $G$ admits a system of $\mu$ collective additive tree $r$-spanners, then $G$ admits an additive $r$-approximate distance labeling scheme with the labels of size $O\left(\mu \log ^{2} n\right)$ bits per vertex and an $O(\mu)$ time distance decoder. Furthermore, $G$ admits an additive $r$-approximate routing labeling scheme with the labels of size $O\left(\mu \log ^{2} n / \log \log n\right)$ bits per vertex. Once computed by the sender in $O(\mu)$ time (by choosing for a given destination an appropriate tree from the collection to perform routing), headers of messages never change, and the routing decision is made in constant time per vertex (see [13,14]).

Other motivations stem from the generic problems of efficient representation of the distances in "complicated" graphs by the tree distances and of algorithmic use of these representations [1,2,6,16]. Approximating a graph distance $d_{G}$ by simpler distances (in particular, by tree distances $d_{T}$ ) is useful in many areas such as communication networks, data analysis, motion planning, image processing, network design, and phylogenetic analysis (see [3-5,8,19,20,22,23,26,28]). An arbitrary metric space (in particular a finite metric defined by a graph) might not have enough structure to be exploited algorithmically.

In this paper, we examine the problem of finding "small" systems of collective additive tree $r$-spanners for small values of $r$ on circle graphs and on polygonal graphs. Circle graphs are known as the intersection graphs of chords in a circle [17]. For any fixed integer $k \geq 2$, the class of $k$-polygonal graphs can be defined as the intersection graphs of chords inside a convex $k$-polygon, where the endpoints of each chord lie on two different sides [15]. Note that permutation graphs are exactly 2-polygonal graphs and any $n$-vertex circle graph is a $k$-polygonal graph for some $k \leq n$. Our results are the following.

- For any constant $c$, there are circle graphs that cannot be collectively $+c$ spanned by any constant number of spanning trees.
- Every $n$-vertex circle graph $G$ admits a system of at most $2 \log _{\frac{3}{2}} n$ collective additive tree 2 -spanners, constructible in polynomial time.
- There are circle graphs on $n$ vertices for which any system of collective additive tree 1 -spanners will require $\Omega(n)$ spanning trees.
- Every $n$-vertex circle graph admits an additive 2 -spanner with at most $O(n \log n)$ edges.
- Every $n$-vertex $k$-polygonal graph admits a system of at most $2 \log _{\frac{3}{2}} k+7$ collective additive tree 2 -spanners, constructible in polynomial time.
- Every $n$-vertex $k$-polygonal graph admits an additive $(k+6)$-spanner with at most $6 n-6$ edges and an additive $(k / 2+8)$ spanner with at most $10 n-10$ edges, constructible in polynomial time.
- Every $n$-vertex 4 -polygonal graph admits a system of at most five collective additive tree 2 -spanners, constructible in linear time.
- Every n-vertex 3-polygonal graph admits a system of at most three collective additive tree 2-spanners and an additive tree 6-spanner, constructible in linear time.


## 2. Preliminaries

All graphs occurring in this paper are connected, finite, undirected, loopless and without multiple edges. In a graph $G=(V, E)$ the length of a path from a vertex $v$ to a vertex $u$ is the number of edges in the path. The distance $d_{G}(u, v)$ between vertices $u$ and $v$ is the length of a shortest path connecting $u$ and $v$. For a vertex $v$ of $G$, the sets $N_{G}(v)=\{w \in V: v w \in E\}$ and $N_{G}[v]=N_{G}(v) \cup\{v\}$ are called the open neighborhood and the closed neighborhood of $v$, respectively. For a set $S \subseteq V$, by $N_{G}[S]=\bigcup_{v \in S} N_{G}[v]$ we denote the closed neighborhood of $S$ and by $G(S)$ the subgraph of $G$ induced by vertices of $S$. Let also $G \backslash S$ be the graph $G(V \backslash S$ ) (which is not necessarily connected).

A graph $G$ is called a circle graph if it is the intersection graph of a finite collection of chords of a circle [17] (see Fig. 1 for an illustration). Without loss of generality, we may assume that no two chords share a common endpoint. For any fixed integer $k \geq 3$, the class of $k$-polygonal (or $k$-gon) graphs is defined as the intersection graphs of chords inside a convex $k$-polygon, where the endpoints of each chord lie on two different sides [15] (see Fig. 3 for an illustration). Permutation graphs can be considered as 2-gon graphs as they are the intersection graphs of chords between two sides (or sides of a degenerate 2-polygon). Again, without loss of generality, we may assume that no two chords share a common endpoint. Clearly, if a graph $G$ is a $k$-gon graph, it is also a $k^{\prime}$-gon graph with $k^{\prime}>k$, but the reverse is not necessarily true. Note also that determining whether a given circle graph $G$ is a $k$-gon graph and producing a $k$-polygon representation of $G$ (if one exists), for any fixed $k$, is a polynomial time solvable problem [15]. In contrast, determining the minimum value of $k$ such that $G$ is a $k$-gon graph is an NP-complete problem [15].

Let $G=(V, E)$ be a permutation graph with a given permutation model $\Pi$. Let $L^{\prime}$ and $L^{\prime \prime}$ be the two sides of $\Pi$. A vertex $s$ of $G$ is called extreme if at least one endpoint of the chord of $\Pi$, corresponding to $s$, is the leftmost or the rightmost endpoint


Fig. 1. A circle graph with an intersection model and two special chords $a$ and $b$. A balanced separator $S=N_{G}[a, b]$ and the connected components of $G \backslash S$ are also shown.
either on $L^{\prime}$ or on $L^{\prime \prime}$. By BFS(s)-tree, we denote a Breadth-First-Search-tree of $G$ starting at vertex $s$. The following result was presented in [13]:

Lemma 1 ([13]). Let G be a permutation graph and let s be an extreme vertex of $G$ in some permutation model. Then, there exists a BFS(s)-tree of $G$, constructible in linear time, which is an additive tree 2-spanner of $G$.

Since an induced cycle on four vertices is a permutation graph, permutation graphs cannot have any additive tree $r$-spanner for $r<2$. Clearly, given any $\operatorname{BFS}(s)$-tree $T_{s}$ of $G, d_{T_{s}}(x, s)=d_{G}(x, s)$ holds for any $x \in V$.

## 3. Additive spanners for circle graphs

In this section, we show that every $n$-vertex circle graph $G$ admits a system of at most $2 \log _{\frac{3}{2}} n$ collective additive tree 2-spanners. This upper bound result is complemented also with two lower bound results.

We start with the main lemma of this section which is also of independent interest.
Lemma 2. Every n-vertex $(n \geq 2)$ circle graph $G=(V, E)$ has two vertices $a$ and $b$ such that $S=N_{G}[a, b]$ is a balanced separator of $G$, i.e. each connected component of $G \backslash S$ has at most $\frac{2}{3} n$ vertices.
Proof. Consider an intersection model $\phi(G)$ of $G$ and let $\mathcal{C}$ be the circle in that model. Let also $\mathcal{P}:=\left(p_{1}, p_{2}, \ldots, p_{2 n}\right)$ be the sequence in clockwise order of the $2 n$ endpoints of the chords representing the vertices of $G$ in $\phi(G)$. We divide the circle $\mathcal{C}$ into three circular arcs $B$ (bottom), $L$ (left) and $R$ (right) each containing at most $\left\lceil\frac{2}{3} n\right\rceil$ consecutive endpoints (see Fig. 1 for an illustration). We say that a chord of $\phi(G)$ is an $X Y$-chord if its endpoints lie on $\operatorname{arcs} X$ and $Y(X, Y \in\{B, L, R\})$ of $\mathcal{C}$. If $v$ is an $X Y$-chord then let $v_{X}$ and $v_{Y}$ be its endpoints on $X$ and $Y$, respectively.

Let $X$ be an arc from the set of arcs $\{B, L, R\}$. Since $G$ is a connected graph, for any $X$, there must exist a chord in $\phi(G)$ with one endpoint in $X$ and the other endpoint not in $X$. Moreover, since we have three $\operatorname{arcs}(B, L, R)$, there must exist an arc $X$ in $\{B, L, R\}$ which has both types of chords: between $X$ and $Y \in\{B, L, R\} \backslash\{X\}$ and between $X$ and $Z \in\{B, L, R\} \backslash\{X, Y\}$. Assume, without loss of generality, that $X=B$. Let $p$ be the point of $\mathcal{C}$ separating arcs $L$ and $R$ (see Fig. 1). Now choose a $B L$-chord $a$ in $\phi(G)$ with endpoint $a_{L}$ closest to $p$ and choose a BR-chord $b$ in $\phi(G)$ with endpoint $b_{R}$ closest to $p$. By $a, b$ we also denote the vertices of $G$ which correspond to chords $a$ and $b$.

Points $a_{B}, a_{L}, b_{R}$ and $b_{B}$ of $\mathcal{C}$ divide $\mathcal{C}$ into four arcs. We name these four arcs $A_{U}, A_{R}, A_{D}$ and $A_{L}$. The arc $A_{U}:=\left(a_{L}, b_{R}\right)$ is formed by all points of $\mathcal{C}$ from $a_{L}$ to $b_{R}$ in clockwise order. If chords $a$ and $b$ intersect, then we set $A_{R}:=\left(b_{R}, a_{B}\right), A_{D}:=\left(a_{B}, b_{B}\right)$, and $A_{L}:=\left(b_{B}, a_{L}\right)$ (all arcs begin at the left arc-endpoint and go clockwise to the right arc-endpoint). If chords $a$ and $b$ do not intersect, then set $A_{R}:=\left(b_{R}, b_{B}\right), A_{D}:=\left(b_{B}, a_{B}\right)$, and $A_{L}:=\left(a_{B}, a_{L}\right)$. We consider these arcs as open arcs, i.e., the points $a_{B}, a_{L}, b_{R}$ and $b_{B}$ do not belong to them.

By our choices of $a$ and $b$, we guarantee that $\phi(G)$ has no chords with one endpoint in $A_{U}$ and the other one in $A_{D}$ (regardless of the adjacency of $a$ and $b$ ). Denote by $V_{Y}$ all chords from $\phi(G)$ (vertices of $G$ ) whose both endpoints are in $A_{Y}$, where $Y$ is either $U$, or $R$, or $D$, or $L$. Then, it is easy to see that in $G$, the set $S:=N_{G}[a, b]$ separates vertices of $V_{Y}$ from vertices of $V_{Y^{\prime}}$, where $Y, Y^{\prime} \in\{U, R, D, L\}, Y \neq Y^{\prime}$. Now, since $A_{L}$ is a sub-arc of arc $B \cup L, A_{U}$ is a sub-arc of arc $L \cup R, A_{R}$ is a sub-arc of arc $R \cup B, A_{D}$ is a sub-arc of arc $B$, and $\operatorname{arcs} A_{U}, A_{R}, A_{D}$ and $A_{L}$ do not contain points $a_{B}, a_{L}, b_{R}$ and $b_{B}$, we conclude that $\left|A_{L} \cap \mathcal{P}\right| \leq \frac{4}{3} n,\left|A_{U} \cap \mathcal{P}\right| \leq \frac{4}{3} n,\left|A_{R} \cap \mathcal{P}\right| \leq \frac{4}{3} n$ and $\left|A_{D} \cap \mathcal{P}\right| \leq \frac{2}{3} n$. Hence, the number of arcs in $\phi(G)$ whose both endpoints are in $A_{L}$ (respectively, in $A_{U}, A_{R}, A_{D}$ ), and therefore the number of vertices in $V_{L}$ (respectively, in $V_{U}, V_{R}, V_{D}$ ), is at most $\frac{2}{3} n$.

In [12], a large class of graphs, called ( $\alpha, \gamma, r$ )-decomposable graphs, was defined as follows.
Let $\alpha$ be a positive real number smaller than $1, \gamma$ be a positive integer and $r$ be a non-negative integer. We say that an $n$-vertex graph $G$ is $(\alpha, \gamma, r)$-decomposable if $n \leq \gamma$ or there is a separator $S \subseteq V$ in $G$, such that the following three


Fig. 2. (a) A house (or $l$-house for $l=1$ ), (b) an $l$-house for $l=3$, (c) a tree of $l$-houses of depth 3 where $l=2$.
conditions hold:

- the removal of $S$ from $G$ leaves no connected component with more than $\alpha n$ vertices;
- there exists a subset $D \subseteq V$ such that $|D| \leq \gamma$ and for any vertex $u \in S, d_{G}(u, D) \leq r$;
- each connected component of $G \backslash S$ is an ( $\alpha, \gamma, r$ )-decomposable graph, too.

The main result of [12] is the following.
Theorem 1 ([12]). Any ( $\alpha, \gamma, r$ )-decomposable graph $G$ with $n$ vertices admits a system of at most $\gamma \log _{1 / \alpha} n$ collective additive tree $2 r$-spanners.

Since any subgraph of a circle graph is also a circle graph, and, by Lemma 2, each $n$-vertex circle graph $G=(V, E)$ admits a separator $S=N_{G}[D]$ (where $D=\{a, b\}, a, b \in V$ ), such that no connected component of $G \backslash S$ has more than $\frac{2}{3} n$ vertices, we immediately conclude

Corollary 1. Every circle graph is $\left(\frac{2}{3}, 2,1\right)$-decomposable.
Theorem 2. Every n-vertex circle graph $G$ admits a system $\mathcal{T}(G)$ of at most $2 \log _{\frac{3}{2}} n$ collective additive tree 2 -spanners.
Note that such a system of spanning trees $\mathcal{T}(G)$ for a $n$-vertex $m$-edge circle graph $G$, given together with an intersection model $\phi(G)$, can be constructed in $O(m \log n)$ time, since a balanced separator $S=N_{G}[a, b]$ of $G$ can be found in linear $O(m)$ time (see [12] for details of the construction). Note that there is an $O\left(n^{2}\right)$ time recognition algorithm for circle graphs that also computes a circle model of the input graph if it is a circle graph (see [27]).

Taking the union of all these spanning trees in $\mathcal{T}(G)$, we also obtain a sparse additive 2 -spanner for a circle graph $G$.
Corollary 2. Every n-vertex circle graph $G$ admits an additive 2 -spanner with at most $2(n-1) \log _{\frac{3}{2}} n$ edges.
We complement our upper bound result with the following lower bounds.
Proposition 1. There are circle graphs on $n$ vertices for which any system of collective additive tree 1 -spanners will require $\Omega$ ( $n$ ) spanning trees.

Proof. Since complete bipartite graphs are circle graphs, we can use the lower bound shown in [14] for complete bipartite graphs. It says that any system of collective additive tree 1 -spanners will need to have $\Omega(n)$ spanning trees for each complete bipartite graph on $n$ vertices.

Proposition 2. For any constant $c$, there are circle graphs that cannot be collectively $+c$ spanned by any constant number of spanning trees.

Proof. Consider a $C_{4}$ 1-2-3-4-1. By identifying the 3,4 vertices of this $C_{4}$ with the 1,2 vertices of a new $C_{4}$ we can build a chain of $C_{4}$ s such that all vertices except the 1,2 vertices of the first $C_{4}$ and the 3,4 vertices of the last $C_{4}$ have degree 3 . By adding a vertex of degree 2 adjacent to the 1,2 vertices of the first $C_{4}$ in a chain of $l C_{4}$ s, we get an l-house (see Fig. 2).

Using the $\left\lceil\frac{c}{2}\right\rceil$-house as a gadget, we form a graph that is a tree of gadgets. We let $r$ be the roof of the $\left\lceil\frac{c}{2}\right\rceil$-house, namely the vertex of degree 2 that is in the triangle, and let the two other vertices of degree 2 be terminals $t_{1}$ and $t_{2}$ of the gadget. The tree is formed by taking one gadget as the root gadget and then building the tree by identifying the roof vertex of a child


Fig. 3. A 6-gon graph with an intersection model and two special chords $a$ and $b$. A balanced separator $S=N_{G}[a, b]$ and the connected components of $G \backslash S$ are also shown.
gadget with a terminal vertex of its parent (see Fig. 2). In this way, we form a complete binary tree of gadgets and consider such graphs to be sufficiently large to establish our lower bounds. Furthermore, when constructing graph $H$ from gadget $G$, we say that $H$ has depth 1 if $G \equiv H$ and has depth $k(k>1)$ if all terminal vertices of the root gadget are the roofs of graphs of depth $k-1$.

In [9] the authors show that a tree of $\left[\frac{c}{2}\right]$-houses with sufficiently large depth cannot be collectively $+c$ spanned by any constant number of spanning trees. Therefore, it is sufficient to prove that such a tree of $\left\lceil\frac{c}{2}\right\rceil$-houses is a circle graph which follows from the next claim, which may have been previously observed.

Claim. Let $x$ be a vertex of circle graph $G$ and let $y$ be a vertex of permutation graph $H$. Then the graph $G^{\prime}$ formed from $G$ and $H$ by identifying vertices $x$ and $y$ is a circle graph.

Proof. To prove this claim, let $\Pi$ be a permutation model of $H$ with upper line $L^{\prime}$ and lower line $L^{\prime \prime}$. Consider the left ends (beyond any chord endpoints) of $L^{\prime}$ and $L^{\prime \prime}$ to be identified, and do the same for the right ends of $L^{\prime}$ and $L^{\prime \prime}$. Let $A$ be the ordered set of endpoints that starts with the endpoint immediately to the left of $y^{\prime}$ s endpoint on $L^{\prime}$ and ends with the endpoint immediately to the left of $y$ 's endpoint on $L^{\prime \prime}$. Similarly, let $B$ be the ordered set of endpoints that starts with the endpoint immediately to the right of $y^{\prime}$ 's endpoint on $L^{\prime \prime}$ and ends with the endpoint immediately to the right of $y^{\prime}$ 's endpoint on $L^{\prime}$. Now take a circle representation of $G$ and let one endpoint of $x$ be at the 6 o'clock position; call this endpoint $x^{\prime}$. Let $u^{\prime}$ (respectively $v^{\prime}$ ) be the first endpoint clockwise (respectively counter-clockwise) from $x^{\prime}$. Insert $A$ in the interval between $u^{\prime}$ and $x^{\prime}$ with the first element of $A$ closest to $u^{\prime}$ and insert $B$ in the interval between $x^{\prime}$ and $v^{\prime}$ with the first element of $B$ closest to $x^{\prime}$.

It is straightforward to see that this new representation corresponds to the graph $G^{\prime}$ formed by identifying vertices $x$ and $y$. In particular, the only vertices of $H$ that are adjacent to $x$ are precisely those vertices that are adjacent to $y$ in $H$. Since $G^{\prime}$ has a circle representation, it is a circle graph as required.

Since any $l$-house is a permutation graph, the proposition follows.

## 4. Additive spanners for $\boldsymbol{k}$-gon graphs

In this section, among other results, we show that every $n$-vertex $k$-gon graph $G$ admits a system of at most $2 \log _{\frac{3}{2}} k+7$ collective additive tree 2 -spanners, an additive ( $k+6$ )-spanner with at most $6 n-6$ edges, and an additive $(k / 2+8)$-spanner with at most $10 n-10$ edges. We will assume, in what follows, that our $k$-gon graph $G$ is given together with its intersection model.

Lemma 3. Every n-vertex $(n \geq 2) k$-gon graph $G=(V, E)$ has two vertices a and $b$ such that $S=N_{G}[a, b]$ is a separator of $G$ and each connected component of $G \backslash S$ is a $k^{\prime}$-gon graph with $k^{\prime} \leq 2\left\lceil\frac{1}{3} k\right\rceil$, when $k>5$, and $k^{\prime}=k-1$ when $k=3,4,5$.
Proof. Consider an intersection model $\rho(G)$ of $G$ and let $\mathcal{P}$ be the closed polygonal chain (the boundary) of the $k$-polygon in that model. The vertices of the $k$-polygon, in what follows, are called the corners. Let $\mathcal{C}:=\left[c_{1}, c_{2}, \ldots, c_{k}\right]$ be the sequence in the clockwise order of the corners of $\mathcal{P}$. The proof of this lemma is similar to the proof of Lemma 2 , but here we operate with the corners rather than with the endpoints of the chords. We divide the closed polygonal chain $\mathcal{P}$ into three polygonal sub-chains $B:=\left(c_{1}, \ldots, c_{k_{1}}\right), L:=\left(c_{k_{1}}, \ldots, c_{k_{2}}\right)$ and $R:=\left(c_{k_{2}}, \ldots, c_{k}, c_{1}\right)$ each containing at most $\left\lceil\frac{1}{3} k\right\rceil+1$ consecutive corners (see Fig. 3 for an illustration). We say that a chord of $\rho(G)$ is an $X Y$-chord if its endpoints lie on polygonal chains $X$ and $Y(X, Y \in\{B, L, R\})$ of $\mathcal{P}$. If $v$ is an $X Y$-chord then let $v_{X}$ and $v_{Y}$ be its endpoints on $X$ and $Y$, respectively.

As in the proof of Lemma 2, we may assume, without loss of generality, that there are both BL- and $B R$-chords in $\rho(G)$. Choose BL-chord $a$ in $\rho(G)$ with endpoint $a_{L}$ closest to the corner $c:=c_{k_{2}}$. Choose $B R$-chord $b$ in $\rho(G)$ with endpoint $b_{R}$ closest to the corner $c$ (see Fig. 3). By $a, b$ we denote also the vertices of $G$ which correspond to chords $a$ and $b$. Using the
same arguments as in the proof of Lemma 2, we can see that the removal of chords $a$ and $b$ and all the chords intersecting them divides the remaining chords of $\rho(G)$ into four pairwise disjoint groups $V_{D}, V_{L}, V_{U}$ and $V_{R}$; no chord of one group intersects any chord of another group. Hence, each connected component of the graph $G \backslash N_{G}[a, b]$ has its vertices entirely in either $V_{D}$ or $V_{L}$ or $V_{U}$ or $V_{R}$. Since all chords of $V_{L}$ are between no more than $2\left\lceil\frac{1}{3} k\right\rceil$ consecutive sides of $\mathcal{P}$, the induced subgraph $G\left(V_{L}\right)$ of $G$ is a $k^{\prime}$-gon graph for $k^{\prime} \leq 2\left\lceil\frac{1}{3} k\right\rceil$ (and $k^{\prime}=k-1$ when $k=3,4,5$ ). The same is true for the induced subgraphs $G\left(V_{U}\right), G\left(V_{R}\right)$ and $G\left(V_{D}\right)$ of $G$.

Consider the following procedure which constructs for any $k$-gon graph $G$ a hierarchy of subgraphs of $G$ and a system of local shortest path trees.

Procedure 1. Construct for a $k$-gon graph $G$ a system of local shortest path trees and a system of $r$-gon subgraphs $(r<k)$.
Input: A $k$-gon graph $G$ with an intersection model $\rho(G)$ and a positive integer $r \geq 2$.
Output: $\mathcal{T}$, a system of local shortest path trees, and $\mathcal{F}$, a system of $r$-gon subgraphs of $G$.
Method:
set $i:=0 ; \mathcal{G}_{i}:=\{G\} ; \mathcal{T}:=\emptyset ; \mathcal{F}:=\emptyset$;
while $g_{i} \neq \emptyset$ do
set $\mathcal{G}_{i+1}:=\emptyset ; \mathcal{T}_{i}^{\prime}:=\emptyset ; \mathcal{T}_{i}^{\prime \prime}:=\emptyset ; \mathcal{F}_{i}:=\emptyset$;
for each $G^{\prime} \in g_{i}$ do
if $G^{\prime}$ is an $r$-gon subgraph of $G$ (i.e., all chords of $\rho\left(G^{\prime}\right)$ are between at most $r$ sides of $\rho(G)$ ) then add $G^{\prime}$ to $\mathcal{F}_{i}$;
else
find special vertices $a$ and $b$ in $G^{\prime}$ as described in the proof of Lemma 3;
construct a shortest path tree of $G^{\prime}$ rooted at $a$ and put it in $\mathcal{T}_{i}^{\prime}$;
construct a shortest path tree of $G^{\prime}$ rooted at $b$ and put it in $\widetilde{T}_{i}^{\prime \prime}$;
put all the connected components of $G^{\prime} \backslash N_{G^{\prime}}[a, b]$ into $\mathcal{G}_{i+1}$;
set $\mathcal{T}:=\mathcal{T} \cup \mathcal{T}_{i}^{\prime} \cup \mathcal{T}_{i}^{\prime \prime}$ and $\mathcal{F}:=\mathcal{F} \cup \mathcal{F}_{i}$;
set $i:=i+1$;
return $\mathcal{T}, \mathcal{F}$, and $\mathcal{T}_{i}^{\prime}, \mathcal{T}_{i}^{\prime \prime}$ and $\mathcal{F}_{i}$ for each $i$.
The following lemma follows from Procedure 1.
Lemma 4. For any two vertices $x, y \in V(G)$, there exists a local shortest path tree $T \in \mathcal{T}$ such that $d_{T}(x, y) \leq d_{G}(x, y)+2$ or an $r$-gon subgraph $F \in \mathcal{F}$ of $G$ such that $d_{F}(x, y)=d_{G}(x, y)$.
Proof. Let $g:=\bigcup_{i}\left\{g_{i}\right\}$ be the family of all subgraphs of $G$ generated by Procedure 1 . Let $G^{\prime}$ be the smallest (by the number of vertices in it) subgraph from $\mathcal{q}$, containing both vertices $x$ and $y$ together with a shortest path of $G$ connecting them. Denote this shortest path by $P^{G}(x, y)$. By the choice of $G^{\prime}$, we know that $d_{G^{\prime}}(x, y)=d_{G}(x, y)$. If $G^{\prime}$ belongs to $\mathcal{F}$, then it is an $r$-gon graph and therefore we are done. If $G^{\prime}$ does not belong to $\mathcal{F}$, then any subgraph $G^{\prime \prime} \in \mathcal{G}$, properly contained in $G^{\prime}$ either does not contain both vertices $x$ and $y$ or $d_{G^{\prime \prime}}(x, y)>d_{G}(x, y)$.

Consider special vertices $a$ and $b$, defined in Lemma 3, of a $k^{\prime}$-gon graph $G^{\prime}$ and let $S:=N_{G^{\prime}}[a, b]$ and $T^{\prime}, T^{\prime \prime} \in \mathcal{T}$ be the two shortest path trees of $G^{\prime}$ rooted at $a$ and $b$, respectively. From the choice of $G^{\prime}$, we have $P^{G}(x, y) \cap S \neq \emptyset$. Let $w$ be a vertex from $P^{G}(x, y) \cap S$ and assume, without loss of generality, that $w$ belongs to $N_{G^{\prime}}[a]$. Since $T^{\prime}$ is a shortest path tree of $G^{\prime}$ rooted at $a$, we have $d_{T^{\prime}}(x, a)=d_{G^{\prime}}(x, a) \leq d_{G^{\prime}}(x, w)+1$ and $d_{T^{\prime}}(y, a)=d_{G^{\prime}}(y, a) \leq d_{G^{\prime}}(y, w)+1$. Combining these inequalities with $d_{G^{\prime}}(x, y)=d_{G^{\prime}}(x, w)+d_{G^{\prime}}(y, w)$ and $d_{T^{\prime}}(x, y) \leq d_{T^{\prime}}(x, a)+d_{T^{\prime}}(y, a)$, we obtain $d_{T^{\prime}}(x, y) \leq d_{G^{\prime}}(x, y)+2$. Since $d_{G}(x, y)=d_{G^{\prime}}(x, y)$, we conclude $d_{T^{\prime}}(x, y) \leq d_{G}(x, y)+2$.

Now we are ready to show how to construct a system of at most $2 \log _{\frac{3}{2}} k+7$ collective additive tree 2 -spanners for a $k$-gon graph $G$ with $n$ vertices and $m$ edges. Let $g_{i}:=\left\{G_{i}^{1}, G_{i}^{2}, \ldots, G_{i}^{p_{i}{ }^{2}}\right.$ be the connected graphs of the ith iteration of the while loop $(i=0,1,2, \ldots)$. We run Procedure 1 with parameter $r=2$. Since, in the start iteration, a $k$-gon graph $G$ is reduced to a set of $k_{1}$-gon graphs with $k_{1} \leq 2\left\lceil\frac{1}{3} k\right\rceil \leq \frac{2}{3} k+\frac{4}{3}$, and generally, at iteration $i-1$, any $k_{i-1}$-gon graph is reduced to a set of $k_{i}$-gon graphs with $k_{i} \leq \frac{2}{3} k_{i-1}+\frac{4}{3}$, we conclude that all graphs of $g_{i}$ are $k_{i}$-gon graphs for $k_{i} \leq\left(\frac{2}{3}\right)^{i} k+2\left(\left(\frac{2}{3}\right)^{i}+\left(\frac{2}{3}\right)^{i-1}+\cdots+\frac{2}{3}\right)=\left(\frac{2}{3}\right)^{i} k+4\left(1-\left(\frac{2}{3}\right)^{i}\right)$. Hence, after at most $\log _{\frac{3}{2}} k+1$ iterations, the input $k$-gon graph $G$ will be reduced to a set of $k^{\prime}$-gon graphs with $k^{\prime} \leq 4$, and all graphs at the beginning of iteration $\log _{\frac{3}{2}} k+4$ will be 2-gon graphs (i.e., permutation graphs).

We use $T_{i}^{\prime j}, T_{i}^{\prime \prime j}$ to denote the two local shortest path trees constructed for a graph $G_{i}^{j} \notin \mathcal{F}_{i}, 1 \leq j \leq p_{i}$, by Procedure 1 . For a permutation graph $G_{i}^{j} \in \mathcal{F}_{i}$, let $T_{i}^{j}$ be an additive tree 2 -spanner of $G_{i}^{j}$, which exists by Lemma 1 , and let $T_{i}^{\prime j}:=T_{i}^{\prime \prime j}:=T_{i}^{j}$. Clearly, for any $j, j^{\prime} \in\left\{1, \ldots, p_{i}\right\}, j \neq j^{\prime}$, we have $G_{i}^{j} \cap G_{i}^{j^{\prime}}=\emptyset$. Therefore, $T_{i}^{j j} \cap T_{i}^{\prime j^{\prime}}=\emptyset$ and $T_{i}^{\prime \prime j} \cap T_{i}^{\prime \prime j^{\prime}}=\emptyset$ hold. We can extend in linear $O(m)$ time the forest $T_{i}^{\prime}{ }_{i}, T_{i}^{\prime 2}, \ldots, T_{i}^{\prime p_{i}}$ of $G$ to a single spanning tree $T_{i}^{\prime}$ of $G$ (using, for example, a variant of Kruskal's Spanning Tree Algorithm). Similarly, we can extend the forest $T^{\prime \prime 1}{ }_{i}^{1}, T_{i}^{\prime \prime 2}, \ldots, T_{i}^{\prime \prime p_{i}}$ to another single spanning tree $T_{i}^{\prime \prime}$ of $G$. We call these trees $T_{i}^{\prime}, T_{i}^{\prime \prime}$ the spanning trees of $G$ corresponding to the ith iteration of the while loop. For the last
iteration, it is sufficient to consider only one spanning tree $T_{\text {last }}$ (an extension of the forest $T_{\text {last }}^{1}, T_{\text {last }}^{2}, \ldots, T_{\text {last }}^{p_{\text {last }}}$ ). Since the while loop has at most $\log _{\frac{3}{2}} k+4$ iterations, in this way we will construct at most $2 \log _{\frac{3}{2}} k+7$ spanning trees for $G$, two for each iteration of the while loop, except for the last iteration, where we have only one spanning tree. Denote the collection of these spanning trees by $\delta \mathcal{T}(G)$. By Lemma 4, it is rather straightforward to show that for any two vertices $x$ and $y$ of $G$, there exists a spanning tree $T \in f \mathcal{T}(G)$ such that $d_{T}(x, y) \leq d_{G}(x, y)+2$. Thus, we have

Theorem 3. Every n-vertex m-edge $k$-gon graph $G$ admits a system of at most $2 \log _{\frac{3}{2}} k+7$ collective additive tree 2 -spanners, constructible in $O(m \log k)$ time. Moreover, every 3-gon graph admits a system of no more than three collective additive tree 2 -spanners, and every 4 -gon graph admits a system of no more than five collective additive tree 2-spanners.

Similar to Corollary 2, we have
Corollary 3. Every $n$-vertex $k$-gon graph $G$ admits an additive 2 -spanner with at most $\left(2 \log _{\frac{3}{2}} k+7\right)(n-1)$ edges. Moreover, every 3-gon graph admits an additive 2-spanner with at most $3(n-1)$ edges, and every 4 -gon graph admits an additive 2-spanner with at most $5(n-1)$ edges.

We can state also the following result.
Theorem 4. Every n-vertex m-edge $k$-gon graph $G$ admits an additive $\left(2\left(\left(\frac{2}{3}\right)^{\ell} k+4\left(1-\left(\frac{2}{3}\right)^{\ell}\right)\right)+1\right)$-spanner with at most $2(\ell+1)(n-1)$ edges, for each $0 \leq \ell \leq \log _{\frac{3}{2}} k+3$. Moreover, such a sparse spanner is constructible in $O(m \log k)$ time.
Proof. We run Procedure 1 with parameter $r:=k_{\ell}=\left(\frac{2}{3}\right)^{\ell} k+4\left(1-\left(\frac{2}{3}\right)^{\ell}\right)$. In this case, we will have only $\ell+1$ iterations of the while loop of Procedure 1. We use also the fact that in any $k^{\prime}$-gon graph the length of a largest induced cycle is at most $2 k^{\prime}$ (see [15]). In [7], it was shown that if the length of the largest induced cycle of a graph $G^{\prime}$ is $c$, then $G^{\prime}$ admits an additive $(c+1)$-spanner with at most $2\left|V\left(G^{\prime}\right)\right|-2$ edges, and such a sparse spanner for $G^{\prime}$ can be constructed in $O\left(\left|E\left(G^{\prime}\right)\right|\right)$ time. Using this, for each $k_{\ell}$-gon graph $G_{i}^{j} \in \mathcal{F}$ we can construct an additive $\left(2 k_{\ell}+1\right)$-spanner $H_{i}^{j}$ with at most $2\left|V\left(G_{i}^{j}\right)\right|-2$ edges.

Now, a spanning subgraph $H=(V, F)$ of $G=(V, E)$ can be defined as follows. The edge-set $F$ of $H$ is empty initially. For each iteration $i(0 \leq i \leq \ell)$, if $G_{i}^{j} \in \mathscr{g}_{i}$ belongs to $\mathcal{F}_{i}$, then add all edges $E\left(H_{i}^{j}\right)$ into $F$, else add into $F$ all edges of local shortest path trees $T_{i}^{\prime j}$ and $T^{\prime \prime j}$. Since for each iteration we add into $F$ at most $2 n-2$ edges of $G$, the final edge-set $F$ will have no more than $(2 n-2)(\ell+1)$ edges. Using Lemma 4, it is easy to see also that for any two vertices $x$ and $y$ of $G$, $d_{H}(x, y) \leq d_{G}(x, y)+2 k_{\ell}+1$ holds.

Choosing $\ell$ equal to $0,1,2,3$ or 4 in Theorem 4, we obtain
Corollary 4. Every $n$-vertex $k$-gon graph $G$ admits an additive $(2 k+1)$-spanner with at most $2 n-2$ edges, an additive $\left(\frac{4}{3} k+4\right)$ spanner with at most $4 n-4$ edges, an additive $\left(\frac{8}{9} k+6\right)$-spanner with at most $6 n-6$ edges, an additive $\left(\frac{16}{27} k+7\right)$-spanner with at most $8 n-8$ edges, and an additive $\left(\frac{32}{81} k+8\right)$-spanner with at most $10 n-10$ edges.

## 5. Additive tree spanners for 3-gon graphs

In this section, we show that any connected 3 -gon graph $G$ admits an additive tree 6 -spanner constructible in linear time. Note that, since an induced cycle on six vertices is a 3-gon graph, 3-gon graphs cannot have any additive tree $r$-spanner for $r<4$. The algorithm will identify permutation graphs in each of the three corners of the 3-gon and use the algorithm presented in Lemma 1 to construct effective tree spanners of each of these subgraphs. These three tree spanners are incorporated into a tree spanner for the entire graph by analyzing the structure in the "center" of the given 3-gon graph.

Let $G=(V, E)$ be a connected 3-gon graph. We may assume that $G$ is not a permutation graph. Consider a 3-gon intersection model $\Delta$ of $G$ and fix an orientation of $\Delta$. Denote by $L$ (left), $R$ (right) and $B$ (bottom) the corresponding sides of the 3-gon $\Delta$, and by $C_{L}, C_{R}$ and $C_{U}$ the left, right and upper corners of $\Delta$. We say that a chord of $\Delta$ is an $X Y$-chord if its endpoints lie on sides $X$ and $Y$ of $\Delta$. If $v$ is an $X Y$-chord then let $v_{X}$ and $v_{Y}$ be its endpoints on $X$ and $Y$, respectively. Since $G$ is not a permutation graph, we must have all three types of chords in $\Delta$ : $L R$-chords, $L B$-chords and $R B$-chords. Let $a$ be the $L B$-chord of $G$ whose endpoint on $L$ is closest to the upper corner $C_{U}$ of $\Delta$. Let $b$ be the $R B$-chord of $G$ whose endpoint on $R$ is closest to the upper corner of $\Delta$ (see the left 3-gon in Fig. 4 for an illustration). Note that $a$ and $b$ may or may not cross. By $a, b$ we also denote the corresponding vertices of $G$.

Let $V_{U}$ be the subset of $L R$-chords of $\Delta$ (of vertices of $G$ ) with endpoints in segments $\left(a_{L}, C_{U}\right)$ and $\left(b_{R}, C_{U}\right)$. We will add at most two more $L R$-chords to $V_{U}$ to form a permutation graph named $G_{U}$. Choose (if it exists) an $L R$-chord $\alpha^{u}$ in $\Delta$ such that $\alpha_{L}^{u}$ belongs to segment $\left(C_{L}, a_{L}\right)$ of $L, \alpha_{R}^{u}$ belongs to segment $\left(C_{U}, b_{R}\right)$ of $R$ and $\alpha_{R}^{u}$ is closest to the corner $C_{U}$. Clearly, if $\alpha^{u}$ exists then it must intersect $a$ (but not $b$ ). Analogously, choose (if it exists) an $L R$-chord $\beta^{u}$ in $\Delta$ such that $\beta_{R}^{u}$ belongs to segment $\left(C_{R}, b_{R}\right)$ of $R, \beta_{L}^{u}$ belongs to segment $\left(C_{U}, a_{L}\right)$ of $L$ and $\beta_{L}^{u}$ is closest to the corner $C_{U}$. Again, if $\beta^{u}$ exists then it must intersect $b$ (but not $a$ ). Note that, if $V_{U} \neq \emptyset$, then at least one of $\left\{\alpha^{u}, \beta^{u}\right\}$ must exist (since otherwise, $G$ is not connected), and if both chords exist then they must intersect each other. See the right picture in Fig. 4. Now, we define our permutation graph $G_{U}$ to be the subgraph of $G$ induced by vertices $V_{U} \cup\left\{\alpha^{u}, \beta^{u}\right\}$, assuming that $V_{U} \neq \emptyset$ (see Fig. 5 for an illustration). If $V_{U}=\emptyset$, then we set $G_{U}$ to be an empty graph.


Fig. 4. A 3-gon intersection model $\Delta$ with special chords $a, b, \alpha^{u}$ and $\beta^{u}$.


Fig. 5. Permutation graph $G_{U}$ extracted from $G$.
The following two propositions hold for $G_{U}$.
Proposition 3. For every $x, y \in V_{U} \cup\left\{\alpha^{u}, \beta^{u}\right\}, d_{G_{U}}(x, y)=d_{G}(x, y)$.
Proof. Clearly if both $\alpha^{u}$ and $\beta^{u}$ exist, then $d_{G}\left(\alpha^{u}, \beta^{u}\right)=d_{G_{U}}\left(\alpha^{u}, \beta^{u}\right)=1$. Let $P_{G}(x, y)$ be a shortest path in $G$ between $x, y \in V_{U}$. If $P_{G}(x, y)$ has no vertices outside $V_{U}$, then this path is in $G_{U}$, too, and therefore $d_{G U}(x, y)=d_{G}(x, y)$. Assume now that $P_{G}(x, y)$ contains vertices from $V \backslash V_{U}$. Consider such a vertex $x^{\prime}$ closest to $x$ and such a vertex $y^{\prime}$ closest to $y$. Let $x^{\prime \prime}$ be the neighbor of $x^{\prime}$ on $P_{G}(x, y)$ closer to $x$, and $y^{\prime \prime}$ be the neighbor of $y^{\prime}$ on $P_{G}(x, y)$ closer to $y$. Necessarily, $x^{\prime \prime}, y^{\prime \prime} \in V_{U}$, $x^{\prime}, y^{\prime}$ belong to $N_{G}[a, b]$ and because of the maximality of $a_{L}$ and $b_{R}$, the corresponding chords $x^{\prime}, y^{\prime}$ are between $\left(C_{L}, a_{L}\right)$ and ( $C_{U}, b_{R}$ ) or between $\left(C_{R}, b_{R}\right)$ and $\left(C_{U}, a_{L}\right)$. If $x^{\prime} \neq y^{\prime}$, then a simple geometric consideration shows that $x^{\prime \prime}$ must be adjacent to $y^{\prime}$ or $y^{\prime \prime}$ must be adjacent to $x^{\prime}$ or $x^{\prime \prime}, y^{\prime \prime}$ are adjacent. Since that is impossible in a shortest path $P_{G}(x, y)$, we conclude $x^{\prime}=y^{\prime}$. Assume, without loss of generality, that the chord $x^{\prime}=y^{\prime}$ is between $\left(C_{L}, a_{L}\right)$ and $\left(C_{U}, b_{R}\right)$. In $P_{G}(x, y)$, by replacing vertex $x^{\prime}$ with vertex $\alpha^{u}$ (note that chord $\alpha^{u}$ crosses both $x^{\prime \prime}$ and $y^{\prime \prime}$ ), one can obtain a shortest ( $x, y$ )-path of $G$ completely contained in $G_{U}$.

Now let $P_{G}\left(x, \alpha^{u}\right)$ be a shortest path in $G$ between $x \in V_{U}$ and $\alpha^{u}$. If $P_{G}\left(x, \alpha^{u}\right)$ has no vertices outside $V_{U} \cup\left\{\alpha^{u}\right\}$, then this path is in $G_{U}$, too, and therefore $d_{G_{U}}\left(x, \alpha^{u}\right)=d_{G}\left(x, \alpha^{u}\right)$. Assume that $P_{G}\left(x, \alpha^{u}\right)$ contains vertices from $V \backslash\left(V_{U} \cup\left\{\alpha^{u}\right\}\right)$. Consider such a vertex $x^{\prime}$ closest to $x$ and let $x^{\prime \prime}$ be the neighbor of $x^{\prime}$ on $P_{G}\left(x, \alpha^{u}\right)$ closer to $x$. Since $\beta^{u}$ and $\alpha^{u}$ are adjacent, $x^{\prime \prime} \neq \beta^{u}$. Necessarily, $x^{\prime \prime} \in V_{U}, x^{\prime}$ belongs to $N_{G}[a, b]$ and the corresponding chord $x^{\prime}$ is between $\left(C_{L}, a_{L}\right)$ and $\left(C_{U}, b_{R}\right)$ or between $\left(C_{R}, b_{R}\right)$ and $\left(C_{U}, a_{L}\right)$. A simple geometric consideration shows that $x^{\prime \prime}$ must be adjacent to $\beta^{u}$. In $P_{G}\left(x, \alpha^{u}\right)$, replacing vertex $x^{\prime}$ with vertex $\beta^{u}$ (note that chord $\beta^{u}$ crosses both $x^{\prime \prime}$ and $\alpha^{u}$ ), one can obtain a shortest ( $x, \alpha^{u}$ )-path of $G$ completely contained in $G_{U}$, i.e., $d_{G}\left(x, \alpha^{u}\right)=d_{G U}\left(x, \alpha^{u}\right)$. Similarly, we can show that $d_{G}\left(x, \beta^{u}\right)=d_{G_{U}}\left(x, \beta^{u}\right)$ for every $x \in V_{U}$.

Consider the case when both $\alpha^{u}$ and $\beta^{u}$ exist. Set $s^{u}:=\alpha^{u}$ if $\alpha^{u}$ has a neighbor in $V_{U}$ which is not a neighbor of $\beta^{u}$ and set $s^{u}:=\beta^{u}$ otherwise.

Proposition 4. There is a linear time constructible BFS $\left(s^{u}\right)$-tree $T_{U}$ of $G_{U}$ such that $d_{G}(x, y) \leq d_{T_{U}}(x, y) \leq d_{G}(x, y)+2$ and $d_{T_{U}}\left(x, s^{u}\right)=d_{G}\left(x, s^{u}\right)$ for any $x, y$ in $V_{U} \cup\left\{\alpha^{u}, \beta^{u}\right\}$.
Proof. Since $G_{U}$ is a permutation graph and $s^{u}$ is extreme, by Lemma 1, there is in $G_{U}$ a linear time constructible $B F S\left(s^{u}\right)$ tree $T_{U}$ such that $d_{T_{U}}(x, y) \leq d_{G_{U}}(x, y)+2$ and $d_{T_{U}}\left(x, s^{u}\right)=d_{G_{U}}\left(x, s^{u}\right)$ for any $x, y$ in $V_{U} \cup\left\{\alpha^{u}, \beta^{u}\right\}$. Moreover, since $T_{U}$ is a subgraph of $G, d_{G}(x, y) \leq d_{T_{U}}(x, y)$ for all $x, y \in V_{U} \cup\left\{\alpha^{u}, \beta^{u}\right\}$. Hence, by Proposition 3, we are done.


Fig. 6. Permutation graph $G_{L}$ obtained from $G$.

Let $V_{L}$ be the subset of all chords of $\Delta$ (of vertices of $G$ ) with endpoints in segments $\left(C_{L}, a_{L}\right)$ and $\left(C_{L}, a_{B}\right) \cap\left(C_{L}, b_{B}\right)$. We will add at most two more chords to $V_{L}$ to form a permutation graph named $G_{L}$. Choose (if it exists) a chord $\alpha^{\ell}$ in $\Delta$ such that one endpoint of $\alpha^{\ell}$ belongs to segment $\left(C_{L}, a_{L}\right)$ of $L$, the other endpoint belongs to $R \cup\left(a_{B}, C_{R}\right) \cup\left(b_{B}, C_{R}\right)$ and $\alpha_{L}^{\ell}$ is closest to the corner $C_{L}$. Equivalently, among all chords of $\Delta$ intersecting $a$ or $b$ (or both), $\alpha^{\ell}$ is chosen to be the chord with an endpoint $\alpha_{L}^{\ell}$ in ( $C_{L}, a_{L}$ ) closest to $C_{L}$. Note that $\alpha^{\ell}$ may or may not cross $b$. Also, choose (if it exists) an $R B$-chord $\beta^{\ell}$ in $\Delta$ such that $\beta_{R}^{\ell}$ belongs to segment $\left(C_{R}, b_{R}\right)$ of $R, \beta_{B}^{\ell}$ belongs to segment $\left(C_{L}, a_{B}\right) \cap\left(C_{L}, b_{B}\right)$ of $B$ and $\beta_{B}^{\ell}$ is closest to the corner $C_{L}$. Notice, if $\beta^{\ell}$ exists then it must intersect both $a$ and $b$. Furthermore, if $V_{L} \neq \emptyset$, then at least one chord from $\left\{\alpha^{\ell}, \beta^{\ell}\right\}$ must exist (since, otherwise, $G$ is not connected). Now, we define our permutation graph $G_{L}$. If $V_{L}=\emptyset$, then set $G_{L}$ to be an empty graph. Otherwise, $G_{L}$ is set to be the subgraph of $G$ induced by vertices $V_{L} \cup\left\{\alpha^{\ell}, \beta^{\ell}\right\}$ with one extra edge ( $\alpha^{\ell}$, $\beta^{\ell}$ ) added if it was not already an edge of $G$ (see Fig. 6 for an illustration).

The following three propositions hold for $G_{L}$.
Proposition 5. Let both $\alpha^{\ell}$ and $\beta^{\ell}$ exist. Then, there is no shortest path in $G_{L}$ between any $x, y \in V_{L}$, which uses the edge ( $\alpha^{\ell}$, $\beta^{\ell}$ ). Moreover, for any vertex $x \in V_{L}$ and $s \in\left\{\alpha^{\ell}, \beta^{\ell}\right\}$, there is a shortest path $P_{G_{L}}(x, s)$ of $G_{L}$ which does not use the edge ( $\alpha^{\ell}, \beta^{\ell}$ ), whenever $\left(N_{G}(s) \backslash N_{G}\left(\left\{\alpha^{\ell}, \beta^{\ell}\right\} \backslash\{s\}\right)\right) \cap V_{L} \neq \emptyset$.
Proof. Let $P_{G_{L}}(x, y)$ be a shortest path of $G_{L}$ between $x$ and $y\left(x, y \in V_{L}\right)$ using the edge ( $\alpha^{\ell}, \beta^{\ell}$ ). Consider the neighbors $f$ and $t\left(f, t \in V_{L}\right)$ in $P_{G_{L}}(x, y)$ of $\alpha^{\ell}$ and $\beta^{\ell}$, respectively. Since $f \in N_{G}\left(\alpha^{\ell}\right) \backslash N_{G}\left(\beta^{\ell}\right), t \in N_{G}\left(\beta^{\ell}\right) \backslash N_{G}\left(\alpha^{\ell}\right)$ and $f, t \in V_{L}$, a simple geometric consideration shows that $f$ and $t$ must be adjacent in $G$ (and hence in $G_{L}$ ), thereby contradicting $P_{G_{L}}(x, y)$ being a shortest $(x, y)$-path in $G_{L}$.

Now let $P_{G_{L}}(x, s)$ be a shortest path of $G_{L}$ between $x \in V_{L}$ and $s \in\left\{\alpha^{\ell}, \beta^{\ell}\right\}$ using the edge ( $\alpha^{\ell}, \beta^{\ell}$ ), and assume that $s$ has a neighbor $f$ in $V_{L}$ which is not adjacent to $g:=\left\{\alpha^{\ell}, \beta^{\ell}\right\} \backslash\{s\}$. Consider the neighbor $t \in V_{L}$ in $P_{G_{L}}(x, s)$ of $g$. Since $t \in N_{G}(g) \backslash N_{G}(s), f \in N_{G}(s) \backslash N_{G}(g)$ and $f, t \in V_{L}$, a simple geometric consideration shows that $f$ and $t$ must be adjacent in $G$ (and hence in $G_{L}$ ). Replacing vertex $g$ in $P_{G_{L}}(x, s)$ with vertex $f$, we obtain a new shortest ( $x, s$ )-path in $G_{L}$, which does not use the edge ( $\alpha^{\ell}, \beta^{\ell}$ ).

Note that, since $G_{L}$ may have edge ( $\alpha^{\ell}, \beta^{\ell}$ ) which may not be an edge of $G$, some distances in $G_{L}$ can be smaller than in $G$.
Proposition 6. For every $x, y \in V_{L}, d_{G_{L}}(x, y)=d_{G}(x, y)$. Moreover, for each $s \in\left\{\alpha^{\ell}, \beta^{\ell}\right\}, d_{G_{L}}(x, s) \leq d_{G}(x, s)$ holds for all $x \in V_{L}$, and if $d_{G_{L}}(x, s)<d_{G}(x, s)$ for some $x \in V_{L}$, then $d_{G_{L}}(x, s)=d_{G}(x, s)-1$ and every neighbor of $s$ in $V_{L}$ is a neighbor of $g:=\left\{\alpha^{\ell}, \beta^{\ell}\right\} \backslash\{s\}$.

Proof. Let $P_{G}(x, y)$ be a shortest path in $G$ between $x, y \in V_{L}$. If $P_{G}(x, y)$ has no vertices outside $V_{L}$, then this path is in $G_{L}$, too, and therefore $d_{G_{L}}(x, y) \leq d_{G}(x, y)$. Hence, by Proposition 5, $d_{G_{L}}(x, y)=d_{G}(x, y)$. Assume now that $P_{G}(x, y)$ contains vertices from $V \backslash V_{L}$. Consider such a vertex $x^{\prime}$ closest to $x$ and such a vertex $y^{\prime}$ closest to $y$. Let $x^{\prime \prime}$ be the neighbor of $x^{\prime}$ on $P_{G}(x, y)$ closer to $x$, and $y^{\prime \prime}$ be the neighbor of $y^{\prime}$ on $P_{G}(x, y)$ closer to $y$. Necessarily, $x^{\prime \prime}, y^{\prime \prime} \in V_{L}, x^{\prime}, y^{\prime}$ belong to $N_{G}[a, b]$ and the corresponding chords $x^{\prime}, y^{\prime}$ are between $\left(C_{L}, a_{B}\right) \cap\left(C_{L}, b_{B}\right)$ and $\left(C_{R}, b_{R}\right)$ or between $\left(C_{L}, a_{L}\right)$ and $R \cup\left(a_{B}, C_{R}\right) \cup\left(b_{B}, C_{R}\right)$. If $x^{\prime} \neq y^{\prime}$, then a simple geometric consideration shows that $x^{\prime \prime}$ must be adjacent to $y^{\prime}$ or $y^{\prime \prime}$ must be adjacent to $x^{\prime}$ or $x^{\prime \prime}$, $y^{\prime \prime}$ are adjacent. Since that is impossible in a shortest path $P_{G}(x, y)$, we conclude $x^{\prime}=y^{\prime}$. If the chord $x^{\prime}=y^{\prime}$ is between $\left(C_{L}, a_{B}\right) \cap\left(C_{L}, b_{B}\right)$ and $\left(C_{R}, b_{R}\right)$, then in $P_{G}(x, y)$ we can replace vertex $x^{\prime}$ with vertex $\beta^{\ell}$ (since chord $\beta^{\ell}$ crosses both $x^{\prime \prime}$ and $\left.y^{\prime \prime}\right)$. If the chord $x^{\prime}=y^{\prime}$ is between $\left(C_{L}, a_{L}\right)$ and $R \cup\left(a_{B}, C_{R}\right) \cup\left(b_{B}, C_{R}\right)$, then in $P_{G}(x, y)$ we can replace vertex $x^{\prime}$ with vertex $\alpha^{\ell}$ (since chord $\alpha^{\ell}$ crosses both $x^{\prime \prime}$ and $y^{\prime \prime}$ ). In both cases, we obtain a shortest ( $x, y$ ) -path of $G$ completely contained in $G_{L}$. Hence, $d_{G_{L}}(x, y) \leq d_{G}(x, y)$, implying $d_{G_{L}}(x, y)=d_{G}(x, y)$, by Proposition 5 .

Consider now a shortest path $P_{G}\left(x, \alpha^{\ell}\right)$ in $G$ between $x \in V_{L}$ and $\alpha^{\ell}$. If $P_{G}\left(x, \alpha^{\ell}\right)$ has no vertices outside $V_{L}$, except $\alpha^{\ell}$ itself, then this path is in $G_{L}$, too, and therefore $d_{G_{L}}\left(x, \alpha^{\ell}\right) \leq d_{G}\left(x, \alpha^{\ell}\right)$. We will have $d_{G_{L}}\left(x, \alpha^{\ell}\right)<d_{G}\left(x, \alpha^{\ell}\right)$ only if there
is a path $P$ in $G_{L}$ shorter than $P_{G}\left(x, \alpha^{\ell}\right)$ where $P$ uses the edge $\left(\alpha^{\ell}, \beta^{\ell}\right) \in E\left(G_{L}\right) \backslash E(G)$. But then, by Proposition 5 , any neighbor of $\alpha^{\ell}$ in $V_{L}$ is a neighbor of $\beta^{\ell}$, too, thereby contradicting the existence of $P$. Assume now that $P_{G}\left(x, \alpha^{\ell}\right)$ contains vertices from $V \backslash\left(V_{L} \cup\left\{\alpha^{\ell}\right\}\right)$. Consider such a vertex $x^{\prime}$ closest to $x$ and let $x^{\prime \prime}$ be the neighbor of $x^{\prime}$ on $P_{G}\left(x, \alpha^{\ell}\right)$ closer to $x$. Necessarily, $x^{\prime \prime} \in V_{L}, x^{\prime}$ belongs to $N_{G}[a, b]$ and the corresponding chord $x^{\prime}$ is between $\left(C_{L}, a_{B}\right) \cap\left(C_{L}, b_{B}\right)$ and $\left(C_{R}, b_{R}\right)$ or between $\left(C_{L}, a_{L}\right)$ and $R \cup\left(a_{B}, C_{R}\right) \cup\left(b_{B}, C_{R}\right)$. A simple geometric consideration shows that $x^{\prime \prime}$ is either adjacent to $\alpha^{\ell}$ (which contradicts $P_{G}\left(x, \alpha^{\ell}\right)$ being a shortest path) or to $\beta^{\ell}$. Since $\beta^{\ell}$ is adjacent in $G_{L}$ to $\alpha^{\ell}$ as well, we get a ( $x, \alpha^{\ell}$ )-path completely contained in $G_{L}$ and of length at most the length of $P_{G}\left(x, \alpha^{\ell}\right)$. Hence, $d_{G}\left(x, \alpha^{\ell}\right) \geq d_{G_{L}}\left(x, \alpha^{\ell}\right)$. Again, $d_{G_{L}}\left(x, \alpha^{\ell}\right)<d_{G}\left(x, \alpha^{\ell}\right)$ can hold only if there is no shortest path between $x$ and $\alpha^{\ell}$ in $G_{L}$ not using the edge ( $\alpha^{\ell}, \beta^{\ell}$ ). But then, by Proposition 5 , any neighbor of $\alpha^{\ell}$ in $V_{L}$ is a neighbor of $\beta^{\ell}$, too. Similarly, we can show that $d_{G_{L}}\left(x, \beta^{\ell}\right) \leq d_{G}\left(x, \beta^{\ell}\right)$ holds for all $x \in V_{L}$, and if $d_{G_{L}}\left(x, \beta^{\ell}\right)<d_{G}\left(x, \beta^{\ell}\right)$ for some $x \in V_{L}$, then every neighbor of $\beta^{\ell}$ in $V_{L}$ is a neighbor of $\alpha^{\ell}$.

To show that $d_{G_{L}}\left(x, \alpha^{\ell}\right)<d_{G}\left(x, \alpha^{\ell}\right)$ implies $d_{G_{L}}\left(x, \alpha^{\ell}\right)=d_{G}\left(x, \alpha^{\ell}\right)-1$, first note that vertex $a$ is adjacent in $G$ to both $\alpha^{\ell}$ and $\beta^{\ell}$, and that $d_{G}\left(x, \beta^{\ell}\right)=d_{G_{L}}\left(x, \beta^{\ell}\right)$, since $\beta^{\ell}$ is adjacent to $x^{\prime \prime}$ in $V_{L}$. Hence, $d_{G}\left(x, \alpha^{\ell}\right) \leq d_{G}\left(x, \beta^{\ell}\right)+2=d_{G_{L}}\left(x, \beta^{\ell}\right)+2$. On the other hand, we have $d_{G}\left(x, \alpha^{\ell}\right) \geq d_{G_{L}}\left(x, \alpha^{\ell}\right)+1=d_{G_{L}}\left(x, \beta^{\ell}\right)+2$. From these two inequalities, $d_{G_{L}}\left(x, \alpha^{\ell}\right)=d_{G}\left(x, \alpha^{\ell}\right)-1$ follows. Similarly, one can show that $d_{G_{L}}\left(x, \beta^{\ell}\right)<d_{G}\left(x, \beta^{\ell}\right)$ implies $d_{G_{L}}\left(x, \beta^{\ell}\right)=d_{G}\left(x, \beta^{\ell}\right)-1$.

Consider the case when both $\alpha^{\ell}$ and $\beta^{\ell}$ exist. Set $s^{\ell}:=\alpha^{\ell}$ if $\alpha^{\ell}$ has a neighbor in $V_{L}$ which is not a neighbor of $\beta^{\ell}$ and set $s^{\ell}:=\beta^{\ell}$ otherwise.

Corollary 5. $d_{G_{L}}\left(x, s^{\ell}\right)=d_{G}\left(x, s^{\ell}\right)$ for every $x \in V_{L}$.
Proof. If $s^{\ell}=\alpha^{\ell}$, i.e., there is a neighbor of $\alpha^{\ell}$ in $V_{L}$ which is not a neighbor of $\beta^{\ell}$, then, by Proposition $6, d_{G_{L}}\left(x, \alpha^{\ell}\right)=$ $d_{G}\left(x, \alpha^{\ell}\right)$. Assume now that $s^{\ell}=\beta^{\ell}$. If there is a neighbor of $\beta^{\ell}$ in $V_{L}$ which is not a neighbor of $\alpha^{\ell}$, then again, by Proposition 6, $d_{G_{L}}\left(x, \beta^{\ell}\right)=d_{G}\left(x, \beta^{\ell}\right)$. Hence, we may assume that $\alpha^{\ell}$ and $\beta^{\ell}$ have the same neighborhood in $V_{L}$. In this case, $d_{G_{L}}\left(x, \alpha^{\ell}\right)=d_{G}\left(x, \alpha^{\ell}\right)=d_{G_{L}}\left(x, \beta^{\ell}\right)=d_{G}\left(x, \beta^{\ell}\right)$, since edge $\left(\alpha^{\ell}, \beta^{\ell}\right)$ is not part of any shortest path of $G_{L}$ between $x \in V_{L}$ and $v \in\left\{\alpha^{\ell}, \beta^{\ell}\right\}$.

Proposition 7. There is a linear time constructible BFS $\left(s^{\ell}\right)$-tree $T_{L}$ of $G_{L}$ such that $d_{G}(x, y)-1 \leq d_{T_{L}}(x, y) \leq d_{G}(x, y)+2$, for any $x, y$ in $V_{L} \cup\left\{\alpha^{\ell}, \beta^{\ell}\right\}$, and $d_{T_{L}}\left(x, s^{\ell}\right)=d_{G}\left(x, s^{\ell}\right)$, for all $x \in V_{L}$. Moreover, $d_{G}(x, y) \leq d_{T_{L}}(x, y)$ for all $x, y \in V_{L}$.
Proof. Since $G_{L}$ is a permutation graph, by Lemma 1, there is in $G_{L}$ a linear time constructible $\operatorname{BFS}\left(s^{\ell}\right)$-tree $T_{L}$ such that $d_{G_{L}}(x, y) \leq d_{T_{L}}(x, y) \leq d_{G_{L}}(x, y)+2$ and $d_{T_{L}}\left(x, s^{\ell}\right)=d_{G_{L}}\left(x, s^{\ell}\right)$ for any $x, y$ in $V_{L} \cup\left\{\alpha^{\ell}, \beta^{\ell}\right\}$. Hence, $d_{T_{L}}\left(x, s^{\ell}\right)=d_{G}\left(x, s^{\ell}\right)$ for every $x \in V_{L}$, by Corollary 5, and $d_{G}(x, y)-1 \leq d_{T_{L}}(x, y) \leq d_{G}(x, y)+2$ for every $x, y \in V_{L} \cup\left\{\alpha^{\ell}\right.$, $\left.\beta^{\ell}\right\}$ (with $d_{G}(x, y) \leq d_{T_{L}}(x, y)$ for all $x, y \in V_{L}$ ), by Proposition 6. Clearly, $d_{G}\left(\alpha^{\ell}, \beta^{\ell}\right) \leq 2$ (since $a$ is adjacent to both $\alpha^{\ell}, \beta^{\ell}$ ) and $d_{T_{L}}\left(\alpha^{\ell}, \beta^{\ell}\right)=d_{G_{L}}\left(\alpha^{\ell}, \beta^{\ell}\right)=1$.

Taking symmetry into account, similar to $\alpha^{\ell}, \beta^{\ell}$ and $G_{L}$, we can define for the corner $C_{R}$ of $\Delta$ two specific chords $\alpha^{r}, \beta^{r}$ and a permutation graph $G_{R}$. We will have $\beta^{r}$ adjacent to both $a$ and $b$, and $\alpha^{r}$ adjacent to $a$ or $b$. Consider the case when both $\alpha^{r}$ and $\beta^{r}$ exist. Set $s^{r}:=\alpha^{r}$ if $\alpha^{r}$ has a neighbor in $V_{R}$ which is not a neighbor of $\beta^{r}$ and set $s^{r}:=\beta^{r}$ otherwise. We can state

Proposition 8. There is a linear time constructible BFS $\left(s^{r}\right)$-tree $T_{R}$ of $G_{R}$ such that $d_{G}(x, y)-1 \leq d_{T_{R}}(x, y) \leq d_{G}(x, y)+2$, for any $x, y$ in $V_{R} \cup\left\{\alpha^{r}, \beta^{r}\right\}$, and $d_{T_{R}}\left(x, s^{r}\right)=d_{G}\left(x, s^{r}\right)$, for all $x \in V_{R}$. Moreover, $d_{G}(x, y) \leq d_{T_{R}}(x, y)$ for all $x, y \in V_{R}$.

We will need also the following straightforward facts.
Proposition 9. We have $d_{G}\left(x, s^{\ell}\right) \leq d_{G}\left(x,\left\{\alpha^{\ell}, \beta^{\ell}\right\}\right)+1$ for every $x \in V_{L}$, and $d_{G}\left(x, s^{r}\right) \leq d_{G}\left(x,\left\{\alpha^{r}, \beta^{r}\right\}\right)+1$ for every $x \in V_{R}$. Proof. We will prove only the first part. The proof of the second part is similar. Let $g:=\left\{\alpha^{\ell}, \beta^{\ell}\right\} \backslash\left\{s^{\ell}\right\}$, and assume $d_{G}(x, g) \leq d_{G}\left(x, s^{\ell}\right)-2$ for some vertex $x \in V_{L}$. Consider a shortest path $P_{G}(x, g)$ in $G$ between $x$ and $g$ and the neighbor $g^{\prime}$ of $g$ in $P_{G}(x, g)$. Since $a$ is adjacent to both $\alpha^{\ell}$ and $\beta^{\ell}, d_{G}(x, g)=d_{G}\left(x, s^{\ell}\right)-2$ thereby implying that $s^{\ell}$ has no neighbors in $P_{G}(x, g)$ and $a$ has only $g$ as a neighbor in $P_{G}(x, g)$. Then, necessarily, $g^{\prime}$ belongs to $V_{L}$ and, by the choice of $s^{\ell}$, there must exist a neighbor $f$ of $s^{\ell}$ in $V_{L}$ which is not adjacent to $g$. A simple geometric consideration then shows that $f$ and $g^{\prime}$ have to be adjacent in $G$. The latter is impossible since $d_{G}\left(x, s^{\ell}\right)=d_{G}(x, g)+2$.

Proposition 10. We have $V=V_{U} \cup V_{L} \cup V_{R} \cup N_{G}[a, b]$ where $V_{U}, V_{L}, V_{R}, N_{G}[a, b]$ are disjoint sets. Moreover, $N_{G}[a, b]$ separates vertices of $V_{X}$ from vertices of $V_{Y}$ for every $X, Y \in\{L, R, U\}, X \neq Y$.

Proposition 11. We have $d_{G}(a, b) \leq 3$.
Proof. If $\beta^{r}$ or $\beta^{\ell}$ exist (say, without loss of generality, that $\beta^{r}$ exists), then $d_{G}(a, b) \leq 2$, since $\beta^{r}$ crosses both $a$ and $b$. Assume now that neither $\beta^{r}$ nor $\beta^{\ell}$ exists. Then both $\alpha^{r}$ and $\alpha^{\ell}$ must exist. If $d_{G}(a, b)>2$ and both $\alpha^{u}$ and $\beta^{u}$ exist, then $\left(a, \alpha^{u}\right),\left(\alpha^{u}, \beta^{u}\right),\left(\beta^{u}, b\right) \in E(G)$ and hence $d_{G}(a, b)=3$. Assume now that $d_{G}(a, b)>2$ and, without loss of generality, $\alpha^{u}$ exists and $\beta^{u}$ does not exist. Choose a $B R$-chord $x$ such that $x_{R}$ belongs to segment $\left(C_{R}, b_{R}\right), x_{B}$ belongs to segment $\left(a_{B}, b_{B}\right)$ and $x_{B}$ is closest to $C_{L}$. Analogously, choose a BL-chord $y$ such that $y_{L}$ belongs to segment $\left(C_{L}, a_{L}\right), y_{B}$ belongs to segment $\left(a_{B}, b_{B}\right)$ and $y_{B}$ is closest to $C_{R}$. Since $\beta^{u}$ does not exist, but there must be a connection in $G$ between vertices of $V_{L}$ and vertices of $V_{R}$, the chords $x$ and $y$ must exist and have to cross each other. Hence, $d_{G}(a, b)=d_{G}(a, y)+d_{G}(y, x)+d_{G}(x, b)=3$.


Fig. 7. Trees $T_{L}, T_{R}$ and $T_{U}$ connected via $N_{G}[a, b]$ to form a tree spanner of $G$.
Now we are ready to state the main result of this section.

## Theorem 5. Any connected 3-gon graph G admits an additive tree 6-spanner constructible in linear time.

Proof. We will create a spanning tree $T$ of $G$ from the trees $T_{U}, T_{L}$ and $T_{R}$ described in Propositions 4,7 and 8 as follows. Initially, $T$ is just the union of $T_{U}, T_{L}$ and $T_{R}$. We know that $\left\{\beta^{\ell}, \beta^{r}, \alpha^{u}\right\} \subseteq N_{G}(a)$ and $\left\{\beta^{\ell}, \beta^{r}, \beta^{u}\right\} \subseteq N_{G}(b)$. Make vertex $a$ adjacent to $\alpha^{u}$ and vertex $b$ adjacent to $\beta^{u}$ in $T$. Denote $M:=\left\{\alpha^{\ell}, \beta^{\ell}, \alpha^{r}, \beta^{r}\right\}$. If $M \subseteq N_{G}(a)$, then make vertex $a$ adjacent in $T$ to each vertex in $M$. If $M \backslash N_{G}(a) \neq \emptyset$ but $M \subseteq N_{G}(b)$, then make vertex $b$ adjacent in $T$ to each vertex in $M$. If neither $M \subseteq N_{G}(a)$ nor $M \subseteq N_{G}(b)$, then make vertices $\alpha^{\ell}, \beta^{\ell}$ adjacent in $T$ to a common neighbor in $\{a, b\}$ and vertices $\alpha^{r}, \beta^{r}$ adjacent in $T$ to a common neighbor in $\{a, b\}$. Remove from $T$ the edge ( $\alpha^{\ell}, \beta^{\ell}$ ) (it was a part of tree $T_{L}$ if both $\alpha^{\ell}$ and $\beta^{\ell}$ existed) and the edge ( $\alpha^{r}, \beta^{r}$ ) (it was a part of tree $T_{R}$ if both $\alpha^{r}$ and $\beta^{r}$ existed).

If $a$ and $b$ are adjacent in $G$, then add edge $(a, b)$ to $T$. If $a$ is not adjacent to $b$ in $G$ but $d_{G}(a, b)=2$, then choose a common neighbor $z$ of $a$ and $b$ in $N_{G}[a, b]$ and add edges $(a, z)$ and $(b, z)$ to $T$. In these cases, i.e., when $d_{G}(a, b) \leq 2$, remove the possible edge ( $\alpha^{u}, \beta^{u}$ ) from $T$ (it was a part of the tree $T_{U}$ if both $\alpha^{u}$ and $\beta^{u}$ existed). If $d_{G}(a, b)>2$ then, by the proof of Proposition 11, $d_{G}(a, b)=3$, chords $\beta^{\ell}, \beta^{r}$ do not exist and the edge ( $\alpha^{u}, \beta^{u}$ ) from $T_{U}$ goes to $T$ if both chords $\alpha^{u}$ and $\beta^{u}$ exist. If one of these chords does not exist, then there must be two vertices $x, y$ that are adjacent in $G$, with $x \in N_{G}(b)$ and $y \in N_{G}(a)$ (see the proof of Proposition 11), and we put the edge ( $x, y$ ) into $T$. Finally, make all vertices from $N_{G}(a) \backslash\left\{\alpha^{\ell}, \beta^{\ell}, \alpha^{u}, \beta^{u}, \alpha^{r}, \beta^{r}, b, z\right\}$ adjacent to $a$ in $T$ and all remaining vertices from $N_{G}(b)$ (i.e., those that are not adjacent to $a$ in $T$ ) adjacent to $b$; see Fig. 7 for an illustration. It is possible that $\alpha^{\ell}=\alpha^{u}, \alpha^{\ell}=\beta^{r}, \alpha^{r}=\beta^{u}, \alpha^{r}=\beta^{\ell}$ and $\alpha^{\ell}=\alpha^{r}$, but our assignment of vertices $\alpha^{\ell}, \beta^{\ell}, \alpha^{u}, \beta^{u}, \alpha^{r}, \beta^{r}$ to $a$ and $b$ in $T$ agrees with that since vertices of each pair are assigned to the same vertex from $\{a, b\}$. Clearly, $T$ constructed this way is a spanning tree of $G$. In what follows we show that $T$ is an additive tree 6 -spanner of $G$.

Consider any vertices $x, y \in V_{L}$. We have $d_{T}(x, y) \leq d_{T_{L}}(x, y)+1$, by construction of $T$, and $d_{T_{L}}(x, y) \leq d_{G}(x, y)+2$, by Proposition 7. Hence, $d_{T}(x, y) \leq d_{G}(x, y)+3$. The same inequality holds for $x, y \in V_{R}$.

Consider any vertices $x, y \in V_{U}$. We have $d_{T}(x, y) \leq d_{T_{U}}(x, y)+3$, by construction of $T$, and $d_{T_{U}}(x, y) \leq d_{G}(x, y)+2$, by Proposition 4. Hence, $d_{T}(x, y) \leq d_{G}(x, y)+5$.

For any two vertices $x, y \in N_{G}(a, b)$, clearly $d_{T}(x, y) \leq 1+d_{T}(a, b)+1 \leq 5$ by Proposition 11 and thus $d_{T}(x, y) \leq$ $d_{G}(x, y)+4$.

Now consider arbitrary vertices $x \in V_{L}$ and $y \in V_{R}$. It is easy to see that $d_{G}\left(x, N_{G}[a, b]\right)=d_{G}\left(x,\left\{\alpha^{\ell}, \beta^{\ell}\right\}\right)$ and, hence, $d_{G}(x, a)=d_{G}\left(x,\left\{\alpha^{\ell}, \beta^{\ell}\right\}\right)+1$. Assuming without loss of generality that $\alpha^{\ell}$ and $\beta^{\ell}$ are attached in $T$ to $a$ we know, by the construction of $T$, that $d_{T}(x, a)$ is equal either to $d_{T_{L}}\left(x, s^{\ell}\right)$ or to $d_{T_{L}}\left(x, s^{\ell}\right)+1$. Hence, by Propositions 7 and 9 , $d_{T}(x, a) \leq d_{T_{L}}\left(x, s^{\ell}\right)+1=d_{G}\left(x, s^{\ell}\right)+1 \leq d_{G}\left(x,\left\{\alpha^{\ell}, \beta^{\ell}\right\}\right)+2=d_{G}\left(x, N_{G}[a, b]\right)+2$. Moreover, $d_{T}(x, a)=d_{G}\left(x, N_{G}[a, b]\right)+2$ only if both $\alpha^{\ell}$ and $\beta^{\ell}$ exist (i.e., $d_{G}(a, b) \leq 2$ ). Similarly, assuming without loss of generality, that $\alpha^{r}$ and $\beta^{r}$ are attached in $T$ to $b$, we see that $d_{T}(y, b) \leq d_{T_{R}}\left(y, s^{r}\right)+1=d_{G}\left(y, s^{r}\right)+1 \leq d_{G}\left(y,\left\{\alpha^{r}, \beta^{r}\right\}\right)+2=d_{G}\left(y, N_{G}[a, b]\right)+2$. Now, $d_{T}(x, y) \leq d_{T}(x, a)+d_{T}(a, b)+d_{T}(y, b) \leq d_{G}\left(x, N_{G}[a, b]\right)+2+d_{T}(a, b)+d_{G}\left(y, N_{G}[a, b]\right)+2 \leq d_{G}(x, y)+6$, since $N_{G}[a, b]$ separates $V_{L}$ from $V_{R}$ and $d_{T}(a, b) \leq 2$ if $d_{T}(x, a)=d_{G}\left(x, N_{G}[a, b]\right)+2$.

For vertices $x \in V_{L}$ and $y \in N_{G}[a, b], d_{T}(x, y) \leq d_{T}(x, a)+d_{T}(a, y) \leq d_{G}\left(x, N_{G}[a, b]\right)+2+d_{T}(a, y) \leq d_{G}(x, y)+5$, since $d_{G}(x, y) \geq d_{G}\left(x, N_{G}[a, b]\right)$ and $d_{T}(a, y) \leq 3$ when $d_{T}(x, a)=d_{G}\left(x, N_{G}[a, b]\right)+2$ (i.e., when both $\alpha^{\ell}$ and $\beta^{\ell}$ exist and hence $d_{G}(a, b) \leq 2$ holds). Similarly, for vertices $x \in V_{R}$ and $y \in N_{G}[a, b]$, we have $d_{T}(x, y) \leq d_{G}(x, y)+5$.

Finally, consider arbitrary vertices $x \in V_{L}$ and $y \in V_{U}$ (the case when $x \in \bar{V}_{R}$ and $y \in V_{U}$ is similar). We know that $d_{T}(x, a) \leq d_{G}\left(x, N_{G}[a, b]\right)+2$. Let $w$ be a vertex from $\left\{\alpha^{u}, \beta^{u}\right\}$ such that $d_{T_{U}}(y, w)=d_{T_{U}}\left(y,\left\{\alpha^{u}, \beta^{u}\right\}\right)$. We have $d_{T}(y, w)=d_{T_{U}}(y, w) \leq d_{T_{U}}\left(y, s^{u}\right)=d_{G}\left(y, s^{u}\right)$ by Proposition 4. Since vertices $\alpha^{u}$ and $\beta^{u}$, if both exist, are adjacent in $G$, we also have $d_{G}\left(y, s^{u}\right) \leq d_{G}\left(y,\left\{\alpha^{u}, \beta^{u}\right\}\right)+1=d_{G}\left(y, N_{G}[x, y]\right)+1$. Now, $d_{T}(x, y) \leq d_{T}(x, a)+d_{T}(a, w)+d_{T}(y, w) \leq$ $d_{G}\left(x, N_{G}[a, b]\right)+2+d_{T}(a, w)+d_{G}\left(y, N_{G}[a, b]\right)+1 \leq d_{G}(x, y)+6$, since $N_{G}[a, b]$ separates $V_{L}$ from $V_{U}$ and $d_{T}(a, w) \leq 3$.

For vertices $x \in N_{G}[a, b]$ and $y \in V_{U}$, using $w$ as defined above, we see that $d_{T}(x, y) \leq d_{T}(y, w)+d_{T}(w, x) \leq$ $d_{G}\left(y, N_{G}[a, b]\right)+1+d_{T}(w, x) \leq d_{G}(x, y)+5$, since $d_{G}(x, y) \geq d_{G}\left(y, N_{G}[a, b]\right)$ and $d_{T}(w, x) \leq 4$.

## 6. Concluding remarks

In this paper, we examined the problem of finding "small" systems of collective additive tree $r$-spanners or sparse additive $r$-spanners for small values of $r$ on circle graphs and on polygonal graphs.

We demonstrated that every $n$-vertex circle graph $G$ admits a system of at most $2 \log _{\frac{3}{2}} n$ collective additive tree 2-spanners, and complemented this by two lower bound results stating that, for any constant $c$, there are circle graphs that cannot be collectively $+c$ spanned by any constant number of spanning trees, and that there are circle graphs on $n$ vertices for which any system of collective additive tree 1 -spanners will require $\Omega(n)$ spanning trees. Additionally, we showed that every $n$-vertex circle graph admits an additive 2 -spanner with at $\operatorname{most} O(n \log n)$ edges. It is interesting to know also whether there is a constant integer $c$ such that every $n$-vertex circle graph admits an additive $c$-spanner with at most $O(n)$ edges.

For $n$-vertex $k$-polygonal graphs, we showed that each such graph admits a system of at most $2 \log _{\frac{3}{2}} k+7$ collective additive tree 2 -spanners and an additive $(k+O(1))$-spanner with at most $O(n)$ edges. It is interesting to know also whether there is a constant integer $c$ (independent of $k$ ) such that every $n$-vertex $k$-polygonal graph admits an additive $c$-spanner with at most $O(n)$ edges.

For 3-gon graphs, we showed that each such graph admits a system of at most three collective additive tree 2 -spanners and an additive tree 6 -spanner. Using ideas from Section 5 , we can show also that any connected $n$-vertex 3 -gon graph $G$ admits an additive 3 -spanner with at most $2 n-2$ edges constructible in linear time. For each of the three corners of $G$, we consider a permutation graph $G_{i}(i \in\{L, R, U\})$. Then, using the algorithm mentioned in Lemma 1 , we construct two tree 2-spanners of $G_{i}$ (which are special BFS-trees started at two particular extreme vertices $u$ and $v$ of $G_{i}$ and built from the "center" of the 3-gon graph $G$ towards the corresponding corner). These trees preserve distances from any vertex of $G_{i}$ to those extreme vertices $u$ and $v$. Further, we take the union of the resulting six BFS-trees (two for each corner) and add some more edges from the "center" of $G$ to obtain a sparse 3 -spanner for the entire graph $G$. We leave details to the reader. Here we state only the result.

Theorem 6. Any connected 3 -gon graph $G$ admits an additive 3 -spanner with at most $2 n-2$ edges constructible in linear time.
It seems to be an interesting open problem whether every 3 -gon graph admits an additive tree 4 -spanner.

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[^0]:    This work was supported by the European Regional Development Fund (ERDF) and by Canada's Natural Sciences and Engineering Research Council (NSERC). Results of this paper were partially presented at the WG'08 conference [11].

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